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Existence and Hyers–Ulam stability for three-point boundary value problems with Riemann–Liouville fractional derivatives and integrals

Lei Xu[1], Qixiang Dong[1*] and Gang Li[1]

*Correspondence: qxdongyz@outlook.com; qxdong@yzu.edu.cn
[1]School of Mathematical Sciences, Yangzhou University, Yangzhou, China

Abstract

This paper is concerned with a class of two-term fractional differential equations. Three-point boundary value problems with mixed Riemann–Liouville fractional differential and integral boundary conditions are discussed. The Green's function is investigated and the existence results are obtained based on some fixed point theorems. The Hyers–Ulam stability is also studied for null boundary conditions. As an auxiliary result, a Gronwall type inequality of fractional order integral is obtained.

Keywords: Fractional differential equation; Boundary value problem; Stability

1 Introduction

In this paper, we consider the following boundary value problem of nonlinear fractional differential equations:

$$\begin{cases} \lambda D_0^{\alpha} x(t) + D_0^{\beta} x(t) = f(t, x(t)), \quad 0 < t < T, \\ x(0) = 0, \qquad \mu D_0^{\gamma_1} x(T) + I_0^{\gamma_2} x(\eta) = \gamma_3, \end{cases} \tag{1}$$

where D_0^{α} and D_0^{β} are Riemann–Liouville fractional derivatives with $1 < \alpha \le 2$ and $1 \le \beta < \alpha$, $0 < \lambda \le 1$, $0 \le \mu \le 1$, $0 \le \gamma_1 \le \alpha - \beta$, $\gamma_2 \ge 0$, $0 < \eta < T$ and $f : [0, T] \times R \to R$ is a given function satisfying some assumptions that will be verified later.

Fractional differential problems have attracted much attention in recent years due to its wide application in many fields of science and engineering, including fractal theory, potential theory, biology, chemistry, diffusion, etc. See, e.g., [1, 2]. There are several kinds of fractional derivatives used in different applied areas, such as Caputo derivative and Riemann–Liouville derivative. Recently, Atangana et al. [3–6] presented the Caputo–Fabrizio and the Atangana–Baleanu fractional derivatives.

In the literature, ${}^cD_{0+}^{\alpha} u(t) + f(t, u(t)) = 0$ is known as a single term equation. This kind of fractional differential equation has many applications and has been studied widely. See, e.g., [1, 2, 7–10]. Equations containing more than one fractional differential terms are called multi-term fractional differential equations; they have some concrete applications

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in many fields. Due to the complexity of such a kind of equations, it seems that there has been no result for a general multi-term fractional differential equation. Only some special cases have been investigated. A classical example is the so-called Bagley–Torvik equation (B–T equation for short) [11],

$$Ay''(t) + B^c D_0^{\frac{3}{2}} y(t) + Cy(t) = f(t),$$

where A, B and C are certain constants and f is a given function. This equation arises from the mathematical model of the motion of a thin plate in a Newtonian fluid. The B–T equation, as well as various generalizations, has wide applications in fluid dynamics and hence has attracted much attention. For example, Cermak et al. [12] investigated the two-term fractional differential equation

$$D_0^{\alpha} y(t) + aD_0^{\beta} y(t) + yb(t) = 0$$

with real coefficients a, b and positive real orders $\alpha > \beta$, which contains some important cases such as the B–T equation for $\alpha = 2$, $\beta = 3/2$ and the Basset equation for $\alpha = 1$, $\beta = 1/2$. The analytic solution and the numerical solution for the B–T equation were studied in [13] and [14], respectively. Various methods were introduced to investigate the approximate solutions such as the finite difference method [15], the variational iteration method [13, 16], the homotopy perturbation method [17] and the generalized differential transform method [18]. Boundary value problems for the B–T equations were studied in [19, 20] and [18, 21–23] for various boundary value conditions. In [18], the authors considered the approximate solution of B–T equations with variable coefficients and three-point boundary value,

$$\begin{cases} y''(x) + p(x)D_a^{\alpha} y + q(x)y = g(x), & x \in [a,b], \\ y(a) = \alpha_1, \quad y(b) + \lambda y(\xi) = \beta_1, & \xi \in [a,b], \end{cases}$$

where $0 < \alpha < 2$, $p(x)$, $q(x)$ and $g(x)$ are known functions, α_1, β_1, λ, μ and ξ are given constants. In [19, 20], Ntouyas et al. studied the generalized B–T equations with multiple integral and differential boundary conditions

$$\begin{cases} \lambda D_0^{\alpha} x(t) + (1-\lambda) D_0^{\beta} x(t) = f(t, x(t)), & 0 < t < T, \\ x(0) = 0, \quad \mu D_0^{\gamma_1}(T) + (1-\mu) D_0^{\gamma_2}(T) = \gamma_3, \end{cases}$$

and

$$\begin{cases} \lambda D_0^{\alpha} x(t) + (1-\lambda) D_0^{\beta} x(t) = f(t, x(t)), & 0 < t < T, \\ x(0) = 0, \quad \mu I_0^{\delta_1} x(T) + (1-\mu) I_0^{\delta_2} x(T) = \delta_3. \end{cases}$$

Green functions for the corresponding problems were investigated and the existence results were obtained by using fixed point theorems. For more detailed information, we refer the reader to [2, 24–26] and the references therein.

Stability analysis is an important respect of differential equations. Hyers–Ulam stability for differential equations was initialed in the 1940s by Ulam [27] and Hyers [28]. Roughly speaking, the Hyers–Ulam stability for a differential equation is the answer to the question whether there is an exact solution near an approximate solution (the solution to the approximate equation) to the differential equation. So it is obviously important for the study of numerical and approximate solutions and real world applications of differential equations. For this reason, many researchers investigate the Hyers–Ulam stability for differential equations of both integer and fractional order [29–32]. There are only a few works on the Ulam stability of fractional differential equations. Recently, Wang, Lv and Zhou [33] presented some Ulam stability results of fractional differential equations by using Hery–Gronwall inequality. By the method of Laplace transform, Wang and Li [34] studied Ulam stability of a fractional order linear differential equation. Later, Wang and Li [35] investigated the Hyers–Ulam stability for a nonlinear fractional Langevin equation and its corresponding impulsive problem. The authors of [36] researched a class of new differential equations with no instantaneous impulses. However, to the best of our knowledge, few results can be found on the Hyers–Ulam type stability for boundary value problems except that of Kumam [37].

Inspired by the above comments, in this paper, we consider the three points boundary value problem of a two-term fractional differential equation, with mixed integral and differential boundary conditions (1). Equation (1) with Riemann–Liouville fractional derivatives of order α and β is a generalization of the B–T equation. To compare with the discussion in [11, 19], there is no coefficient $(1-\lambda)$ or $(1-\mu)$ here, and the boundary conditions with mixed derivatives and integrals are different from the boundary value conditions in [11, 19]. So, our results can be regarded as an extension of B–T equation and partially extend the results in [11, 19, 38] and [20]. We also investigate the Hyers–Ulam stability for Eq. (1) with the boundary value conditions $x(0) = 0$ and $x(T) = \gamma$. To deal with the Hyers–Ulam stability, we prove a fractional Gronwall-type integral inequality by the method of iteration, which is a generalization of the main result in [39] and Lemma 3.4 in [10].

2 Preliminaries

Let $C([0,T],\mathbf{R})$ be the Banach space of all continuous functions from $[0,T]$ into $\mathbf{R}$ with the norm $\|x\|_\infty = \max_{t\in[0,T]} |x(t)|$, and $L^1([0,T],\mathbf{R})$ denotes the Banach space of functions $x:[0,T]\to\mathbf{R}$ that are Lebesgue integrable with norm $\|x\|_{L^1} = \int_0^T |x(t)|\,dt$.

Definition 2.1 ([1]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f:[a,b]\to\mathbf{R}$ at the point t is defined by

$$I_a^\alpha f(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s)\,ds,$$

provided the right side is point-wisely defined, where Γ is the Gamma function.

Definition 2.2 ([1]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $f:[a,b]\to\mathbf{R}$ at the point t is defined by

$$D_a^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\alpha-1} f(s)\,ds,$$

provided the right side is point-wisely defined, where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Lemma 2.3 ([1]) *Let $\alpha > 0$, and $x \in C[0,T] \cap L[0,T]$. Then the fractional differential equation $D_0^\alpha x(t) = 0$ has the unique solution*

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \quad c_i \in \mathbf{R}, i = 0,1,2,\ldots,n-1, n = [\alpha]+1.$$

Lemma 2.4 ([1]) *Let $\alpha > 0$. Then for $x \in C(0,T) \cap L(0,T)$, we have*

$$I_0^\alpha D_0^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbf{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\alpha]+1$.

Lemma 2.5 *The solution of the boundary value problem (1) satisfies the integral equation*

$$x(t) = \frac{\gamma_3}{\Theta} t^{\alpha-1} + \int_0^T G_1(t,s)x(s)\,ds + \int_0^T G_2(t,s)f\bigl(s,x(s)\bigr)\,ds,$$

where

$$\Theta = \frac{\mu\Gamma(\alpha)T^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)} + \frac{\Gamma(\alpha)\eta^{\alpha+\gamma_2-1}}{\Gamma(\alpha+\gamma_2)}, \quad 0 < t < T, \tag{2}$$

$$G_{11}(t,s) = \begin{cases} -\frac{1}{\lambda\Gamma(\alpha-\beta)}(t-s)^{\alpha-\beta-1} \\ \quad + \frac{t^{\alpha-1}}{\Theta\lambda}[\frac{\mu}{\Gamma(\alpha-\beta-\gamma_1)}(T-s)^{\alpha-\beta-\gamma_1-1} \\ \quad + \frac{1}{\Gamma(\alpha-\beta+\gamma_2)}(\eta-s)^{\alpha-\beta+\gamma_2-1}], & 0 < s \le t \le \eta < T, \\ \frac{t^{\alpha-1}}{\Theta\lambda}[\frac{\mu}{\Gamma(\alpha-\beta-\gamma_1)}(T-s)^{\alpha-\beta-\gamma_1-1} \\ \quad + \frac{1}{\Gamma(\alpha-\beta+\gamma_2)}(\eta-s)^{\alpha-\beta+\gamma_2-1}], & 0 < t \le s \le \eta < T, \\ \frac{\mu t^{\alpha-1}}{\Theta\lambda\Gamma(\alpha-\beta-\gamma_1)}(T-s)^{\alpha-\beta-\gamma_1-1}, & 0 < t \le \eta \le s < T, \end{cases} \tag{3}$$

$$G_{12}(t,s) = \begin{cases} -\frac{1}{\lambda\Gamma(\alpha-\beta)}(t-s)^{\alpha-\beta-1} \\ \quad + \frac{t^{\alpha-1}}{\Theta\lambda}[\frac{\mu}{\Gamma(\alpha-\beta-\gamma_1)}(T-s)^{\alpha-\beta-\gamma_1-1} \\ \quad + \frac{1}{\Gamma(\alpha-\beta+\gamma_2)}(\eta-s)^{\alpha-\beta+\gamma_2-1}], & 0 < s \le \eta \le t < T, \\ -\frac{1}{\lambda\Gamma(\alpha-\beta)}(t-s)^{\alpha-\beta-1} + \frac{\mu t^{\alpha-1}}{\Theta\lambda\Gamma(\alpha-\beta-\gamma_1)}(T-s)^{\alpha-\beta-\gamma_1-1}, & 0 < \eta \le s \le t < T, \\ \frac{\mu t^{\alpha-1}}{\Theta\lambda\Gamma(\alpha-\beta-\gamma_1)}(T-s)^{\alpha-\beta-\gamma_1-1}, & 0 < \eta \le t \le s < T, \end{cases} \tag{4}$$

$$G_{21}(t,s) = \begin{cases} \frac{1}{\lambda\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{t^{\alpha-1}}{\Theta\lambda}[\frac{\mu}{\Gamma(\alpha-\gamma_1)}(T-s)^{\alpha-\gamma_1-1} \\ \quad + \frac{1}{\Gamma(\alpha+\gamma_2)}(\eta-s)^{\alpha+\gamma_2-1}], & 0 < s \le t \le \eta < T, \\ -\frac{t^{\alpha-1}}{\Theta\lambda}[\frac{\mu}{\Gamma(\alpha-\gamma_1)}(T-s)^{\alpha-\gamma_1-1} \\ \quad + \frac{1}{\Gamma(\alpha+\gamma_2)}(\eta-s)^{\alpha+\gamma_2-1}], & 0 < t \le s \le \eta < T, \\ -\frac{\mu t^{\alpha-1}}{\Theta\lambda\Gamma(\alpha-\gamma_1)}(T-s)^{\alpha-\gamma_1-1}, & 0 < t \le \eta \le s < T, \end{cases} \tag{5}$$

$$G_{22}(t,s) = \begin{cases} \frac{1}{\lambda\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{t^{\alpha-1}}{\Theta\lambda}[\frac{\mu}{\Gamma(\alpha-\gamma_1)}(T-s)^{\alpha-\gamma_1-1} \\ \quad + \frac{1}{\Gamma(\alpha+\gamma_2)}(\eta-s)^{\alpha+\gamma_2-1}], & 0 < s \le \eta \le t < T, \\ \frac{1}{\lambda\Gamma(\alpha)}(t-s)^{\alpha-1} - \frac{\mu t^{\alpha-1}}{\Theta\lambda\Gamma(\alpha-\gamma_1)}(T-s)^{\alpha-\gamma_1-1}, & 0 < \eta \le s \le t < T, \\ -\frac{\mu t^{\alpha-1}}{\Theta\lambda\Gamma(\alpha-\gamma_1)}(T-s)^{\alpha-\gamma_1-1}, & 0 < \eta \le t \le s < T, \end{cases} \tag{6}$$

$$G_1(t,s) = \begin{cases} G_{11}(t,s), & 0 < t \le \eta < T, \\ G_{12}(t,s), & 0 < \eta \le t < T, \end{cases} \tag{7}$$

$$G_2(t,s) = \begin{cases} G_{21}(t,s), & 0 < t \le \eta < T, \\ G_{22}(t,s), & 0 < \eta \le t < T, \end{cases} \tag{8}$$

Proof From Eq. (1), we have

$$D_0^{\alpha} x(t) = -\frac{1}{\lambda} D_0^{\beta} x(t) + \frac{1}{\lambda} f\bigl(t, x(t)\bigr), \quad t \in [0,T]. \tag{9}$$

Taking the Riemann–Liouville fractional integral of order α on both sides of (9), we get

$$x(t) = -\frac{1}{\lambda \Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s)\,ds + \frac{1}{\lambda \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\bigl(s, x(s)\bigr)\,ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}.$$

Then, $x(0) = 0$ implies $C_2 = 0$. Hence

$$x(t) = -\frac{1}{\lambda \Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} x(s)\,ds + \frac{1}{\lambda \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\bigl(s, x(s)\bigr)\,ds + C_1 t^{\alpha-1}. \tag{10}$$

Applying the Riemann–Liouville fractional derivative of order γ_1 to both sides of (10), we deduce that

$$\begin{aligned} D_0^{\gamma_1} x(t) = &-\frac{1}{\lambda \Gamma(\alpha-\beta-\gamma_1)} \int_0^t (t-s)^{\alpha-\beta-\gamma_1-1} x(s)\,ds \\ &+ \frac{1}{\lambda \Gamma(\alpha-\gamma_1)} \int_0^t (t-s)^{\alpha-\gamma_1-1} f\bigl(s, x(s)\bigr)\,ds \\ &+ C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)} t^{\alpha-\gamma_1-1}. \end{aligned}$$

Taking the Riemann–Liouville integral of order γ_2 to both sides of (10), we obtain

$$\begin{aligned} I_0^{\gamma_2} x(t) = &-\frac{1}{\lambda \Gamma(\alpha-\beta+\gamma_2)} \int_0^t (t-s)^{\alpha-\beta+\gamma_2-1} x(s)\,ds \\ &+ \frac{1}{\lambda \Gamma(\alpha+\gamma_2)} \int_0^t (t-s)^{\alpha+\gamma_2-1} f\bigl(s, x(s)\bigr)\,ds \\ &+ C_1 \frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma_2)} t^{\alpha+\gamma_2-1}. \end{aligned}$$

Let

$$\Theta := \frac{\mu \Gamma(\alpha) T^{\alpha-\gamma_1-1}}{\Gamma(\alpha-\gamma_1)} + \frac{\Gamma(\alpha) \eta^{\alpha+\gamma_2-1}}{\Gamma(\alpha+\gamma_2)}.$$

By using the second condition of (1), we get

$$\begin{aligned}\gamma_3 &= -\frac{\mu}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)\,ds \\ &\quad + \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}f\bigl(s,x(s)\bigr)\,ds \\ &\quad + C_1\frac{\mu\Gamma(\alpha)}{\Gamma(\alpha-\gamma_1)}T^{\alpha-\gamma_1-1} - \frac{1}{\lambda\Gamma(\alpha-\beta+\gamma_2)}\int_0^\eta (\eta-s)^{\alpha-\beta+\gamma_2-1}x(s)\,ds \\ &\quad + \frac{1}{\lambda\Gamma(\alpha+\gamma_2)}\int_0^\eta (\eta-s)^{\alpha+\gamma_2-1}f\bigl(s,x(s)\bigr)\,ds + C_1\frac{\Gamma(\alpha)}{\Gamma(\alpha+\gamma_2)}\eta^{\alpha+\gamma_2-1},\end{aligned}$$

which leads to

$$\begin{aligned}C_1 &= \frac{1}{\Theta}\biggl[\gamma_3 + \frac{\mu}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)\,ds \\ &\quad - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}f\bigl(s,x(s)\bigr)\,ds \\ &\quad + \frac{1}{\lambda\Gamma(\alpha-\beta+\gamma_2)}\int_0^\eta (\eta-s)^{\alpha-\beta+\gamma_2-1}x(s)\,ds \\ &\quad - \frac{1}{\lambda\Gamma(\alpha+\gamma_2)}\int_0^\eta (\eta-s)^{\alpha+\gamma_2-1}f\bigl(s,x(s)\bigr)\,ds\biggr].\end{aligned}$$

Substituting the value of C_1 into (10), we have

$$\begin{aligned}x(t) &= -\frac{1}{\lambda\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}x(s)\,ds + \frac{1}{\lambda\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f\bigl(s,x(s)\bigr)\,ds \\ &\quad + \frac{t^{\alpha-1}}{\Theta}\biggl[\gamma_3 + \frac{\mu}{\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}x(s)\,ds \\ &\quad - \frac{\mu}{\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}f\bigl(s,x(s)\bigr)\,ds \\ &\quad + \frac{1}{\lambda\Gamma(\alpha-\beta+\gamma_2)}\int_0^\eta (\eta-s)^{\alpha-\beta+\gamma_2-1}x(s)\,ds \\ &\quad - \frac{1}{\lambda\Gamma(\alpha+\gamma_2)}\int_0^\eta (\eta-s)^{\alpha+\gamma_2-1}f\bigl(s,x(s)\bigr)\,ds\biggr].\end{aligned}$$

In the two cases of $0 < t \leq \eta < T$ and $0 < \eta \leq t < T$, we can verify

$$x(t) = \frac{\gamma_3}{\Theta}t^{\alpha-1} + \int_0^T G_1(t,s)x(s)\,ds + \int_0^T G_2(t,s)f\bigl(s,x(s)\bigr)\,ds,$$

which completes the proof. □

It is easily seen that G_2 is continuous, and hence is bounded on $[0,T]\times[0,T]$. When $\alpha-\beta-1<0$ or $\alpha-\beta-\gamma_1-1<0$, G_1 is unbounded. Fortunately, we have

$$\int_0^T \bigl|G_1(t,s)\bigr|\,ds \leq \frac{1}{|\lambda\Gamma(\alpha-\beta)|}\int_0^t (t-s)^{\alpha-\beta-1}\,ds$$

$$
\begin{aligned}
&+\frac{t^{\alpha-1}}{\Theta\lambda}\left[\frac{\mu}{|\Gamma(\alpha-\beta-\gamma_1)|}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}\,ds\right.\\
&\left.+\frac{1}{|\Gamma(\alpha-\beta+\gamma_2)|}\int_0^\eta (\eta-s)^{\alpha-\beta+\gamma_2-1}\,ds\right]\\
=&\ \frac{1}{\lambda\Gamma(\alpha-\beta+1)}t^{\alpha-\beta}\\
&+\frac{t^{\alpha-1}}{\Theta\lambda}\left[\frac{\mu}{\Gamma(\alpha-\beta-\gamma_1+1)}T^{\alpha-\beta-\gamma_1}\right.\\
&\left.+\frac{1}{\Gamma(\alpha-\beta+\gamma_2+1)}\eta^{\alpha-\beta+\gamma_2}\right]\\
\le&\ \frac{1}{\lambda\Gamma(\alpha-\beta+1)}T^{\alpha-\beta}\\
&+\frac{T^{\alpha-1}}{\Theta\lambda}\left[\frac{\mu}{\Gamma(\alpha-\beta-\gamma_1+1)}T^{\alpha-\beta-\gamma_1}\right.\\
&\left.+\frac{1}{\Gamma(\alpha-\beta+\gamma_2+1)}\eta^{\alpha-\beta+\gamma_2}\right],
\end{aligned}
$$

which means $\int_0^T |G_1(t,s)|\,ds$ is uniformly bounded for $t \in [0,T]$. We denote

$$
M_1 := \max_{t\in(0,T)} \int_0^T \left|G_1(t,s)\right|\,ds
$$

and

$$
M_2 := \max_{t\in(0,T)} \int_0^T \left|G_2(t,s)\right|\,ds.
$$

Next, we introduce an integral inequality which can be regarded as a generalization of the Gronwall inequality. It is also the generalization of the main result in [39] and Lemma 3.4 of [10].

Lemma 2.6 *Suppose $\alpha > 0$, $a > 0$, $g(t,s)$ is a nonnegative continuous function defined on $[0,T]\times[0,T]$ with $g(t,s) \le M$, and $g(t,s)$ is nondecreasing w.r.t. the first variable and non-increasing w.r.t. the second variable. Assume that $u(t)$ is nonnegative and integrable on $[0,T]$ with*

$$
u(t) \le a + \int_0^t g(t,s)(t-s)^{\alpha-1}u(s)\,ds, \quad t \in [0,T].
$$

Then

$$
u(t) \le a + a\int_0^t \sum_{n=1}^{\infty} \frac{(g(t,s)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\,ds. \tag{11}
$$

Proof Let $Au(t) = \int_0^t g(t,s)(t-s)^{\alpha-1}u(s)\,ds$, where $u(t)$ is nonnegative and locally integrable on $t \in [0,T]$. It follows that $u(t) \le a + Au(t)$, which implies

$$
u(t) \le \sum_{k=0}^{n-1} A^k a + A^n u(t) \quad \bigl(A^0 a = a\bigr). \tag{12}
$$

We now prove that

$$A^n u(t) \le \int_0^t \frac{(g(t,s)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}u(s)\,ds \tag{13}$$

by induction. For $n = 1$, the proof is trivial. Assume that Eq. (13) holds for $n = k$. Then, for $n = k + 1$, we obtain

$$\begin{aligned} A^{k+1}u(t) &= A\bigl(A^k u(t)\bigr) = \int_0^t g(t,s)(t-s)^{\alpha-1}A^k u(s)\,ds \\ &\le \int_0^t g(t,s)(t-s)^{\alpha-1}\int_0^s \frac{(g(s,\tau)\Gamma(\alpha))^k}{\Gamma(k\alpha)}(s-\tau)^{k\alpha-1}u(\tau)\,d\tau\,ds \\ &\le \int_0^t g(t,\tau)\int_0^s \frac{(g(t,\tau)\Gamma(\alpha))^k}{\Gamma(k\alpha)}(t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}u(\tau)\,d\tau\,ds \\ &= \int_0^t g(t,\tau)^{k+1}\biggl[\int_\tau^t \frac{\Gamma(\alpha)^k}{\Gamma(k\alpha)}(t-s)^{\alpha-1}(s-\tau)^{k\alpha-1}\,ds\biggr]u(\tau)\,d\tau \\ &= \int_0^t \frac{g(t,\tau)^{k+1}(\Gamma(\alpha))^k}{\Gamma(k\alpha)}B(\alpha,k\alpha)(t-\tau)^{(k+1)\alpha-1}u(\tau)\,d\tau \\ &= \int_0^t \frac{(g(t,s)\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)}(t-s)^{(k+1)\alpha-1}u(s)\,ds. \end{aligned}$$

By mathematical induction, the inequality (13) holds for all $n \in \mathbf{N}$. Replaced $u(t)$ by a in (13) we deduce that $A^k a \le a\int_0^t \frac{(g(t,s)\Gamma(\alpha))^k}{\Gamma(k\alpha)}(t-s)^{k\alpha-1}\,ds$, $k = 1,2,\ldots$. Similar to the proof of Lemma 3.4 of [10] we can verify that

$$A^n u(t) \le \int_0^t \frac{(M\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}u(s)\,ds \to 0$$

as $n \to \infty$ uniformly in $t \in [0,T]$. Finally, letting $n \to \infty$ in (12), we get

$$u(t) \le \sum_{n=0}^{\infty} A^n a \le a + a\int_0^t \sum_{n=1}^{\infty} \frac{(g(t,s)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\,ds,$$

i.e., the inequality (11) holds. The lemma is proved. □

3 Existence results

In this section, we investigate the existence for *BVP* (1). For convenience we list the hypothesis.

(H_1) $f : [0,T] \times \mathbf{R} \to \mathbf{R}$ is continuous.

(H_2) There exists a constant $L > 0$ such that

$$\bigl|f(t,x_1) - f(t,x_2)\bigr| \le L|x_1 - x_2|$$

for all $x_1, x_2 \in \mathbf{R}$ and $t \in [0,T]$.

(H_3) There exists $u \in C([0,T],\mathbf{R}^+)$ such that

$$\bigl|f(t,x)\bigr| \le u(t)$$

for each $t \in [0,T]$ and $x \in \mathbf{R}$.

(H_4) There exist a continuous function $\phi_1(t) \in C([0,T],\mathbf{R}^+)$ and a nondecreasing function $\phi_2 \in C([0,T],\mathbf{R}^+)$ such that

$$\left|f(t,x)\right| \le \phi_1(t)\phi_2\left(|x|\right)$$

for each $t \in [0,T]$ and $x \in \mathbf{R}$.

Theorem 3.1 *Assume that (H_1) and (H_2) hold. If*

$$M_1 + LM_2 < 1,$$

then the boundary value problem (1) has a unique solution in $C([0,T],\mathbf{R})$.

Proof We define an operator $S : C([0,T],\mathbf{R}) \to C([0,T],\mathbf{R})$ by

$$Sx(t) = \frac{\gamma_3}{\Theta}t^{\alpha-1} + \int_0^T G_1(t,s)x(s)\,ds + \int_0^T G_2(t,s)f\left(s,x(s)\right)ds,$$

for each $t \in (0,T)$.

By Lemma 2.3, $x \in C([0,T],\mathbf{R})$ is a solution of problem (1) if and only if x is a fixed point of S. We now prove that S has a fixed point. Taking $x_1, x_2 \in C([0,T],\mathbf{R})$ arbitrary, according to (H_2), we find

$$\begin{aligned}
&\left|Sx_1(t) - Sx_2(t)\right| \\
&\quad \le \int_0^T \left|G_1(t,s)\right|\left|x_1(s) - x_2(s)\right|ds + \int_0^T \left|G_2(t,s)\right|\left|f\left(s,x_1(s)\right) - f\left(s,x_2(s)\right)\right|ds \\
&\quad \le \|x_1 - x_2\| \int_0^T \left|G_1(t,s)\right|ds + L\|x_1 - x_2\| \int_0^T \left|G_2(t,s)\right|ds \\
&\quad \le (M_1 + LM_2)\|x_1 - x_2\|,
\end{aligned}$$

and hence

$$\|Sx_1 - Sx_2\| \le (M_1 + LM_2)\|x_1 - x_2\|.$$

Since $M_1 + LM_2 < 1$, S is a contraction. By Banach contraction principle, S has a unique fixed point in $C([0,T],\mathbf{R})$, i.e., the boundary value problem (1) has a unique solution. This completes the proof. □

Theorem 3.2 *Assume that (H_1) and (H_3) hold. If*

$$M_1 < 1,$$

then the boundary value problem (1) has at least one solution in $C([0,T],\mathbf{R})$.

Proof We define mappings A and B from $C((0,T),\mathbf{R})$ into itself by

$$Ax(t) = \frac{\gamma_3}{\Theta}t^{\alpha-1} + \int_0^T G_1(t,s)x(s)\,ds$$

and

$$Bx(t) = \int_0^T G_2(t,s)f\bigl(s,x(s)\bigr)\,ds.$$

It is easy to verify that B is continuous on $C([0,T],\mathbf{R})$, since G_2 and f are continuous. Further, since $M_1 < 1$, we can choose a constant $r > 0$ large enough such that $M_1 + \frac{M_2\|u\|}{r} + \frac{\gamma_3 T^{\alpha-1}}{r\Theta} < 1$, i.e., $rM_1 + M_2\|u\| + \frac{\gamma_3 T^{\alpha-1}}{\Theta} < r$, where u belongs to $C([0,T],\mathbf{R}^+)$ such that $|f(t,x)| \le u(t)$, according to (H_3). Set $B_r = \{x \in C((0,T),\mathbf{R}) : \|x\| \le r\}$. Then B_r is nonempty, bounded, closed and convex. Moreover, for any $x, y \in B_r$ and $t \in [0,T]$, we have

$$\begin{aligned}\bigl|Ax(t)\bigr| &\le \left|\frac{\gamma_3 t^{\alpha-1}}{\Theta}\right| + \int_0^T \bigl|G_1(t,s)\bigr|\bigl|x(s)\bigr|\,ds \\ &\le rM_1 + \frac{\gamma_3 T^{\alpha-1}}{\Theta}\end{aligned}$$

and

$$\begin{aligned}\bigl|By(t)\bigr| &\le \int_0^T \bigl|G_2(t,s)\bigr|\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\ &\le M_2\|u\|.\end{aligned}$$

Thus, $|Ax(t) + By(t)| \le rM_1 + M_2\|u\| + \frac{\gamma_3 T^{\alpha-1}}{\Theta} < r$, i.e., $Ax + By \in B_r$.

Next we prove that A is a contraction. In fact, for any $x, y \in C((0,T),\mathbf{R})$ and $t \in [0,1]$,

$$\bigl|Ax(t) - Ay(t)\bigr| \le \int_0^T \bigl|G_1(t,s)\bigr|\bigl|x(t) - y(t)\bigr|\,ds \le M_1\|x - y\|.$$

It follows that $\|Ax - Ay\| \le M_1\|x - y\|$. Since $M_1 < 1$, we know that A is a contraction.

Finally, we have to show that B is compact. Take any bounded subset $U \subseteq C([0,T],\mathbf{R})$. Then there exists a constant $r_0 > 0$ such that $U = \{u \in U : \|u\| \le r_0\}$. We prove that BU is bounded and equicontinuous. In fact, for any $x \in U$, we have

$$\bigl|Bx(t)\bigr| \le \int_0^T \bigl|G_2(t,s)\bigr|\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \le M_2\|u\| \le M_2 r_0$$

for each $t \in (0,T)$. Hence BU is bounded. Further, for any $0 \le t_1 < t_2 \le T$ and $x \in U$, we have

$$\begin{aligned}\bigl|Bx(t_2) - Bx(t_1)\bigr| &= \left|\int_0^T G_2(t_2,s)f\bigl(s,x(s)\bigr) - \int_0^T G_2(t_1,s)f\bigl(s,x(s)\bigr)\,ds\right| \\ &\le \left|\int_0^T \frac{\mu}{\Theta\lambda\Gamma(\alpha-\gamma_1)}\bigl(t_1^{\alpha-1} - t_2^{\alpha-1}\bigr)(T-s)^{\alpha-\gamma_1-1}f\bigl(s,x(s)\bigr)\,ds\right| \\ &\quad + \left|\int_0^\eta \frac{1}{\Theta\lambda\Gamma(\alpha+\gamma_2)}\bigl(t_1^{\alpha-1} - t_2^{\alpha-1}\bigr)(\eta-s)^{\alpha+\gamma_2-1}f\bigl(s,x(s)\bigr)\,ds\right| \\ &\quad + \left|\int_0^{t_1} \frac{1}{\lambda\Gamma(\alpha)}\bigl[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr]f\bigl(s,x(s)\bigr)\,ds\right| \\ &\quad + \left|\int_{t_1}^{t_2} \frac{1}{\lambda\Gamma(\alpha)}(t_2-s)^{\alpha-1}f\bigl(s,x(s)\bigr)\,ds\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\mu(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Theta\lambda\Gamma(\alpha-\gamma_1)}\int_0^T (T-s)^{\alpha-\gamma_1-1}\,ds\|u\| \\
&\quad + \frac{t_2^{\alpha-1}-t_1^{\alpha-1}}{\Theta\lambda\Gamma(\alpha+\gamma_2)}\int_0^\eta (\eta-s)^{\alpha+\gamma_2-1}\,ds\|u\| \\
&\quad + \frac{1}{\lambda\Gamma(\alpha)}\int_0^{t_1}\bigl|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\bigr|\,ds\|u\| \\
&\quad + \frac{1}{\lambda\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\|u\| \\
&= \frac{\mu T^{\alpha-\gamma_1}(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Theta\lambda\Gamma(\alpha-\gamma_1+1)}r_0 \\
&\quad + \frac{\eta^{\alpha+\gamma_2}(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Theta\lambda\Gamma(\alpha+\gamma_2+1)}r_0 \\
&\quad + \frac{1}{\lambda\Gamma(\alpha+1)}\bigl[t_2^\alpha-t_1^\alpha+2(t_2-t_1)^\alpha\bigr]r_0.
\end{aligned}
$$

We can see from the above inequality that $|Bx(t_2)-Bx(t_1)|\to 0$ as $t_2-t_1\to 0$, and the convergence is independent on x, t_1 and t_2. This shows that BU is equicontinuous. An employment of Arzelà–Ascoli theorem shows that B is compact. Now, we apply Krasnoselskii's fixed point theorem on operator A and B to deduce that there exists at least one x such that $Ax+Bx=x$, which is the solution of the boundary value problem (1). □

Theorem 3.3 *Assume that* (H_1) *and* (H_4) *hold. If*

$$
M_1 + M_2\|\phi_1\|\limsup_{r\to\propto}\frac{\phi_2(r)}{r} < 1,
$$

then the boundary value problem (1) *has at least one solution in* $C([0,T],\mathbf{R})$.

Proof Define a mapping $S: C([0,T],\mathbf{R})\to C([0,T],\mathbf{R})$ by

$$
Sx(t) = \frac{\gamma_3}{\Theta}t^{\alpha-1} + \int_0^T G_1(t,s)x(s)\,ds + \int_0^T G_2(t,s)f\bigl(s,x(s)\bigr)\,ds
$$

for $t\in[0,T]$. It is easy to prove that S is continuous. In order to apply Schauder's fixed point theorem, we only need to show that S is compact.

Firstly, we take any bounded subset $Q\subseteq C([0,T],\mathbf{R})$. Then there exists q ($q>0$) satisfying that $Q\subseteq B_q=\{x\in C([0,T],\mathbf{R}):\|x\|\leq q\}$. Notice that B_q is a closed convex and bounded subset. For any $x\in B_q$, we have

$$
\begin{aligned}
\bigl|Sx(t)\bigr| &\leq \left|\frac{\gamma_3 t^{\alpha-1}}{\Theta}\right| + \int_0^T \bigl|G_1(t,s)\bigr|\bigl|x(s)\bigr|\,ds + \int_0^T \bigl|G_2(t,s)\bigr|\bigl|f\bigl(s,x(s)\bigr)\bigr|\,ds \\
&\leq \frac{\gamma_3 T^{\alpha-1}}{\Theta} + \|x\|\int_0^T\bigl|G_1(t,s)\bigr|\,ds + \|\phi_1\|\phi_2\bigl(\|x\|\bigr)\int_0^T\bigl|G_2(t,s)\bigr|\,ds, \\
&\leq \frac{\gamma_3 T^{\alpha-1}}{\Theta} + TM_1q + TM_2\|\phi_1\|\phi_2(q),
\end{aligned}
$$

which means SB_q, and therefore SQ, is uniformly bounded in $C((0,T),\mathbf{R})$. For $0 \le t_1 < t_2 \le T$ and any $x \in B_q$,

$$\begin{aligned}
\bigl|Sx(t_1) - Sx(t_2)\bigr| &\le \frac{\gamma_3}{\Theta}\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr) + \left|\int_0^T G_1(t_2,s)x(s) - \int_0^T G_1(t_1,s)x(s)\,ds\right| \\
&\quad + \left|\int_0^T G_2(t_2,s)x(s) - \int_0^T G_2(t_1,s)x(s)\,ds\right| \\
&= P_1 + P_2 + P_3;
\end{aligned}$$

$$\begin{aligned}
P_1 &= \frac{\gamma_3}{\Theta}\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr), \\
P_2 &= \left|\int_0^T G_1(t_2,s)x(s) - \int_0^T G_1(t_1,s)x(s)\,ds\right| \\
&\le \left|\int_0^T \frac{\mu}{\Theta\lambda\Gamma(\alpha-\beta-\gamma_1)}\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)(T-s)^{\alpha-\beta-\gamma_1-1}x(s)\,ds\right| \\
&\quad + \left|\int_0^\eta \frac{1}{\Theta\lambda\Gamma(\alpha-\beta+\gamma_2)}\bigl(t_2^{\alpha-1} - t_1^{\alpha-1}\bigr)(\eta-s)^{\alpha-\beta+\gamma_2-1}x(s)\,ds\right| \\
&\quad + \left|\int_0^{t_1} \frac{1}{\lambda\Gamma(\alpha-\beta)}\bigl[(t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}\bigr]x(s)\,ds\right| \\
&\quad + \left|\int_{t_1}^{t_2} \frac{1}{\lambda\Gamma(\alpha-\beta)}(t_2-s)^{\alpha-\beta-1}x(s)\,ds\right| \\
&\le \frac{\mu(t_2^{\alpha-1} - t_1^{\alpha-1})}{\Theta\lambda\Gamma(\alpha-\beta-\gamma_1)}\int_0^T (T-s)^{\alpha-\beta-\gamma_1-1}\,ds\|x\| \\
&\quad + \frac{t_2^{\alpha-1} - t_1^{\alpha-1}}{\Theta\lambda\Gamma(\alpha-\beta+\gamma_2)}\int_0^\eta (\eta-s)^{\alpha-\beta+\gamma_2-1}\,ds\|x\| \\
&\quad + \frac{1}{\lambda\Gamma(\alpha-\beta)}\int_0^{t_1} \bigl|(t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}\bigr|\,ds\|x\| \\
&\quad + \frac{1}{\lambda\Gamma(\alpha-\beta)}\int_{t_1}^{t_2} (t_2-s)^{\alpha-\beta-1}\,ds\|x\| \\
&= \quad \frac{\mu(t_2^{\alpha-1} - t_1^{\alpha-1})T^{\alpha-\beta-\gamma_1}q}{\Theta\lambda\Gamma(\alpha-\beta-\gamma_1)} \\
&\quad + \frac{(t_2^{\alpha-1} - t_1^{\alpha-1})\eta^{\alpha-\beta-\gamma_1}q}{\Theta\lambda\Gamma(\alpha-\beta+\gamma_2)} \\
&\quad + \frac{q}{\lambda\Gamma(\alpha-\beta)}\bigl[t_2^{\alpha-\beta} - t_1^{\alpha-\beta} + 2(t_2-t_1)^{\alpha-\beta}\bigr], \\
P_3 &= \left|\int_0^T G_2(t_2,s)f\bigl(s,x(s)\bigr) - \int_0^T G_2(t_1,s)f\bigl(s,x(s)\bigr)\,ds\right| \\
&\le \left|\int_0^T \frac{\mu}{\Theta\lambda\Gamma(\alpha-\gamma_1)}\bigl(t_1^{\alpha-1} - t_2^{\alpha-1}\bigr)(T-s)^{\alpha-\gamma_1-1}f\bigl(s,x(s)\bigr)\,ds\right| \\
&\quad + \left|\int_0^\eta \frac{1}{\Theta\lambda\Gamma(\alpha+\gamma_2)}\bigl(t_1^{\alpha-1} - t_2^{\alpha-1}\bigr)(\eta-s)^{\alpha+\gamma_2-1}f\bigl(s,x(s)\bigr)\,ds\right| \\
&\quad + \left|\int_0^{t_1} \frac{1}{\lambda\Gamma(\alpha)}\bigl[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\bigr]f\bigl(s,x(s)\bigr)\,ds\right| \\
&\quad + \left|\int_{t_1}^{t_2} \frac{1}{\lambda\Gamma(\alpha)}(t_2-s)^{\alpha-1}f\bigl(s,x(s)\bigr)\,ds\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\mu(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Theta\lambda\Gamma(\alpha-\gamma_1)}\left[\int_0^T (T-s)^{\alpha-\gamma_1-1}\,ds\right]\|\phi_1\|\phi_2\bigl(\|x\|\bigr)\\
&\quad+\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Theta\lambda\Gamma(\alpha+\gamma_2)}\left[\int_0^\eta (\eta-s)^{\alpha+\gamma_2-1}\,ds\right]\|\phi_1\|\phi_2\bigl(\|x\|\bigr)\\
&\quad+\frac{1}{\lambda\Gamma(\alpha)}\left[\int_0^{t_1}\bigl|(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}\bigr|\,ds\right]\|\phi_1\|\phi_2\bigl(\|x\|\bigr)\\
&\quad+\frac{1}{\lambda\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\,ds\|\phi_1\|\phi_2\bigl(\|x\|\bigr)\\
&= \frac{\mu T^{\alpha-\gamma_1}(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Theta\lambda\Gamma(\alpha-\gamma_1+1)}\|\phi_1\|\phi_2(q)\\
&\quad+\frac{\eta^{\alpha+\gamma_2}(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Theta\lambda\Gamma(\alpha+\gamma_2+1)}\|\phi_1\|\phi_2(q)\\
&\quad+\frac{1}{\lambda\Gamma(\alpha+1)}\bigl[t_2^\alpha-t_1^\alpha+2(t_2-t_1)^\alpha\bigr]\|\phi_1\|\phi_2(q).
\end{aligned}
$$

It is trivial that P_1, P_2 and P_3 all tend to 0 as $t_2-t_1\to 0$. Hence, $|Sx(t_2)-Sx(t_1)|\to 0$ ($t_2-t_1\to 0$). Notice that the convergence is independent on x, t_1 and t_2. It follows that SB_q is equicontinuous. An application of the Arzelà–Ascoli Theorem yields that SB_q is compact. Therefore, we have shown that S maps bounded subsets into compact subset, i.e., S is a compact mapping.

Now, from the condition $M_1+M_2\|\phi_1\|\lim_{r\to\propto}\sup\frac{\phi_2(r)}{r}<1$, we can choose a positive r large enough, such that

$$
M_1+M_2\|\phi_1\|\frac{\phi_2(r)}{r}+\frac{\gamma_3T^{\alpha-1}}{r\Theta}<1,
$$

i.e., $M_1r+M_2\|\phi_1\|\phi_2(r)+\frac{\gamma_3T^{\alpha-1}}{\Theta}<r$. Hence, we can take $C>0$ such that $M_1C+M_2\|\phi_1\|\phi_2(C)+\frac{\gamma_3T^{\alpha-1}}{\Theta}<C$. Let $U=\{x\in C((0,T),\mathbf{R}):\|x\|\leq C\}$. Then $S:\overline{U}\to C((0,T),\mathbf{R})$ is compact and continuous. If there exist $\lambda\in(0,1)$ and $x\in\overline{U}$ such that $x=\lambda Sx$, then, for every $t\in(0,T)$,

$$
\begin{aligned}
\bigl|x(t)\bigr| &= \bigl|\lambda Sx(t)\bigr|\leq\bigl|Sx(t)\bigr|\\
&\leq\left|\frac{\gamma_3t^{\alpha-1}}{\Theta}\right|+\left|\int_0^T G_1(t,s)x(s)\,ds\right|+\left|\int_0^T G_2(t,s)f\bigl(s,x(s)\bigr)\,ds\right|\\
&\leq\frac{\gamma_3T^{\alpha-1}}{\Theta}+M_1C+M_2\|\phi_1\|\phi_2\bigl(\|x\|\bigr)\\
&\leq M_1C+M_2\|\phi_1\|\phi_2(C)+\frac{\gamma_3T^{\alpha-1}}{\Theta}.
\end{aligned}
$$

It then follows that $C=\|x\|\leq M_1C+M_2\|\phi_1\|\phi_2(C)+\frac{\gamma_3T^{\alpha-1}}{\Theta}<C$, which contradict to the fact that $x\in\overline{U}$. Thus, $x\neq\lambda Sx$ for each $\lambda\in(0,1)$ and $x\in\overline{U}$. By the Leray Schauder alternative, S has at least one fixed point, which is the solution to the boundary value problem (1). This completes the proof. □

4 Stability analysis

In this section, we study Hyers–Ulam stability of the boundary value problem

$$\begin{cases} \lambda D_0^\alpha x(t) + D_0^\beta x(t) = f(t,x(t)), & 0 \le t \le T, \\ x(0) = 0, \qquad x(T) = \gamma. \end{cases} \tag{14}$$

Definition 4.1 The boundary value problem (14) is Hyers–Ulam stable if there exists a real constant $c > 0$, such that for any $\varepsilon > 0$, and for every solution $y(t) \in C([0,T],\mathbf{R})$ of the inequality

$$\left|\lambda D_0^\alpha y(t) + D_0^\beta y(t) - f\bigl(t,y(t)\bigr)\right| \le \varepsilon, \quad t \in [0,T],$$

there exists a solution $x(t) \in C([0,T],\mathbf{R})$ of problem (14) with

$$\left|y(t) - x(t)\right| \le c\varepsilon, \quad t \in [0,T].$$

Theorem 4.2 *Assume that* (H_1) *and* (H_2) *hold. Then the solution of the boundary value problem* (14) *is Hyers–Ulam stable.*

Proof For $\varepsilon > 0$, and each solution $y(t) \in C([0,T],\mathbf{R})$ of the inequality

$$\left|\lambda D_0^\alpha y(t) + D_0^\beta y(t) - f\bigl(t,y(t)\bigr)\right| \le \varepsilon, \quad t \in [0,T],$$

we can find a function $g(t)$ satisfying $\lambda D_0^\alpha y(t) + D_0^\beta y(t) = f(t,y(t)) + g(t)$ and $|g(t)| \le \varepsilon$. It follows that

$$\begin{aligned} y(t) = {}& -\frac{1}{\lambda} I_0^{\alpha-\beta} y(t) + \frac{1}{\lambda} I_0^\alpha f\bigl(t,y(t)\bigr) + \frac{1}{\lambda} I_0^\alpha g(t) \\ & + \frac{t^{\alpha-1}}{T^{\alpha-1}} \left[y(T) + \frac{1}{\lambda} I_0^{\alpha-\beta} y(T) - \frac{1}{\lambda} I_0^\alpha f\bigl(T,y(T)\bigr) - \frac{1}{\lambda} I_0^\alpha g(T) \right]. \end{aligned}$$

Let $x(t) \in C([0,T],\mathbf{R})$ be the unique solution of (14). Then $x(t)$ is given by

$$x(t) = -\frac{1}{\lambda} I_0^{\alpha-\beta} x(t) + \frac{1}{\lambda} I_0^\alpha f\bigl(t,x(t)\bigr) + \frac{t^{\alpha-1}}{T^{\alpha-1}} \left[x(T) + \frac{1}{\lambda} I_0^{\alpha-\beta} x(T) - \frac{1}{\lambda} I_0^\alpha f\bigl(T,x(T)\bigr) \right].$$

Then we have

$$\begin{aligned} \left|y(t) - x(t)\right| \le {}& \frac{1}{\lambda} I_0^{\alpha-\beta} \left|y(t) - x(t)\right| + \frac{1}{\lambda} I_0^\alpha \left|f\bigl(t,y(t)\bigr) - f\bigl(t,x(t)\bigr)\right| \\ & + \frac{t^{\alpha-1}}{T^{\alpha-1}} \left[\frac{1}{\lambda} I_0^{\alpha-\beta} \left|y(T) - x(T)\right| + \frac{1}{\lambda} I_0^\alpha \left|f\bigl(T,y(T)\bigr) - f\bigl(T,x(T)\bigr)\right| \right] \\ & + \frac{1}{\lambda} I_0^\alpha \left|g(t)\right| + \frac{t^{\alpha-1}}{\lambda T^{\alpha-1}} I_0^\alpha \left|g(T)\right| \\ \le {}& \frac{1}{\lambda} \int_0^t \left[\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + L \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \right] \left|y(s) - x(s)\right| \, ds \\ & + \frac{t^{\alpha-1}}{\lambda T^{\alpha-1}} \int_0^T \left[\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} + L \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)} \right] \left|y(s) - x(s)\right| \, ds \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\lambda}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\big|g(s)\big|\,ds+\frac{t^{\alpha-1}}{\lambda T^{\alpha-1}}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\big|g(s)\big|\,ds \\
\le{}& \frac{1}{\lambda}\int_0^t \biggl[\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+L\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\biggr]\big|y(s)-x(s)\big|\,ds \\
&+\frac{1}{\lambda}\int_0^T \biggl[\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+L\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\biggr]\big|y(s)-x(s)\big|\,ds \\
&+2\varepsilon\frac{1}{\lambda}\int_0^T \frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}\,ds \\
\le{}& \frac{1}{\lambda}\int_0^t \biggl[\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+L\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\biggr]\big|y(s)-x(s)\big|\,ds \\
&+M(\varepsilon)\varepsilon+\frac{2T^{\alpha}}{\lambda\Gamma(\alpha+1)}\varepsilon,
\end{aligned}
$$

where $\frac{1}{\lambda}\int_0^T[\frac{(T-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}+L\frac{(T-s)^{\alpha-1}}{\Gamma(\alpha)}]|y(s)-x(s)|\,ds\le M(\varepsilon)\varepsilon$ and $M(\varepsilon)$ is a constant dependent on ε. Let $g(t,s)=\frac{1}{\lambda\Gamma(\alpha-\beta)}+L\frac{(t-s)^{\beta}}{\lambda\Gamma(\alpha)}$ and $\overline{M}=M(\varepsilon)+\frac{2T^{\alpha}}{\lambda\Gamma(\alpha+1)}$. Then

$$
\big|y(t)-x(t)\big|\le\overline{M}\varepsilon+\int_0^t g(t,s)(t-s)^{\alpha-\beta-1}\big|y(s)-x(s)\big|\,ds.
$$

We note that $g(t,s)\le\frac{1}{\lambda\Gamma(\alpha-\beta)}+L\frac{T^{\beta}}{\lambda\Gamma(\alpha)}$ $(=M)$. Hence, in view of Lemma 3.4,

$$
\begin{aligned}
\big|y(t)-x(t)\big|&\le\overline{M}\varepsilon+\overline{M}\varepsilon\int_0^t\sum_{n=1}^{\infty}\frac{(g(t,s)\Gamma(\alpha-\beta))^n}{\Gamma(n(\alpha-\beta))}(t-s)^{n(\alpha-\beta)-1}\,ds \\
&\le\overline{M}\varepsilon+\overline{M}\varepsilon\int_0^t\sum_{n=1}^{\infty}\frac{(M\Gamma(\alpha-\beta))^n}{\Gamma(n(\alpha-\beta))}(t-s)^{n(\alpha-\beta)-1}\,ds \\
&=\overline{M}\varepsilon+\overline{M}\varepsilon\sum_{n=1}^{\infty}\frac{(M\Gamma(\alpha-\beta))^n}{\Gamma(n(\alpha-\beta)+1)}T^{n(\alpha-\beta)} \\
&\le\overline{M}\varepsilon E_{\alpha-\beta}\bigl(MT^{(\alpha-\beta)}\Gamma(\alpha-\beta)\bigr).
\end{aligned}
$$

Let $c=\overline{M}E_{\alpha-\beta}(MT^{(\alpha-\beta)}\Gamma(\alpha-\beta))$. The inequality

$$
\big|y(t)-x(t)\big|\le c\varepsilon
$$

holds. Thus, the boundary value problem (14) is Hyers–Ulam stable. □

5 Examples

Example 5.1 Let us consider the following multiply three-term fractional differential equation

$$
\begin{cases}
\frac{4}{5}D_0^{\frac{3}{2}}x(t)+D_0^{\frac{5}{4}}x(t)=t^2\sin(x(t)), & 0<t<\frac{1}{4}, \\
x(0)=0, & \frac{1}{8}D_0^{\frac{1}{8}}x(\frac{1}{4})+I_0^{\frac{3}{4}}x(\frac{1}{8})=\frac{1}{16}.
\end{cases}
\tag{15}
$$

Here $\alpha=\frac{3}{2}$, $\beta=\frac{5}{4}$, $\lambda=\frac{4}{5}$, $\mu=\frac{1}{8}$, $\gamma_1=\frac{1}{8}<\alpha-\beta$, $\gamma_2=\frac{3}{4}$, $\gamma_3=\frac{1}{16}$, $T=\frac{1}{4}$, $\eta=\frac{1}{8}$ and $f(t,x)=t^2\sin x$. It is evident that $|f(t,x_1)-f(t,x_2)|<(\frac{1}{4})^2|x_1-x_2|<|x_1-x_2|$, which satisfies condition (H_2). So, we can select $L=1$. We can calculate that $\Theta=0.9090913801766$,

$M_1 \le 0.4588930235784$, $M_2 \le 0.0465184974808$. So, $M_1 + LM_2 \le 0.5054115210592 < 1$. Therefore, by applying Theorem 3.1, we deduce that the boundary value problem (15) has a unique solution on $(0, \frac{1}{4})$.

Let $g(t,s) = \frac{1}{\lambda\Gamma(\alpha-\beta)} + L\frac{(t-s)^{\beta}}{\lambda\Gamma(\alpha)}$. Noting that $g(t,s) \le \frac{1}{\lambda\Gamma(\alpha-\beta)} + L\frac{T^{\beta}}{\lambda\Gamma(\alpha)} \approx 4.7314835278026$ $(= M)$. Let $x(\frac{1}{4}) = 1$. By Theorem 4.2, the problem (15) with $x(\frac{1}{4}) = 1$ is Hyers–Ulam stable.

Example 5.2 Considering the following boundary value problem which contains Riemann–Liouville fractional derivatives of two orders in a differential equation and the condition

$$\begin{cases} \frac{4}{5}D_0^{\frac{3}{2}}x(t) + D_0^{\frac{5}{4}}x(t) = t^2x^{\frac{1}{2}}, & 0 < t < \frac{1}{8}, \\ x(0) = 0, & \frac{1}{16}D_0^{\frac{1}{8}}x(\frac{1}{4}) + I_0^{\frac{3}{4}}x(\frac{1}{16}) = \frac{1}{24}. \end{cases} \tag{16}$$

Here $\alpha = \frac{3}{2}$, $\beta = \frac{5}{4}$, $\lambda = \frac{4}{5}$, $\mu = \frac{1}{16}$, $\gamma_1 = \frac{1}{8} < \alpha - \beta$, $\gamma_2 = \frac{3}{4}$, $\gamma_3 = \frac{1}{24}$, $T = \frac{1}{8}$ and $\eta = \frac{1}{16}$. By direct computation, we have $\Theta \approx 0.053013292298$, $M_1 \le 0.2209565547363$, $M_2 \le 0.0111290158488$. Choosing $\phi_1(t) = t^2$ and $\phi_2 = x^{\frac{1}{2}}$, we can show that $M_1 + M_2\|\phi_1\| \lim_{r\to\propto} \sup \frac{\phi_2(r)}{r} < 1$. Hence, by Theorem 3.3, the problem (16) has at least one solution on $(0, \frac{1}{8})$.

6 Conclusion

In this paper, we study a class of two-term fractional differential equations. We first investigate the Green function of a three-point boundary value problems with mixed fractional differential and integral boundary conditions. Existence results are obtained based on some fixed point theorems. We also study the Hyers–Ulam stability for null boundary conditions. Our results improve and generalize the results in [11, 19, 38] and [20]. Moreover, we present a Gronwall type inequality of fractional order integral by the method of iteration.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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