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Weak convergence of the complex fractional Brownian motion

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Abstract
In this paper, we obtain two approximations in law of the complex fractional Brownian motion by processes constructed from a Poisson process and a Lévy process, respectively.

MSC: 60F05; 60G15; 60G22

Keywords: Complex fractional Brownian motion; Poisson process; Lévy process; Weak convergence

1 Introduction

Statistics and econometrics have made great progress in the last two decades. It became necessary to study the distributions or the asymptotic distributions of some complex statistics, so the weak convergence of a stochastic process has been widely studying as an important subject of modern probability theory.

Kac [13] described the solution of the telegrapher's equation in terms of a Poisson process. Later, Stroock [18] gave that the law of the continuous processes $\{X_\varepsilon(t), t \geq 0\}$ given by

$$X_\epsilon(t) = \varepsilon \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N(r)}\, dr = \frac{1}{\varepsilon} \int_0^t (-1)^{N(\frac{r}{\varepsilon^2})}\, dr, \quad t \geq 0, \tag{1}$$

where $\{N(t), t \geq 0\}$ is a standard Poisson process, weakly converges when ε tends to zero, in the Banach space $\mathcal{C}([0,T])$ of continuous functions on $[0,T]$, to the law of a standard Brownian motion.

This result of Stroock [18] has been extended to obtain approximations of other processes such as, among others: Brownian sheet (cf. Bardina and Jolis [5]), m-dimensional Brownian motion (cf. Bardina and Rovira [7]), fractional Brownian motion (cf. Delgado and Jolis [11], Li and Dai [14]), fractional Brownian sheet (cf. Tudor [19], Bardina *et al.* [6], Wang *et al.* [20]) and so on. We can refer to Dai [10], Mishura and Banna [16], Nieminen [12], Ouahra [17], Wang *et al.* [21] and the references therein for more information about weak convergence.

On the other hand, by the fact $(-1)^{N(r)} = \cos(\pi N(r)) = e^{i\pi N(r)}$, equality (1) can be rewritten by

$$X_\epsilon(t) = \frac{1}{\varepsilon}\int_0^t e^{i\pi N(\frac{r}{\varepsilon^2})}\,dr, \quad t \geq 0. \tag{2}$$

In this case, $X_\epsilon(t)$ can also be written by Euler's formula as $X_\epsilon(t) = \operatorname{Re} X_\epsilon(t) + i \operatorname{Im} X_\epsilon(t)$, where

$$\operatorname{Re} X_\epsilon(t) = \frac{1}{\varepsilon}\int_0^t \cos\left(\pi N\left(\frac{r}{\varepsilon^2}\right)\right)dr \quad \text{and} \quad \operatorname{Im} X_\epsilon(t) = \frac{1}{\varepsilon}\int_0^t \sin\left(\pi N\left(\frac{r}{\varepsilon^2}\right)\right)dr$$

are the real part and the imaginary part, respectively.

Then, some authors considered the weak convergence to the complex Brownian motion by the angles θ replacing the π, where $\theta \in (0, 2\pi)$ in equation (2). For example, Bardina [2] and Bardina *et al.* [4] constructed the process from a standard Poisson process which respectively weakly or strongly converges in law to a complex Brownian motion, and got that the real part and the imaginary part of this process are two independent Brownian motions. Bardina and Bascompte [3] obtained the weak convergence towards two independent Gaussian processes from a Poisson process (see also Bardina and Rovira [7], a d-dimensional Brownian motion of this result). In addition, it is well known that some properties of the Poisson process can be found from a Lévy process (cf. Applebaum [1]), so there are some literature works which research an approximation of a complex Brownian motion from the Lévy process (cf. Bardina and Rovira [8]).

Inspired by all the above works, the purpose of this paper is to research a weak approximation of a complex fractional Brownian motion from a standard Poisson process and from a Lévy process, respectively, by the method in Delgado and Jolis [11].

Let $\{M_t, t \geq 0\}$ be a Poisson process of parameter 2. We define $\{N_t, t \geq 0\}$ and $\{N'_t, t \geq 0\}$ two other counter processes where, at each jump of M, each of them jumps or does not jump with probability $\frac{1}{2}$, independently of the jumps of the other process and of its past. In Bardina *et al.* [4], they proved that N and N' are Poisson processes of parameter 1 with independent increments on disjoint intervals.

Then, for $\theta \in (0, \pi) \cup (\pi, 2\pi)$, we consider the first processes $Z_\varepsilon^\theta = \{Z_\varepsilon^\theta(t), t \in [0, 1]\}$ with

$$Z_\varepsilon^\theta(t) = \frac{2}{\varepsilon}(-1)^G \int_0^1 K_H(t, r)(-1)^{N'_{\frac{2r}{\varepsilon^2}}} e^{i\theta N_{\frac{2r}{\varepsilon^2}}}\,dr, \quad t \in [0, 1], \tag{3}$$

where N and N' are the processes defined above, G is a random variable independent of N and N', with Bernoulli distribution of parameter $\frac{1}{2}$, and

$$K_H(t, r) = c_H r^{\frac{1}{2}-H}\int_r^t (u - r)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\,du \tag{4}$$

which is the kernel of fractional Brownian motion with $H \in (\frac{1}{2}, 1)$ and c_H is the following normalizing constant:

$$c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right]^{\frac{1}{2}}$$

(cf. Mandelbrot and Van Ness [15]).

And consider the other processes $Y_\varepsilon^\theta = \{Y_\varepsilon^\theta(t), t \in [0,1]\}$ from a Lévy process with

$$Y_\varepsilon^\theta(t) = \frac{2}{\varepsilon} c(\theta) \int_0^1 K_H(t,r) e^{i\theta X_{\frac{2r}{\varepsilon^2}}}\, dr, \quad t \geq 0, \tag{5}$$

where $\{X_s, s \geq 0\}$ is a Lévy process with Lévy exponent $\psi_X(u)$, $c(\theta) = \sqrt{\frac{\|\psi_X(\theta)\|^2}{2\,\mathrm{Re}[\psi_X(\theta)]}}$ is a constant depending on θ, $\theta \in (0, 2\pi)$, and $K_H(t,r)$ is defined in (4).

It is well known that Lévy exponent $\psi_X(u)$ can be expressed by the Lévy–Khinchine formula as follows:

$$\psi_X(u) = -aiu + \frac{1}{2}\sigma^2 u^2 - \int_{R\setminus\{0\}} \bigl(e^{iux} - 1 - iuxI_{|x|<1}\bigr)\eta(dx),$$

where $a \in R$, $\sigma \geq 0$, and η is a Lévy measure, that is, $\int_{R\setminus\{0\}} \min\{x^2, 1\}\eta(dx) < \infty$. For the sake of simplicity, let

$$a(u) := \mathrm{Re}\,\psi_X(u) = \frac{1}{2}\sigma^2 u^2 - \int_{R\setminus\{0\}} \bigl(\cos(ux) - 1\bigr)\eta(dx) \tag{6}$$

and

$$b(u) := \mathrm{Im}\,\psi_X(u) = -au - \frac{1}{2}\sigma^2 u^2 - \int_{R\setminus\{0\}} \bigl(\sin(ux) - uxI_{|x|<1}\bigr)\eta(dx). \tag{7}$$

In addition, denote $\phi_{X_t}(u) = \mathrm{E}(e^{iuX_t}) = e^{-t\psi_X(u)}$ as the characteristic function of a Lévy process. According to (6) and (7), it is easy to get

$$\bigl\|\phi_{X_t}(u)\bigr\| = e^{-ta(u)}. \tag{8}$$

The one aim of this paper is to extend the result in Bardina *et al.* [4] to the case of the complex fractional Brownian motion from the unique standard Poisson process and a sequence of independent random variables with common distribution Bernoulli $\frac{1}{2}$, that is:

Theorem 1.1 *Let $\{P_\varepsilon^1, \varepsilon > 0\}$ be the family of laws of the processes Z_ε^θ given by (3) in the Banach space $\mathcal{C}([0,1],\mathbb{C})$. Then P_ε^1 converges weakly as ε tends to zero to the law P^θ in the Banach space $\mathcal{C}([0,1],\mathbb{C})$ of a complex fractional Brownian motion $Z = \{Z(t), t \in [0,1]\}$:*

$$Z(t) = \int_0^1 K_H(t,r)\, dW(r) = \int_0^1 K_H(t,r)\, dW^1(r) + i\int_0^1 K_H(t,r)\, dW^2(r), \tag{9}$$

where $W(r) = W^1(r) + iW^2(r)$ is a complex Brownian motion, $W^1(r)$ and $W^2(r)$ are two independent standard Brownian motions.

The other aim of this paper is to extend the result of Bardina and Rovira [8] to a slightly more general setting applicable to the complex fractional Brownian motion. So, for our processes Y_ε^θ, we get the following weak convergence of realizations of these processes, which is stated as follows.

Theorem 1.2 *The family* $\{P_\varepsilon^2, \varepsilon > 0\}$ *of laws of the processes* Y_ε^θ *in* $\mathcal{C}([0,1],\mathbb{C})$ *converges weakly when* ε *tends to zero to the family* P^θ *of laws of a complex fractional Brownian motion* Z.

The rest of the paper is organized as follows. Section 2 is devoted to proving the tightness of the family $\{P_\varepsilon^1, \varepsilon > 0\}$ and $\{P_\varepsilon^2, \varepsilon > 0\}$. In Sect. 3, we give the proof of our main result.

In addition, throughout the paper C denotes positive constants, not depending on ε, which may change from one expression to another.

2 Main lemmas

In order to prove that the family P_ε^1 is tight, we need to prove that the laws corresponding to the real part and the imaginary part of processes Z_ε^θ are tight. Using the Billingsley criterium (see Billingsley [9]) and that our processes are null on the origin, it suffices to prove the following.

Lemma 2.1 *For any* $t > s$, $\varepsilon > 0$, *there exists a constant* C *such that*

$$\sup_\varepsilon \bigl(\mathrm{E}\bigl(\operatorname{Re} Z_\varepsilon^\theta(t) - \operatorname{Re} Z_\varepsilon^\theta(s)\bigr)^4 + \mathrm{E}\bigl(\operatorname{Im} Z_\varepsilon^\theta(t) - \operatorname{Im} Z_\varepsilon^\theta(s)\bigr)^4\bigr) \le C(t-s)^{4H}. \tag{10}$$

Proof From the definition and the independence of N and N', it is easy to calculate (see Bardina *et al.* [4]) that, for any $0 \le x_1 \le x_2$,

$$\mathrm{E}\bigl[(-1)^{N'_{x_2} - N'_{x_1}} e^{i\theta(N_{x_2} - N_{x_1})}\bigr] = e^{-2(x_2 - x_1)}. \tag{11}$$

Following the representation of complex fractional Brownian motions Z_ε^θ, the real part and the imaginary part can be written as follows:

$$\operatorname{Re} Z_\varepsilon^\theta(t) = \frac{2}{\varepsilon}(-1)^G \int_0^1 K_H(t,r)(-1)^{N'_{\frac{2r}{\varepsilon^2}}} \cos(\theta N_{\frac{2r}{\varepsilon^2}})\,dr \tag{12}$$

and

$$\operatorname{Im} Z_\varepsilon^\theta(t) = \frac{2}{\varepsilon}(-1)^G \int_0^1 K_H(t,r)(-1)^{N'_{\frac{2r}{\varepsilon^2}}} \sin(\theta N_{\frac{2r}{\varepsilon^2}})\,dr, \tag{13}$$

respectively. Then $Z_\varepsilon^\theta(t) = \operatorname{Re} Z_\varepsilon^\theta(t) + i \operatorname{Im} Z_\varepsilon^\theta(t)$. Furthermore, the increments of the real part and the imaginary part of the processes Z_ε^θ can be expressed as follows:

$$\begin{aligned}\operatorname{Re} Z_\varepsilon^\theta(t) - \operatorname{Re} Z_\varepsilon^\theta(s) &= \frac{2}{\varepsilon}(-1)^G \int_0^1 \bigl(K_H(t,r) - K_H(s,r)\bigr)(-1)^{N'_{\frac{2r}{\varepsilon^2}}} \cos(\theta N_{\frac{2r}{\varepsilon^2}})\,dr \\ &= \frac{2}{\varepsilon}(-1)^G \int_0^1 \Delta K_H(t,s,r)(-1)^{N'_{\frac{2r}{\varepsilon^2}}} \cos(\theta N_{\frac{2r}{\varepsilon^2}})\,dr, \end{aligned} \tag{14}$$

and

$$\begin{aligned}\operatorname{Im} Z_\varepsilon^\theta(t) - \operatorname{Im} Z_\varepsilon^\theta(s) &= \frac{2}{\varepsilon}(-1)^G \int_0^1 \bigl(K_H(t,r) - K_H(s,r)\bigr)(-1)^{N'_{\frac{2r}{\varepsilon^2}}} \sin(\theta N_{\frac{2r}{\varepsilon^2}})\,dr \\ &= \frac{2}{\varepsilon}(-1)^G \int_0^1 \Delta K_H(t,s,r)(-1)^{N'_{\frac{2r}{\varepsilon^2}}} \sin(\theta N_{\frac{2r}{\varepsilon^2}})\,dr,\end{aligned} \tag{15}$$

where $\Delta K(t,s,r) = K_H(t,r) - K_H(s,r)$.

Considering

$$\begin{aligned}&2(\cos x_1 \cos x_2 \cos x_3 \cos x_4 + \sin x_1 \sin x_2 \sin x_3 \sin x_4) \\ &\quad = \cos(x_4 - x_3)\cos(x_2 - x_1) + \cos(x_4 + x_3)\cos(x_2 + x_1)\end{aligned} \tag{16}$$

and the independent increments of N', we can get equality (17):

$$\begin{aligned}(-1)^{N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_2}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}} + N'_{\frac{2r_4}{\varepsilon^2}}} &= (-1)^{(N'_{\frac{2r_2}{\varepsilon^2}} - N'_{\frac{2r_1}{\varepsilon^2}}) + (N'_{\frac{2r_4}{\varepsilon^2}} - N'_{\frac{2r_3}{\varepsilon^2}}) + 2(N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}})} \\ &= (-1)^{(N'_{\frac{2r_2}{\varepsilon^2}} - N'_{\frac{2r_1}{\varepsilon^2}}) + (N'_{\frac{2r_4}{\varepsilon^2}} - N'_{\frac{2r_3}{\varepsilon^2}})}.\end{aligned} \tag{17}$$

Therefore, the left-hand side of inequality (10) can be calculated as follows:

$$\begin{aligned}&\mathrm{E}\bigl(\operatorname{Re} Z_\varepsilon^\theta(t) - \operatorname{Re} Z_\varepsilon^\theta(s)\bigr)^4 + \mathrm{E}\bigl(\operatorname{Im} Z_\varepsilon^\theta(t) - \operatorname{Im} Z_\varepsilon^\theta(s)\bigr)^4 \\ &\quad = \frac{16}{\varepsilon^4}\mathrm{E}\Biggl[(-1)^{4G} \int_{[0,1]^4} \prod_{i=1}^4 \Delta K_H(t,s,r_i)(-1)^{N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_2}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}} + N'_{\frac{2r_4}{\varepsilon^2}}} \\ &\qquad \times \cos(\theta N_{\frac{2r_1}{\varepsilon^2}})\cos(\theta N_{\frac{2r_2}{\varepsilon^2}})\cos(\theta N_{\frac{2r_3}{\varepsilon^2}})\cos(\theta N_{\frac{2r_4}{\varepsilon^2}}) \bigotimes_{i=1}^4 dr_i\Biggr] \\ &\qquad + \frac{16}{\varepsilon^4}\mathrm{E}\Biggl[(-1)^{4G} \int_{[0,1]^4} \prod_{i=1}^4 \Delta K_H(t,s,r_i)(-1)^{N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_2}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}} + N'_{\frac{2r_4}{\varepsilon^2}}} \\ &\qquad \times \sin(\theta N_{\frac{2r_1}{\varepsilon^2}})\sin(\theta N_{\frac{2r_2}{\varepsilon^2}})\sin(\theta N_{\frac{2r_3}{\varepsilon^2}})\sin(\theta N_{\frac{2r_4}{\varepsilon^2}}) \bigotimes_{i=1}^4 dr_i\Biggr] \\ &\quad = \frac{16}{\varepsilon^4} \times 4!\mathrm{E}\Biggl[(-1)^{4G} \int_{[0,1]^4} \mathbf{1}_{\{r_1 \le r_2 \le r_3 \le r_4\}} \prod_{i=1}^4 \Delta K_H(t,s,r_i) \\ &\qquad \times (-1)^{N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_2}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}} + N'_{\frac{2r_4}{\varepsilon^2}}} \\ &\qquad \times \bigl[\cos(\theta N_{\frac{2r_1}{\varepsilon^2}})\cos(\theta N_{\frac{2r_2}{\varepsilon^2}})\cos(\theta N_{\frac{2r_3}{\varepsilon^2}})\cos(\theta N_{\frac{2r_4}{\varepsilon^2}}) \\ &\qquad + \sin(\theta N_{\frac{2r_1}{\varepsilon^2}})\sin(\theta N_{\frac{2r_2}{\varepsilon^2}})\sin(\theta N_{\frac{2r_3}{\varepsilon^2}})\sin(\theta N_{\frac{2r_4}{\varepsilon^2}})\bigr] \bigotimes_{i=1}^4 dr_i\Biggr] \\ &\quad = \frac{16}{\varepsilon^4} \times \frac{4!}{2}\mathrm{E}\Biggl[\int_{[0,1]^4} \mathbf{1}_{\{r_1 \le r_2 \le r_3 \le r_4\}} \prod_{i=1}^4 \Delta K_H(t,s,r_i) \\ &\qquad \times (-1)^{N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_2}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}} + N'_{\frac{2r_4}{\varepsilon^2}}}\end{aligned}$$

$$
\begin{aligned}
&\times \cos(\theta N_{\frac{2r_4}{\varepsilon^2}} - \theta N_{\frac{2r_3}{\varepsilon^2}}) \cos(\theta N_{\frac{2r_2}{\varepsilon^2}} - \theta N_{\frac{2r_1}{\varepsilon^2}}) \bigotimes_{i=1}^{4} dr_i \Bigg] \\
&+ \frac{16}{\varepsilon^4} \times \frac{4!}{2} \mathrm{E}\Bigg[\int_{[0,1]^4} \mathbf{1}_{\{r_1 \le r_2 \le r_3 \le r_4\}} \prod_{i=1}^{4} \Delta K_H(t,s,r_i) \\
&\times (-1)^{N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_2}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}} + N'_{\frac{2r_4}{\varepsilon^2}}} \\
&\times \cos(\theta N_{\frac{2r_4}{\varepsilon^2}} + \theta N_{\frac{2r_3}{\varepsilon^2}}) \cos(\theta N_{\frac{2r_2}{\varepsilon^2}} + \theta N_{\frac{2r_1}{\varepsilon^2}}) \bigotimes_{i=1}^{4} dr_i \Bigg] \\
:={}& I_1 + I_2,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
I_1 = {}& \frac{16}{\varepsilon^4} \times \frac{4!}{2} \mathrm{E}\Bigg[\int_{[0,1]^4} \mathbf{1}_{\{r_1 \le r_2 \le r_3 \le r_4\}} \prod_{i=1}^{4} \Delta K_H(t,s,r_i) \\
&\times (-1)^{N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_2}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}} + N'_{\frac{2r_4}{\varepsilon^2}}} \\
&\times \cos(\theta N_{\frac{2r_4}{\varepsilon^2}} - \theta N_{\frac{2r_3}{\varepsilon^2}}) \cos(\theta N_{\frac{2r_2}{\varepsilon^2}} - \theta N_{\frac{2r_1}{\varepsilon^2}}) \bigotimes_{i=1}^{4} dr_i \Bigg]
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
I_2 = {}& \frac{16}{\varepsilon^4} \times \frac{4!}{2} \mathrm{E}\Bigg[\int_{[0,1]^4} \mathbf{1}_{\{r_1 \le r_2 \le r_3 \le r_4\}} \prod_{i=1}^{4} \Delta K_H(t,s,r_i) \\
&\times (-1)^{N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_2}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}} + N'_{\frac{2r_4}{\varepsilon^2}}} \\
&\times \cos(\theta N_{\frac{2r_4}{\varepsilon^2}} + \theta N_{\frac{2r_3}{\varepsilon^2}}) \cos(\theta N_{\frac{2r_2}{\varepsilon^2}} + \theta N_{\frac{2r_1}{\varepsilon^2}}) \bigotimes_{i=1}^{4} dr_i \Bigg].
\end{aligned} \tag{20}
$$

According to equality (11) and $|\cos\theta x| \le |e^{i\theta x}|$, it is easy to get

$$
\mathrm{E}(-1)^{N'_{\frac{2r_2}{\varepsilon^2}} - N'_{\frac{2r_1}{\varepsilon^2}}} e^{i\theta(N_{\frac{2r_2}{\varepsilon^2}} - N_{\frac{2r_1}{\varepsilon^2}})} = e^{-2(\frac{2r_2}{\varepsilon^2} - \frac{2r_1}{\varepsilon^2})} = e^{-\frac{4(r_2 - r_1)}{\varepsilon^2}}. \tag{21}
$$

Then

$$
\begin{aligned}
I_1 = {}& \frac{16}{\varepsilon^4} \times \frac{4!}{2} \mathrm{E}\Bigg[\int_{[0,1]^4} \mathbf{1}_{\{r_1 \le r_2 \le r_3 \le r_4\}} \prod_{i=1}^{4} \Delta K_H(t,s,r_i) \\
&\times (-1)^{N'_{\frac{2r_1}{\varepsilon^2}} + N'_{\frac{2r_2}{\varepsilon^2}} + N'_{\frac{2r_3}{\varepsilon^2}} + N'_{\frac{2r_4}{\varepsilon^2}}} \\
&\times \cos(\theta N_{\frac{2r_4}{\varepsilon^2}} - \theta N_{\frac{2r_3}{\varepsilon^2}}) \cos(\theta N_{\frac{2r_2}{\varepsilon^2}} - \theta N_{\frac{2r_1}{\varepsilon^2}}) \bigotimes_{i=1}^{4} dr_i \Bigg] \\
= {}& \frac{16}{\varepsilon^4} \times \frac{4!}{2} \int_{[0,1]^4} \mathbf{1}_{\{r_1 \le r_2 \le r_3 \le r_4\}} \prod_{i=1}^{4} \Delta K_H(t,s,r_i)
\end{aligned}
$$

$$
\begin{aligned}
&\times \mathrm{E}\Big[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}}\cos(\theta N_{\frac{2r_2}{\varepsilon^2}}-\theta N_{\frac{2r_1}{\varepsilon^2}})\Big] \\
&\times \mathrm{E}\Big[(-1)^{N'_{\frac{2r_4}{\varepsilon^2}}-N'_{\frac{2r_3}{\varepsilon^2}}}\cos(\theta N_{\frac{2r_4}{\varepsilon^2}}-\theta N_{\frac{2r_3}{\varepsilon^2}})\Big]\bigotimes_{i=1}^{4} dr_i \\
\leq{}& \frac{16}{\varepsilon^4}\times\frac{4!}{2}\Bigg\{\int_{[0,1]^2}\mathbf{1}_{\{r_1\leq r_2\}}\prod_{i=1}^{2}\Delta K_H(t,s,r_i) \\
&\times \mathrm{E}\Big[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}}\cos(\theta N_{\frac{2r_2}{\varepsilon^2}}-\theta N_{\frac{2r_1}{\varepsilon^2}})\Big]\,dr_1\,dr_2\Bigg\}^2 \\
\leq{}& \frac{16}{\varepsilon^4}\times\frac{4!}{2}\Bigg\{\int_{[0,1]^2}\mathbf{1}_{\{r_1\leq r_2\}}\prod_{i=1}^{2}\Delta K_H(t,s,r_i) \\
&\times \mathrm{E}\Big[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}}e^{i\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}})}\Big]\,dr_1\,dr_2\Bigg\}^2 \\
={}& \frac{16}{\varepsilon^4}\times\frac{4!}{2}\Bigg\{\int_{[0,1]^2}\mathbf{1}_{\{r_1\leq r_2\}}\prod_{i=1}^{2}\Delta K_H(t,s,r_i)e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_1\,dr_2\Bigg\}^2. \qquad (22)
\end{aligned}
$$

Using the inequality $|ab|\leq\frac{1}{2}(a^2+b^2)$, the last expression of (22) is easily bounded by

$$
\begin{aligned}
&\frac{16}{\varepsilon^4}\times\frac{4!}{2}\times\frac{1}{2^2}\Bigg\{\int_{[0,1]^2}\mathbf{1}_{\{r_1\leq r_2\}}\Delta K_H^2(t,s,r_1)e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_1\,dr_2 \\
&\quad+\int_{[0,1]^2}\mathbf{1}_{\{r_1\leq r_2\}}\Delta K_H^2(t,s,r_2)e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_1\,dr_2\Bigg\}^2 \\
&=\frac{4!}{2}\times\frac{1}{2^2}\Bigg\{\int_0^1\Delta K_H^2(t,s,r_1)\,dr_1\int_{r_1}^1\frac{4}{\varepsilon^2}e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_2 \\
&\quad+\int_0^1\Delta K_H^2(t,s,r_2)\,dr_2\int_0^{r_2}\frac{4}{\varepsilon^2}e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_1\Bigg\}^2 \\
&\leq\frac{4!}{2}\times\frac{1}{2^2}\Bigg\{\int_0^1\Delta K_H^2(t,s,r)\,dr\Bigg\}^2 \qquad (23)
\end{aligned}
$$

as ε tends to zero. It is well known that

$$\int_0^1\big(K_H(t,r)-K_H(s,r)\big)^2\,dr=E\big[B_H(t)-B_H(s)\big]^2=(t-s)^{2H}. \qquad (24)$$

Then there exists a constant C_1 such that

$$\mathrm{E}\big(\operatorname{Re}Z_\varepsilon^\theta(t)-\operatorname{Re}Z_\varepsilon^\theta(s)\big)^4\leq C_1(t-s)^{4H}. \qquad (25)$$

Moreover, considering the fact that

$$
\begin{aligned}
&\cos(x_4+x_3)\cos(x_2+x_1) \\
&\quad=\big[\cos\big((x_4-x_3)+(x_2-x_1)+(2x_3+x_1-x_2)\big)\cos(x_2+x_1)\big]
\end{aligned}
$$

$$
\begin{aligned}
&= \bigl[\cos\bigl((x_4-x_3)+(x_2-x_1)\bigr)\bigr]\bigl[\cos(2x_3+x_1-x_2)\cos(x_2+x_1)\bigr] \\
&\quad - \bigl[\sin\bigl((x_4-x_3)+(x_2-x_1)\bigr)\bigr]\bigl[\sin(2x_3+x_1-x_2)\cos(x_2+x_1)\bigr] \\
&\le \bigl|\bigl[\cos\bigl((x_4-x_3)+(x_2-x_1)\bigr)\bigr]\bigr| + \bigl|\bigl[\sin\bigl((x_4-x_3)+(x_2-x_1)\bigr)\bigr]\bigr| \\
&\le \bigl[\bigl|\cos(x_4-x_3)\bigr| + \bigl|\sin(x_4-x_3)\bigr|\bigr] \times \bigl[\bigl|\cos(x_2-x_1)\bigr| + \bigl|\sin(x_2-x_1)\bigr|\bigr]
\end{aligned}
\tag{26}
$$

and using the inequality $|x+y| \le \sqrt{2(x^2+y^2)}$, we get

$$
\begin{aligned}
\bigl|\cos(x_4-x_3)+\sin(x_4-x_3)\bigr| &\le \sqrt{2}\bigl(\cos^2(x_4-x_3)+\sin^2(x_4-x_3)\bigr)^{\frac{1}{2}} \\
&= \sqrt{2}\bigl|e^{i(x_4-x_3)}\bigr|.
\end{aligned}
\tag{27}
$$

Then, for the term I_2, we have that

$$
\begin{aligned}
I_2 &= \frac{16}{\varepsilon^4} \times \frac{4!}{2} \mathrm{E}\biggl[\int_{[0,1]^4} \mathbf{1}_{\{r_1\le r_2\le r_3\le r_4\}} \prod_{i=1}^{4} \Delta K_H(t,s,r_i) \\
&\quad \times (-1)^{N'_{\frac{2r_1}{\varepsilon^2}}+N'_{\frac{2r_2}{\varepsilon^2}}+N'_{\frac{2r_3}{\varepsilon^2}}+N'_{\frac{2r_4}{\varepsilon^2}}} \\
&\quad \times \cos(\theta N_{\frac{2r_4}{\varepsilon^2}}+\theta N_{\frac{2r_3}{\varepsilon^2}})\cos(\theta N_{\frac{2r_2}{\varepsilon^2}}+\theta N_{\frac{2r_1}{\varepsilon^2}}) \bigotimes_{i=1}^{4} dr_i \biggr] \\
&\le \frac{16}{\varepsilon^4} \times \frac{4!}{2} \mathrm{E}\biggl[\int_{[0,1]^4} \mathbf{1}_{\{r_1\le r_2\le r_3\le r_4\}} \prod_{i=1}^{4} \Delta K_H(t,s,r_i) \\
&\quad \times (-1)^{N'_{\frac{2r_1}{\varepsilon^2}}+N'_{\frac{2r_2}{\varepsilon^2}}+N'_{\frac{2r_3}{\varepsilon^2}}+N'_{\frac{2r_4}{\varepsilon^2}}} \\
&\quad \times \bigl[\cos(\theta N_{\frac{2r_4}{\varepsilon^2}}-\theta N_{\frac{2r_3}{\varepsilon^2}})+\sin(\theta N_{\frac{2r_4}{\varepsilon^2}}-\theta N_{\frac{2r_3}{\varepsilon^2}})\bigr] \\
&\quad \times \bigl[\cos(\theta N_{\frac{2r_2}{\varepsilon^2}}-\theta N_{\frac{2r_1}{\varepsilon^2}})+\sin(\theta N_{\frac{2r_2}{\varepsilon^2}}-\theta N_{\frac{2r_1}{\varepsilon^2}})\bigr] \bigotimes_{i=1}^{4} dr_i \biggr] \\
&\le \frac{16}{\varepsilon^4} \times \frac{4!}{2} \int_{[0,1]^4} \mathbf{1}_{\{r_1\le r_2\le r_3\le r_4\}} \prod_{i=1}^{4} \Delta K_H(t,s,r_i) \bigotimes_{i=1}^{4} dr_i \\
&\quad \times \mathrm{E}\Bigl[(-1)^{N'_{\frac{2r_4}{\varepsilon^2}}-N'_{\frac{2r_3}{\varepsilon^2}}}\bigl(\cos(\theta N_{\frac{2r_4}{\varepsilon^2}}-\theta N_{\frac{2r_3}{\varepsilon^2}})+\sin(\theta N_{\frac{2r_4}{\varepsilon^2}}-\theta N_{\frac{2r_3}{\varepsilon^2}})\bigr)\Bigr] \\
&\quad \times \mathrm{E}\Bigl[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}}\bigl(\cos(\theta N_{\frac{2r_2}{\varepsilon^2}}-\theta N_{\frac{2r_1}{\varepsilon^2}})+\sin(\theta N_{\frac{2r_2}{\varepsilon^2}}-\theta N_{\frac{2r_1}{\varepsilon^2}})\bigr)\Bigr] \\
&= \frac{16}{\varepsilon^4} \times \frac{4!}{2} \biggl\{\int_{[0,1]^2} \mathbf{1}_{\{r_1\le r_2\}} \prod_{i=1}^{2} \Delta K_H(t,s,r_i) \mathrm{E}\Bigl[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}} \\
&\quad \times \bigl(\cos(\theta N_{\frac{2r_2}{\varepsilon^2}}-\theta N_{\frac{2r_1}{\varepsilon^2}})+\sin(\theta N_{\frac{2r_2}{\varepsilon^2}}-\theta N_{\frac{2r_1}{\varepsilon^2}})\bigr)\Bigr] dr_1\, dr_2 \biggr\}^2 \\
&\le \frac{16}{\varepsilon^4} \times \frac{4!}{2} \biggl\{\int_{[0,1]^2} \mathbf{1}_{\{r_1\le r_2\}} \prod_{i=1}^{2} \Delta K_H(t,s,r_i)
\end{aligned}
$$

$$\times \mathrm{E}\Big[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}} e^{i\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}})}\Big]\,dr_1\,dr_2\Bigg\}^2$$

$$\leq \frac{16}{\varepsilon^4}\times\frac{4!}{2}\times\frac{1}{2^2}\bigg\{\int_{[0,1]^2}\mathbf{1}_{\{r_1\leq r_2\}}\Delta K_H^2(t,s,r_1)e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_1\,dr_2\bigg\}^2. \tag{28}$$

Similar to the proof of the term I_1, using equality (23), we easily get the bound of the last integral of (28) as follows:

$$\frac{4!}{2}\times\frac{1}{2^2}\bigg\{\int_0^1\Delta K_H^2(t,s,r)\,dr\bigg\}^2\leq C_2(t-s)^{4H}, \tag{29}$$

where C_2 is a constant. Combining (25) and (29), we obtain that there exists a constant C such that

$$\sup_{\varepsilon}\Big(\mathrm{E}\big(\operatorname{Re}Z_\varepsilon^\theta(t)-\operatorname{Re}Z_\varepsilon^\theta(s)\big)^4+\mathrm{E}\big(\operatorname{Im}Z_\varepsilon^\theta(t)-\operatorname{Im}Z_\varepsilon^\theta(s)\big)^4\Big)\leq C(t-s)^{4H}. \tag{30}$$

This completes the proof. □

Next, we consider the tightness of the processes Y_ε^θ.

Lemma 2.2 *For any $t>s$, $\varepsilon>0$, there exists a constant C such that*

$$\sup_{\varepsilon}\Big(\mathrm{E}\big[\operatorname{Re}Y_\varepsilon^\theta(t)-\operatorname{Re}Y_\varepsilon^\theta(s)\big]^4+\mathrm{E}\big[\operatorname{Im}Y_\varepsilon^\theta(t)-\operatorname{Im}Y_\varepsilon^\theta(s)\big]^4\Big)\leq C(t-s)^{4H}. \tag{31}$$

Proof For the complex function $a(t)=e^{i\theta t}$, we can obtain by using fundamental operations:

$$(\cos\theta t-\cos\theta s)^4+(\sin\theta t-\sin\theta s)^4\leq\big|a(t)-a(s)\big|^4. \tag{32}$$

By (32), we have

$$\begin{aligned}
&\mathrm{E}\big[\operatorname{Re}Y_\varepsilon^\theta(t)-\operatorname{Re}Y_\varepsilon^\theta(s)\big]^4+\mathrm{E}\big[\operatorname{Im}Y_\varepsilon^\theta(t)-\operatorname{Im}Y_\varepsilon^\theta(s)\big]^4\\
&\quad\leq\mathrm{E}\big|Y_\varepsilon^\theta(t)-Y_\varepsilon^\theta(s)\big|^4\\
&\quad=c^4(\theta)\frac{2^4}{\varepsilon^4}\mathrm{E}\bigg(\int_0^1\Delta K_H(t,s,r_1)e^{i\theta X_{\frac{2r_1}{\varepsilon^2}}}\,dr_1\int_0^1\Delta K_H(t,s,r_2)e^{-i\theta X_{\frac{2r_2}{\varepsilon^2}}}\,dr_2\bigg)^2\\
&\quad=c^4(\theta)\frac{2^4}{\varepsilon^4}\bigg(\int_{[0,1]^4}\prod_{i=1}^4\Delta K_H(t,s,r_i)\mathrm{E}\Big[e^{i\theta(X_{\frac{2r_1}{\varepsilon^2}}-X_{\frac{2r_2}{\varepsilon^2}})}e^{i\theta(X_{\frac{2r_3}{\varepsilon^2}}-X_{\frac{2r_4}{\varepsilon^2}})}\Big]\bigotimes_{i=1}^4 dr_i\bigg)\\
&\quad=c^4(\theta)\frac{2^4}{\varepsilon^4}\times 4!\int_{[0,1]^4}\mathbf{1}_{\{r_1\leq r_2\leq r_3\leq r_4\}}\prod_{i=1}^4\Delta K_H(t,s,r_i)\\
&\qquad\times\mathrm{E}\Big[e^{i\theta(X_{\frac{2r_2}{\varepsilon^2}}-X_{\frac{2r_1}{\varepsilon^2}})}e^{i\theta(X_{\frac{2r_4}{\varepsilon^2}}-X_{\frac{2r_3}{\varepsilon^2}})}\Big]\bigotimes_{i=1}^4 dr_i.
\end{aligned} \tag{33}$$

Because the process X has independent increments, we have

$$\mathrm{E}\Big[e^{i\theta[(X_{\frac{2r_4}{\varepsilon^2}}-X_{\frac{2r_3}{\varepsilon^2}})+(X_{\frac{2r_2}{\varepsilon^2}}-X_{\frac{2r_1}{\varepsilon^2}})]}\Big]=\big\|\phi_{X_{\frac{2r_4}{\varepsilon^2}}-X_{\frac{2r_3}{\varepsilon^2}}}(\theta)\big\|\big\|\phi_{X_{\frac{2r_2}{\varepsilon^2}}-X_{\frac{2r_1}{\varepsilon^2}}}(\theta)\big\|. \tag{34}$$

So, the last expression of (33) is easily bounded by

$$c^4(\theta)\frac{2^4}{\varepsilon^4}\times 4!\int_{[0,1]^4}\mathbf{1}_{\{r_1\le r_2\le r_3\le r_4\}}\prod_{i=1}^4\Delta K_H(t,s,r_i)\left\|\phi_{X_{\frac{2r_2}{\varepsilon^2}}-X_{\frac{2r_1}{\varepsilon^2}}}(\theta)\right\| \\ \times\left\|\phi_{X_{\frac{2r_4}{\varepsilon^2}}-X_{\frac{2r_3}{\varepsilon^2}}}(\theta)\right\|\bigotimes_{i=1}^4 dr_i. \tag{35}$$

According to equality (8) of the Lévy process, we get $\|\phi_{X_u-X_v}(\theta)\| = e^{-(u-v)a(\theta)}$. Thus, expression (35) is equal to

$$\begin{aligned}
&c^4(\theta)\frac{2^4}{\varepsilon^4}\times 4!\int_{[0,1]^4}\mathbf{1}_{\{r_1\le r_2\le r_3\le r_4\}}\prod_{i=1}^4\Delta K_H(t,s,r_i)e^{-\frac{2(r_2-r_1)}{\varepsilon^2}a(\theta)} \\
&\quad\times e^{-\frac{2(r_4-r_3)}{\varepsilon^2}a(\theta)}\bigotimes_{i=1}^4 dr_i \\
&\quad= c^4(\theta)\frac{2^3}{\varepsilon^4}\times 4!\left(\int_{[0,1]^2}\mathbf{1}_{\{r_1\le r_2\}}\prod_{i=1}^2\Delta K_H(t,s,r_i)e^{-\frac{2(r_2-r_1)}{\varepsilon^2}a(\theta)}\,dr_1\,dr_2\right)^2 \\
&\quad\le c^4(\theta)\frac{2^4}{\varepsilon^4}\times 4!\left(\int_{[0,1]^2}\mathbf{1}_{\{r_1\le r_2\}}\Delta K_H^2(t,s,r_1)e^{-\frac{2(r_2-r_1)}{\varepsilon^2}a(\theta)}\,dr_1\,dr_2\right. \\
&\qquad\left.+\int_{[0,1]^2}\mathbf{1}_{\{r_1\le r_2\}}\Delta K_H^2(t,s,r_2)e^{-\frac{2(r_2-r_1)}{\varepsilon^2}a(\theta)}\,dr_1\,dr_2\right)^2 \\
&\quad= c^4(\theta)2\times 4!\left(\int_0^1\Delta K_H^2(t,s,r_1)\left(\int_{r_1}^1\frac{2}{\varepsilon^2}e^{-\frac{2(r_2-r_1)}{\varepsilon^2}a(\theta)}\,dr_2\right)dr_1\right. \\
&\qquad\left.+\int_0^1\Delta K_H^2(t,s,r_2)\left(\int_0^{r_2}\frac{2}{\varepsilon^2}e^{-\frac{2(r_2-r_1)}{\varepsilon^2}a(\theta)}\,dr_1\right)dr_2\right)^2 \\
&\quad\le 4c^4(\theta)4!\left(\int_0^1\Delta K_H^2(t,s,r)\,dr\right)^2.
\end{aligned}\tag{36}$$

By equality (24), there exists a constant C such that the last integral of (36) can be bounded by $C(t-s)^{4H}$. The proof has been completed. □

For the proof of Theorem 1.1, we need the following lemma.

Lemma 2.3 *For any $f(r)\in L^2([0,1])$ and $\varepsilon>0$, let*

$$F(t)=\frac{2}{\varepsilon}(-1)^G\int_0^t f(r)(-1)^{N'_{\frac{2r}{\varepsilon^2}}}e^{i\theta N_{\frac{2r}{\varepsilon^2}}}\,dr,\quad t\in[0,1].$$

Then there exists a constant C such that $\mathrm{E}[\operatorname{Re}F(t)]^2\le C\int_0^t f^2(r)\,dr$ *and* $\mathrm{E}[\operatorname{Im}F(t)]^2\le C\int_0^t f^2(r)\,dr$.

Proof Following the definition of $F(t)$, we get

$$\begin{aligned}
\mathrm{E}\big[\operatorname{Re}F(t)\big]^2 &= \mathrm{E}\biggl[\frac{2}{\varepsilon}(-1)^G \int_0^t f(r)(-1)^{N'_{\frac{2r}{\varepsilon^2}}} \cos(\theta N_{\frac{2r}{\varepsilon^2}})\,dr\biggr]^2 \\
&= \mathrm{E}\biggl[\frac{4}{\varepsilon^2}(-1)^{2G} \int_0^t f(r_1)(-1)^{N'_{\frac{2r_1}{\varepsilon^2}}} \cos(\theta N_{\frac{2r_1}{\varepsilon^2}})\,dr_1 \\
&\quad \times \int_0^t f(r_2)(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}} \cos(\theta N_{\frac{2r_2}{\varepsilon^2}})\,dr_2\biggr] \\
&= \mathrm{E}\biggl[\frac{4}{\varepsilon^2} \times 2! \int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f(r_1)f(r_2)(-1)^{N'_{\frac{2r_1}{\varepsilon^2}}+N'_{\frac{2r_2}{\varepsilon^2}}} \\
&\quad \times \cos(\theta N_{\frac{2r_1}{\varepsilon^2}})\cos(\theta N_{\frac{2r_2}{\varepsilon^2}})\,dr_1\,dr_2\biggr].
\end{aligned} \tag{37}$$

Because

$$(-1)^{N'_{\frac{2r_1}{\varepsilon^2}}+N'_{\frac{2r_2}{\varepsilon^2}}} = (-1)^{(N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}})+2N'_{\frac{2r_1}{\varepsilon^2}}} = (-1)^{(N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}})}$$

and

$$\cos(\theta N_{\frac{2r_1}{\varepsilon^2}})\cos(\theta N_{\frac{2r_2}{\varepsilon^2}}) = \frac{1}{2}\big[\cos\theta(N_{\frac{2r_2}{\varepsilon^2}} - N_{\frac{2r_1}{\varepsilon^2}}) + \cos\theta(N_{\frac{2r_2}{\varepsilon^2}} + N_{\frac{2r_1}{\varepsilon^2}})\big].$$

Then equation (37) is equal to

$$\begin{aligned}
&\mathrm{E}\biggl[\frac{4}{\varepsilon^2}\int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f(r_1)f(r_2)(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}} \big[\cos\theta(N_{\frac{2r_2}{\varepsilon^2}} - N_{\frac{2r_1}{\varepsilon^2}}) \\
&\quad + \cos\theta(N_{\frac{2r_2}{\varepsilon^2}} + N_{\frac{2r_1}{\varepsilon^2}})\big]\,dr_1\,dr_2\biggr] \\
&= \frac{4}{\varepsilon^2}\int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f(r_1)f(r_2)\mathrm{E}\big[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}}\big[\cos\theta(N_{\frac{2r_2}{\varepsilon^2}} - N_{\frac{2r_1}{\varepsilon^2}})\big]\big]\,dr_1\,dr_2 \\
&\quad + \frac{4}{\varepsilon^2}\int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f(r_1)f(r_2)\mathrm{E}\big[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}}\big[\cos\theta(N_{\frac{2r_2}{\varepsilon^2}} + N_{\frac{2r_1}{\varepsilon^2}})\big]\big]\,dr_1\,dr_2 \\
&:= I_3 + I_4.
\end{aligned}$$

Using (21), we get

$$\begin{aligned}
I_3 &= \frac{4}{\varepsilon^2}\int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f(r_1)f(r_2)e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_1\,dr_2 \\
&= \frac{4}{\varepsilon^2}\times\frac{1}{2}\biggl[\int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f^2(r_1)e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_1\,dr_2 \\
&\quad + \int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f^2(r_2)e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_1\,dr_2\biggr] \\
&= \frac{1}{2}\biggl[\int_0^t f^2(r_1)\,dr_1 \int_{r_1}^t \frac{4}{\varepsilon^2}e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_2 + \int_0^t f^2(r_2)\,dr_2 \int_0^{r_2} \frac{4}{\varepsilon^2}e^{-\frac{4(r_2-r_1)}{\varepsilon^2}}\,dr_1\biggr]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\int_0^t f^2(r_1)\left(1 - e^{-\frac{4(t-r_1)}{\varepsilon^2}}\right) dr_1 + \int_0^t f^2(r_2)\left(1 - e^{-\frac{4r_2}{\varepsilon^2}}\right) dr_2\right] \\
&\le \frac{1}{2}\left[\int_0^t f^2(r_1)\, dr_1 + \int_0^t f^2(r_2)\, dr_2\right] = \int_0^t f^2(r)\, dr
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \frac{4}{\varepsilon^2}\int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f(r_1)f(r_2)\mathrm{E}\Big[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}}\big[\cos\theta(N_{\frac{2r_2}{\varepsilon^2}}+N_{\frac{2r_1}{\varepsilon^2}})\big]\Big]\, dr_1\, dr_2 \\
&\le \frac{4}{\varepsilon^2}\sqrt{2}\int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f(r_1)f(r_2)\Big|\mathrm{E}\Big[(-1)^{N'_{\frac{2r_2}{\varepsilon^2}}-N'_{\frac{2r_1}{\varepsilon^2}}} e^{i\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}})}\Big]\Big|\, dr_1\, dr_2 \\
&\le \sqrt{2}\int_0^t f^2(r)\, dr
\end{aligned}
$$

since

$$
\begin{aligned}
&\cos\theta(N_{\frac{2r_2}{\varepsilon^2}}+N_{\frac{2r_2}{\varepsilon^2}}) \\
&\quad= \cos\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}}+2N_{\frac{2r_1}{\varepsilon^2}}) \\
&\quad= \cos\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}})\cos(2N_{\frac{2r_1}{\varepsilon^2}}) - \sin\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}})\sin(2N_{\frac{2r_1}{\varepsilon^2}}) \\
&\quad\le \big|\cos\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}})\big| + \big|\sin\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}})\big| \\
&\quad\le \sqrt{2}\big(\cos^2\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}}) + \sin^2\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}})\big) \\
&\quad= \sqrt{2}\big|e^{i\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}})}\big|.
\end{aligned}
$$

For the term $\mathrm{E}[\mathrm{Im}\, F(t)]^2$, we have

$$
\begin{aligned}
\mathrm{E}\big[\mathrm{Im}\, F(t)\big]^2 &= \mathrm{E}\left[\frac{2}{\varepsilon}(-1)^G\int_0^t f(r)(-1)^{N'_{\frac{2r}{\varepsilon^2}}}\sin(\theta N_{\frac{2r}{\varepsilon^2}})\, dr\right]^2 \\
&= \mathrm{E}\bigg[\frac{4}{\varepsilon^2}\times 2!\int_{[0,t]^2} \mathbf{1}_{\{r_1\le r_2\}} f(r_1)f(r_2)(-1)^{N'_{\frac{2r_1}{\varepsilon^2}}+N'_{\frac{2r_2}{\varepsilon^2}}}\sin(\theta N_{\frac{2r_1}{\varepsilon^2}}) \\
&\qquad\times \sin(\theta N_{\frac{2r_2}{\varepsilon^2}})\, dr_1\, dr_2\bigg].
\end{aligned}
$$

Using the equation

$$
\sin(\theta N_{\frac{2r_1}{\varepsilon^2}})\sin(\theta N_{\frac{2r_2}{\varepsilon^2}}) = \frac{1}{2}\big(\cos\theta(N_{\frac{2r_2}{\varepsilon^2}}-N_{\frac{2r_1}{\varepsilon^2}}) - \cos\theta(N_{\frac{2r_2}{\varepsilon^2}}+N_{\frac{2r_2}{\varepsilon^2}})\big),
$$

similar to the calculation of $\mathrm{E}[\mathrm{Re}\, F(t)]^2$, we can get $\mathrm{E}[\mathrm{Im}\, F(t)]^2 \le C\int_0^t f^2(r)\, dr$. The proof of this lemma is accomplished. □

3 Weak convergence to the complex fractional Brownian motion

In this section, we give the proof of Theorem 1.1 and Theorem 1.2 by checking that the families of laws of the processes Z_ε^θ and Y_ε^θ are tight respectively and that any weakly convergent subsequence converges to the law of the complex fractional Brownian motion.

Proof of Theorem 1.1 Firstly, following Lemma 2.1, we have proved the tightness of the family P^1_ε of laws of the processes Z^θ_ε by applying Theorem 12.3 of Billingsley [9].

Next we will prove that the family of stochastic processes Z^θ_ε converges in the sense of finite dimensional distribution function to the process Z. That is, for any integer number $N \geq 1$, considering arbitrary real numbers $a_1, \dots, a_N \in \mathbb{R}$ and $t_1, \dots, t_N \in [0,1]$, we have

$$S^\theta_\varepsilon := \sum_{k=1}^N a_k Z^\theta_\varepsilon(t_k) \to S := \sum_{k=1}^N a_k Z(t_k)$$

as ε tends to zero. To prove this, the convergence of the corresponding characteristic functions must be checked.

For the sake of simplicity, we denote

$$\begin{aligned}\theta(r) &= \frac{2}{\varepsilon}(-1)^{G+N'_{\frac{2r}{\epsilon^2}}} e^{i\theta N_{\frac{2r}{\epsilon^2}}} \\ &= \frac{2}{\varepsilon}(-1)^{G+N'_{\frac{2r}{\epsilon^2}}} \cos(\theta N_{\frac{2r}{\epsilon^2}}) + i\frac{2}{\varepsilon}(-1)^{G+N'_{\frac{2r}{\epsilon^2}}} \sin(\theta N_{\frac{2r}{\epsilon^2}}) \\ &:= \operatorname{Re}\theta(r) + i\operatorname{Im}\theta(r).\end{aligned}$$

Note that

$$S^\theta_\varepsilon = \int_0^1 K^*(r)\theta(r)\,dr = \int_0^1 K^*(r)\operatorname{Re}\theta(r)\,dr + i\int_0^1 K^*(r)\operatorname{Im}\theta(r)\,dr \tag{38}$$

and

$$S = \int_0^1 K^*(r)\,dW(r) = \int_0^1 K^*(r)\,dW^1(r) + i\int_0^1 K^*(r)\,dW^2(r), \tag{39}$$

where $K^*(r) = \sum_{k=1}^N a_k K(t_k, r)$, $W(r) = W^1(r) + iW^2(r)$ is a complex Brownian motion, $W^1(r)$ and $W^2(r)$ are two independent standard Brownian motions.

The function $K^*(r) \in L^2([0,1])$ can be approximated by a sequence of step functions of the form

$$K^n(r) = \sum_{i=0}^{m_n-1} K^n_i 1_{(r^n_i, r^n_{i+1}]}(r), \tag{40}$$

with $0 = r^n_0 < r^n_1 < \cdots < r^n_{m_n-1} < r^n_{m_n} = 1$ and K^n_i, $i = 0, \dots, m_n - 1$ being constants that are chosen such that

$$\int_0^1 \bigl(K^*(r) - K^n(r)\bigr)^2\,dr \leq \frac{1}{n} \quad \text{for any } n \in \mathbb{N}. \tag{41}$$

Define now

$$S^n_\varepsilon = \int_0^1 K^n(r)\theta_\varepsilon(r)\,dr = \int_0^1 K^n(r)\operatorname{Re}\theta_\varepsilon(r)\,dr + i\int_0^1 K^n(r)\operatorname{Im}\theta_\varepsilon(r)\,dr \tag{42}$$

and

$$S^n = \int_0^1 K^n(r)\,dW(r) = \int_0^1 K^n(r)\,dW^1(r) + i\int_0^1 K^n(r)\,dW^2(r). \tag{43}$$

By taking $f(r) = K^*(r) - K^n(r)$ in Lemma 2.3, we have that there exists a positive constant C, which does not depend on n, such that

$$\mathrm{E}\bigl[\bigl(\operatorname{Re} S_\varepsilon^\theta - \operatorname{Re} S_\varepsilon^n\bigr)^2\bigr] \le C\int_0^1 \bigl(K^*(r) - K^n(r)\bigr)^2\,dr \le C\frac{1}{n} \tag{44}$$

and

$$\mathrm{E}\bigl[\bigl(\operatorname{Im} S_\varepsilon^\theta - \operatorname{Im} S_\varepsilon^n\bigr)^2\bigr] \le C\int_0^1 \bigl(K^*(r) - K^n(r)\bigr)^2\,dr \le C\frac{1}{n} \tag{45}$$

for any $\varepsilon > 0$.

On the other hand, for fixed $n \in \mathbb{N}$,

$$\begin{aligned} S_\varepsilon^n &= \sum_{i=0}^{m_n-1}\int_0^1 K_i^n \theta_\varepsilon(r)\,dr \\ &= \sum_{i=0}^{m_n-1}\int_0^1 K_i^n \operatorname{Re}\theta_\varepsilon(r)\,dr + i\sum_{i=0}^{m_n-1}\int_0^1 K_i^n \operatorname{Im}\theta_\varepsilon(r)\,dr \end{aligned}$$

converges in law as ε tends to zero to

$$\begin{aligned} S^n &= \int_0^1 K^n(r)\,dW^1(r) + i\int_0^1 K^n(r)\,dW^2(r) \\ &= \sum_{i=0}^{m_n-1}\int_0^1 K_i^n\,dW^1(r) + i\sum_{i=0}^{m_n-1}\int_0^1 K_i^n\,dW^2(r) \end{aligned}$$

due to the result established by Bardina *et al.* [4]. Then we have the convergence of the corresponding characteristic function: for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$\mathrm{E}\bigl[e^{ixS_\varepsilon^n}\bigr] \to \mathrm{E}\bigl[e^{ixS^n}\bigr] \quad \text{as } \varepsilon \to 0. \tag{46}$$

From Bardina *et al.* [4], it is easy to get that $\operatorname{Re} Z_\varepsilon^\theta(t)$ and $\operatorname{Im} Z_\varepsilon^\theta(t)$ are two independent centered Gaussian processes. So, we get that

$$\begin{aligned} &\bigl|\mathrm{E}\bigl[e^{i\lambda S_\varepsilon^\theta}\bigr] - \mathrm{E}\bigl[e^{i\lambda S}\bigr]\bigr| \\ &\quad = \bigl|\mathrm{E}\bigl[e^{i\lambda(\operatorname{Re} S_\varepsilon^\theta + i\operatorname{Im} S_\varepsilon^\theta)}\bigr] - \mathrm{E}\bigl[e^{i\lambda(\operatorname{Re} S + i\operatorname{Im} S)}\bigr]\bigr| \\ &\quad = \bigl|\mathrm{E}\bigl[e^{i\lambda\operatorname{Re} S_\varepsilon^\theta} e^{-\lambda\operatorname{Im} S_\varepsilon^\theta}\bigr] - \mathrm{E}\bigl[e^{i\lambda\operatorname{Re} S} e^{-\lambda\operatorname{Im} S}\bigr]\bigr| \\ &\quad = \bigl|\mathrm{E}\bigl[\bigl(e^{i\lambda\operatorname{Re} S_\varepsilon^\theta} - e^{i\lambda\operatorname{Re} S}\bigr)e^{-\lambda\operatorname{Im} S_\varepsilon^\theta}\bigr] \\ &\qquad + \mathrm{E}\bigl[e^{i\lambda\operatorname{Re} S}\bigl(e^{-\lambda\operatorname{Im} S_\varepsilon^\theta} - e^{-\lambda\operatorname{Im} S}\bigr)\bigr]\bigr| \\ &\quad \le \bigl|\mathrm{E}\bigl[e^{-\lambda\operatorname{Im} S_\varepsilon^\theta}\bigr]\bigr| I_5 + \bigl|\mathrm{E}\bigl[e^{i\lambda\operatorname{Re} S}\bigr]\bigr| I_6, \end{aligned} \tag{47}$$

where $I_5 := |\mathrm{E}(e^{i\lambda\operatorname{Re} S_\varepsilon^\theta} - e^{i\lambda\operatorname{Re} S})|$ and $I_6 := |\mathrm{E}(e^{-\lambda\operatorname{Im} S_\varepsilon^\theta} - e^{-\lambda\operatorname{Im} S})|$.

Using the mean value theorem, there exists $\xi \in (a,b)$ for $a \le b$ such that

$$e^{ib} - e^{ia} = \int_a^b ie^{ix}\,dx = ie^{i\xi}(b-a) = e^{i(\frac{\pi}{2}+\xi)}(b-a).$$

Then it is easy to get

$$\begin{aligned}\left|\mathrm{E}e^{iX_2} - \mathrm{E}e^{iX_1}\right| &= \left|\mathrm{E}\bigl(e^{iX_2} - e^{iX_1}\bigr)\right| = \left|\mathrm{E}e^{i(\frac{\pi}{2}+\xi)}(X_2 - X_1)\right| \\ &\le \mathrm{E}\bigl[\bigl|e^{i(\frac{\pi}{2}+\xi)}\bigr||X_2 - X_1|\bigr] \le \mathrm{E}|X_2 - X_1|,\end{aligned}$$

where X_1, X_2 are two random variables. So, there exists a constant $C > 0$ for the term I_5 such that

$$I_5 \le |\lambda|\bigl\{\mathrm{E}\bigl|\mathrm{Re}\,S_\varepsilon^\theta - \mathrm{Re}\,S\bigr|\bigr\} \le C\bigl\{\mathrm{E}\bigl|\mathrm{Re}\,S_\varepsilon^\theta - \mathrm{Re}\,S\bigr|\bigr\}.$$

Meanwhile, by (38) and (39), we have

$$\mathrm{E}\bigl|\mathrm{Re}\,S_\varepsilon^\theta - \mathrm{Re}\,S\bigr| \le I_{51} + I_{52} + I_{53},$$

where $I_{51} = \mathrm{E}|\,\mathrm{Re}\,S_\varepsilon^\theta - \mathrm{Re}\,S_\epsilon^n|$, $I_{52} = \mathrm{E}|\,\mathrm{Re}\,S_\epsilon^n - \mathrm{Re}\,S^n|$, and $I_{53} = \mathrm{E}|\,\mathrm{Re}\,S^n - \mathrm{Re}\,S|$.

Using the Schwarz inequality $(\mathrm{E}|\xi\eta|)^2 \le \mathrm{E}|\xi|^2\mathrm{E}|\eta|^2$ and (44), we can get, for any $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\begin{aligned}I_{51} &= \mathrm{E}\bigl|\mathrm{Re}\,S_\varepsilon^\theta - \mathrm{Re}\,S_\epsilon^n\bigr| \\ &= \mathrm{E}\left|\int_0^1 \bigl(K^*(r) - K^n(r)\bigr)\,\mathrm{Re}\,\theta(r)\,dr\right| \\ &\le \left(\int_0^1 \bigl(K^*(r) - K^n(r)\bigr)^2\,\mathrm{Re}\,\theta(r)\,dr\right)^{\frac{1}{2}} \left(\mathrm{E}\int_0^1 \bigl(\mathrm{Re}\,\theta(r)\bigr)^2\,dr\right)^{\frac{1}{2}} \\ &\le C\left(\int_0^1 \bigl(K^*(r) - K^n(r)\bigr)^2\,dr\right)^{\frac{1}{2}} \le C\frac{1}{\sqrt{n}}.\end{aligned}$$

By Bardina *et al.* [4], we easily get that the real part and the imaginary part of the processes Z_ε^θ are two independent Brownian motions. So, for the term I_{52}, we have

$$\begin{aligned}I_{52} &= \mathrm{E}\bigl|\mathrm{Re}\,S_\epsilon^n - \mathrm{Re}\,S^n\bigr| \\ &= \mathrm{E}\left|\int_0^1 K^n(r)\,\mathrm{Re}\,\theta(r)\,dr - \int_0^1 K^n(r)\,dW(r)\right| \\ &\to 0, \quad \epsilon \to 0.\end{aligned}$$

According to the Schwarz inequality and the isometric property of the Wiener integral with respect to the term I_{53}, we obtain that, for any $n \in \mathbb{N}$,

$$I_{53} \le \bigl(\mathrm{E}\bigl|\mathrm{Re}\,S^n - \mathrm{Re}\,S\bigr|^2\bigr)^{\frac{1}{2}} \le C\left(\int_0^1 \bigl(K^*(r) - K^n(r)\bigr)^2\,dr\right)^{1/2} \le C\frac{1}{\sqrt{n}}.$$

Then both I_{51} and I_{53} become arbitrarily small by taking $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Similarly, we can prove I_6 converging to 0 as n tends to infinity. This completes the proof. □

Proof of Theorem 1.2 The tightness of the processes Y_ε^θ comes from Lemma 2.3. Next, we identify the limit law by proving that the family of stochastic processes Y_ε^θ converges in the sense of finite dimensional distribution function to the process Z as ε tends to zero, that is, we prove

$$T_\varepsilon := \sum_{k=1}^{N} a_k Y_\varepsilon^\theta(t_k) \to T := \sum_{k=1}^{N} a_k Z(t_k)$$

in distribution when ε tends to zero, where $a_1, \dots, a_N \in \mathbb{R}$ and $t_1, \dots, t_N \in [0,1]$.

Similar to the proof of Theorem 1.1, note that

$$T_\varepsilon = \int_0^1 K^*(r) \frac{2}{\varepsilon} c(\theta) e^{i\theta X_{\frac{2r}{\varepsilon^2}}}\, dr \quad \text{and} \quad T = \int_0^1 K^*(r)\, dW(r),$$

where $K^*(r) = \sum_{k=1}^{N} a_k K(t_k, r)$.

Because $K^*(r) \in L_2([0,1])$, there exists a simple function

$$K^n(r) = \sum_{k=0}^{m_n - 1} K_k^n \mathbf{1}_{(r_k^n, r_{k+1}^n]}(r) \tag{48}$$

with $0 = r_0^n < r_1^n < \cdots < r_{m_n-1}^n < r_{m_n}^n = 1$ such that

$$\int_0^1 \bigl(K^*(r) - K^n(r)\bigr)^2\, dr \leq \frac{1}{n}, \quad n \geq 1.$$

Now, define two variables $T_\varepsilon^n = \int_0^1 K^n(r) \frac{2}{\varepsilon} c(\theta) e^{i\theta X_{\frac{2r}{\varepsilon^2}}}\, dr$ and $T^n = \int_0^1 K^n(r)\, dW(r)$, then

$$\begin{aligned} \mathrm{E}\bigl|T_\varepsilon^n - T_\varepsilon\bigr| &\leq \mathrm{E} \int_0^1 \left| \bigl(K^n(r) - K^*(r)\bigr) \frac{2}{\varepsilon} c(\theta) e^{i\theta X_{\frac{2r}{\varepsilon^2}}} \right| dr \\ &\leq c(\theta) \left(\int_0^1 \bigl|\bigl(K^n(r) - K^*(r)\bigr)\bigr|^2\, dr \right)^{\frac{1}{2}} \\ &\leq c(\theta) \frac{1}{\sqrt{n}}. \end{aligned} \tag{49}$$

By (48), we have

$$\begin{aligned} T_\varepsilon^n &= \int_0^1 K^n(r) \frac{2}{\varepsilon} c(\theta) e^{i\theta X_{\frac{2r}{\varepsilon^2}}}\, dr \\ &= \sum_{k=0}^{m_n-1} K_k^n \int_{r_k^n}^{r_{k+1}^n} \frac{2}{\varepsilon} c(\theta) e^{i\theta X_{\frac{2r}{\varepsilon^2}}}\, dr. \end{aligned} \tag{50}$$

According to the result obtained by Bardina and Rovira [8], it is easy to get the weak convergence of T_ε^n to T^n, where $T^n = \int_0^1 K^n(r)\, dW(r) = \sum_{k=0}^{m_n-1} K_k^n \int_{r_k^n}^{r_{k+1}^n} dW(r)$.

So,

$$\mathrm{E}\big(e^{iuT_\varepsilon^n}\big) \to \mathrm{E}\big(e^{iuT^n}\big), \quad \varepsilon \to 0. \tag{51}$$

By the triangle inequality, we have

$$\big|\mathrm{E}\big(e^{iuT_\varepsilon}\big) - \mathrm{E}\big(e^{iuT}\big)\big| \le \alpha_\varepsilon^n + \beta_\varepsilon^n + \gamma^n, \quad u \in R, \varepsilon > 0, n \in N, \tag{52}$$

where $\alpha_\varepsilon^n = |\mathrm{E}(e^{iuT_\varepsilon}) - \mathrm{E}(e^{iuT_\varepsilon^n})|$, $\beta_\varepsilon^n = |\mathrm{E}(e^{iuT_\varepsilon^n}) - \mathrm{E}(e^{iuT^n})|$, $\gamma^n = |\mathrm{E}(e^{iuT^n}) - \mathrm{E}(e^{iuT})|$.

Using the mean value theorem and inequality (49), we obtain $\alpha_\varepsilon^n \le u\mathrm{E}|T_\varepsilon - T_\varepsilon^n|$ converging to 0 as n tends to infinity. With respect to the term β_ε^n, it is easy to get the convergence of β_ε^n to 0 as ε tends to 0 from (51).

Next we consider the term γ^n. By the Schwarz inequality and the isometric property of the Wiener integer, we get

$$\begin{aligned}
\gamma^n &\le u\mathrm{E}\big|T^n - T\big| \\
&\le u\big[\mathrm{E}\big|\mathrm{Re}\,T^n - \mathrm{Re}\,T\big| + \mathrm{E}\big|\mathrm{Im}\,T^n - \mathrm{Im}\,T\big|\big] \\
&\le |u|\big[\big(\mathrm{E}\big|\mathrm{Re}\,T^n - \mathrm{Re}\,T\big|^2\big)^{\frac{1}{2}} + \big(\mathrm{E}\big|\mathrm{Im}\,T^n - \mathrm{Im}\,T\big|^2\big)^{\frac{1}{2}}\big] \\
&= |u|\left(\int_0^1 \big(K^*(r) - K^n(r)\big)^2\,dr\right)^{\frac{1}{2}} \\
&\le |u|\frac{1}{\sqrt{n}} \to 0
\end{aligned} \tag{53}$$

as n tends to infinity. This completes the proof. □

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Competing interests
The authors declare that they have no competing interests.

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