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# Fujita-type theorems for a class of coupled semilinear parabolic systems with gradient terms

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**Abstract**

This paper concerns the asymptotic behavior of the solution to a class of coupled semilinear parabolic systems with gradient terms. The Fujita-type blow-up theorems are established and the critical Fujita curve is determined not only by the behavior of the coefficients of the gradient term and the source terms at infinity, but also by the spacial dimension.

**MSC:** 35B33; 35K20; 35K58

**Keywords:** Critical Fujita curve; Semilinear parabolic system; Large time behavior; Gradient term

## 1 Introduction

In this paper, we deal with the following Cauchy problem of the coupled semilinear parabolic system:

$$\frac{\partial u}{\partial t} = \Delta u + b(|x|)x \cdot \nabla u + (|x|+1)^{\lambda_1} v^p, \quad x \in \mathbb{R}^n, t > 0, \tag{1.1}$$

$$\frac{\partial v}{\partial t} = \Delta v + b(|x|)x \cdot \nabla v + (|x|+1)^{\lambda_2} u^q, \quad x \in \mathbb{R}^n, t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \quad x \in \mathbb{R}^n, \tag{1.3}$$

where $p, q > 1$, $\lambda_1, \lambda_2 \geq 0$, $u_0, v_0 \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ are nonnegative nontrivial, $b \in C^1([0,+\infty))$ satisfies

$$\lim_{s\to+\infty} s(s+1)b(s) = \kappa \quad (-\infty \leq \kappa \leq +\infty), \tag{1.4}$$

and additionally, in the case that $-n < \kappa \leq +\infty$, $b$ also satisfies

$$\kappa_0 = \inf\{s(s+1)b(s) : s > 0\} > -n. \tag{1.5}$$

It was Fujita [1] who first proved that the Cauchy problem of the semilinear equation

$$\frac{\partial u}{\partial t} = \Delta u + u^p, \quad x \in \mathbb{R}^n, t > 0 \tag{1.6}$$

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admits no nonnegative nontrivial global solutions if $1 < p < p_c = 1 + 2/n$, whereas it admits both nontrivial global (with small initial data) and non-global nonnegative (with large initial data) solutions if $p > p_c$. Later, the fact that the critical case $p = p_c$ belongs to the blow-up case was shown in [2–4]. From then on, many mathematicians have focused on the extensions of Fujita's results (see, e.g., [5–23] and the references therein).

Studies on equations with a gradient term are relatively rich. Meier [5] investigated the Cauchy problem of

$$\frac{\partial u}{\partial t} = \Delta u + \mathbf{b}(x) \cdot \nabla u + u^p, \quad x \in \mathbb{R}^n, t > 0, \tag{1.7}$$

with $\mathbf{b} \in L^\infty(\mathbb{R}^n; \mathbb{R}^n)$ and proved that the critical Fujita exponent is

$$p_c = 1 + \frac{1}{\omega^*},$$

where $\omega^*$ is the maximal decay rate for solutions to (1.7) without $u^p$. For constant vector $\mathbf{b}$, $\omega^* = n/2$, while for nonconstant vector $\mathbf{b}$, $\omega^*$ is unknown generally. Nevertheless, there are still some results for some special nonconstant vectors $\mathbf{b}$. In [20], Zheng et al. studied the Neumann exterior problem for (1.7) with

$$\mathbf{b}(x) = \frac{\kappa}{|x|^2} x, \quad x \in \mathbb{R}^n \ (-\infty < \kappa < +\infty)$$

and formulated its critical Fujita exponent as

$$p_c = \begin{cases} +\infty, & -\infty \le \kappa \le -n, \\ 1 + \frac{2}{n+\kappa}, & -n < \kappa < +\infty, \\ 1, & \kappa = +\infty. \end{cases} \tag{1.8}$$

Also, it was shown in [22] that the critical Fujita exponent to the Cauchy problem for (1.7) with $\mathbf{b}(x) = b(|x|)x$ is still (1.8), where $b$ satisfies (1.4) and (1.5). A more general case that the coefficients of the derivative of $u$ with respect to time $t$ and source term depend on spatial position was considered in [23]. For more studies about the quasilinear equations with gradient terms, one can see [14, 18, 20], etc.

The results of Fujita type for coupled systems are also fairly rich. In 1991, Escobedo et al. [6] formulated the critical Fujita curve for (1.1)–(1.3) with $b \equiv 0$ and $\lambda_1 = \lambda_2 = 0$ as follows:

$$(pq)_c = 1 + \frac{2}{n} \max\{p + 1, q + 1\}. \tag{1.9}$$

Moreover, Guo [9] studied the Neumann exterior problem of the system

$$\frac{\partial u}{\partial t} = \Delta u + \frac{\kappa}{|x|^2} x \cdot \nabla u + |x|^\lambda v^p, \quad x \in \mathbb{R}^n \setminus B_1, t > 0, \tag{1.10}$$

$$\frac{\partial v}{\partial t} = \Delta v + \frac{\kappa}{|x|^2} x \cdot \nabla v + |x|^\lambda u^q, \quad x \in \mathbb{R}^n \setminus B_1, t > 0 \tag{1.11}$$

with $\kappa \in \mathbb{R}$, $\lambda \geq 0$, and proved that the corresponding critical Fujita curve is

$$(pq)_c = \begin{cases} +\infty, & \kappa \leq -n, \\ 1 + \frac{2+\lambda}{n+\kappa} \max\{p+1, q+1\}, & \kappa > -n. \end{cases}$$

Na et al. [24] showed that the critical Fujita curve for problem (1.1)–(1.3) with $\lambda_1 = \lambda_2 = 0$ and nonnegative $b$ is

$$(pq)_c = \begin{cases} 1 + \frac{2}{n+\kappa} \max\{p+1, q+1\}, & 0 \leq \kappa < +\infty, \\ 1, & \kappa = +\infty. \end{cases}$$

There are also some results about coupled parabolic systems involving time-weighted sources. For example, Cao et al. [25] investigated the Cauchy problem of the following systems:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f_1(t) v^p, \quad x \in \mathbb{R}^n, t > 0, \\ \frac{\partial v}{\partial t} &= \Delta v + f_2(t) u^q, \quad x \in \mathbb{R}^n, t > 0, \end{aligned}$$

where $f_i(t) \in C^\mu([0,+\infty))$, $f_i(t) \sim t^{\sigma_i} (t \to +\infty)$ with $\sigma_i \in \mathbb{R}$, $i = 1, 2$, and showed that the critical Fujita curve is

$$(pq)_c = 1 + \frac{2}{n} \max\bigl\{(\sigma_2 + 1)p + (\sigma_1 + 1), (\sigma_1 + 1)q + (\sigma_2 + 1)\bigr\}.$$

In this paper, we formulate the critical Fujita curve to problem (1.1)–(1.3) as follows:

$$(pq)_c = \begin{cases} +\infty, & -\infty \leq \kappa \leq -n, \\ 1 + \frac{1}{n+\kappa} \max\{(2+\lambda_1) + p(2+\lambda_2), & \\ \quad q(2+\lambda_1) + (2+\lambda_2)\}, & -n < \kappa < +\infty, \\ 1, & \kappa = +\infty, \end{cases} \tag{1.12}$$

and prove the following Fujita-type blow-up theorems.

**Theorem 1.1** *Assume that $b \in C^1([0,+\infty))$ satisfies (1.4) and (1.5) with $-\infty \leq \kappa < +\infty$. Let $p, q > 1$ satisfy*

$$p \geq 1 + \frac{\lambda_1}{n+\kappa}, \qquad q \geq 1 + \frac{\lambda_2}{n+\kappa}, \quad 1 < pq < (pq)_c$$

*with $(pq)_c$ defined in (1.12). Then every nonnegative nontrivial solution to problem (1.1)–(1.3) must blow up in a finite time.*

**Theorem 1.2** *Assume that $b \in C^1([0,+\infty))$ satisfies (1.4) and (1.5) with $-n < \kappa \leq +\infty$. Let $pq > (pq)_c$ with $(pq)_c$ defined in (1.12), then there exist both nonnegative nontrivial global and nonnegative blow-up solutions to problem (1.1)–(1.3).*

The difference between (1.12) and (1.9) shows that the asymptotic behavior of the coefficients of the gradient term and the exponents in the coefficients of source terms can affect the properties of solutions essentially. The method used in the paper is mainly inspired by [16, 18, 20, 22, 23]. To prove the blow-up of solutions, we determine the interactions between the diffusion terms and the gradient terms by energy estimates instead of the pointwise comparison principle. A nontrivial global supersolution is constructed to show the global existence of nontrivial solutions. What is noteworthy is that the non-self-similarity of (1.1) and (1.2) brings a difficult challenge for constructing the supersolution.

The paper is divided into three parts. In Section 2, we list some preliminaries such as the well-posedness of problem (1.1)–(1.3). Later, in Section 3, we illustrate several auxiliary lemmas to be used later. Finally, in Section 4, the Fujita-type blow-up theorems are proved. We will always assume that

$$(2+\lambda_1)+p(2+\lambda_2)\geq q(2+\lambda_1)+(2+\lambda_2)$$

without loss of generality.

## 2 Preliminaries

The solution to problem (1.1)–(1.3) is defined as follows.

**Definition 2.1** Let $0<T\leq+\infty$. $(u,v)$ is called a solution to problem (1.1)–(1.3) in $(0,T)$ if

$$0\leq u,v\in C\bigl([0,T),L^1_{\mathrm{loc}}\bigl(\mathbb{R}^n\bigr)\bigr)\cap L^\infty_{\mathrm{loc}}\bigl(0,T;L^\infty\bigl(\mathbb{R}^n\bigr)\bigr)$$

and the integral identities

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}u(x,t)\frac{\partial\varphi}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbb{R}^n}u(x,t)\bigl(\Delta\varphi(x,t)-\operatorname{div}\bigl(b\bigl(|x|\bigr)\varphi(x,t)x\bigr)\bigr)\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^T\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_1}v^p(x,t)\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^n}u_0(x)\varphi(x,0)\,\mathrm{d}x=0\end{aligned}$$

and

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}v(x,t)\frac{\partial\psi}{\partial t}(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbb{R}^n}v(x,t)\bigl(\Delta\psi(x,t)-\operatorname{div}\bigl(b\bigl(|x|\bigr)\psi(x,t)x\bigr)\bigr)\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^T\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^q(x,t)\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}^n}v_0(x)\psi(x,0)\,\mathrm{d}x=0\end{aligned}$$

are fulfilled for any $\varphi,\psi\in C^{2,1}(\mathbb{R}^n\times[0,T))$ vanishing when $t$ is near $T$ or $|x|$ are sufficiently large.

**Definition 2.2** $(u,v)$ is said to be a blow-up solution to problem (1.1)–(1.3) if

$$\bigl\|u(\cdot,t)\bigr\|_{L^\infty(\mathbb{R}^n)}+\bigl\|v(\cdot,t)\bigr\|_{L^\infty(\mathbb{R}^n)}\to+\infty\quad\text{as } t\to T_*^-$$

with $0<T_*<+\infty$, which is called blow-up time. Otherwise, $(u,v)$ is said to be a global solution.

The existence theorem and the comparison principle to problem (1.1)–(1.3) can be found in [26, 27] and the references therein.

## 3 Auxiliary lemmas

As mentioned in [20, 22, 24], one can prove the following lemma and remarks which will be used later.

**Lemma 3.1** *Assume that $b \in C^1([0,+\infty))$ satisfies (1.4) and (1.5) with $-\infty \le \kappa < +\infty$. Let $(u,v)$ be a solution to problem (1.1)–(1.3). Then there exist $R_0 > 0$, $\delta > 1$, and $M_0 > 0$, depending only on $n$ and $b$, such that for any $R > R_0$,*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} u(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x \ge{}& -M_0R^{-2}\int_{B_{\delta R}\setminus B_R} u(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x \\ &+\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_1}v^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x, \quad t>0\end{aligned} \tag{3.1}$$

*and*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n} v(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x \ge{}& -M_0R^{-2}\int_{B_{\delta R}\setminus B_R} v(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x \\ &+\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^q(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x, \quad t>0\end{aligned} \tag{3.2}$$

*in the distribution sense, where*

$$\eta_R(s) = \begin{cases} h(s), & 0 \le s \le R, \\ \frac{1}{2}h(s)(1+\cos\frac{(s-R)\pi}{(\delta-1)R}), & R < s < \delta R, \\ 0, & s \ge \delta R \end{cases}$$

*with*

$$h(s) = \exp\biggl\{\int_0^s \tilde{s}b(\tilde{s})\,\mathrm{d}\tilde{s}\biggr\}, \quad r \ge 0,$$

*while $B_r$ denotes the open ball in $\mathbb{R}^n$ with radius $r$ and centered at the origin.*

*Remark* 3.1 If $\kappa = +\infty$, then Lemma 3.1 holds for any fixed $R > 0$, but $\delta > 1$ and $M_0 > 0$ depend also on $R$.

Let us seek the self-similar supersolutions to system (1.1) and (1.2) of the following form:

$$\begin{aligned} &u(x,t) = (t+t_0)^{-\mu}\mathcal{U}\bigl((t+t_0)^{-1/2}\bigl(|x|+1\bigr)\bigr), \\ &v(x,t) = (t+t_0)^{-\nu}\mathcal{V}\bigl((t+t_0)^{-1/2}\bigl(|x|+1\bigr)\bigr), \\ &\quad x \in \mathbb{R}^n, t \ge 0 \end{aligned} \tag{3.3}$$

with

$$\mu = \frac{(2+\lambda_1)+p(2+\lambda_2)}{2(pq-1)}, \qquad \nu = \frac{q(2+\lambda_1)+(2+\lambda_2)}{2(pq-1)},$$

and $t_0 > 0$ to be determined later. If $\mathcal{U}, \mathcal{V} \in C^1([0,+\infty))$ solve

$$\begin{aligned}&\mathcal{U}''(r) + \frac{n-1}{r}\mathcal{U}'(r) + (t+t_0)^{1/2}\big((t+t_0)^{1/2}r-1\big)b\big((t+t_0)^{1/2}r-1\big)\mathcal{U}'(r) + \frac{1}{2}r\mathcal{U}'(r)\\&\quad + \mu\mathcal{U}(r) + r^{\lambda_1}\mathcal{U}^p(r) \le 0, \quad r > (t+t_0)^{-1/2},\end{aligned} \tag{3.4}$$

$$\begin{aligned}&\mathcal{V}''(r) + \frac{n-1}{r}\mathcal{V}'(r) + (t+t_0)^{1/2}\big((t+t_0)^{1/2}r-1\big)b\big((t+t_0)^{1/2}r-1\big)\mathcal{V}'(r) + \frac{1}{2}r\mathcal{V}'(r)\\&\quad + \nu\mathcal{V}(r) + r^{\lambda_2}\mathcal{V}^q(r) \le 0, \quad r > (t+t_0)^{-1/2},\end{aligned} \tag{3.5}$$

for fixed $t > 0$, respectively. Then $(u, v)$ given by (3.3) is a supersolution to system (1.1) and (1.2).

**Lemma 3.2** *Assume that $b \in C^1([0,+\infty))$ satisfies (1.4) and (1.5) with $-n < \kappa \le +\infty$. Let $pq > (pq)_c$ with $(pq)_c$ defined in (1.12) and*

$$\mathcal{U}(r) = \mathcal{V}(r) = \sigma\,\mathrm{e}^{-\omega(r)}, \quad r \ge 0, \tag{3.6}$$

*with $\omega \in C^{1,1}([0,+\infty))$ satisfies $\omega(0) = 0$ and*

$$\omega'(r) = \begin{cases} \omega_1 r, & 0 \le r \le l^2,\\ (\omega_2 + (\omega_1-\omega_2)\frac{l^{2(n+\kappa_2)}}{r^{n+\kappa_2}})r, & l^2 < r < l,\\ (\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2})r, & r \ge l,\end{cases}$$

*where $0 < l < 1$ will be determined,*

$$\omega_1 = \frac{2(pq-1)\mu}{(n+\kappa_1)(pq+(pq)_c-2)}, \qquad \omega_2 = \frac{2(pq-1)\mu}{(n+\kappa_2)(pq+(pq)_c-2)}$$

*with $\kappa_1, \kappa_2$ satisfying*

$$\kappa_1 < \kappa_0, \qquad -n < \kappa_1 < \frac{2((2+\lambda_1)+p(2+\lambda_2))}{pq+(pq)_c-2} - n < \kappa_2 < \kappa.$$

*Then there exist $\sigma > 0$, $0 < l < 1$, and $t_0 > 0$ such that $(u, v)$ given by (3.3) and (3.6) is a supersolution to (1.1)–(1.3).*

*Proof* In the case that $0 < r < l^2$ and $t > 0$,

$$\begin{aligned}&\mathcal{U}''(r) + \frac{n-1}{r}\mathcal{U}'(r) + (t+t_0)^{1/2}\big((t+t_0)^{1/2}r-1\big)b\big((t+t_0)^{1/2}r-1\big)\mathcal{U}'(r)\\&\quad + \frac{1}{2}r\mathcal{U}'(r) + \mu\mathcal{U}(r)\\&= \left(-n\omega_1 - \omega_1(t+t_0)^{1/2}r\big((t+t_0)^{1/2}r-1\big)b\big((t+t_0)^{1/2}r-1\big) + \mu + \omega_1\left(\omega_1 - \frac{1}{2}\right)r^2\right)\\&\quad \times \mathcal{U}(r)\\&\le \big(-(n+\kappa_0)\omega_1 + \mu + \omega_1^2 l\big)\mathcal{U}(r)\\&\le \left(-(\kappa_0-\kappa_1)\omega_1 - \frac{(pq-(pq)_c)\mu}{pq+(pq)_c-2} + \omega_1^2 l\right)\mathcal{U}(r),\end{aligned}$$

where $\kappa_0 = \inf\{s(s+1)b(s) : s > 0\}$. We can take $0 < l_1 < 1$ such that, for any $0 < l < l_1$,

$$\begin{aligned}
&\mathcal{U}''(r) + \frac{n-1}{r}\mathcal{U}'(r) + (t+t_0)^{1/2}\bigl((t+t_0)^{1/2}r - 1\bigr)b\bigl((t+t_0)^{1/2}r - 1\bigr)\mathcal{U}'(r) \\
&\quad + \frac{1}{2}r\mathcal{U}'(r) + \mu\mathcal{U}(r) \\
&\quad \le -\frac{(pq-(pq)_c)\mu}{2(pq+(pq)_c-2)}\mathcal{U}(r), \quad 0 < r < l^2, t > 0.
\end{aligned} \tag{3.7}$$

From the definition of the function $\omega$, one gets

$$\begin{aligned}
&\mathcal{U}''(r) + \frac{n+\kappa_2-1}{r}\mathcal{U}'(r) + \frac{1}{2}r\mathcal{U}'(r) + \mu\mathcal{U}(r) \\
&\quad = \left(\bigl(\omega'(r)\bigr)^2 - \omega''(r) - \frac{n+\kappa_2-1}{r}\omega'(r) - \frac{1}{2}r\omega'(r) + \mu\right)\mathcal{U}(r) \\
&\quad = \left(\left(\omega_2 + (\omega_1-\omega_2)\frac{l^{2(n+\kappa_2)}}{r^{n+\kappa_2}}\right)\left(\omega_2 + (\omega_1-\omega_2)\frac{l^{2(n+\kappa_2)}}{r^{n+\kappa_2}} - \frac{1}{2}\right)r^2 - \frac{(pq-(pq)_c)\mu}{pq+(pq)_c-2}\right)\mathcal{U}(r) \\
&\quad \le \left(-\frac{(pq-(pq)_c)\mu}{pq+(pq)_c-2} + \omega_1^2 l\right)\mathcal{U}(r), \quad l^2 < r < l,
\end{aligned}$$

which implies that one can take $0 < l_2 < l_1$ such that, for any $0 < l < l_2$,

$$\begin{aligned}
&\mathcal{U}''(r) + \frac{n+\kappa_2-1}{r}\mathcal{U}'(r) + \frac{1}{2}r\mathcal{U}'(r) + \mu\mathcal{U}(r) \\
&\quad \le -\frac{(pq-(pq)_c)\mu}{2(pq+(pq)_c-2)}\mathcal{U}(r), \quad l^2 < r < l, t > 0.
\end{aligned} \tag{3.8}$$

Finally, for $r > l$, it holds that

$$\begin{aligned}
&\mathcal{U}''(r) + \frac{n+\kappa_2-1}{r}\mathcal{U}'(r) + \frac{1}{2}r\mathcal{U}'(r) + \mu\mathcal{U}(r) \\
&\quad = \bigl(\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2}\bigr)\left(\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2} - \frac{1}{2}\right)r^2\mathcal{U}(r) \\
&\qquad + \bigl(\mu - (n+\kappa_2)\bigl(\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2}\bigr)\bigr)\mathcal{U}(r) \\
&\quad \le \bigl(\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2}\bigr)\left(\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2} - \frac{1}{2}\right)r^2\mathcal{U}(r) + \bigl(\mu - (n+\kappa_2)\omega_2\bigr)\mathcal{U}(r) \\
&\quad = \bigl(\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2}\bigr)\left(\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2} - \frac{1}{2}\right)r^2\mathcal{U}(r) - \frac{(pq-(pq)_c)\mu}{pq+(pq)_c-2}\mathcal{U}(r).
\end{aligned}$$

The choice of $\kappa_1, \kappa_2$ leads to

$$\lim_{l\to 0^+}\bigl(\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2}\bigr) = \omega_2 < \frac{1}{2},$$

which yields that there exists $0 < l_3 < l_2$ such that, for any $0 < l < l_3$,

$$\omega_2 + (\omega_1-\omega_2)l^{n+\kappa_2} < \frac{1}{2},$$

and thus

$$\mathcal{U}''(r)+\frac{n+\kappa_2-1}{r}\mathcal{U}'(r)+\frac{1}{2}r\mathcal{U}'(r)+\mu\mathcal{U}(r)\le-\frac{(pq-(pq)_c)\mu}{2(pq+(pq)_c-2)}\mathcal{U}(r),\quad r>l,t>0. \tag{3.9}$$

Fix $0<l<l_3$, it follows from (1.4) that, for $t_0>0$ sufficiently large,

$$(t+t_0)^{1/2}r\big((t+t_0)^{1/2}r-1\big)b\big((t+t_0)^{1/2}r-1\big)\ge\frac{\kappa_2}{r},\quad r>l^2,t>0. \tag{3.10}$$

It follows from (3.7)–(3.10) that

$$\begin{aligned}
&\mathcal{U}''(r)+\frac{n-1}{r}\mathcal{U}'(r)+(t+t_0)^{1/2}\big((t+t_0)^{1/2}r-1\big)b\big((t+t_0)^{1/2}r-1\big)\mathcal{U}'(r)\\
&\quad+\frac{1}{2}r\mathcal{U}'(r)+\mu\mathcal{U}(r)\\
&\le\mathcal{U}''(r)+\frac{n+\kappa_2-1}{r}\mathcal{U}'(r)+\frac{1}{2}r\mathcal{U}'(r)+\mu\mathcal{U}(r)\\
&\le-\frac{(pq-(pq)_c)\mu}{2(pq+(pq)_c-2)}\mathcal{U}(r),\quad r\in\big(0,l^2\big)\cup\big(l^2,l\big)\cup(l,+\infty),t>0.
\end{aligned} \tag{3.11}$$

Similarly, one can show that

$$\begin{aligned}
&\mathcal{V}''(r)+\frac{n-1}{r}\mathcal{V}'(r)+(t+t_0)^{1/2}\big((t+t_0)^{1/2}r-1\big)b\big((t+t_0)^{1/2}r-1\big)\mathcal{V}'(r)\\
&\quad+\frac{1}{2}r\mathcal{V}'(r)+\nu\mathcal{V}(r)\\
&\le-\left(\frac{(pq-(pq)_c)\mu}{2(pq+(pq)_c-2)}+\frac{((2+\lambda_1)+p(2+\lambda_2))-(q(2+\lambda_1)+(2+\lambda_2))}{2(pq-1)}\right)\mathcal{V}(r)\\
&\le-\frac{(pq-(pq)_c)\mu}{2(pq+(pq)_c-2)}\mathcal{V}(r),\quad r\in\big(0,l^2\big)\cup\big(l^2,l\big)\cup(l,+\infty),t>0.
\end{aligned} \tag{3.12}$$

Due to $\lambda_1,\lambda_2\ge0$, $p,q>1$, and the definition of the function $\omega$,

$$0<K_0=\sup_{r>0}\big(r^{\lambda_1}\mathrm{e}^{-(p-1)\omega(r)}+r^{\lambda_2}\mathrm{e}^{-(q-1)\omega(r)}\big)<+\infty.$$

Choose $\sigma>0$ sufficiently small such that

$$\max\big\{\sigma^{p-1},\sigma^{q-1}\big\}\le\frac{(pq-(pq)_c)\mu}{2K_0(pq+(pq)_c-2)}.$$

Then (3.11) and (3.12) yield that

$$\begin{aligned}
&\mathcal{U}''(r)+\frac{n-1}{r}\mathcal{U}'(r)+(t+t_0)^{1/2}\big((t+t_0)^{1/2}r-1\big)b\big((t+t_0)^{1/2}r-1\big)\mathcal{U}'(r)\\
&\quad+\frac{1}{2}r\mathcal{U}'(r)+\mu\mathcal{U}(r)+r^{\lambda_1}\mathcal{V}^p(r)\le0,\\
&\quad r\in\big(0,l^2\big)\cup\big(l^2,l\big)\cup(l,+\infty),t>0
\end{aligned}$$

and

$$\begin{aligned}&\mathcal{V}''(r)+\frac{n-1}{r}\mathcal{V}'(r)+(t+t_0)^{1/2}\big((t+t_0)^{1/2}r-1\big)b\big((t+t_0)^{1/2}r-1\big)\mathcal{V}'(r)\\&\quad+\frac{1}{2}r\mathcal{V}'(r)+\nu\mathcal{V}(r)+r^{\lambda_2}\mathcal{U}^q(r)\le 0,\\&\quad r\in\big(0,l^2\big)\cup\big(l^2,l\big)\cup(l,+\infty),t>0.\end{aligned}$$

Therefore, $(u,v)$ given by (3.3) and (3.6) is a supersolution to system (1.1) and (1.2). □

## 4 Proofs of Fujita-type blow-up theorems

In the final section, we will prove the blow-up theorems of Fujita type for problem (1.1)–(1.3). Let $\eta_R$, $h$, $R_0$, $\delta$, and $M_0$ be given as Lemma 3.1 in this section.

Let us prove Theorem 1.1 firstly.

*Proof of Theorem* 1.1 It follows from $-\infty\le\kappa<+\infty$ and $1<pq<(pq)_c$ that

$$\kappa<\frac{(2+\lambda_1)+p(2+\lambda_2)}{pq-1}-n.$$

Fix $\tilde\kappa\ge\kappa$ to satisfy

$$-n\le\tilde\kappa<\frac{(2+\lambda_1)+p(2+\lambda_2)}{pq-1}-n, \tag{4.1}$$

which, together with (1.4), yields that there exists $R_1>1$ such that

$$s^2b(s)<\tilde\kappa,\quad s>R_1.$$

For any $R>R_1$, one can get that

$$\int_0^s\tilde sb(\tilde s)\,\mathrm{d}\tilde s\le\begin{cases}K_0, & 0\le s\le R_1,\\ K_0+\ln s^{\tilde\kappa}, & s>R_1,\end{cases}$$

and

$$h(s)=\exp\left\{\int_0^s\tilde sb(\tilde s)\,\mathrm{d}\tilde s\right\}\le\begin{cases}\mathrm{e}^{K_0}, & 0\le s\le R_1,\\ \mathrm{e}^{K_0}s^{\tilde\kappa}, & s>R_1\end{cases}\le K(s+1)^{\tilde\kappa},\quad s\ge 0,$$

where

$$K=\max\left\{\sup_{0\le s\le R_1}\frac{\mathrm{e}^{K_0}}{(s+1)^{\tilde\kappa}},\sup_{s>R_1}\frac{\mathrm{e}^{K_0}s^{\tilde\kappa}}{(s+1)^{\tilde\kappa}}\right\},\quad K_0=|\tilde\kappa|\ln R_1+\sup_{0\le s\le R_1}\int_0^s\tilde sb(\tilde s)\,\mathrm{d}\tilde s.$$

Therefore,

$$0\le\eta_R(s)\le h(s)\chi_{[0,\delta R]}(s)=K(s+1)^{\tilde\kappa}\chi_{[0,\delta R]}(s),\quad s\ge 0, \tag{4.2}$$

where $\chi_{[0,\delta R]}$ is the characteristic function of the interval $[0,\delta R]$ and $K>0$ depends only on $n$, $b$, $R_1$, $\delta$, and $\tilde{\kappa}$. Let $(u,v)$ be the solution to problem (1.1)–(1.3), and denote

$$w_R(t)=\int_{\mathbb{R}^n}\bigl(u(x,t)+R^\theta v(x,t)\bigr)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x,\quad t\geq 0$$

with some constant $\theta$ to be determined. For any $R>\max\{R_0,R_1\}$, Lemma 3.1 shows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}w_R(t)\geq{}&-M_0R^{-2}w_R(t)+R^\theta\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^q(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\\&+\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_1}v^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x,\quad t>0.\end{aligned}\tag{4.3}$$

From the Hölder inequality and (4.2), one gets

$$\begin{aligned}&\int_{\mathbb{R}^n}u(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\\&\quad\leq\biggl(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{-\lambda_2/(q-1)}\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{(q-1)/q}\biggl(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^q(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{1/q}\\&\quad\leq\biggl(K\int_{B_{\delta R}}\bigl(|x|+1\bigr)^{\tilde{\kappa}-\lambda_2/(p-1)}\,\mathrm{d}x\biggr)^{(q-1)/q}\biggl(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{1/q}\\&\quad\leq\biggl(K\omega_n\int_0^{\delta R}(r+1)^{n+\tilde{\kappa}-1-\lambda_2/(q-1)}\,\mathrm{d}r\biggr)^{(q-1)/q}\biggl(\int_{B_{\delta R}}\bigl(|x|+1\bigr)^{\lambda_2}u^q(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{1/q}\\&\quad\leq M_1^{(q-1)/q}R^{n+\tilde{\kappa}-(n+\tilde{\kappa}+\lambda_2)/q}\biggl(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^q(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{1/q},\quad t>0.\end{aligned}\tag{4.4}$$

Similarly, we have

$$\begin{aligned}&\int_{\mathbb{R}^n}v(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\leq M_1^{(p-1)/p}R^{n+\tilde{\kappa}-(n+\tilde{\kappa}+\lambda_1)/p}\biggl(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_1}v^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\biggr)^{1/p},\\&\quad t>0,\end{aligned}\tag{4.5}$$

while $M_1>0$ depends only on $n$, $b$, $R_1$, $\delta$, $\lambda_1$, $\lambda_2$, $p$, $q$, and $\tilde{\kappa}$. Substituting (4.4) and (4.5) into (4.3) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}w_R(t)\geq{}&-M_0R^{-2}w_R(t)+M_1^{-(q-1)}R^{-(q-1)(n+\tilde{\kappa})+\lambda_2+\theta}\biggl(\int_{\mathbb{R}^n}u(x,t)\eta_R(x)\,\mathrm{d}x\biggr)^q\\&+M_1^{-(p-1)}R^{-(p-1)(n+\tilde{\kappa})+\lambda_1-p\theta}\biggl(R^\theta\int_{\mathbb{R}^n}v(x,t)\eta_R(x)\,\mathrm{d}x\biggr)^p,\quad t>0.\end{aligned}\tag{4.6}$$

Choosing $\theta=\frac{(q-p)(n+\tilde{\kappa})+\lambda_1-\lambda_2}{p+1}$, then

$$-(q-1)(n+\tilde{\kappa})+\lambda_2+\theta=-(p-1)(n+\tilde{\kappa})+\lambda_1-p\theta=\frac{(1-pq)(n+\tilde{\kappa})+\lambda_1+p\lambda_2}{p+1}.$$

Lemma 3.6 in [24] and (4.6) lead to

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}w_R(t) &\geq -M_0R^{-2}w_R(t) + M_2R^{-(pq-1)(n+\tilde{\kappa}+\lambda_1)/(p+1)} \\ &\quad \cdot \left\{\left(\int_{\mathbb{R}^n} u(x,t)\eta_R(x)\,\mathrm{d}x\right)^q + \left(R^\theta \int_{\mathbb{R}^n} v(x,t)\eta_R(x)\,\mathrm{d}x\right)^p\right\} \\ &\geq -M_0R^{-2}w_R(t) + 2^{-p}M_2R^{[(1-pq)(n+\tilde{\kappa})+\lambda_1+p\lambda_2]/(p+1)} \cdot \min\left\{w_R^p(t), w_R^q(t)\right\} \\ &= w_R(t)\left(-M_0R^{-2} + 2^{-p}M_2R^{[(1-pq)(n+\tilde{\kappa})+\lambda_1+p\lambda_2]/(p+1)} \cdot \min\left\{w_R^{p-1}(t), w_R^{q-1}(t)\right\}\right), \\ &\quad t > 0 \end{aligned} \tag{4.7}$$

with $M_2 = \min\{M_1^{1-p}, M_1^{1-q}\}$. Note that (4.1) implies

$$\frac{(1-pq)(n+\tilde{\kappa}) + \lambda_1 + p\lambda_2}{p+1} > -2,$$

while $w_R(0)$ is nondecreasing with respect to $R \in (0,+\infty)$ and

$$\sup\left\{w_R(0) : R > 0\right\} > 0.$$

Therefore, there exists $R_2 > 0$ such that, for any $R > R_2$,

$$M_0R^{-2} \leq 2^{-(p+1)}M_2R^{[(1-pq)(n+\tilde{\kappa})+\lambda_1+p\lambda_2]/(p+1)} \cdot \min\left\{w_R^{p-1}(0), w_R^{q-1}(0)\right\}. \tag{4.8}$$

Fix $R > \max\{R_0, R_1, R_2\}$. Then (4.7) and (4.8) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}w_R(t) \geq 2^{-(p+1)}M_2R^{[(1-pq)(n+\tilde{\kappa})+\lambda_1+p\lambda_2]/(p+1)} \cdot \min\left\{w_R^p(t), w_R^q(t)\right\}, \quad t > 0.$$

It follows from $p, q > 1$ that there exists $T_* > 0$ such that

$$w_R(t) = \int_{\mathbb{R}^n} \left(u(x,t) + R^\theta v(x,t)\right)\eta_R(x)\,\mathrm{d}x \to +\infty \quad \text{as } t \to T_*^-.$$

From $\operatorname{supp}\eta_R(|x|) = \overline{B}_{\delta R}$, one gets

$$\left\|u(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)} + \left\|v(\cdot,t)\right\|_{L^\infty(\mathbb{R}^n)} \to +\infty \quad \text{as } t \to T_*^-.$$

That is to say, $(u,v)$ blows up in a finite time. □

Let us turn to proving Theorem 1.2.

*Proof of Theorem* 1.2 The fact that problem (1.1)–(1.3) with small initial data admits a nontrivial global solution follows from the comparison principle and Lemma 3.2. Then let us show the blow-up of the solution to problem (1.1)–(1.3) with large initial data.

Fix $R > R_0$ and let $(u,v)$ be the solution to problem (1.1)–(1.3). Denote

$$\tilde{w}_R(t) = \int_{\mathbb{R}^n} \left(u(x,t) + v(x,t)\right)\eta_R(x)\,\mathrm{d}x, \quad t \geq 0.$$

From Lemma 3.1, Remark 3.1, the Hölder inequality, and Lemma 3.6 in [24], it follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{w}_R(t) &\geq -M_0R^{-2}\tilde{w}_R(t) + \int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_2}u^q(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{\lambda_1}v^p(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x \\
&\geq -M_0R^{-2}\tilde{w}_R(t) + \left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{-\lambda_2/(q-1)}\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1-q} \\
&\quad \times \left(\int_{\mathbb{R}^n}u(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^q \\
&\quad + \left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{-\lambda_1/(p-1)}\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1-p}\left(\int_{\mathbb{R}^n}v(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^p \\
&\geq -M_0R^{-2}\tilde{w}_R(t) + M_3\left\{\left(\int_{\mathbb{R}^n}u(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^q + \left(\int_{\mathbb{R}^n}v(x,t)\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^p\right\} \\
&\geq -M_0R^{-2}\tilde{w}_R(t) + 2^{-p}M_3\cdot\min\bigl\{\tilde{w}_R^p(t),\tilde{w}_R^q(t)\bigr\} \\
&= \tilde{w}_R(t)\bigl(-M_0R^{-2} + 2^{-p}M_3\cdot\min\bigl\{\tilde{w}_R^{p-1}(t),\tilde{w}_R^{q-1}(t)\bigr\}\bigr), \quad t>0
\end{aligned} \tag{4.9}$$

with

$$M_3 = \min\left\{\left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{-\lambda_2/(q-1)}\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1-q}, \left(\int_{\mathbb{R}^n}\bigl(|x|+1\bigr)^{-\lambda_1/(p-1)}\eta_R\bigl(|x|\bigr)\,\mathrm{d}x\right)^{1-p}\right\}$$

depending only on $n$, $\delta$, $p$, $q$, and $R$. If $(u_0,v_0)$ is so large that

$$2^{-(p+1)}M_3\cdot\min\bigl\{\tilde{w}_R^{p-1}(0),\tilde{w}_R^{q-1}(0)\bigr\} \geq M_0R^{-2},$$

then (4.9) leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{w}_R(t) \geq 2^{-(p+1)}M_3\cdot\min\bigl\{\tilde{w}_R^p(t),\tilde{w}_R^q(t)\bigr\}, \quad t>0.$$

The same argument as the proof of Theorem 1.1 shows that $(u,v)$ must blow up in a finite time. □

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**Authors' contributions**
All the authors contributed to each part of this study equally and approved the final version of the manuscript.

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