# Approximation by quaternion ( $p, q$ )-Bernstein polynomials and Voronovskaja type result on compact disk 

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#### Abstract

In this paper, we define the $(p, q)$-Bernstein polynomials of degree $m$ of a quaternion variable. We obtain some approximation results, and also the Voronovskaja type result with quantitative upper estimates is proved.


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## 1 Introduction

The quaternion field is an extension of the class of complex numbers, i.e., $\mathbb{C} \subset \mathbb{H}$. It is a non-commutative field defined by

$$
\mathbb{H}=\left\{w=y_{1}+y_{2} i+y_{3} j+y_{4} k: y_{1}, y_{2}, y_{3}, y_{4} \in \mathbb{R}\right\},
$$

where the complex units $i, j, k \notin \mathbb{R}$ satisfy

$$
i^{2}=j^{2}=k^{2}=-1, \quad i j=-j i=k, \quad j k=-k j=i, \quad k i=-i k=j .
$$

For $\omega=y_{1}+y_{2} i+y_{3} j+y_{4} k$, the norm is defined as $\|w\|=\sqrt{y_{1}^{2}+y_{2}^{2}+y_{3}^{2}+y_{4}^{2}}$ on $\mathbb{H}$.
We need to give some basic details of analyticity of a function of quaternion variable and some properties of $(p, q)$-calculus for our purpose.

Definition 1.1 (see Gal [7], p. 296) A function $\mathcal{G}: \mathbb{D}_{\mathcal{R}} \longrightarrow \mathbb{H}$ is left Weierstrass analytic (or left $\mathcal{W}$-analytic) in $\mathbb{D}_{\mathcal{R}}$ if $\mathcal{G}(w)=\sum_{l=0}^{\infty} c_{l} w^{l}$ for all $w \in \mathbb{D}_{\mathcal{R}}$, where $\mathbb{D}_{\mathcal{R}}$ denotes the open ball, i.e., $\mathbb{D}_{\mathcal{R}}=\{w \in \mathbb{H}:\|w\|<\mathcal{R}\}$ and $c_{l} \in \mathbb{H}$ for all $l=0,1,2, \ldots$. Similarly, $\mathcal{G}$ is called right $\mathcal{W}$-analytic in $\mathbb{D}_{\mathcal{R}}$ if $\mathcal{G}(w)=\sum_{l=0}^{\infty} w^{l} c_{l}$ for all $w \in \mathbb{D}_{\mathcal{R}}$.

It is understood here that in any closed ball $\overline{\mathbb{D}}_{r}=\{\omega \in \mathbb{H}:\|w\| \leq r\}, 0<r<\mathcal{R}$, the partial sums $\sum_{l=0}^{n} c_{l} w^{l}$ and $\sum_{l=0}^{n} w^{l} c_{l}$ converge to $\mathcal{G}$ uniformly, with respect to the metric $d(w, z)=$ $\|w-z\|$.

Remark The two concepts given in Definition 1.1 coincide with the Weierstrass type analyticity in the case of complex variable and that is equivalent to the holomorphy concept given by Cauchy. Also, it is better known that the only functions of the trivial form $\mathcal{G}(w)=C w+D$ are analytic in the Cauchy sense in the case of quaternion variable (Mejlihzon [17]), while the classic classes of left (or right) monogenic functions at each $w=y_{1}+y_{2} i+y_{3} j+y_{4} k \in \mathbb{D}_{\mathcal{R}} \subset \mathbb{H}$ were introduced by Moisil [18] as the class of functions $\mathcal{G}=\mathcal{G}_{1}+\mathcal{G}_{2} i+\mathcal{G}_{3} j+\mathcal{G}_{4} k$ satisfying $T \mathcal{G}(w)=0\left(\right.$ or $\mathcal{G} T(w)=0$ respectively) at each $w \in \mathbb{D}_{\mathcal{R}}$, where $T=\frac{\partial}{\partial y_{1}}+\frac{\partial}{\partial y_{2}} i+\frac{\partial}{\partial y_{3}} j+\frac{\partial}{\partial y_{4}} k$ does not coincide with the class of left $\mathcal{W}$-analytic (or right $\mathcal{W}$-analytic, respectively) function defined in Definition 1.1. For more details concerning the properties of the left (or right) $\mathcal{W}$-analytic functions, see [11], [10].

Many authors have worked on $q$-analogue of different operators, for instance, in [9, 23, $25,26]$. Recently in several areas of mathematical sciences the $(p, q)$-calculus has many interesting applications (see $[5,12,16])$. For $p=1$, the notion of $(p, q)$-calculus is reduced to $q$-calculus and the transition from the $q$-case to the $(p, q)$-case with an extra parameter $p$ is fairly straightforward. One advantage of using the parameter $p$ has been mentioned in approximation by $(p, q)$ Lorentz operators in compact disk [20]. For more $(p, q)$ approximation, we refer to [1-4, 13-15] and [27].

Most recently, Mursaleen et al. introduced and studied approximation properties of the ( $p, q$ )-analogues of many well-known operators, such as Bernstein operators [19], Lorentz operators on compact disk [20], Bleimann-Butzar-Hahn operators [21], divideddifference and Bernstein operators [22], and many more.
Now recall some notations and definitions on $(p, q)$-calculus.
For $0<q<p$ and any positive integer $m,(p, q)$ integers are defined as

$$
[m]_{p, q}= \begin{cases}\frac{p^{m}-q^{m}}{p-q}, & p \neq q \\ m, & p=q=1\end{cases}
$$

The $(p, q)$-binomial expansion is defined as

$$
\begin{aligned}
& (a x+b y)_{p, q}^{m}=\sum_{k=0}^{m} p^{\frac{(m-k)(m-k-1)}{2}} q^{\frac{k(k-1)}{2}}\binom{m}{k}_{p, q} a^{m-k} b^{k} x^{m-k} y^{k}, \\
& (x+y)_{p, q}^{m}=(x+y)(p x+q y)(p x+q y)\left(p^{2} x+q^{2} y\right) \cdots\left(p^{n-1} x+q^{n-1} y\right), \\
& (1-x)_{p, q}^{m}=(1-x)(p-q x)\left(p^{2}-q^{2} x\right) \cdots\left(p^{n-1}-q^{n-1} x\right),
\end{aligned}
$$

and for the integers $0 \leq k \leq m,(p, q)$-binomial coefficients are defined by

$$
\binom{m}{k}_{p, q}=\frac{[m]_{p, q}!}{[k]_{p, q}![m-k]_{p, q}!},
$$

where

$$
[m]_{p, q}!=[m]_{p, q}[m-1]_{p, q} \cdots[1]_{p, q} .
$$

The $(p, q)$-analogue of Bernstein operators is defined as follows [22]:

$$
\begin{equation*}
\mathcal{B}_{p, q}^{m}(f ; x)=\frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{k=0}^{m}\binom{m}{k}_{p, q} p^{\frac{k(k-1)}{2}} f\left(\frac{[k]_{p, q}}{p^{k-n}[n]_{p, q}}\right) x^{k} \prod_{s=0}^{m-k-1}\left(p^{s}-q^{s} x\right) . \tag{1.1}
\end{equation*}
$$

The Euler identity is defined by

$$
\begin{equation*}
\prod_{s=0}^{m-1}\left(p^{s}-q^{s} x\right)=\sum_{k=0}^{m} p^{\frac{(m-k)(m-k-1)}{2}} q^{\frac{k(k-1)}{2}}\binom{m}{k}_{p, q} x^{k} \tag{1.2}
\end{equation*}
$$

We introduce the following.

Definition 1.2 Let $q>p \geq 1$ and $\mathcal{R}>1$. For a function $\mathcal{G}: \mathbb{D}_{\mathcal{R}} \longrightarrow \mathbb{H}$, because of noncommutativity, we define three distinct $(p, q)$-Bernstein polynomials of a quaternion variable:

$$
\begin{aligned}
& \mathcal{B}_{p, q}^{m}(\mathcal{G})(w)=\frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{l=0}^{m} \mathcal{G}\left(\frac{[l]_{p, q}}{p^{l-m}[m]_{p, q}}\right)\binom{m}{l}_{p, q} p^{\frac{l(l-1)}{2}} w^{l} \prod_{s=0}^{m-1-l}\left(p^{s}-q^{s} w\right), \quad w \in \mathbb{H}, \\
& \mathcal{B}_{p, q}^{m *}(\mathcal{G})(w)=\frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{l=0}^{m}\binom{m}{l}_{p, q} p^{\frac{l(l-1)}{2}} w^{l} \prod_{s=0}^{m-1-l}\left(p^{s}-q^{s} w\right) \mathcal{G}\left(\frac{[l]_{p, q}}{p^{l-m}[m]_{p, q}}\right), \quad w \in \mathbb{H}, \\
& \mathcal{B}_{p, q}^{m * *}(\mathcal{G})(w)=\frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{l=0}^{m}\binom{m}{l}_{p, q} p^{\frac{l(l-1)}{2}} w^{l} \mathcal{G}\left(\frac{[l]_{p, q}}{p^{l-m}[m]_{p, q}}\right) \prod_{s=0}^{m-1-l}\left(p^{s}-q^{s} w\right), \quad w \in \mathbb{H},
\end{aligned}
$$

by labeling them as the left $(p, q)$-Bernstein polynomials, the right $(p, q)$-Bernstein polynomials, and the middle $(p, q)$-Bernstein polynomials, respectively.

## 2 Approximation results

Firstly, we show that for any continuous function $\mathcal{G}$ these three kinds of $(p, q)$-Bernstein polynomials do not converge. For example, if we take $\mathcal{G}=i w i$, we get easily

$$
\begin{aligned}
\left\|\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-i w i\right\| & =\left\|\mathcal{B}_{p, q}^{m *}(\mathcal{G})(w)-i w i\right\|=\left\|\mathcal{B}_{p, q}^{m * *}(\mathcal{G})(w)-i w i\right\| \\
& =\|-w-i w i\|=\|-i w+w i\|>0 \quad \text { for all } w \neq i .
\end{aligned}
$$

We can obtain convergence result for the classes of functions in Definition 1.1. For this we need some auxiliary results for ( $p, q$ )-operators in complex plane similar as done in [28] for $q$-operators.

Lemma 2.1 Let $q \geq p \geq 1$ be fixed. Then, for $n \geq 2$,

$$
\begin{equation*}
\mathcal{B}_{p, q}^{m}\left(t^{n}, w\right)=c_{1} w+c_{2} w^{2}+\cdots+c_{i} w^{i}, \quad i=\min (n, m) \tag{2.1}
\end{equation*}
$$

where $c_{j} \geq 0(j=1,2, \ldots, i)$ and $c_{1}+c_{2}+\cdots+c_{i}=1$. Besides, if $m \geq n$, then

$$
\begin{align*}
& c_{n}=\prod_{j=1}^{n-1}\left(1-p^{m-j} \frac{[j]_{p, q}}{[m]_{p, q}}\right),  \tag{2.2}\\
& c_{n-1}=\frac{[1]_{p, q}+[2]_{p, q}+\cdots+[n-1]_{p, q}}{[m]_{p, q}} \prod_{j=1}^{n-2}\left(1-p^{m-j} \frac{[j]_{p, q}}{[m]_{p, q}}\right) .
\end{align*}
$$

Also, for any $r \geq 1$,

$$
\begin{equation*}
\left\|\mathcal{B}_{p, q}^{m}\left(t^{n}, w\right)-w^{n}\right\| \leq 2(n-1)[n-1]_{p, q} r^{n} \quad \text { for }\|w\| \leq r . \tag{2.3}
\end{equation*}
$$

Proof The proof is simple, one can prove this lemma with the help of Lemma 3 of [24]. So we skip it.

Lemma 2.2 Let $c_{1}, c_{2}, \ldots, c_{k} \in(0,1)$. Then

$$
\begin{equation*}
1-\prod_{i=1}^{k}\left(1-c_{i}\right) \leq \sum_{i=1}^{k} c_{i} \tag{2.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|1-\prod_{i=1}^{k}\left(1-c_{i}\right)-\sum_{i=1}^{k} c_{i}\right\| \leq \sum_{1 \leq i<j \leq k} c_{i} c_{j} . \tag{2.5}
\end{equation*}
$$

Proof For the proof, see (Wang [28], Lemma 2).

Lemma 2.3 Let $q \geq p \geq 1$ be fixed. If $m \geq n \geq 2$ and $r \geq 1$, then for any $w,\|w\| \leq r$,

$$
\begin{align*}
& \left\|[m]_{p, q}\left(\mathcal{B}_{p, q}^{m}\left(t^{n}, w\right)-w^{n}\right)-\left([1]_{p, q}+[2]_{p, q}+\cdots+[n-1]_{p, q}\right)\left(w^{n-1}-w^{n}\right)\right\| \\
& \quad \leq \frac{4 p^{m-n+1}(n-1)^{2}[n-1]_{p, q}^{2}}{[m]_{p, q}} r^{n} . \tag{2.6}
\end{align*}
$$

Proof It follows from (2.1) and (2.2) that, for $\|w\| \leq r$,

$$
\begin{aligned}
K & =\left\|[m]_{p, q}\left(\mathcal{B}_{p, q}^{m}\left(t^{n}, w\right)-w^{n}\right)-\left([1]_{p, q}+[2]_{p, q}+\cdots+[n-1]_{p, q}\right)\left(w^{n-1}-w^{n}\right)\right\| \\
& \leq r^{n}[m]_{p, q} \sum_{i=1}^{n-2} c_{i}+r^{n}\left\|[m]_{p, q} c_{n-1}-\sum_{i=1}^{n-1}[i]_{p, q}\right\|+\left\|[m]_{p, q}\left(1-c_{n}\right)-\sum_{i=1}^{n-1}[i]_{p, q}\right\| \\
& =r^{n}[m]_{p, q}\left(1-c_{n}-c_{n-1}\right)+r^{n}\left\|[m]_{p, q} c_{n-1}-\sum_{i=1}^{n-1}[i]_{p, q}\right\|+\left\|[m]_{p, q}\left(1-c_{n}\right)-\sum_{i=1}^{n-1}[i]_{p, q}\right\| \\
& \leq 2 r^{n}\left\|[m]_{p, q} c_{n-1}-\sum_{i=1}^{n-1}[i]_{p, q}\right\|+\left\|[m]_{p, q}\left(1-c_{n}\right)-\sum_{i=1}^{n-1}[i]_{p, q}\right\| \\
& \left.=2 r^{n}\left(\sum_{i=1}^{n-1}[i]_{p, q}\right)\left(1-\prod_{i=1}^{n-2}\left(1-p^{m-i} \frac{[i]_{p, q}}{[m]_{p, q}}\right)\right)\right)
\end{aligned}
$$

$$
\left.+2 r^{n}[m]_{p, q} \| 1-\prod_{i=1}^{n-2}\left(1-p^{m-i} \frac{[i]_{p, q}}{[m]_{p, q}}\right)\right)-\sum_{i=1}^{n-1} \frac{[i]_{p, q}}{[m]_{p, q}} \| .
$$

Using (2.4) and (2.5), we get

$$
\begin{aligned}
K & \leq 2 r^{n}\left(\sum_{i=1}^{n-1}[i]_{p, q}\right)\left(\sum_{i=1}^{n-2} \frac{p^{m-i}[i]_{p, q}}{[m]_{p, q}}\right)+2 r^{n} \sum_{1 \leq i<j \leq n-1} \frac{p^{m-i}[i]_{p, q}}{[m]_{p, q}} \frac{[j]_{p, q}}{[m]_{p, q}} \\
& \leq \frac{4 p^{m-n+1}(n-1)^{2}[n-1]_{p, q}^{2}}{[m]_{p, q}} r^{n} .
\end{aligned}
$$

Hence the lemma is proved.

Theorem 2.4 Let $q>p \geq 1$. Suppose that $\mathcal{G}: \mathbb{D}_{\mathcal{R}} \longrightarrow \mathbb{H}$ such that $\mathcal{G}(w) \in \mathbb{R}$ for all $w \in$ $[0,1]$. We have the following representation formula:

$$
\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)=\frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{n=0}^{m}\binom{m}{n}_{p, q} p^{\frac{(m-n)(m-n-1)}{2}}\left[\Delta^{n} \mathcal{G}(0)\right]_{p, q} w^{n} \quad \text { for all } w \in \mathcal{H}
$$

where $\left[\Delta^{k} \mathcal{G}(0)\right]_{p, q}=\sum_{k=0}^{n}(-1)^{k} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}}\binom{n}{k}_{p, q} \mathcal{G}(n-k)$.
Proof In the expression of $\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)$, the real values $\mathcal{G}\left(\frac{\left[[]_{p, q}\right.}{p^{L-m}[m]_{p, q}}\right)$ commute with the other terms. As we take the condition on $\mathcal{G}$, so that $w^{l} \prod_{s=0}^{m-1-l}\left(p^{s}-q^{s} w\right)=\prod_{s=0}^{m-1-l}\left(p^{s}-\right.$ $\left.q^{s} w\right) w^{l}, \alpha w=w \alpha$, for all $\alpha \in \mathbb{R}, w \in \mathbb{D}$ and that we can interchange the order of terms in the product $\prod_{s=0}^{m-1-l}\left(p^{s}-q^{s} w\right)$. By the same reason in the case of $(p, q)$-Bernstein polynomials of real variable [22], we obtain that the coefficient of $w^{m}$ in the expression of $\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)$ is

$$
\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)=\frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{n=0}^{m}\binom{m}{n}_{p, q} p^{\frac{(n)(n-1)}{2}} \mathcal{G}\left(\frac{[l]_{p, q}}{p^{l-m}[m]_{p, q}}\right) w^{n} \prod_{s=0}^{m-n-l}\left(p^{s}-q^{s} w\right) .
$$

By using the Euler identity based on $(p, q)$-analogue, we get the required result.

Remark Clearly, Theorem 2.4 holds also for middle $(p, q)$-Bernstein operators $\mathcal{B}_{p, q}^{m * *}(\mathcal{G})(w)$ and right $(p, q)$-Bernstein operators $\mathcal{B}_{p, q}^{m *}(\mathcal{G})(w)$. In [7] and [8] upper estimates by $q$ Bernstein operators, $q \geq 1$, of quaternion variable were proved.

Theorem 2.5 (Gal [7]) Suppose that $\mathcal{G}: \mathbb{D}_{\mathcal{R}} \longrightarrow \mathbb{H}$ is left $\mathcal{W}$-analytic in $\mathbb{D}_{\mathcal{R}}$. Then, for all $1 \leq r<\mathcal{R},\|w\| \leq r$, and $m \in \mathbb{N}$, we have

$$
\left\|\mathcal{B}^{m}(\mathcal{G})(w)-\mathcal{G}(w)\right\| \leq \frac{2}{m} \sum_{k=2}^{\infty}\left\|c_{k}\right\| k(k-1) r^{k}=O(1 / m) .
$$

Theorem 2.6 (Gal [8]) Let $1<q<\mathcal{R}$ and suppose that $\mathcal{G}: \mathbb{D}_{\mathcal{R}} \longrightarrow \mathbb{H}$ is left $\mathcal{W}$-analytic in $\mathbb{D}_{\mathcal{R}}$. Then, for all $1 \leq r<\frac{\mathcal{R}}{q},\|w\| \leq r$, and $m \in \mathbb{N}$, we have

$$
\left\|\mathcal{B}_{q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)\right\| \leq \frac{2}{(q-1)[m]_{q}} \sum_{k=1}^{\infty}\left\|c_{k}\right\| k(q r)^{k}=O\left(1 / q^{m}\right) .
$$

Remark By the right Bernstein polynomials $\mathcal{B}_{q}^{m *}(\mathcal{G})(w)$, a similar upper estimate in approximation can be obtained if $\mathcal{G}$ is supposed to be right $\mathcal{W}$-analytic for $q \geq 1$.

Theorem 2.7 Let $\mathcal{R}>q>p>1$ and $\mathcal{G}: \mathbb{D}_{\mathcal{R}} \rightarrow \mathbb{H}$ be left $\mathcal{W}$-analytic in $\mathbb{D}_{\mathcal{R}}$, i.e., $\mathcal{G}(w)=$ $\sum_{i=0}^{\infty} c_{i} w^{i}$ for all $w \in \mathbb{D}_{\mathcal{R}}$, where $c_{i} \in \mathbb{H}$ for all $i=0,1,2, \ldots$. Then, for all $1<r<\frac{p \mathcal{R}}{q},\|w\| \leq r$, and $m \in \mathbb{N}$, we have

$$
\left\|\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)\right\| \leq \frac{2 p^{m}}{[m]_{p, q}} \sum_{i=0}^{\infty}\left\|c_{i}\right\| i[i-1]_{p, q} r^{i}=O(p / q)^{m} .
$$

Proof Denoting $e_{i}(w)=w^{i}$. Firstly we will show that

$$
\begin{equation*}
\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)=\sum_{i=0}^{\infty} c_{i} \mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w) \quad \text { for all }\|w\| \leq r \tag{2.7}
\end{equation*}
$$

Here $\mathcal{G}_{n}(w)=\sum_{i=0}^{n} c_{i} e_{i}(w), n \in \mathbb{N}$, is the partial sum of the expansion of $\mathcal{G}$, due to the linearity of $\mathcal{B}_{p, q}^{m}$, we get

$$
\mathcal{B}_{p, q}^{m}\left(\mathcal{G}_{n}\right)(w)=\sum_{i=0}^{n} c_{i} \mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w) \quad \text { for all }\|w\| \leq r
$$

it is enough to prove that $\lim _{n \rightarrow \infty} \mathcal{B}_{p, q}^{m}\left(\mathcal{G}_{n}\right)(w)=\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)$ for all $\|w\| \leq r$ and $m \in \mathbb{N}$.
By Theorem 2.4 we have

$$
\mathcal{B}_{p, q}^{m}\left(\mathcal{G}_{n}\right)(w)=\frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{k=0}^{m}\binom{m}{k}_{p, q} p^{\frac{(m-k)(m-k-1)}{2}}\left[\Delta^{k} \mathcal{G}_{n}(0)\right]_{p, q} e_{k}(w),
$$

for all $m, n \in \mathbb{N}$ and $\|w\| \leq \mathcal{R}$, it follows

$$
\begin{aligned}
&\left\|\mathcal{B}_{p, q}^{m}\left(\mathcal{G}_{n}\right)(w)-\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)\right\| \\
& \leq \frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{k=0}^{m}\binom{m}{k}_{p, q} p^{\frac{(m-k)(m-k-1)}{2}}\left\|\left[\left(\Delta^{k} \mathcal{G}_{n}-\mathcal{G}\right)(0)\right]_{p, q}\right\| \cdot\left\|e_{k}(w)\right\| \\
& \leq \frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{k=0}^{m}\binom{m}{k}_{p, q} p^{\frac{(m-k)(m-k-1)}{2}} \\
& \times \sum_{j=0}^{k}(-1)^{j} p^{\frac{(k-j)(k-j-1)}{2}} q^{\frac{j(j-1)}{2}}\binom{k}{j}_{p, q}\left\|\left(\mathcal{G}_{n}-\mathcal{G}\right)\left(\frac{p^{m-k+j}[k-j]_{p, q}}{[m]_{p, q}}\right)\right\| \cdot\left\|e_{m}(w)\right\| \\
& \leq \frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{k=0}^{m}\binom{m}{k}_{p, q} p^{\frac{(m-k)(m-k-1)}{2}} \sum_{j=0}^{k} p^{\frac{(k-j)(k-j-1)}{2}} q^{\frac{j(j-1)}{2}}\binom{k}{j}_{p, q} \mathcal{C}_{j, \beta}^{p, q}\left\|\left(\mathcal{G}_{n}-\mathcal{G}\right)\right\|_{r} \cdot\left\|e_{k}(w)\right\| \\
& \leq Z_{m, j}^{p, q}\left\|\left(\mathcal{G}_{n}-\mathcal{G}\right)\right\|_{r},
\end{aligned}
$$

which by $\lim _{n \rightarrow \infty}\left\|\left(\mathcal{G}_{n}-\mathcal{G}\right)\right\|_{r}=0$ implies the desired conclusion.
Here $\left\|\left(\mathcal{G}_{n}-\mathcal{G}\right)\right\|_{r}=\max \|\left(\mathcal{G}_{n}(w)-\mathcal{G}(w)\|;\| w \| \leq r\right.$.

Consequently, we obtain

$$
\begin{aligned}
& \left\|\mathcal{B}_{p, q}^{m}\left(\mathcal{G}_{n}\right)(w)-(\mathcal{G})(w)\right\| \\
& \quad \leq \sum_{i=0}^{\infty}\left\|c_{i}\right\| \cdot\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-e_{i}(w)\right\| \\
& \quad=\sum_{i=0}^{m}\left\|c_{i}\right\| \cdot\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-e_{i}(w)\right\|+\sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-e_{i}(w)\right\| .
\end{aligned}
$$

It remains to estimate $\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-e_{i}(w)\right\|$, firstly for all $0 \leq i \leq m$ and secondly for $i \geq$ $m+1$, where

$$
\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)=\frac{1}{p^{\frac{m(m-1)}{2}}} \sum_{k=0}^{m}\binom{m}{k}_{p, q} p^{\frac{(m-k)(m-k-1)}{2}}\left[\Delta^{k} e_{i}(0)\right]_{p, q} e_{k}(w) .
$$

Set

$$
\mathcal{A}_{p, q}^{m, k, i}=\frac{1}{p^{\frac{m(m-1)}{2}}}\binom{m}{k}_{p, q} p^{\frac{(m-k)(m-k-1)}{2}}\left[\Delta^{k} e_{i}(0)\right]_{p, q^{\prime}}
$$

by relationship given in [22], we can write

$$
\begin{equation*}
\mathcal{A}_{p, q}^{m, k, i}=p^{\frac{(m-k)(m-k-1)}{2}} q^{\frac{k(k-1)}{2}}\binom{m}{k}_{p, q} \frac{[k]_{p, q}!}{[m]_{p, q}}\left[0, \frac{p^{m-1}[1]_{p, q}}{[m]_{p, q}}, \ldots, \frac{p^{m-k}[k]_{p, q}}{[m]_{p, q}} ; e_{i}\right], \tag{2.8}
\end{equation*}
$$

where $\left[0, \frac{p^{m-1}[1]_{p, q}}{[m]_{p, q}}, \ldots, \frac{p^{m-k}[k]_{p, q}}{[m]_{p, q}} ; e_{i}\right]$ denotes the divided difference of $e_{i}(w)=w^{i}$.
Recall that the divided difference of a function $\mathcal{F}$ on the knots $y_{0}, y_{1}, \ldots, y_{j}$ is given by

$$
\left[y_{0}, y_{1}, \ldots, y_{j} ; \mathcal{F}\right]=\sum_{i=0}^{j} \frac{\mathcal{F}\left(y_{i}\right)}{\left(y_{i}-y_{0}\right) \ldots\left(y_{i}-y_{i-1}\right)\left(y_{i}-y_{i+1}\right) \ldots\left(y_{i}-y_{j}\right)}
$$

therefore it follows

$$
\begin{equation*}
\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)=\sum_{k=0}^{m} \mathcal{A}_{p, q}^{m, k, i} e_{k}(w) . \tag{2.9}
\end{equation*}
$$

However, by the relationship given in [22], we get the formula

$$
\begin{aligned}
& \binom{m}{k}_{p, q} \frac{[k]_{p, q}!}{[m]_{p, q}} p^{\frac{(m-k)(m-k-1)}{2}} q^{\frac{k(k-1)}{2}} \\
& \quad=\left(1-\frac{p^{m-1}[1]_{p, q}}{[m]_{p, q}}\right)\left(1-\frac{p^{m-2}[2]_{p, q}}{[m]_{p, q}}\right) \ldots\left(1-\frac{p^{m-k+1}[k-1]_{p, q}}{[m]_{p, q}}\right),
\end{aligned}
$$

which combined with the above relationship (2.8) implies

$$
\mathcal{A}_{p, q}^{m, k, i}=\left(1-\frac{p^{m-1}[1]_{p, q}}{[m]_{p, q}}\right)\left(1-\frac{p^{m-2}[2]_{p, q}}{[m]_{p, q}}\right) \cdots
$$

$$
\begin{equation*}
\times\left(1-\frac{p^{m-k+1}[k-1]_{p, q}}{[m]_{p, q}}\right)\left[0, \frac{p^{m-1}[1]_{p, q}}{[m]_{p, q}}, \ldots, \frac{p^{m-k}[k]_{p, q}}{[m]_{p, q}} ; e_{i}\right] . \tag{2.10}
\end{equation*}
$$

Since each $e_{i}$ is convex of any order and $\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(1)=e_{i}(1)=1$ for all $i$, it follows that all $\mathcal{A}_{p, q}^{m, k, i} \geq 0$ and $\sum_{k=0}^{m} \mathcal{A}_{p, q}^{m, k, i}=1$ for all $i$ and $m$.
Also, note that $\mathcal{A}_{p, q}^{m, i, i}=\left(1-\frac{p^{m-1}[1]_{p, q}}{[m]_{p, q}}\right)\left(1-\frac{p^{m-2}[2]_{p, q}}{[m]_{p, q}}\right) \ldots\left(1-\frac{p^{m-i+1}[i-1]_{p, q}}{[m]_{p, q}}\right)$ for all $i \geq 1$ and that $\mathcal{A}_{p, q}^{m, 0,0}=1$.

In the estimation of $\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-e_{i}(w)\right\|$, we distinguish two cases: (1) $0 \leq i \leq m$; (2) $i>$ $m$.

Case 1. We have

$$
\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-e_{i}(w)\right\| \leq\left\|e_{i}(w)\right\| \cdot\left|1-\mathcal{A}_{p, q}^{m, i, i}\right|+\sum_{k=0}^{i-1} \mathcal{A}_{p, q}^{m, k, i}\left\|e_{k}(w)\right\|
$$

Since $\left\|e_{k}(w)\right\| \leq r^{k}$ for all $\|w\| \leq r$ and $k \geq 0$, by [24] we immediately get

$$
\begin{aligned}
\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-e_{i}(w)\right\| & \leq 2\left[1-\mathcal{A}_{p, q}^{m, i, i}\right] r^{i} \\
& \leq 2 p^{m-i+1} \frac{(i-1)[i-1]_{p, q}}{[m]_{p, q}} r^{i} \\
& \leq 2 p^{m-i+1} \frac{i[i-1]_{p, q}}{[m]_{p, q}} r^{i} \\
& \leq 2 p^{m} \frac{i[i-1]_{p, q}}{[m]_{p, q}} r^{i}
\end{aligned}
$$

for all $\|z\|<r$.
Case 2. Here we have

$$
\begin{aligned}
\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-e_{i}(w)\right\| & \leq\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)\right\|+\left\|e_{i}(w)\right\| \\
& \leq 2 r^{i} \\
& \leq 2 p^{m-i+1} \frac{(i-1)[i-1]_{p, q}}{[m]_{p, q}} r^{i} \\
& \leq 2 p^{m} \frac{i[i-1]_{p, q}}{[m]_{p, q}} r^{i}
\end{aligned}
$$

From both of the above cases we conclude

$$
\left\|\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)\right\| \leq \frac{2 p^{m}}{[m]_{p, q}} \cdot \sum_{i=0}^{\infty}\left\|c_{i}\right\| i[i-1]_{p, q} r^{i},
$$

where $\|w\| \leq r, m \in \mathbb{N}$, which proves the desired result.

Remark Our results generalize the results of Gal [8] (see also [9]), which can be obtained by taking $p=1$ in our results. Taking an extra parameter $p$ gives more flexibility to study a general class of positive linear operators. By taking $q>p=1$ in Theorem 2.7, we get the estimate of Theorem 2.6.

## 3 Voronovskaja type result

Theorem 3.1 Suppose that $1 \leq p \leq q<\mathcal{R}$ and $\mathcal{G}: \mathbb{D}_{\mathcal{R}} \longrightarrow \mathbb{H}$ is left $\mathcal{W}$-analytic in $\mathbb{D}_{\mathcal{R}}$, i.e., $\mathcal{G}(w)=\sum_{k=0}^{\infty} c_{k} w^{k}$, for all $w \in \mathbb{D}_{\mathcal{R}}$, where $c_{k} \in \mathbb{H}$ for all $k=0,1,2, \ldots$. Also, denote $\mathcal{S}_{p, q}^{i}=$ $[1]_{p, q}+[2]_{p, q}+\cdots+[i-1]_{p, q}, i \geq 2$.
(i) If $q>p \geq 1,1<r<\frac{p \mathcal{R}}{q^{2}},\|w\| \leq r$, and $m \in \mathbb{N}$, then the following upper estimate

$$
\left\|\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)-\sum_{i=2}^{\infty} c_{i} \cdot \frac{\mathcal{S}_{p, q}^{i}}{[m]_{p, q}}\left[w^{i-1}-w^{i}\right]\right\| \leq \frac{\mathcal{C}_{p, q}^{r}(\mathcal{G})}{[m]_{p, q}^{2}}
$$

holds, where $\mathcal{C}_{p, q}^{r}(\mathcal{G})=\max \left\{\frac{p^{m-n+1}}{(q-p)(q-1)}, \frac{p^{m-n+1}}{(q-p)^{2}(q-1)}\right\} \sum_{i=2}^{\infty}\left\|c_{i}\right\| .(i-1)^{2}\left(q^{2} r\right)^{i}$.
(ii) If $q \geq p \geq 1$, then for any $1<r<\frac{p \mathcal{R}}{q}$, we have

$$
\lim _{m \rightarrow \infty}[m]_{p, q}\left(\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)\right)=E_{p, q}(\mathcal{G})(w),
$$

uniformly in $\overline{\mathbb{D}}_{r}$, where $E_{p, q}(\mathcal{G})(w)=\sum_{i=2}^{\infty} c_{i} \mathcal{S}_{p, q}^{i}\left[w^{i-1}-w^{i}\right]$, $w \in \mathbb{H}$.
Proof First we recall some important relationship for our proof. Let $1<r<\frac{p \mathcal{R}}{q}$.

$$
\begin{align*}
& \mathcal{S}_{p, q}^{i}=\frac{i(i-1)}{2}, \quad \text { for } p=q=1, \\
& \mathcal{S}_{p, q}^{i}=\frac{q^{i}-i(q-1)-1}{(q-1)^{2}}, \quad \text { for } p=1 \text { and } q>1,  \tag{3.1}\\
& \mathcal{S}_{p, q}^{i}=\frac{1}{(q-p)}\left[\frac{q^{i}-q}{(q-1)}-\frac{p^{i}-p}{(p-1)}\right], \quad \text { for } q>p>1 .
\end{align*}
$$

From Theorem 2.7, we get

$$
\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)=\sum_{k=0}^{m} \mathcal{A}_{p, q}^{m, k, i} e_{k}(w),
$$

where

$$
\mathcal{A}_{p, q}^{m, k, i}=\binom{m}{k}_{p, q} \frac{[k]_{p, q}!}{[m]_{p, q}} p^{\frac{(m-k)(m-k-1)}{2}} q^{\frac{k(k-1)}{2}}\left[0, \frac{p^{m-1}[1]_{p, q}}{[m]_{p, q}}, \ldots, \frac{p^{m-k}[k]_{p, q}}{[m]_{p, q}} ; e_{i}\right],
$$

here we know $\mathcal{A}_{p, q}^{m, k, i} \geq 0$ for all $0 \leq k \leq m, i \geq 0$ and $\sum_{k=0}^{i} \mathcal{A}_{p, q}^{m, k, i}=1$ for all $0 \leq i \leq m$

$$
\begin{align*}
& \mathcal{A}_{p, q}^{m, i, i}=\prod_{l=1}^{i-1}\left(1-\frac{p^{m-i+1}[l]_{p, q}}{[m]_{p, q}}\right), \\
& \mathcal{A}_{p, q}^{m, i-1, i}=\frac{\mathcal{S}_{p, q}^{i}}{[m]_{p, q}} \cdot \prod_{l=1}^{i-2}\left(1-\frac{p^{m-i+2}[l]_{p, q}}{[m]_{p, q}}\right), \quad i \leq m . \tag{3.2}
\end{align*}
$$

First, we need to prove that $E_{p, q}(\mathcal{G})(w)$ is left $\mathcal{W}$-analytic in $\overline{\mathbb{D}}_{r}$, where

$$
E_{p, q}(\mathcal{G})(w)=\sum_{i=2}^{\infty} c_{i} \cdot \mathcal{S}_{p, q}^{i} \cdot\left[w^{i-1}-w^{i}\right]
$$

for $1<r<\frac{p \mathcal{R}}{q}$, using the inequality

$$
\left\|E_{p, q}(\mathcal{G})(w)\right\| \leq \sum_{i=0}^{\infty}\left\|c_{i}\right\| \cdot \mathcal{S}_{p, q}^{i} \cdot\left[\left\|w^{i-1}\right\|+\left\|w^{i}\right\|\right] .
$$

By (3.1), $\mathcal{S}_{p, q}^{i} \leq \frac{q^{i}}{(q-p)(q-1)}$ for $q>p>1$, it immediately follows

$$
\begin{aligned}
& \left\|E_{p, q}(\mathcal{G})(w)\right\| \leq \frac{2}{(q-p)(q-1)} \sum_{i=0}^{\infty}\left\|c_{i}\right\|(q r)^{i}<\infty, \quad \text { if } q>p>1, \\
& \left\|E_{p, q}(\mathcal{G})(w)\right\| \leq \frac{2}{(q-1)^{2}} \sum_{i=0}^{\infty}\left\|c_{i}\right\|(q r)^{i}<\infty, \quad \text { if } q>1 \text { and } p=1, \\
& \left\|E_{p, q}(\mathcal{G})(w)\right\| \leq \sum_{i=0}^{\infty}\left\|c_{i}\right\| i(i-1)(q r)^{i}<\infty, \quad \text { if } q=p=1,
\end{aligned}
$$

for all $w \in \overline{\mathbb{D}}_{r}$. These show that, for $q \geq p \geq 1$, the function $E_{p, q}(\mathcal{G})(w)$ is well-defined and left $\mathcal{W}$-analytic in $\overline{\mathbb{D}}_{r}$.

By (2.7) we obtain

$$
\left\|\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)-\sum_{i=2}^{\infty} c_{i} \cdot \frac{\mathcal{S}_{p, q}^{i}}{[m]_{p, q}}\left[w^{i-1}-w^{i}\right]\right\| \leq \sum_{i=0}^{\infty}\left\|c_{i}\right\| \cdot\left\|L_{p, q}^{i, m}(w)\right\|,
$$

where

$$
L_{p, q}^{i, m}=\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-w^{i}-\frac{\mathcal{S}_{p, q}^{i}}{[m]_{p, q}}\left[w^{i-1}-w^{i}\right]
$$

and

$$
L_{p, q}^{0, m}=L_{p, q}^{1, m}=L_{p, q}^{2, m}=0 .
$$

We have to estimate the expression

$$
\sum_{i=3}^{\infty}\left\|c_{i}\right\| \cdot\left\|L_{p, q}^{i, m}(w)\right\|=\sum_{i=3}^{m}\left\|c_{i}\right\| \cdot\left\|L_{p, q}^{i, m}(w)\right\|+\sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left\|L_{p, q}^{i, m}(w)\right\| .
$$

To estimate $\left\|L_{p, q}^{i, m}(w)\right\|$, we discuss two cases: 1$) 3 \leq i \leq m$; 2) $i \geq m+1$.
Case (1). We obtain

$$
\begin{aligned}
{[m]_{p, q}\left\|L_{p, q}^{i, m}(w)\right\|=} & \left\|[m]_{p, q}\left(\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-w^{i}\right)-\mathcal{S}_{p, q}^{i} \cdot\left(w^{i-1}-w^{i}\right)\right\| \\
\leq & r^{i}[m]_{p, q} \sum_{l=1}^{i-2} \mathcal{A}_{p, q}^{m, l, i}+\left|[m]_{p, q} \mathcal{A}_{p, q}^{m, i-1, i}-\mathcal{S}_{p, q}^{i}\right| r^{i} \\
& +\left|[m]_{p, q}\left(1-\mathcal{A}_{p, q}^{m, i, i}\right)-\mathcal{S}_{p, q}^{i}\right| r^{i} .
\end{aligned}
$$

Taking into account (3.1), (3.2) and following Lemma 2.3, we arrive at

$$
\begin{align*}
& \left\|L_{p, q}^{i, m}(w)\right\| \leq \frac{4 p^{m-n+1}(i-1)^{2}[i-1]_{p, q}^{2}}{[m]_{p, q}^{2}} \cdot r^{i}  \tag{3.3}\\
& \sum_{i=3}^{m}\left\|c_{i}\right\| \cdot\left\|L_{p, q}^{i, m}(w)\right\| \leq \frac{4 p^{m-n+1}}{[m]_{p, q}^{2}} \cdot \sum_{i=3}^{m}\left\|c_{i}\right\|(i-1)^{2}[i-1]_{p, q}^{2} q^{i} \tag{3.4}
\end{align*}
$$

for all $w \in \overline{\mathbb{D}}_{r}$ and $m \in \mathbb{N}$.
In the case of complex variable, the estimate in (3.4) remains exactly the same, with $\|\cdot\|$ replaced by |.|, because all the calculations and estimates are made with real numbers as calculated in [6].

Case (2). Here we get

$$
\begin{align*}
\sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left\|L_{p, q}^{i, m}(w)\right\| \leq & \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)\right\|+\sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left\|w^{i}\right\| \\
& +\frac{1}{[m]_{p, q}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot S_{p, q}^{i} \cdot\left\|w^{i-1}\right\|+\frac{1}{[m]_{p, q}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot \mathcal{S}_{p, q}^{i} \cdot\left\|w^{i}\right\| \\
= & D_{p, q}^{1}(w)+D_{p, q}^{2}(w)+D_{p, q}^{3}(w)+D_{p, q}^{4}(w) . \tag{3.5}
\end{align*}
$$

By (3.2), for all $w \in \overline{\mathbb{D}}_{r}$, it follows

$$
\begin{aligned}
D_{p, q}^{1}(w) & \leq \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot \sum_{k=0}^{m} \mathcal{A}_{p, q}^{m, k i}\left\|w^{k}\right\| \\
& \leq \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot r^{i} \\
& \leq \frac{1}{[m]_{p, q}^{2}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\|[i-1]_{p, q}^{2} \cdot r^{i} \\
& \leq \frac{1}{(q-p)^{2}[m]_{p, q}^{2}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left(q^{2} r\right)^{i}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
D_{p, q}^{2}(w) & \leq \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot r^{i} \\
& \leq \frac{1}{[m]_{p, q}^{2}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\|[i-1]_{p, q}^{2} \cdot r^{i} \\
& \leq \frac{1}{(q-p)^{2}[m]_{p, q}^{2}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left(q^{2} r\right)^{i} .
\end{aligned}
$$

Also, by (3.1) we get $\mathcal{S}_{p, q}^{i} \leq \frac{q^{i}}{(q-p)(q-1)}$, for all $w \in \overline{\mathbb{D}}_{r}$ it follows

$$
\begin{aligned}
D_{p, q}^{3}(w) & \leq \frac{1}{(q-p)(q-1)[m]_{p, q}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot q^{i} r^{i} \\
& =\frac{1}{(q-p)(q-1)[m]_{p, q}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot(q r)^{i} \\
& \leq \frac{1}{(q-p)^{2}(q-1)[m]_{p, q}^{2}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left(q^{2} r\right)^{i}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
D_{p, q}^{4}(w) & \leq \frac{1}{[m]_{p, q}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot q^{i} r^{i} \\
& \leq \frac{1}{(q-p)^{2}(q-1)[m]_{p, q}^{2}} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left(q^{2} r\right)^{i}
\end{aligned}
$$

By (3.5) we obtain

$$
\begin{equation*}
\sum_{i=m+1}^{\infty}\left|c_{i}\right| \cdot\left|L_{p, q}^{i, m}(w)\right| \leq \frac{\mathcal{C}_{p, q}^{r}(\mathcal{G})}{[m]_{p, q}^{2}} \tag{3.6}
\end{equation*}
$$

for all $w \in \overline{\mathbb{D}}_{r}$, where

$$
\mathcal{C}_{p, q}^{r}(\mathcal{G})=\max \left\{\frac{1}{(q-p)(q-1)}, \frac{1}{(q-p)^{2}(q-1)}\right\} \sum_{i=m+1}^{\infty}\left\|c_{i}\right\| \cdot\left(q^{2} r\right)^{r} .
$$

Since $[i-1]_{p, q}^{2} \leq[i]_{p, q}^{2} \leq \frac{q^{2 i}}{(q-p)^{2}(q-1)^{2}}$ for $w \in \overline{\mathbb{D}}_{r}$ with $1<r<\frac{p \mathcal{R}}{q}$, we obtain by (3.4)

$$
\begin{equation*}
\sum_{i=3}^{m}\left\|c_{i}\right\| \cdot\left\|L_{p, q}^{i, m}(w)\right\| \leq \frac{4 p^{m-n+1}}{(q-p)^{2}(q-1)^{2}[m]_{p, q}^{2}} \cdot \sum_{i=3}^{m}\left\|c_{i}\right\|(i-1)^{2}\left(q^{2} r\right)^{i} \tag{3.7}
\end{equation*}
$$

We immediately obtain the upper estimate in (i) by collecting (3.6) and (3.7).
(ii) For the case $1<r<\frac{p \mathcal{R}}{q^{2}}$ and $q>p \geq 1$, we can get the desired conclusion by multiplying (i) with $[m]_{p, q}$ and passing the limit $m \longrightarrow \infty$. But (ii) holds under a more general condition $1<r<\frac{p \mathcal{R}}{q}$.

$$
\begin{aligned}
& \left\|[m]_{p, q}\left(\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)\right)-E_{p, q}(\mathcal{G})(w)\right\| \\
& \quad \leq \sum_{i=2}^{m_{0}}\left\|c_{i}\right\| \cdot\left\|[m]_{p, q}\left(\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-e_{i}(w)\right)-\mathcal{S}_{p, q}^{i}\left(w^{i-1}-w^{i}\right)\right\| \\
& \quad+\sum_{i=m_{0}+1}^{\infty}\left\|c_{i}\right\| \cdot\left([m]_{p, q}\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-w^{i}\right\|+\mathcal{S}_{p, q}^{i}\left\|w^{i-1}-w^{i}\right\|\right) \\
& \quad \leq \sum_{i=2}^{m_{0}}\left\|c_{i}\right\| \cdot \frac{4(i-1)^{2}[i-1]_{p, q}^{2}}{[m]_{p, q}} \cdot r^{i}
\end{aligned}
$$

$$
+\sum_{i=m_{0}+1}^{\infty}\left\|c_{i}\right\| \cdot\left([m]_{p, q}\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-w^{i}\right\|+\mathcal{S}_{p, q}^{i}\left\|w^{i-1}-w^{i}\right\|\right)
$$

But by Theorem 2.7,

$$
\left\|\mathcal{B}_{p, q}^{m}\left(e^{i}\right)(w)-w^{i}\right\| \leq \frac{i[i-1]_{p, q}}{[m]_{p, q}} . r^{i},
$$

while for $i>m$, using (3.2), we have

$$
\begin{aligned}
\left\|\mathcal{B}_{p, q}^{m}\left(e^{i}\right)(w)-w^{i}\right\| & \leq\left\|\mathcal{B}_{p, q}^{m}\left(e^{i}\right)(w)\right\|+\left\|w^{i}\right\| \\
& \leq \sum_{k=0}^{m} \mathcal{A}_{p, q}^{m, k, i}\left\|w^{k}\right\|+\left\|w^{i}\right\| \\
& \leq r^{m}+r^{i} \leq 2 r^{i} \leq 2 \frac{i[i-1]_{p, q}}{[m]_{p, q}} \cdot r^{i}
\end{aligned}
$$

for all $w \in \overline{\mathbb{D}}_{r}$.
Also, since $\mathcal{S}_{p, q}^{i} \leq(i-1)[i-1]_{p, q}$, it is immediate that

$$
\mathcal{S}_{p, q}^{i}\left\|w^{i-1}-w^{i}\right\| \leq \mathcal{S}_{p, q}^{i} \cdot\left\|w^{i-1}\right\|+\left\|w^{i}\right\| \leq 2(i-1)[i-1]_{p, q} r^{i} .
$$

Therefore, we easily obtain

$$
\begin{aligned}
& \sum_{i=m_{0}+1}^{\infty}\left\|c_{i}\right\| \cdot\left([m]_{p, q}\left\|\mathcal{B}_{p, q}^{m}\left(e_{i}\right)(w)-w^{i}\right\|+\mathcal{S}_{p, q}^{i}\left\|w^{i-1}-w^{i}\right\|\right) \\
& \quad \leq 2 \sum_{i=m_{0}+1}^{\infty}\left\|c_{i}\right\| \cdot(i-1)[i-1]_{p, q} r^{i},
\end{aligned}
$$

valid for all $w \in \overline{\mathbb{D}}_{r}$.
For all $w \in \overline{\mathbb{D}}_{r}$ and $m>m_{0}$, we have

$$
\begin{aligned}
& \left\|[m]_{p, q} \mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)-E_{p, q}(\mathcal{G})(w)\right\| \\
& \leq \sum_{i=2}^{m_{0}}\left\|c_{i}\right\| \cdot \frac{4(i-1)^{2}[i-1]_{p, q}^{2}}{[m]_{p, q}} \cdot r^{i} \\
& \quad+2 \sum_{i=m_{0}+1}^{\infty}\left\|c_{i}\right\| \cdot(i-1)[i-1]_{p, q} r^{i} \\
& \leq \frac{4}{[m]_{p, q}^{t}} \cdot \sum_{i=2}^{m_{0}}\left\|c_{i}\right\| \cdot i^{2}[i-1]_{p, q}^{1+t} \cdot r^{i} \\
& \quad+2 \sum_{i=m_{0}+1}^{\infty}\left\|c_{i}\right\| \cdot i^{2} q^{i} r^{i} \\
& \leq \frac{4}{[m]_{p, q}^{t}} \cdot \sum_{i=2}^{m_{0}}\left\|c_{i}\right\| \cdot i^{4} q^{(1+t)^{i}} . r^{i}+2 \epsilon .
\end{aligned}
$$

Now, since $\frac{4}{[m]_{p, q}^{t}} \longrightarrow 0$ as $m \longrightarrow \infty$ and $\sum_{i=2}^{m_{0}}\left\|c_{i}\right\| . i^{4} q^{(1+t)^{i} .} .^{i}<\infty$, for given $\epsilon>0$, there exists an index $m_{1}$ such that $\frac{4}{[m]_{p, q}^{t}} \cdot \sum_{i=2}^{m_{0}}\left\|c_{i}\right\| . i^{4} q^{(1+t)^{i}} . r^{i}<\epsilon$ for all $m>m_{1}$.

Finally, for all $m>\max \left\{m_{0}, m_{1}\right\}$ and $w \in \overline{\mathbb{D}}_{r}$, we get

$$
\left\|[m]_{p, q} \mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)-E_{p, q}(\mathcal{G})(w)\right\| \leq 3 \epsilon
$$

which shows that

$$
\lim _{m \rightarrow \infty}[m]_{p, q}\left(\mathcal{B}_{p, q}^{m}(\mathcal{G})(w)-\mathcal{G}(w)\right)=E_{p, q}(\mathcal{G})(w), \quad \text { uniformly in } \overline{\mathbb{D}}_{r} .
$$

The theorem is proved.

Corollary 3.2 Let $1<p<q<\mathcal{R}$ and $\mathcal{G}: \mathbb{D}_{\mathcal{R}} \longrightarrow \mathbb{H}$ be right $\mathcal{W}$-analytic in $\mathbb{D}_{\mathcal{R}}$, i.e., $\mathcal{G}(w)=$ $\sum_{k=0}^{\infty} w^{k} c_{k}$, for all $w \in \mathbb{D}_{\mathcal{R}}$, where $c_{k} \in \mathbb{H}$ for all $k=0,1,2, \ldots$. Then, for all $1 \leq r<\frac{p \mathcal{R}}{q},\|w\| \leq$ $r$, and $n \in \mathbb{N}$, we have

$$
\left\|\mathcal{B}_{p, q}^{m *}(\mathcal{G})(w)-\mathcal{G}(w)\right\| \leq \frac{2 p^{m}}{[m]_{p, q}} \cdot \sum_{i=1}^{\infty}\left\|c_{i}\right\| i[i-1]_{p, q} r^{i}
$$

Remarks (i) However, it is easy to observe that the middle $(p, q)$-Bernstein type polynomials $\mathcal{B}_{p, q}^{m * *}(\mathcal{G})(w)$ cannot be obtained as an estimate of the form in Theorem 2.7, because when $\mathcal{G}$ is right $\mathcal{W}$-analytic or left $\mathcal{W}$-analytic, it cannot be written for the middle $(p, q)$ Bernstein type polynomials $\mathcal{B}_{p, q}^{m * *}(\mathcal{G})(w)=\sum_{k=0}^{\infty} c_{k} \mathcal{B}_{p, q}^{m * *}\left(e_{k}\right)(w)$.
(ii) Since the choice of $p>1$ assures that $\frac{p \mathcal{R}}{q}>\frac{\mathcal{R}}{q}$, this implies that the approximation estimated by $(p, q)$-Bernstein operators in Theorem 2.7 holds in larger disks than those in the case when $p=1$.

For the case $p=q=1$, the ordinary Bernstein operators for quaternion variable have order of approximation $\frac{1}{m}$ (see Gal [6]), which is weaker than the case of $q>p=1$, i.e., $\frac{1}{q^{m}}$ (see [8]).
However, $\frac{p^{m}}{[m]_{p, q}}=(q-p) \frac{p^{m}}{q^{m}-p^{m}}$ implies that the order of approximation of $(p, q)$-Bernstein operators for quaternion variable is $\left(\frac{p}{q}\right)^{m}$, which is also weaker than $\frac{1}{q^{m}}$.

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## Competing interests

The authors declare that they have no competing interests.

## Authors' contributions

The first author (HBJ) has 50\% contribution, the third author (AN) has also 50\% contribution while the second author (MM) checked and gave the final shape to the paper. All authors read and approved the final manuscript.

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