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Existence of periodic solutions for a higher-order neutral difference equation

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Abstract

Based on a continuation theorem of Mawhin, the existence of a periodic solution for a higher-order nonlinear neutral difference equation is studied. Our conclusion is new and interesting.

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1 Introduction

The periodic solution theory of differential equation and difference equation has important academic value and application background. It has aroused people's great concern, and many good results have been achieved. For example, see articles [1–8] and the references therein. However, as far as we know, the results of the periodic solutions of neutral difference equations are relatively few (see [7, 8]).

In this paper, we study the periodic solutions of a higher-order nonlinear neutral difference equation of the form

$$\Delta^k[u_n + cu_{n-\sigma}] = g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l}), \quad n \in \mathbb{Z}, \quad (1)$$

where k is a positive integer, c is a real number different from -1 and 1 , σ and τ_i are integers for $i \in \{1, 2, \dots, l\}$, $g_n \in C(R^l, R)$ for $n \in \mathbb{Z}$ and $g_n = g_{n+\omega}$, where ω is a positive integer which satisfies $\omega \geq 2$. We use a continuity theorem to give some criteria for the existence of a periodic solution of (1), and our conclusion is new and interesting.

A solution of (1) is a real sequence of the form $x = \{x_n\}_{n \in \mathbb{Z}}$ which renders (1) into an identity after substitution. As usual, a solution of (1) of the form $x = \{x_n\}_{n \in \mathbb{Z}}$ is said to be ω -periodic if $x_{n+\omega} = x_n$ for $n \in \mathbb{Z}$.

We also state Mawhin's continuation theorem (see [1]). Let X and Y be two Banach spaces, and $L : \text{Dom } L \subset X \rightarrow Y$ is a linear mapping and $N : X \rightarrow Y$ is a continuous mapping. The mapping L is called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim Im } L < +\infty$, and $\text{Im } L$ is closed in Y . If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \rightarrow X$ and $Q : Y \rightarrow Y$ such that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q = \text{Im}(I - Q)$. It follows that $L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$ has an inverse which is denoted by K_p . If Ω is an open and bounded subset of X , then the mapping N is called

L -compact on $\overline{\Omega}$ when $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N : \overline{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J : \text{Im } Q \rightarrow \text{Ker } L$.

Theorem A (Mawhin's continuation theorem [1]) *Let L be a Fredholm mapping of index zero, and let N be L -compact on $\overline{\Omega}$. Suppose*

- (I) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega$, $Lx \neq \lambda Nx$; and*
- (II) *for each $x \in \partial\Omega \cap \text{Ker } L$, $QNx \neq 0$ and $\deg(JQN, \Omega \cap \text{Ker } L, 0) \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \text{Dom } L$.

2 Main result

The main result of this paper is as follows:

Theorem 2.1 *Let $|c| \neq 1$. Assume that there exist constants $D > 0$, $\alpha \geq 0$, and $\beta \geq 0$ such that*

- (I) $|g_n(x_1, x_2, \dots, x_l)| \leq \beta \max_{1 \leq i \leq l} |x_i| + \alpha$ for $n \in \mathbb{Z}$ and $(x_1, x_2, \dots, x_l)^T \in \mathbb{R}^l$,
- (II) $g_n(x_1, x_2, \dots, x_l) > 0$ for $n \in \mathbb{Z}$ and $x_1, x_2, \dots, x_l \geq D$,
- (III) $g_n(x_1, x_2, \dots, x_l) < 0$ for $n \in \mathbb{Z}$ and $x_1, x_2, \dots, x_l \leq -D$.

Then the higher-order neutral difference equation (1) has an ω -periodic solution when $\omega^k \beta < 2^k |1 - |c||$.

Remark 2.1 When g_n in (1) is replaced by $-g_n$, the result of Theorem 2.1 still holds.

Next, some preparations are presented to prove our theorems. Let X_ω be the Banach space of all real ω -periodic sequences of the form $x = \{x_n\}_{n \in \mathbb{Z}}$ and endowed with the usual linear structure as well as the norm $|x|_\infty = \max_{1 \leq i \leq \omega} |x_i|$.

Define the mappings $L : X_\omega \rightarrow X_\omega$ and $N : X_\omega \rightarrow X_\omega$ respectively by

$$(Lx)_n = \Delta^k [x_n + cx_{n-\sigma}], \quad n \in \mathbb{Z}, \quad (2)$$

and

$$(Nx)_n = g_n(x_{n-\tau_1}, x_{n-\tau_2}, \dots, x_{n-\tau_l}), \quad n \in \mathbb{Z}. \quad (3)$$

It is easy to see that L is a linear mapping. Similar to the paper [8], in case $|c| \neq 1$, direct calculation shows that $\text{Ker } L = \{x \in X_\omega | x_n = x_0, n \in \mathbb{Z}\}$. Since $\dim X_\omega = \omega$ and $L : X_\omega \rightarrow X_\omega$ is a linear mapping, by the knowledge of linear algebra, we know that $\dim \text{Ker } L \oplus \dim \text{Im } L = \dim X_\omega$. It is easy to see that $\dim \text{Ker } L = \text{codim Im } L = 1$ and $\dim \text{Im } L = \omega - 1$. It follows that $\text{Im } L$ is closed in X_ω . Thus L is a Fredholm mapping of index zero. Now, we assert that

$$\text{Ker } L \oplus \text{Im } L = X_\omega. \quad (4)$$

To do that, we just have to prove that

$$\text{Ker } L \cap \text{Im } L = 0. \quad (5)$$

Indeed, if $y = \{y_n\}_{n \in \mathbb{Z}} \in \text{Im } L$, then there is $x = \{x_n\}_{n \in \mathbb{Z}} \in X_\omega$ such that

$$y_n = \Delta^k [x_n + cx_{n-\sigma}], \quad n \in \mathbb{Z}. \quad (6)$$

Thus

$$\sum_{i=1}^{\omega} y_i = \sum_{i=1}^{\omega} \Delta^k [x_i + cx_{i-\sigma}]. \quad (7)$$

Note that $x = \{x_n\}_{n \in \mathbb{Z}} \in X_{\omega}$. It follows that $\{\Delta^{k-1}x_n\}_{n \in \mathbb{Z}} \in X_{\omega}$. Furthermore, direct calculation shows that

$$\begin{aligned} \sum_{i=1}^{\omega} \Delta^k x_i &= [\Delta^{k-1}x_2 - \Delta^{k-1}x_1] + [\Delta^{k-1}x_3 - \Delta^{k-1}x_2] + \cdots \\ &\quad + [\Delta^{k-1}x_{\omega} - \Delta^{k-1}x_{\omega-1}] + [\Delta^{k-1}x_{\omega+1} - \Delta^{k-1}x_{\omega}] \\ &= \Delta^{k-1}x_{\omega+1} - \Delta^{k-1}x_1 = \Delta^{k-1}x_1 - \Delta^{k-1}x_1 = 0. \end{aligned} \quad (8)$$

By (7) and (8), we have

$$\sum_{i=1}^{\omega} y_i = \sum_{i=1}^{\omega} \Delta^k [x_i + cx_{i-\sigma}] = \sum_{i=1}^{\omega} \Delta^k x_i + c \sum_{i=1}^{\omega} \Delta^k x_{i-\sigma} = 0 + 0 = 0. \quad (9)$$

We see that for any $u = \{u_n\}_{n \in \mathbb{Z}} \in \text{Ker } L \cap \text{Im } L$, then $u = \{u_n\}_{n \in \mathbb{Z}} \in \text{Ker } L$ and $u = \{u_n\}_{n \in \mathbb{Z}} \in \text{Im } L$. Because of $u = \{u_n\}_{n \in \mathbb{Z}} \in \text{Ker } L$ and $\text{Ker } L = \{x \in X_{\omega} | x_n = x_0, n \in \mathbb{Z}\}$, thus for any $n \in \mathbb{Z}$, we have

$$u_n = \frac{1}{\omega} \sum_{i=1}^{\omega} u_i. \quad (10)$$

On the other hand, since $u = \{u_n\}_{n \in \mathbb{Z}} \in \text{Im } L$, by (9), we have

$$\sum_{i=1}^{\omega} u_i = 0. \quad (11)$$

By (10) and (11), we see that, for any $n \in \mathbb{Z}$, $u_n = 0$. This implies that (5) is true, that (4) is true. Now, for any $u = \{u_n\}_{n \in \mathbb{Z}} \in X_{\omega}$, if

$$u = x + y,$$

where $x = \{x_n\}_{n \in \mathbb{Z}} \in \text{Ker } L$ and $y = \{y_n\}_{n \in \mathbb{Z}} \in \text{Im } L$, then

$$x_n = \frac{1}{\omega} \sum_{i=1}^{\omega} u_i, \quad n \in \mathbb{Z}, \quad (12)$$

and

$$y_n = u_n - \frac{1}{\omega} \sum_{i=1}^{\omega} u_i, \quad n \in \mathbb{Z}.$$

As in paper [8], we define $P = Q : X_{\omega} \rightarrow X_{\omega}$ by

$$(Px)_n = (Qx)_n = \frac{1}{\omega} \sum_{i=1}^{\omega} x_i, \quad n \in \mathbb{Z}. \quad (13)$$

The operators P and Q are projections. We have $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$, and $X_\omega = \text{Ker } P \oplus \text{Ker } L = \text{Im } L \oplus \text{Im } Q$. It follows that $L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)X_\omega \rightarrow \text{Im } L$ has an inverse which is denoted by K_p . By (3) and (13), we see that, for any $x = \{x_n\}_{n \in \mathbb{Z}} \in X_\omega$,

$$(QNx)_n = \frac{1}{\omega} \sum_{i=1}^{\omega} g_i(x_{i-\tau_1}, x_{i-\tau_2}, \dots, x_{i-\tau_l}), \quad n \in \mathbb{Z}, \quad (14)$$

and

$$((I - Q)Nx)_n = g_n(x_{n-\tau_1}, x_{n-\tau_2}, \dots, x_{n-\tau_l}) - \frac{1}{\omega} \sum_{i=1}^{\omega} g_i(x_{i-\tau_1}, x_{i-\tau_2}, \dots, x_{i-\tau_l}), \quad n \in \mathbb{Z}. \quad (15)$$

Since the Banach space X_ω is finite dimensional, K_p is linear. By relations (14) and (15), we see that QN and $K_p(I - Q)N$ are continuous on X_ω and take bounded sets into bounded sets respectively. Thus, we know that if Ω is an open and bounded subset of X_ω , then the mapping N is called L -compact on $\overline{\Omega}$.

Lemma 2.1 (see [7]) *Let $\{u_n\}_{n \in \mathbb{Z}}$ be a real ω -periodic sequence, then we have*

$$\max_{1 \leq s, t \leq \omega} |u_s - u_t| \leq \frac{1}{2} \sum_{n=1}^{\omega} |\Delta u_n|, \quad (16)$$

where the constant factor $1/2$ is the best possible.

Lemma 2.2 (see [7]) *Let $\{u_n\}_{n \in \mathbb{Z}}$ be a real ω -periodic sequence, then*

$$\sum_{n=1}^{\omega} |\Delta u_n| \leq \frac{\omega}{2} \sum_{n=1}^{\omega} |\Delta^2 u_n|. \quad (17)$$

Lemma 2.3 (see [7]) *If $|c| \neq 1$ and $\{u_n\}_{n \in \mathbb{Z}}$ is a real ω -periodic sequence, then*

$$\max_{1 \leq s, t \leq \omega} |u_s - u_t| \leq \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |\Delta^k [u_n + cu_{n-\sigma}]|. \quad (18)$$

Proof Note that

$$\begin{aligned} \sum_{n=1}^{\omega} |\Delta^k [u_n + cu_{n-\sigma}]| &\geq \left| \sum_{n=1}^{\omega} |\Delta^k u_n| - |c| \sum_{n=1}^{\omega} |\Delta^k u_{n-\sigma}| \right| \\ &= |1 - |c|| \sum_{n=1}^{\omega} |\Delta^k u_n|. \end{aligned}$$

It follows that

$$\sum_{n=1}^{\omega} |\Delta^k u_n| \leq \frac{1}{|1 - |c||} \sum_{n=1}^{\omega} |\Delta^k [u_n + cu_{n-\sigma}]|. \quad (19)$$

If $k = 1$, by Lemma 2.1 and (19), then

$$\max_{1 \leq s, t \leq \omega} |u_s - u_t| \leq \frac{1}{2|1 - |c||} \sum_{n=1}^{\omega} |\Delta[u_n + cu_{n-\sigma}]|.$$

If $k \geq 2$, by Lemma 2.2, then

$$\begin{aligned} \sum_{i=1}^{\omega} |\Delta u_i| &\leq \frac{\omega}{2} \sum_{n=1}^{\omega} |\Delta^2 u_n| \\ &\leq \frac{\omega^2}{2^2} \sum_{n=1}^{\omega} |\Delta^3 u_n| \leq \dots \\ &\leq \frac{\omega^{k-1}}{2^{k-1}} \sum_{n=1}^{\omega} |\Delta^k u_n|. \end{aligned} \quad (20)$$

In view of (19) and (20), we have

$$\max_{1 \leq s, t \leq \omega} |u_s - u_t| \leq \frac{\omega^{k-1}}{2^k} \sum_{n=1}^{\omega} |\Delta^k u_n| \leq \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |\Delta^k [u_n + cu_{n-\sigma}]|.$$

The proof is completed. $\square$

Proof of Theorem 2.1 Consider the system

$$(Lx)_n = \lambda(Nx)_n, \quad n \in Z, \quad (21)$$

where $\lambda \in (0, 1)$ is a parameter. Let $u \in X_{\omega}$ be a solution of (21). By (2), (3), and (21),

$$\Delta^k [u_n + cu_{n-\sigma}] = \lambda g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l}), \quad n \in Z. \quad (22)$$

Let $u_{\xi} = \max_{1 \leq n \leq \omega} u_n$ and $u_{\eta} = \min_{1 \leq n \leq \omega} u_n$. By Lemma 2.3 and (21),

$$\begin{aligned} u_{\xi} - u_{\eta} &= \max_{1 \leq s, t \leq \omega} |u_s - u_t| \\ &\leq \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |\Delta^k [u_n + cu_{n-\sigma}]| \\ &\leq \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})|. \end{aligned} \quad (23)$$

If there exists a constant $m \in \{1, 2, \dots, \omega\}$ such that $|u_m| < D$, by (23), for any $n \in Z$, then

$$\begin{aligned} |u_n| &\leq |u_m| + |u_n - u_m| \\ &\leq D + \max_{1 \leq s, t \leq \omega} |u_s - u_t| \\ &\leq D + \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})|. \end{aligned} \quad (24)$$

Otherwise by (22),

$$\sum_{n=1}^{\omega} g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l}) = 0. \quad (25)$$

In view of conditions (II), (III) and (25), we know $u_{\xi} \geq D$ and $u_{\eta} \leq -D$. By (23),

$$\begin{aligned} u_{\xi} &\leq u_{\eta} + \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})| \\ &\leq -D + \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})|, \end{aligned} \quad (26)$$

and

$$\begin{aligned} u_{\eta} &\geq u_{\xi} - \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})| \\ &\geq D - \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})|. \end{aligned} \quad (27)$$

By (26) and (27), for any $n \in Z$,

$$\begin{aligned} D - \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})| &\leq u_{\eta} \leq u_n \leq u_{\xi} \\ &\leq -D + \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})|. \end{aligned} \quad (28)$$

From (24) and (28), for any $n \in Z$, we have

$$|u_n| \leq D + \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})|.$$

It follows that

$$|u|_{\infty} \leq D + \frac{\omega^{k-1}}{2^k |1 - |c||} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})|. \quad (29)$$

By condition (I),

$$\begin{aligned} \sum_{n=1}^{\omega} |g_n(u_{n-\tau_1}, u_{n-\tau_2}, \dots, u_{n-\tau_l})| &\leq \omega \beta \max_{1 \leq i \leq l} |u_{n-\tau_i}| + \omega \alpha \\ &\leq \omega \beta |u|_{\infty} + \omega \alpha. \end{aligned} \quad (30)$$

By (29) and (30),

$$|u|_{\infty} \leq \frac{C}{1 - \rho}, \quad (31)$$

where $C = D + \frac{\omega^k \alpha}{2^k |1-c|}$ and $\rho = \frac{\omega^k \beta}{2^k |1-c|}$.

Set

$$\Omega = \{u \in X_\omega : |u|_\infty < \bar{D}\},$$

where $\bar{D}$ is a fixed number which satisfies $\bar{D} > D + \frac{C}{1-\rho}$. We have that Ω is an open and bounded subset of X_ω . By (31), for each $\lambda \in (0, 1)$, $u \in \partial\Omega$, $Lu \neq \lambda Nu$. If $u \in \partial\Omega \cap \text{Ker } L$, then $u = \{\bar{D}\}_{n \in \mathbb{Z}}$ or $u = \{-\bar{D}\}_{n \in \mathbb{Z}}$. By (13),

$$(QNu)_n = \frac{1}{\omega} \sum_{i=1}^{\omega} g_i(u_{i-\tau_1}, u_{i-\tau_2}, \dots, u_{i-\tau_l}) \neq 0, \quad n \in \mathbb{Z}.$$

In particular, we see that if $u = \{\bar{D}\}_{n \in \mathbb{Z}}$, then

$$(QNu)_n = \frac{1}{\omega} \sum_{i=1}^{\omega} g_i(\bar{D}, \bar{D}, \dots, \bar{D}) > 0, \quad n \in \mathbb{Z},$$

and if $u = \{-\bar{D}\}_{n \in \mathbb{Z}}$, then

$$(QNu)_n = \frac{1}{\omega} \sum_{i=1}^{\omega} g_i(-\bar{D}, -\bar{D}, \dots, -\bar{D}) < 0, \quad n \in \mathbb{Z}.$$

This indicates

$$\deg(QN, \Omega \cap \text{Ker } L, 0) \neq 0.$$

By Theorem A, we see that the equation $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \text{Dom } L$. In other words, (1) has an ω -periodic solution. The proof is completed. $\square$

3 Example

Example 3.1 The difference equation

$$\Delta^4 \left[u_n + \frac{1}{3} u_{n-2} \right] = \frac{10}{243} \left[2 - \sin \left(\frac{2n\pi}{3} \right) \right] (u_{n-2})^{\frac{1}{3}} (u_{n-1})^{\frac{1}{3}} (u_n)^{\frac{1}{3}} \quad (32)$$

is one of the form (1), where $k = 4$, $c = \frac{1}{3}$, $\sigma = 2$, $l = 3$, $\tau_1 = 2$, $\tau_2 = 1$, $\tau_3 = 0$, and

$$g_n(u_1, u_2, u_3) = \frac{10}{243} \left[2 - \sin \left(\frac{2n\pi}{3} \right) \right] (u_1)^{\frac{1}{3}} (u_2)^{\frac{1}{3}} (u_3)^{\frac{1}{3}}.$$

We can prove that (32) has a 3-periodic nontrivial solution. Indeed, let $D = 1$, $\beta = \frac{10}{81}$, and $\alpha = 1$. Then the conditions of Theorem 2.1 are satisfied. Therefore (32) has a 3-periodic solution. Furthermore, the solution is nontrivial since $g_n(0, 0, 0)$ is not identically zero.

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Authors' contributions

JLZ and GQW worked together in the derivation of the mathematical results. Both authors read and approved the final manuscript.

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