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New oscillation theorems for second order quasi-linear difference equations with sub-linear neutral term

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Abstract

In this paper, the authors obtain some new sufficient conditions for the oscillation of all solutions of the second order neutral difference equation

$$\Delta(a_n(\Delta z_n)^{\beta}) + q_n x_{n-\ell}^{\gamma} = 0, \quad n \geq n_0,$$

where $z_n = x_n + p_n x_{n-k}^{\alpha}$. The established results extend, unify and improve some of the results reported in the literature. Examples are provided to illustrate the importance of the main results.

MSC: 39A10

Keywords: Oscillation; Quasi-linear difference equations; Sub-linear neutral term

1 Introduction

Consider a quasi-linear neutral delay difference equation of the form

$$\Delta\big(a_n(\Delta z_n)^{\beta}\big) + q_n x_{n-\ell}^{\gamma} = 0, \quad n \in \mathbb{N}(n_0), \tag{1.1}$$

where $z_n = x_n + p_n x_{n-k}^{\alpha}$, and $\mathbb{N}(n_0) = \{n_0, n_0 + 1, \ldots\}$, n_0 is a non-negative integer, subject to the following conditions:

(H_1) $\{a_n\}$ is a positive real sequence such that $\sum_{n=n_0}^{\infty} \frac{1}{a_n^{1/\beta}} = \infty$;

(H_2) $\{p_n\}$ and $\{q_n\}$ are positive real sequences for all $n \in \mathbb{N}(n_0)$ and $p_n \to 0$ as $n \to \infty$;

(H_3) k and ℓ are positive integers;

(H_4) $\alpha \in (0, 1]$, β and γ are ratio of odd positive integers.

Let $\theta = \max\{k, \ell\}$. By a solution of Eq. (1.1) we mean a real sequence $\{x_n\}$ defined for $n \geq n_0 - \theta$ and satisfying Eq. (1.1) for all $n \in \mathbb{N}(n_0)$. As usual, a nontrivial solution of Eq. (1.1) is said to be oscillatory if the terms of the sequence are neither eventually positive nor eventually negative and nonoscillatory otherwise.

Neutral type equations arise in a number of important applications in natural sciences and technology; see [7, 13]. Hence, in recent years there has been great interest in studying the oscillation of such type of equations. From the review of literature, one can see that many oscillation results are available for the equation when $\alpha = 1$; see [1, 2, 5, 8–11, 14, 15, 18, 20], and the references cited therein. Also few results available for the oscillation of Eq. (1.1) while $\beta = 1$; see [4, 12, 17, 19, 21, 22]. And as far as the authors knowledge there are no results available in the literature for the oscillatory behavior of Eq. (1.1).

Our purpose in this paper is to establish some new oscillation criteria for Eq. (1.1) which includes many of the known results as special cases when $\alpha = 1$ or $\alpha = 1$ and $\beta = 1$ in Eq. (1.1). Further the methods used in this paper improve and extend some of the known results that are reported in the literature [3, 8–12, 14, 15, 17–21] and this is almost illustrated via examples.

2 Oscillation results

In this section, we obtain sufficient conditions for the oscillation of all solutions of Eq. (1.1). Due to the assumptions and the form of our equation, we need only to give proofs for the case of eventually positive solution since the proofs for eventually negative solutions would be similar.

For convenience, for any real positive sequence $\{\mu_n\}$ which is decreasing to zero, we set

$$\begin{aligned}
B_n &= \left(1 - p_n \mu_n^{\alpha-1}\right),\\
Q_n &= q_n B_{n-\ell}^{\gamma},\\
R_n &= \sum_{s=n_1}^{n-1} a_s^{-1/\beta},\\
\overline{R}_n &= R_n + \frac{1}{\beta}\sum_{s=n_1}^{n-1} Q_s R_{s+1} R_{s-l}^{\beta} \mu_{s-\ell}^{\gamma-\beta}
\end{aligned}$$

and

$$C_n = \frac{R_{n-\ell}}{R_n}$$

for $n \geq n_1$, where $n_1 \in \mathbb{N}(n_0)$ is large enough.

Lemma 2.1 *Let $\{x_n\}$ be a positive solution of Eq. (1.1) for all $n \in \mathbb{N}(n_0)$. Then there exists a $n_1 \in \mathbb{N}(n_0)$ such that for all $n \geq n_1$*

$$z_n > 0, \qquad \Delta z_n > 0, \qquad \Delta\left(a_n(\Delta z_n)^{\beta}\right) \leq 0. \tag{2.1}$$

Proof The proof of the lemma can be found in [3] and hence details are omitted. □

Lemma 2.2 *Let $\{x_n\}$ be a positive solution of Eq. (1.1) for all $n \in \mathbb{N}(n_0)$ and suppose Eq. (2.1) holds. Then there exists a $n_1 \in \mathbb{N}(n_0)$ such that*

$$z_n \geq R_n a_n^{1/\beta} \Delta z_n, \quad n \geq n_1, \tag{2.2}$$

and

$$\left\{\frac{z_n}{R_n}\right\} \text{ is decreasing for } n \geq n_1. \tag{2.3}$$

Proof From (2.1), we see that $a_n^{1/\beta}\Delta z_n$ is decreasing and therefore

$$z_n \geq \sum_{s=n_1}^{n-1} \frac{a_{s+1}^{1/\beta}\Delta z_{s+1}}{a_s^{1/\beta}} \geq R_n a_n^{1/\beta}\Delta z_n.$$

Further, from the last in equality, we have

$$\Delta\left(\frac{z_n}{R_n}\right) \leq 0, \quad t \geq t_1,$$

and so $\frac{z_n}{R_n}$ is decreasing for all $n \geq n_1$. This proof is now complete. □

Lemma 2.3 *Assume that, for large n,* $(p_n, p_{n+1}, \ldots, p_{n+k-1}) \not\equiv 0$. *Then*

$$\Delta x_n + p_n x_{n-\ell}^{\alpha} = 0, \quad n = 0, 1, 2, \ldots$$

has an eventually positive solution if and only if the corresponding inequality

$$\Delta x_n + p_n x_{n-\ell}^{\alpha} \leq 0, \quad n = 0, 1, 2, \ldots$$

has an eventually positive solution.

Proof The proof of the lemma can be found in [21] and hence details are omitted. □

Lemma 2.4 *If* $0 < \alpha < 1, \ell$ *is a positive integer and* $\{p_n\}$ *is a positive real sequence with* $\sum_{n=n_0}^{\infty} p_n = \infty$, *then every solution of eqution* $\Delta x_n + p_n x_{n-\ell}^{\alpha} = 0$, *is oscillatory.*

Lemma 2.5 *If* $\alpha > 1$. *If there exists a* $\lambda > \frac{1}{l}\ln\alpha$ *such that* $\lim_{n\to\infty}\inf[p_n \exp(-e^{\lambda n})] > 0$, *then every solution of eqution* $\Delta x_n + p_n x_{n-\ell}^{\alpha} = 0$ *is oscillatory.*

The proof of the Lemmas 2.4 and 2.5 can be found in [16] and hence details are omitted. Next we state and prove some new oscillation results for Eq. (1.1).

Theorem 2.1 *Let* $\gamma \geq \beta$ *be holds. Assume that there exists a positive real sequence* $\{\mu_n\}$ *tending to zero such that* $B_n > 0$ *for all* $n \in \mathbb{N}(n_0)$. *If the first order delay difference equation*

$$\Delta w_n + Q_n \overline{R}_n w_{n-\ell}^{\gamma/\beta} = 0 \tag{2.4}$$

is oscillatory, then every solution of Eq. (1.1) *is oscillatory.*

Proof Let $\{x_n\}$ be a positive solution of Eq. (1.1) for all $n \in \mathbb{N}(n_0)$. Then there exists a $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0, x_{n-k} > 0$ and $x_{n-\ell} > 0$ for all $n \geq n_1$. By Lemma 2.1, the sequence $\{z_n\}$ satisfies conditions (2.1) for all $n \geq n_1$. From the definition of z_n we have

$$x_n = z_n - p_n x_{n-k}^{\alpha} \geq z_n - p_n z_{n-\ell}^{\alpha}. \tag{2.5}$$

Since $\{z_n\}$ is increasing and $\{\mu_n\}$ is positive, decreasing and tending to zero, we have $z_n \geq \mu_n$ for all $n \geq n_1$. Using this and $0 < \alpha \leq 1$ in (2.5), one obtains

$$x_n \geq B_n z_n,$$

which together with Eq. (1.1)

$$\Delta\big(a_n(\Delta z_n)^\beta\big) \le -Q_n z_{n-\ell}^\gamma. \tag{2.6}$$

Now a simple computation shows that

$$\Delta\big(z_n - R_n a_n^{1/\beta}\Delta z_n\big) = -R_{n+1}\Delta\big(a_n^{1/\beta}\Delta z_n\big). \tag{2.7}$$

By the discrete mean value theorem [1, Theorem 1.7.2], we have

$$\begin{aligned}\Delta\big(a_n(\Delta z_n)^\beta\big) &= \big(a_{n+1}^{\frac{1}{\beta}}\Delta z_{n+1}\big)^\beta - \big(a_n^{\frac{1}{\beta}}\Delta z_n\big)^\beta \\ &\ge \beta\frac{a_n(\Delta z_n)^\beta}{a_{n+1}^{1/\beta}\Delta z_{n+1}}\Delta\big(a_n^{\frac{1}{\beta}}\Delta z_n\big),\end{aligned} \tag{2.8}$$

where we have used $a_n^{\frac{1}{\beta}}\Delta z_n$ is positive and decreasing. Now from (2.6), (2.7) and (2.8), one obtains

$$\begin{aligned} z_n &\ge R_n a_n^{\frac{1}{\beta}}\Delta z_n + \frac{1}{\beta}\sum_{s=n_1}^{n-1}\frac{R_{s+1}Q_s z_{s-l}^\gamma a_{s+1}^{\frac{1}{\beta}}\Delta z_{s+1}}{a_s(\Delta z_s)^\beta} \\ &\ge a_n^{\frac{1}{\beta}}\Delta z_n\left(R_n + \frac{1}{\beta}\sum_{s=n_1}^{n-1}\frac{R_{s+1}Q_s z_{s-l}^\gamma}{a_s(\Delta z_s)^\beta}\right),\end{aligned} \tag{2.9}$$

where we have used $a_n^{\frac{1}{\beta}}\Delta z_n$ is positive and decreasing. From Lemma 2.2 we have

$$\frac{z_{n-l}}{R_{n-l}} \ge \frac{z_n}{R_n} \ge a_n^{\frac{1}{\beta}}\Delta z_n, \quad n \ge n_1. \tag{2.10}$$

Substituting (2.10) in (2.9), we obtain

$$z_n \ge a_n^{\frac{1}{\beta}}\Delta z_n\left(R_n + \frac{1}{\beta}\sum_{s=n_1}^{n-1}R_{s+1}R_{s-l}^\beta Q_s z_{s-l}^\gamma\right). \tag{2.11}$$

Since $\gamma \ge \beta$, we have $z_n^{\gamma-\beta} \ge \mu_n^{\gamma-\beta}$ for all $n \ge n_1$, and using this in (2.11), one obtains

$$z_{n-l}^\gamma \ge \bar{R}_{n-l}\big(a_{n-l}^{1/\beta}\Delta z_{n-l}\big)^\gamma, \quad n \ge n_1.$$

Using (2.11) in (2.6), and in view of (2.1), one can see that $w_n = a_n(\Delta z_n)^\beta$ is a positive solution of the first order delay difference inequality

$$\Delta w_n + Q_n\overline{R}_{n-\ell}^\gamma w_{n-\ell}^{\gamma/\beta} \le 0. \tag{2.12}$$

But by Lemma 2.3, the associated difference equation

$$\Delta w_n + Q_n\overline{R}_{n-\ell}^\gamma w_{n-\ell}^{\gamma/\beta} = 0$$

also has a positive solution, which is a contradiction. Hence we complete the proof. □

Corollary 2.2 *Let all conditions of Theorem 2.1 hold with $\gamma = \beta$ for all $n \in \mathbb{N}(n_0)$. If*

$$\lim_{n\to\infty} \inf \sum_{s=n-\ell}^{n-1} Q_s \overline{R}_{s-\ell}^{\gamma} > \left(\frac{\ell}{\ell+1}\right)^{\ell+1} \tag{2.13}$$

then every solution of Eq. (1.1) is oscillatory.

Proof The proof follows from Theorem 2.1 and Theorem 7.6.1 of [6]. □

Corollary 2.3 *Let all conditions of Theorem 2.1 hold with $\gamma > \beta$ for all $n \in \mathbb{N}(n_0)$. If $\ell > k$ and there exists a $\lambda > \frac{1}{\ell-k} \ln \frac{\gamma}{\beta}$ such that*

$$\lim_{n\to\infty} \inf\bigl[Q_n \overline{R}_{n-l}^{\gamma} \exp\bigl(-e^{\lambda n}\bigr)\bigr] > 0, \quad n \geq n_1. \tag{2.14}$$

Then every solution of Eq. (1.1) is oscillatory.

Proof The proof follows from Theorem 2.1 and Lemma 2.5. □

Theorem 2.4 *Let $\gamma < \beta$ be holds. Assume that there exists a positive decreasing real sequence $\{\mu_n\}$ tending to zero such that $B_n > 0$ for all $n \in \mathbb{N}(n_0)$. If for all $N \geq n_0$,*

$$\sum_{n=N}^{\infty} Q_n \left(R_{n-\ell} + \frac{M^{\gamma-\beta}}{\beta} \sum_{s=n_1}^{n-\ell-1} Q_s R_{s+1} R_{s-\ell}^{\gamma}\right)^{\gamma} = \infty \tag{2.15}$$

for any constant $M > 0$, then every solution of Eq. (1.1) is oscillatory.

Proof Assume that Eq. (1.1) has a positive solution such that there exists a $n_1 \in \mathbb{N}(n_0)$ with $x_n > 0, x_{n-k} > 0$ and $x_{n-\ell} > 0$ for all $n \geq n_1$. Proceeding as in the proof of Theorem 2.1 we have

$$z_n \geq a_n^{\frac{1}{\beta}} \Delta z_n \left(R_n + \frac{1}{\beta} \sum_{s=n_1}^{n-1} Q_s R_{s+1} R_{s-l}^{\beta} z_{s-\ell}^{\gamma-\beta}\right). \tag{2.16}$$

Since z_n/R_n is decreasing, there exists a constant $M > 0$ such that $z_n/R_n \leq M$ for all $n \geq n_1$, and from $\gamma < \beta$, we have $z_{n-\ell}^{\gamma-\beta} \geq M^{\gamma-\beta} R_{n-\ell}^{\gamma-\beta}$ for all $n \geq n_1$. Using this inequality in (2.16), we obtain

$$z_{n-\ell}^{\gamma} \geq \bigl(a_{n-\ell}(\Delta z_{n-\ell})^{\beta}\bigr)^{\gamma/\beta} \left(R_{n-\ell} + \frac{M^{\gamma-\beta}}{\beta} \sum_{s=n_1}^{n-\ell-1} Q_s R_{s+1} R_{s-\ell}^{\gamma}\right)^{\gamma}, \quad n \geq n_1.$$

Using the last inequality in (2.6) and set $w_n = a_n(\Delta z_n)^{\beta} > 0$, we have

$$\Delta w_n + Q_n \left(R_{n-\ell} + \frac{M^{\gamma-\beta}}{\beta} \sum_{s=n_1}^{n-\ell-1} Q_s R_{s+1} R_{s-\ell}^{\gamma}\right)^{\gamma} w_{n-\ell}^{\gamma/\beta} \leq 0.$$

But by Lemma 2.3, the associated difference equation

$$\Delta w_n + Q_n \left(R_{n-\ell} + \frac{M^{\gamma-\beta}}{\beta} \sum_{s=n_1}^{n-\ell-1} Q_s R_{s+1} R_{s-\ell}^{\gamma}\right)^{\gamma} w_{n-\ell}^{\gamma/\beta} = 0 \tag{2.17}$$

also has a positive solution. But Lemma 2.4 and condition (2.15) imply that Eq. (2.17) is oscillatory. This contradiction completes the proof. □

In the following by employing the Riccati substitution technique, we obtain new oscillation criteria for Eq. (1.1).

Theorem 2.5 *Let $\gamma \geq \beta$ hold. Assume that there exists a positive decreasing real sequence $\{\mu_n\}$ tending to zero, such that $B_n > 0$ for all $n \in \mathbb{N}(n_0)$. If there exists a positive, nondecreasing a real sequence $\{\rho_n\}$ such that*

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n} \left(\rho_s Q_s C_s^{\gamma} \mu_s^{\gamma-\beta} - \frac{a_s(\Delta\rho_s)^{1+\beta}}{(\beta+1)^{\beta+1}\rho_s^{\beta}} \right) = \infty, \tag{2.18}$$

for sufficiently large $N > n_1$, then every solution of Eq. (1.1) is oscillatory.

Proof Let $\{x_n\}$ be a positive solution of Eq. (1.1) for all $n \in \mathbb{N}(n_0)$. Then there exists a $n_1 \in \mathbb{N}(n_0)$ such that $x_n > 0, x_{n-k} > 0$ and $x_{n-\ell} > 0$ for all $n \geq n_1$. Then, by Lemma 2.1, z_n satisfies conditions (2.1) for all $n \geq n_1$. Define the Riccati transformation by

$$w_n = \rho_n a_n \left(\frac{\Delta z_n}{z_n} \right)^{\beta}, \quad n \geq n_1. \tag{2.19}$$

Then $w_n > 0$, for all $n \geq n_1$, and

$$\Delta w_n = \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} + \rho_n \frac{\Delta(a_n(\Delta z_n)^{\beta})}{z_n^{\beta}} - \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z_n^{\beta}}{z_n^{\beta}}, \quad n \geq n_1. \tag{2.20}$$

By the discrete mean value theorem, we have

$$\Delta z_n^{\beta} = z_{n+1}^{\beta} - z_n^{\beta} = \beta \frac{z_n^{\beta}\Delta z_n}{z_{n+1}}, \tag{2.21}$$

where we have used z_n is positive and increasing. Using (2.21) in (2.20), we obtain

$$\begin{aligned}
\Delta w_n &\leq \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \beta \frac{\rho_n}{\rho_{n+1}} w_{n+1} \frac{\Delta z_n}{z_{n+1}} - \rho_n Q_n \frac{z_{n-\ell}^{\gamma}}{z_n^{\beta}} \\
&\leq \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \beta \frac{\rho_n}{\rho_{n+1}} \frac{w_{n+1}}{a_n^{1/\beta}} \frac{a_n^{1/\beta}\Delta z_n}{z_{n+1}} - \rho_n Q_n \frac{z_{n-\ell}^{\gamma}}{z_n^{\beta}} \\
&\leq \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \beta \frac{\rho_n}{\rho_{n+1} a_n^{1/\beta}} w_{n+1} \frac{a_{n+1}^{1/\beta}\Delta z_{n+1}}{z_{n+1}} - \rho_n Q_n \frac{z_{n-\ell}^{\gamma}}{z_n^{\beta}},
\end{aligned}$$

where we have used $a_n^{1/\beta}\Delta z_n$ is positive and decreasing. Using (2.19) in the last inequality, we obtain

$$\Delta w_n \leq \frac{\Delta\rho_n}{\rho_{n+1}} w_{n+1} - \beta \frac{\rho_n}{\rho_{n+1}^{1+1/\beta} a_n^{1/\beta}} w_{n+1}^{1+1/\beta} - \rho_n Q_n \frac{z_{n-\ell}^{\gamma}}{z_n^{\beta}}. \tag{2.22}$$

From (2.3) we have

$$\frac{z_{n-\ell}}{R_{n-\ell}} \geq \frac{z_n}{R_n}$$

or

$$z_{n-\ell} \geq \frac{R_{n-\ell}}{R_n} z_n$$

and using this in (2.21) yields

$$\Delta w_n \leq \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta \rho_n}{\rho_{n+1}^{1+\frac{1}{\beta}} a_n^{\frac{1}{\beta}}} w_{n+1}^{1+\frac{1}{\beta}} - \rho_n Q_n C_n^\gamma \mu_n^{\gamma-\beta}, \tag{2.23}$$

where we have used $\gamma \geq \beta$ and $z_n \geq \mu_n$, for all $n \geq n_1$. Letting $A = \frac{\Delta \rho_n}{\rho_{n+1}}$ and $B = \frac{\beta \rho_n}{\rho_{n+1}^{1+\frac{1}{\beta}} a_n^{\frac{1}{\beta}}}$ and using the inequality given in Lemma 2.6 of [15], it follows from (2.23) that

$$\Delta w_n \leq -\rho_n Q_n C_n^\gamma \mu_n^{\gamma-\beta} + \frac{a_n (\Delta \rho_n)^{\beta+1}}{(\beta+1)^{\beta+1} \rho_n^\beta}. \tag{2.24}$$

Let $N \geq n_1$ be sufficiently large and summing (2.24) from N to n, we obtain

$$\sum_{s=N}^{n} \left[\rho_s Q_s C_s^\gamma \mu_s^{\gamma-\beta} - \frac{a_s (\Delta \rho_s)^{\beta+1}}{(\beta+1)^{\beta+1} \rho_s^\beta} \right] \leq w_N,$$

which contradicts (2.18). This completes the proof. □

Theorem 2.6 *Let $\gamma < \beta$ be holds. Assume that there exists a positive, nondecreasing real sequence $\{\mu_n\}$ tending to zero, such that $B_n > 0$ for all $n \in \mathbb{N}(n_0)$. If there exists a positive, nondecreasing real sequence $\{\rho_n\}$ such that, for some sufficiently large $N \geq n_1$,*

$$\lim_{n\to\infty} \sup \sum_{s=N}^{n} \left(\rho_s Q_s C_s^\gamma \mu_s^{\gamma-\beta} - \frac{M^{\beta-\gamma} a_s (\Delta \rho_s)^{1+\beta}}{(\beta+1)^{\beta+1} \rho_s^\beta} \right) = \infty$$

for any constant $M > 0$, then every solution of Eq. (1.1) is oscillatory.

Proof The proof is similar to that of Theorem 2.5 except the inequality (2.23) is replaced by

$$\Delta w_n \leq \frac{\Delta \rho_n}{\rho_{n+1}} w_{n+1} - \frac{\beta \rho_n}{\rho_{n+1}^{1+\frac{1}{\beta}} a_n^{\frac{1}{\beta}}} w_{n+1}^{1+\frac{1}{\beta}} - M^{\gamma-\beta} \rho_n Q_n C_n^\gamma R_n^{\gamma-\beta},$$

where we have used $\frac{z_n}{R_n} \leq M$, for all $n \geq n_1$ and $\gamma < \beta$, and hence the details are omitted. This completes the proof. □

3 Examples

In this section, we present three examples to illustrate the main results.

Example 3.1 Consider the second order neutral difference equation

$$\Delta\big((\Delta z_n)^3\big) + \frac{q_0}{n^3}x_{n-1}^3 = 0, \quad n \geq 1, \tag{3.1}$$

where $z_n = x_n + \frac{1}{2n^{\frac{2}{3}}}x_{n-2}^{\frac{1}{3}}$ and $q_0 > 0$. Comparing with Eq. (1.1), we have $a_n = 1, p_n = \frac{1}{2n^{\frac{2}{3}}}, q_n = \frac{q_0}{n^3}$, $\ell = 1, k = 2, \alpha = \frac{1}{3}$, and $\beta = \gamma = 3$. A simple calculation yields $R_n = n - 1$. By choosing $\mu_n = \frac{1}{n^{\frac{2}{3}}}$, we see that $Q_n = \frac{q_0}{8n^3}$ and $\overline{R}_n = (n-1) + \frac{q_0}{96n^2}(n^2-5n+8)(n^2-5n+4)$. The condition (2.13) becomes

$$\begin{aligned}
&\liminf_{n\to\infty} \sum_{s=n-1}^{n-1} \frac{q_0}{8s^3}\left[s - 2 + \frac{q_0}{96(s-1)^2}\big(s^2 - 7s + 14\big)\big(s^2 - 7s + 10\big)\right]^3 \\
&\quad = \liminf_{n\to\infty} \frac{q_0}{8}\left[1 - \frac{2}{n-1} + \frac{q_0}{96(n-2)^2}\big(n^2 - 9n + 22\big)\big(n^2 - 9n + 18\big)\right]^3 = \infty
\end{aligned}$$

and therefore by Corollary 2.2, we see that every solution of Eq. (3.1) is oscillatory.

Example 3.2 Consider the second order neutral difference equation

$$\Delta\big((\Delta z_n)^3\big) + \frac{q_0}{n}x_{n-1}^5 = 0, \quad n \geq 1, \tag{3.2}$$

where $z_n = x_n + \frac{1}{3n^{\frac{2}{3}}}x_{n-2}^{\frac{1}{3}}$ and $q_0 > 0$. Compared with Eq. (1.1), we have $a_n = 1$, $p_n = \frac{1}{3n^{\frac{2}{3}}}$, $q_n = \frac{q_0}{n}$, $\ell = 1, k = 2, \alpha = \frac{1}{3}, \beta = 3$ and $\gamma = 5$. Simple calculation shows that $R_n = n - 1$. By choosing $\mu_n = \frac{1}{n^{\frac{2}{3}}}$, we see that $Q_n = \frac{32q_0}{243n}$ and $C_n = \frac{n-2}{n-1}$. By taking $\rho_n = n^2$, the condition (2.18) becomes

$$\limsup_{n\to\infty} \sum_{s=N}^{n}\left(\frac{32}{243}\frac{q_0}{s^{\frac{1}{3}}}\left(\frac{s-2}{s-1}\right)^5 - \frac{(2s+1)^4}{256s^6}\right) = \infty$$

and hence by Theorem 2.5, every solution of Eq. (3.2) is oscillatory.

Example 3.3 Consider the second order neutral difference equation

$$\Delta\big((\Delta z_n)^3\big) + \frac{q_0}{n}x_{n-1} = 0, \quad n \geq 1, \tag{3.3}$$

where $z_n = x_n + \frac{p_0}{n^{\frac{2}{3}}}x_{n-2}^{\frac{1}{3}}$, and $p_0 \in [0,1)$ and $q_0 > 0$. Comparing with Eq. (1.1), we have $a_n = 1, p_n = \frac{p_0}{n^{2/3}}, q_n = \frac{q_0}{n}, \ell = 1, k = 2, \alpha = \frac{1}{3}, \beta = 3$, and $\gamma = 1$. Simple calculation shows that $R_n = n - 1$. By taking $\mu_n = \frac{1}{n}$, we have $Q_n = \frac{q_0}{n}(1 - p_0)$. The condition (2.14) becomes

$$\sum_{n=N}^{\infty} \frac{q_0}{n}(1-p_0)\left(n - 2 + \frac{q_0(1-p_0)}{3M^2}\sum_{s=3}^{n-2}(s-2)\right) \geq \sum_{n=N}^{\infty}(1-p_0)q_0\frac{(n-2)}{n} = \infty$$

and hence by Theorem 2.4, every solution of Eq. (3.3) is oscillatory.

4 Conclusion

In this paper, by using a Riccati type transformation and the discrete mean value theorem we have established some new oscillation criteria for more general second order neutral difference equations. The obtained results include similar results to the ones established for second order difference equations with linear neutral terms or nonlinear neutral terms reported in the literature. Further none of the results in the papers [3–5, 8–12, 14, 15, 17–22] can be applied to Eqs. (3.1) to (3.3) to yield any conclusion.

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