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Homoclinic solutions for n-dimensional p-Laplacian neutral differential systems with a time-varying delay

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Abstract

In this paper, we investigate the existence of a set with $2kT$-periodic solutions for n-dimensional p-Laplacian neutral differential systems with a time-varying delay $(\varphi_p(u(t) - Cu(t-\tau))')' + \frac{d}{dt}\nabla F(u(t)) + G(u(t-\gamma(t))) = e_k(t)$ based on the coincidence degree theory of Mawhin. Combining this with the conclusion about uniform convergence and limit, we obtain the corresponding results on the existence of homoclinic solutions.

Keywords: Homoclinic solutions; Coincidence degree theory; Periodic solutions; Delay

1 Introduction

This paper focuses on the existence of homoclinic solutions for n-dimensional p-Laplacian neutral differential systems with a time-varying delay of the following form:

$$\bigl(\varphi_p\bigl(u(t) - Cu(t-\tau)\bigr)'\bigr)' + \frac{d}{dt}\nabla F\bigl(u(t)\bigr) + G\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) = e(t), \tag{1.1}$$

where $p \in (1,+\infty)$, $\varphi_p : \mathbb{R}^n \to \mathbb{R}^n$, $\varphi_p(u) = (|u_1|^{p-2}u_1, |u_2|^{p-2}u_2, \dots, |u_n|^{p-2}u_n)$ for $u \neq \mathbf{0} = (0,0,\dots,0)$, $F \in C^2(\mathbb{R}^n,\mathbb{R})$, $G \in C(\mathbb{R}^n,\mathbb{R}^n)$, $e \in C(\mathbb{R},\mathbb{R}^n)$, $C = \operatorname{diag}(c_1,c_2,\dots,c_n)$, $|c_i| \neq 1$ ($i = 1,2,\dots,n$), τ and $T > 0$ are given constants, $\gamma \in (\mathbb{R},\mathbb{R})$, $\gamma(t+T) = \gamma(t)$ with $\gamma(t) \geq 0$.

In the past few decades, the existence of homoclinic solutions for second-order differential equations has been widely investigated by using critical point theory, the methods of bifurcation theory, or Mawhin's continuation theorem (see [1–8]). However, the corresponding results on the existence of homoclinic solutions to a neutral differential equation are relatively infrequent. For example, the existence of homoclinic solutions to a kind of second-order neutral functional differential systems was considered in [9]:

$$\bigl(u(t) - Cu(t-\tau)\bigr)'' + \frac{d}{dt}\nabla F\bigl(u(t)\bigr) + G\bigl(u(t)\bigr) + H\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) = e(t), \tag{1.2}$$

where $C = [c_{ij}]_{n\times n}$ is a real constant symmetric matrix, $F \in C^2(\mathbb{R}^n,\mathbb{R})$, $G,H \in C^1(\mathbb{R}^n,\mathbb{R})$, $e \in C(\mathbb{R},\mathbb{R}^n)$, $\gamma \in (\mathbb{R},\mathbb{R})$, $\gamma(t+T) = \gamma(t)$ with $\gamma(t) \geq 0$ and given constant $T > 0$. Meanwhile, Du [10] discussed the system

$$\big(u(t) - Cu(t-\tau)\big)'' + \frac{d}{dt}\nabla F\big(u(t)\big) + \nabla G\big(u(t)\big) = e(t), \tag{1.3}$$

where $F \in C^2(\mathbb{R}^n, \mathbb{R})$, $G \in C^1(\mathbb{R}^n, \mathbb{R})$. $e \in C(\mathbb{R}, \mathbb{R}^n)$, $C = \operatorname{diag}(c_1, c_2, \ldots, c_n)$, c_i $(i = 1, 2, \ldots, n)$ and τ are given constants. The existence of homoclinic solutions for Eq. (1.3) is obtained. Then Chen [11] studied the existence of homoclinic solutions for the class of neutral Duffing differential systems

$$\big(u(t) - Cu(t-\tau)\big)'' + \beta(t)x'(t) + g\big(u\big(t - \gamma(t)\big)\big) = p(t), \tag{1.4}$$

where $\beta \in C^1(\mathbb{R}, \mathbb{R})$ with $\beta(t+T) \equiv \beta(t)$, $g \in C(\mathbb{R}^n, \mathbb{R}^n)$, $p \in C(\mathbb{R}, \mathbb{R}^n)$, $\gamma \in (\mathbb{R}, \mathbb{R})$, $\gamma(t+T) = \gamma(t)$ with $\gamma(t) \geq 0$, $T > 0$ and τ are given constants; $\beta(t)$ is allowed to change sign, and $C = [c_{ij}]_{n\times n}$ is a constant symmetric matrix.

It is not hard to find that Eq. (1.1) can be converted to second-order neutral functional differential systems (1.2)–(1.4) when $p = 2$. To our knowledge, there are few results reported in the literature regarding the existence of homoclinic solutions for n-dimensional p-Laplacian neutral differential systems with time-varying delay. Because of the term $(\varphi_p(u(t) - Cu(t-\tau))')'$ in Eq. (1.1), the method of Lemma 2.5 in [12] cannot be applied directly to prove that $|u_0'(t)| \to 0$ as $|t| \to +\infty$. In this paper, we solve this problem by combining the conclusion about uniform convergence and Lemma 2.3 in [13].

Similarly to [9–11], we obtain the existence of a homoclinic solution for the equation by taking a series of the $2kT$-periodic limit for the following equation:

$$\big(\varphi_p\big(u(t) - Cu(t-\tau)\big)'\big)' + \frac{d}{dt}\nabla F\big(u(t)\big) + G\big(u\big(t - \gamma(t)\big)\big) = e_k(t), \tag{1.5}$$

where $k \in \mathbb{N}$, and $e_k : \mathbb{R} \to \mathbb{R}^n$ is a $2kT$-periodic function such that

$$e_k(t) = \begin{cases} e(t), & t \in [-kT, kT - \varepsilon_0), \\ e(kT - \varepsilon_0) + \frac{e(-kT) - e(kT-\varepsilon_0)}{\varepsilon_0}(t - kT + \varepsilon_0), & t \in [kT - \varepsilon_0, kT], \end{cases} \tag{1.6}$$

with a constant $\varepsilon_0 \in (0, T)$ independent of k.

2 Preliminaries

Lemma 2.1 ([12]) *If $u : \mathbb{R} \to \mathbb{R}^n$ is continuously differentiable on $\mathbb{R}$, $a > 0$, $\mu > 1$, and $p > 1$ are constants, then for every $t \in \mathbb{R}$, we have the following inequality*:

$$\big|u(t)\big| \leq (2a)^{-\frac{1}{\mu}} \left(\int_{t-a}^{t+a} \big|u(s)\big|^{\mu}\, ds \right)^{\frac{1}{\mu}} + a(2a)^{-\frac{1}{p}} \left(\int_{t-a}^{t+a} \big|u'(s)\big|^{p}\, ds \right)^{\frac{1}{p}}.$$

Lemma 2.2 ([13]) *Let $s \in C(\mathbb{R}, \mathbb{R})$ with $s(t+\omega) \equiv s(t)$ and $s(t) \in [0, \omega]$ for $t \in \mathbb{R}$. Suppose $p \in (1, +\infty)$, $|s|_0 = \max_{t\in[0,\omega]} s(t)$, and $u \in C^1(\mathbb{R}, \mathbb{R})$ with $u(t+\omega) \equiv u(t)$. Then*

$$\int_0^{\omega} \big|u(t) - u\big(t - s(t)\big)\big|^p\, dt \leq |s|_0^p \int_0^{\omega} \big|u'(t)\big|^p\, dt.$$

Lemma 2.3 ([14]) *If $x \in (0,+\infty)$ satisfies the inequality $x^s \le \alpha x^q + \beta x^r$ for some constants $s > q > r \ge 0$, $\alpha > 0$, and $\beta > 0$, then*

$$0 < x \le \inf_{\varepsilon \in (0,1)} \max\left\{\left(\frac{\beta}{\varepsilon}\right)^{\frac{1}{s-r}}, \left(\frac{\alpha}{1-\varepsilon}\right)^{\frac{1}{s-q}}\right\}.$$

Lemma 2.4 ([15]) *Suppose $\tau \in C^1(\mathbb{R},\mathbb{R})$ with $\tau(t+\omega) \equiv \tau(t)$ and $\tau'(t) < 1$ for $t \in [0,\omega]$. Then the function $t - \tau(t)$ has an inverse $\mu \in C(\mathbb{R},\mathbb{R})$ such that $\mu(t+\omega) \equiv \mu(t) + \omega$ for $t \in \mathbb{R}$.*

Lemma 2.5 ([16]) *Suppose that Ω is an open bounded set in X such that the following conditions are satisfied*:

[A_1] *For each $\lambda \in (0,1)$, the equation*

$$\bigl(\varphi_p\bigl(u(t) - Cu(t-\tau)\bigr)'\bigr)' + \lambda \frac{d}{dt}\nabla F\bigl(u(t)\bigr) + \lambda G\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) = \lambda e_k(t)$$

has no solution on $\partial\Omega$.

[A_2] *The equation*

$$\triangle(a) := \frac{1}{2kT}\int_{-kT}^{kT}\bigl[G(a) - e_k(t)\bigr]\,dt = 0$$

has no solution on $\partial\Omega \cap \mathbb{R}^n$.

[A_3] *The Brouwer degree*

$$d_B\bigl\{\triangle, \Omega \cap \mathbb{R}^n, 0\bigr\} \neq 0.$$

Then Eq. (1.5) has a $2kT$-periodic solution in $\bar{\Omega}$.

Lemma 2.6 ([16]) *Suppose that $c_1, c_2, \ldots, c_n$ are eigenvalues of a matrix C. If $|c_i| \neq 1$ ($i = 1,2,\ldots,n$), then A has a continuous bounded inverse with the following properties*:

(1) $\|A^{-1}f\| \le (\sum_{i=1}^n \frac{1}{|1-|c_i||})\|f\|$ *for all $f \in C_T$,*

(2) $\int_0^T |(A^{-1}f)(t)|^p\,dt \le \alpha \int_0^T |f(t)|^p\,dt$ *for all $f \in C_T$ and $p \ge 1$, where*

$$\alpha = \begin{cases} \max(\frac{1}{(1-|c_i|)^2}), & p = 2, \\ (\sum_{i=1}^n \frac{1}{(1-|c_i|)^{\frac{2p}{2-p}}})^{\frac{2-p}{2}}, & p \in [1,2), \\ (\sum_{i=1}^n \frac{1}{1-|c_i|^q})^{\frac{p}{q}}, & p \in [2,+\infty), \end{cases}$$

and q is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

(3) $(Ax)' = Ax'$ *for all $x \in C_T^1$.*

Throughout this paper, for convenience, we list the following conditions and corresponding mathematical notation.

[H_1] There are constants $m_0 > 0$ and $m_1 > 0$ such that

$$\bigl\langle (E-C)x, G(x)\bigr\rangle \le -m_0|x|^p \quad \text{for all } x \in \mathbb{R}^n,$$

$$\left|G(x)\right| \le m_1|x|^{p-1} \quad \text{for all } x \in \mathbb{R}^n,$$

and

$$\left|\nabla F(x)\right| \le m_2|x|^{p-1} \quad \text{for all } x \in \mathbb{R}^n.$$

$[H_2]$ $e \in C(\mathbb{R}, \mathbb{R}^n)$ is a bounded function with $e(t) \not\equiv \mathbf{0} = (0, 0, \dots, 0)^T$ and

$$B := \left(\int_{\mathbb{R}} \left|e(t)\right|^q \, dt\right)^{\frac{1}{q}} + \sup_{t\in\mathbb{R}}\left|e(t)\right| < +\infty.$$

By (1.6) we know that $|e_k(t)| \le \sup_{t\in\mathbb{R}} |e(t)|$. So for each $k \in \mathbb{N}$, $(\int_{-kT}^{kT} |e_k(t)|^q \, dt)^{\frac{1}{q}} < B$ if $[H_2]$ holds. Let $C_{2kT} = \{x | x \in C(\mathbb{R}, \mathbb{R}^n), x(t + 2kT) \equiv x(t)\}$, $C^1_{2kT} = \{x | x \in C^1(\mathbb{R}, \mathbb{R}^n), x(t + 2kT) \equiv x(t)\}$, and $|x|_0 = \max_{t\in[0,2kT]} |x(t)|$. If the norms of C_{2kT} and C^1_{2kT} are respectively defined by $\|\cdot\|_{C_{2kT}} = |\cdot|_0$ and $\|\cdot\|_{C^1_{2kT}} = \max\{|x|_0, |x'|_0\}$, then C_{2kT} and C^1_{2kT} are Banach spaces. By $\langle\cdot,\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ we denote the standard inner product, and by $|\cdot|$ we denote the absolute value and the Euclidean norm on $\mathbb{R}^n$. For $\varphi \in C_{2kT}$, set $\|\varphi\|_r = (\int_{-kT}^{kT} |\varphi(t)|^r \, dt)^{\frac{1}{r}}$, $r > 1$. Let $\gamma \in C^1(\mathbb{R}, \mathbb{R})$ with $\gamma'(t) < 1$ for all $t \in [0, T]$. Let $\sigma_0 = \min_{t\in[0,T]} \gamma'(t)$ and $\sigma_1 = \max_{t\in[0,T]} \gamma'(t)$. Define the linear operator

$$A : C_T \to C_T, \qquad [Ax](t) = x(t) - Cx(t - \tau).$$

3 Main results

First, we study some properties of all possible $2kT$-periodic solutions of the following equation:

$$\left(\varphi_p\left(u(t) - Cu(t-\tau)\right)'\right)' + \lambda \frac{d}{dt}\nabla F\left(u(t)\right) + \lambda G\left(u\left(t - \gamma(t)\right)\right) = \lambda e_k(t), \quad \lambda \in (0, 1]. \tag{3.1}$$

Let $\Sigma \subset C^1_{2kT}$, $k \in \mathbb{N}$, be the set of all the $2kT$-periodic solutions to Eq. (3.1).

Theorem 3.1 *If assumptions* $[H_1]$–$[H_2]$ *hold and*

$$\frac{(1-\sigma_0)^{p-1}\lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0 + |\tau|)(1-\sigma_1)^{-\frac{1}{q}} + m_2]^p}{m_0^{p-1}} < 1,$$

where $\lambda_M = \max\{c_i^2\}$, $|c_i| \ne 1$, $i = 1, 2, \dots, n$, *and* $u \in \Sigma$ *for each* $k \in \mathbb{N}$, *then*

$$\|u\|_p \le A_0, \qquad \left\|u'\right\|_p \le A_1, \qquad |u|_0 \le \rho_0, \qquad \left|u'\right|_0 \le \rho_1,$$

where A_0, A_1, ρ_0, *and* ρ_1 *are positive constants independent of* λ *and* k.

Proof If $u \in \Sigma$ and $k \in \mathbb{N}$, then u satisfies

$$\left(\varphi_p\left(u(t) - Cu(t-\tau)\right)'\right)' + \lambda \frac{d}{dt}\nabla F\left(u(t)\right) + \lambda G\left(u\left(t - \gamma(t)\right)\right) = \lambda e_k(t), \quad \lambda \in (0, 1]. \tag{3.2}$$

Multiplying both sides of Eq. (3.2) by $[Au](t)$ and integrating from $-kT$ to kT, we get

$$\begin{aligned}&-\|Au'\|_p^p + \lambda \int_{-kT}^{kT} \left\langle [Au](t), \frac{d}{dt}\nabla F\bigl(u(t)\bigr)\right\rangle dt + \lambda \int_{-kT}^{kT} \bigl\langle [Au](t), G\bigl(u\bigl(t-\gamma(t)\bigr)\bigr)\bigr\rangle dt \\ &\quad = \lambda \int_{-kT}^{kT} \bigl\langle [Au](t), e_k(t)\bigr\rangle dt.\end{aligned}$$

Since

$$\int_{-kT}^{kT} \left\langle [Au](t), \frac{d}{dt}\nabla F\bigl(u(t)\bigr)\right\rangle dt = \int_{-kT}^{kT} \bigl\langle Cu'(t-\tau), \nabla F\bigl(u(t)\bigr)\bigr\rangle dt,$$

we have

$$\begin{aligned}&\lambda \int_{-kT}^{kT} \bigl\langle [Au](t), e_k(t)\bigr\rangle dt \\ &\quad = -\|Au'\|_p^p + \lambda \int_{-kT}^{kT} \bigl\langle Cu'(t-\tau), \nabla F\bigl(u(t)\bigr)\bigr\rangle dt \\ &\qquad + \lambda \int_{-kT}^{kT} \bigl\langle u(t) - u\bigl(t-\gamma(t)\bigr), G\bigl(u\bigl(t-\gamma(t)\bigr)\bigr)\bigr\rangle dt \\ &\qquad + \lambda \int_{-kT}^{kT} \bigl\langle (E-C)u\bigl(t-\gamma(t)\bigr), G\bigl(u\bigl(t-\gamma(t)\bigr)\bigr)\bigr\rangle dt \\ &\qquad - \lambda \int_{-kT}^{kT} \bigl\langle Cu(t-\tau) - Cu\bigl(t-\gamma(t)\bigr), G\bigl(u\bigl(t-\gamma(t)\bigr)\bigr)\bigr\rangle dt,\end{aligned}$$

and by assumption $[H_1]$

$$\begin{aligned}&\|Au'\|_p^p + \lambda m_0 \int_{-kT}^{kT} \bigl|u\bigl(t-\gamma(t)\bigr)\bigr|^p dt \\ &\quad \le \lambda m_1 \int_{-kT}^{kT} \bigl|u(t) - u\bigl(t-\gamma(t)\bigr)\bigr| \bigl|u\bigl(t-\gamma(t)\bigr)\bigr|^{p-1} dt \\ &\qquad + \lambda m_1 \lambda_M^{\frac{1}{2}} \int_{-kT}^{kT} \bigl|u(t-\tau) - u\bigl(t-\gamma(t)\bigr)\bigr| \bigl|u\bigl(t-\gamma(t)\bigr)\bigr|^{p-1} dt \\ &\qquad + \left|\lambda \int_{-kT}^{kT} \bigl\langle [Au](t), e_k(t)\bigr\rangle dt\right| + \left|\lambda \int_{-kT}^{kT} \bigl\langle Cu'(t-\tau), \nabla F\bigl(u(t)\bigr)\bigr\rangle dt\right|, \end{aligned} \tag{3.3}$$

where $\lambda_M = \max\{c_i^2\}$, $i = 1, 2, \ldots, n$.

By applying Lemma 2.2, Lemma 2.4, $[H_1]$, and $[H_2]$ we get

$$\begin{aligned}\frac{1}{1-\sigma_0}\|u\|_p^p &\le \int_{-kT}^{kT} \bigl|u\bigl(t-\gamma(t)\bigr)\bigr|^p dt = \int_{-kT}^{kT} \frac{1}{1-\gamma'(\mu(t))} \bigl|u(t)\bigr|^p dt \\ &\le \frac{1}{1-\sigma_1}\|u\|_p^p \end{aligned} \tag{3.4}$$

and

$$\int_{-kT}^{kT} \bigl|u(t) - u\bigl(t-\gamma(t)\bigr)\bigr| \bigl|u\bigl(t-\gamma(t)\bigr)\bigr|^{p-1} dt$$

$$
\begin{aligned}
&\leq \left(\int_{-kT}^{kT} \left|u(t)-u\bigl(t-\gamma(t)\bigr)\right|^{p} dt\right)^{p} \left(\int_{-kT}^{kT} \left|u\bigl(t-\gamma(t)\bigr)\right|^{p} dt\right)^{\frac{p-1}{p}} \\
&\leq |\gamma|_{0} \frac{1}{(1-\sigma_{1})^{\frac{p-1}{p}}} \left\|u'\right\|_{p} \|u\|_{p}^{p-1}.
\end{aligned}
\tag{3.5}
$$

Using the same method as for (3.5), we have

$$
\begin{aligned}
&\int_{-kT}^{kT} \left|u(t-\tau)-u\bigl(t-\gamma(t)\bigr)\right| \left|u\bigl(t-\gamma(t)\bigr)\right|^{p-1} dt \\
&\quad\leq \bigl(|\gamma|_{0}+|\tau|\bigr) \frac{1}{(1-\sigma_{1})^{\frac{p-1}{p}}} \left\|u'\right\|_{p} \|u\|_{p}^{p-1}
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
&\left|\int_{-kT}^{kT} \bigl\langle [Au](t), e_{k}(t)\bigr\rangle dt\right| \\
&\quad\leq \|e_{k}\|_{q} \|u\|_{p} + \|e_{k}\|_{q} \|u\|_{p} \\
&\quad\leq B\bigl(1+\lambda_{M}^{\frac{1}{2}}\bigr) \|u\|_{p}.
\end{aligned}
\tag{3.7}
$$

Furthermore, by $[H_1]$ we have

$$
\begin{aligned}
&\left|\int_{-kT}^{kT} \bigl\langle Cu'(t-\tau), \nabla F\bigl(u(t)\bigr)\bigr\rangle dt\right| \\
&\quad\leq \left(\int_{-kT}^{kT} \left|Cu'(t-\tau)\right|^{p} dt\right)^{\frac{1}{p}} \left(\int_{-kT}^{kT} \left|\nabla F\bigl(u(t)\bigr)\right|^{q} dt\right)^{\frac{1}{q}} \\
&\quad\leq \lambda_{M}^{\frac{1}{2}} m_{2} \left\|u'\right\|_{p} \|u\|_{p}^{p-1}.
\end{aligned}
\tag{3.8}
$$

Applying (3.4)–(3.8) to (3.3), we obtain

$$
\begin{aligned}
&\left\|Au'\right\|_{p}^{p} + \lambda m_{0} \frac{1}{1-\sigma_{0}} \|u\|_{p}^{p} \\
&\quad\leq \lambda \lambda_{M}^{\frac{1}{2}} \bigl[m_{1}\bigl(2|\gamma|_{0}+|\tau|\bigr)(1-\sigma_{1})^{-\frac{1}{q}} \\
&\qquad + \lambda m_{2}\bigr] \left\|u'\right\|_{p} \|u\|_{p}^{p-1} + \lambda B\bigl(1+\lambda_{M}^{\frac{1}{2}}\bigr) \|u\|_{p}.
\end{aligned}
\tag{3.9}
$$

By (3.9) we get

$$
\begin{aligned}
\|u\|_{p}^{p} \leq{}& \frac{1-\sigma_{0}}{m_{0}} \lambda_{M}^{\frac{1}{2}} \bigl[m_{1}\bigl(2|\gamma|_{0}+|\tau|\bigr)(1-\sigma_{1})^{-\frac{1}{q}} + m_{2}\bigr] \left\|u'\right\|_{p} \|u\|_{p}^{p-1} \\
&+ \frac{1-\sigma_{0}}{m_{0}} B\bigl(1+\lambda_{M}^{\frac{1}{2}}\bigr) \|u\|_{p}.
\end{aligned}
\tag{3.10}
$$

Since

$$
\frac{(1-\sigma_{0})^{p-1} \lambda_{M}^{\frac{p}{2}} [m_{1}(2|\gamma|_{0}+|\tau|)(1-\sigma_{1})^{-\frac{1}{q}} + m_{2}]^{p}}{m_{0}^{p-1}} < 1,
$$

there exists a constant $\varepsilon_0 \in (0,1)$ such that

$$\frac{(1-\sigma_0)^{p-1}\lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]^p}{(1-\varepsilon_0)^{p-1}m_0^{p-1}} < 1. \tag{3.11}$$

Applying Lemma 2.3 and (3.10), we get

$$\begin{aligned}
&\|u\|_p^p \\
&\quad \le \max\biggl\{\frac{(1-\sigma_0)^p\lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]^p}{(1-\varepsilon_0)^p m_0^p}\|u'\|_p^p, \\
&\qquad \biggl[\frac{1-\sigma_0}{\varepsilon_0 m_0}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)\biggr]^{\frac{p}{p-1}}\biggr\}.
\end{aligned} \tag{3.12}$$

If

$$\frac{(1-\sigma_0)^p\lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]^p}{(1-\varepsilon_0)^p m_0^p}\|u'\|_p^p \le \biggl[\frac{1-\sigma_0}{\varepsilon_0 m_0}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)\biggr]^{\frac{p}{p-1}},$$

then

$$\begin{aligned}
&\|u\|_p^p \le \biggl[\frac{1-\sigma_0}{\varepsilon_0 m_0}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)\biggr]^{\frac{p}{p-1}}, \qquad \|u\|_p^{p-1} \le \frac{1-\sigma_0}{\varepsilon_0 m_0}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr), \\
&\|u\|_p \le \biggl[\frac{1-\sigma_0}{\varepsilon_0 m_0}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)\biggr]^{\frac{1}{p-1}}.
\end{aligned}$$

By Lemma 2.6 we have $\|u'\|_p = \|A^{-1}Au'\|_p \le \alpha^{\frac{1}{p}}\|Au'\|_p$. From (3.9) and Lemma 2.3 with $\varepsilon = \frac{1}{2}$ we get

$$\begin{aligned}
&\|Au'\|_p^p \\
&\quad \le \alpha^{\frac{1}{p}}\lambda_M^{\frac{1}{2}}\bigl[m_1\bigl(2|\gamma|_0+|\tau|\bigr)(1-\sigma_1)^{-\frac{1}{q}}+m_2\bigr]\frac{1-\sigma_0}{\varepsilon_0 m_0}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)\|Au'\|_p \\
&\qquad + \biggl(\frac{1-\sigma_0}{\varepsilon_0 m_0}\biggr)^{\frac{1}{p-1}}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)^{\frac{p}{p-1}}
\end{aligned}$$

and

$$\begin{aligned}
&\|Au'\|_p \\
&\quad \le \max\biggl\{2^{\frac{1}{p-1}}\biggl[\alpha^{\frac{1}{p}}\lambda_M^{\frac{1}{2}}\bigl[m_1\bigl(2|\gamma|_0+|\tau|\bigr)(1-\sigma_1)^{-\frac{1}{q}}+m_2\bigr]\frac{1-\sigma_0}{\varepsilon_0 m_0}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)\biggr]^{\frac{1}{p-1}}, \\
&\qquad 2^{\frac{1}{p}}\biggl(\frac{1-\sigma_0}{\varepsilon_0 m_0}\biggr)^{\frac{1}{p(p-1)}}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)^{\frac{1}{p-1}}\biggr\} := M_1.
\end{aligned}$$

If

$$\frac{(1-\sigma_0)^p\lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]^p}{(1-\varepsilon_0)^p m_0^p}\|u'\|_p^p \ge \biggl[\frac{1-\sigma_0}{\varepsilon_0 m_0}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)\biggr]^{\frac{p}{p-1}},$$

then

$$\|u\|_p^p \le \frac{(1-\sigma_0)^p \lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]^p}{(1-\varepsilon_0)^p m_0^p}\|u'\|_p^p,$$

$$\|u\|_p^{p-1} \le \left[\frac{(1-\sigma_0)^p \lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]^p}{(1-\varepsilon_0)^p m_0^p}\right]^{\frac{p-1}{p}}\|u'\|_p^{p-1},$$

and

$$\|u\|_p \le \left[\frac{(1-\sigma_0)^p \lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]^p}{(1-\varepsilon_0)^p m_0^p}\right]^{\frac{1}{p}}\|u'\|_p.$$

From (3.9) we have

$$\begin{aligned}
&\|Au'\|_p^p \\
&\quad\le \frac{(1-\sigma_0)^{p-1} \lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]^p}{(1-\varepsilon_0)^{p-1} m_0^{p-1}}\|Au'\|_p^p \\
&\qquad+\alpha^{\frac{1}{p}}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)\frac{(1-\sigma_0)\lambda_M^{\frac{1}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]}{(1-\varepsilon_0)m_0^p}\|Au'\|_p.
\end{aligned}$$

Combining this with (3.11), we see that there exists a constant $M_2 > 0$ such that

$$\|Au'\|_p \le M_2.$$

Obviously,

$$\|Au'\|_p \le \max\{M_1, M_2\} := M, \tag{3.13}$$

$$\|u'\|_p \le \alpha^{\frac{1}{p}}\|Au'\|_p \le \alpha^{\frac{1}{p}} M := A_1, \tag{3.14}$$

$$\begin{aligned}
&\|u\|_p \\
&\quad\le \max\left\{\left[\frac{1-\sigma_0}{\varepsilon_0 m_0}B\bigl(1+\lambda_M^{\frac{1}{2}}\bigr)\right]^{\frac{1}{p-1}},\right. \\
&\qquad\left.\left[\frac{(1-\sigma_0)^p \lambda_M^{\frac{p}{2}}[m_1(2|\gamma|_0+|\tau|)(1-\sigma_1)^{-\frac{1}{q}}+m_2]^p}{(1-\varepsilon_0)^p m_0^p}\right]^{\frac{1}{p}}A_1\right\} := A_0.
\end{aligned} \tag{3.15}$$

By (3.15) we can easily notice that A_0 and A_1 are constants independent of λ and k. By Lemma 2.1, for $t \in [-kT, kT]$, we obtain

$$\begin{aligned}
|u(t)| &\le (2T)^{-\frac{1}{p}}\left(\int_{t-T}^{t+T}|u(s)|^p\,ds\right)^{\frac{1}{p}} + T(2T)^{-\frac{1}{p}}\left(\int_{t-T}^{t+T}|u'(s)|^p\,ds\right)^{\frac{1}{p}} \\
&\le (2T)^{-\frac{1}{p}}\left(\int_{t-kT}^{t+kT}|u(s)|^p\,ds\right)^{\frac{1}{p}} + T(2T)^{-\frac{1}{p}}\left(\int_{t-kT}^{t+kT}|u'(s)|^p\,ds\right)^{\frac{1}{p}} \\
&= (2T)^{-\frac{1}{p}}\left(\int_{-kT}^{kT}|u(s)|^p\,ds\right)^{\frac{1}{p}} + T(2T)^{-\frac{1}{p}}\left(\int_{-kT}^{kT}|u'(s)|^p\,ds\right)^{\frac{1}{p}}.
\end{aligned}$$

From (3.13) and (3.14) we have

$$\begin{aligned}|u|_0 &\leq (2T)^{-\frac{1}{p}}\|u\|_p + T(2T)^{-\frac{1}{p}}\left\|u'\right\|_p \\ &\leq (2T)^{-\frac{1}{p}}A_0 + T(2T)^{-\frac{1}{p}}A_1 := \rho_0.\end{aligned} \tag{3.16}$$

Furthermore, setting $F_{\rho_0} := \max_{|x|\leq\rho_0}|\nabla F(x)|$ and $G_{\rho_0} := \max_{|x|\leq\rho_0}|G(x)|$, by Eq. (3.2) we get

$$\left|\frac{d}{dt}\left[\varphi_p\left(\left[Au'\right](t)\right) + \lambda\nabla F\left(u(t)\right)\right]\right| \leq G_{\rho_0} + \sup_{t\in\mathbb{R}}\left|e(t)\right| := \tilde{\rho}, \quad t\in[-kT,kT]. \tag{3.17}$$

Combining the continuity of $[Au'](t)$ and (3.13), we find that there exists $t_i \in [iT,(i+1)T]$, $i = -k,-k+1,\ldots,k-1$, such that

$$\begin{aligned}\left|\left[Au'\right](t_i)\right| &= \left|\frac{1}{T}\int_{iT}^{(i+1)T}\left[Au'\right](s)\,ds\right| \\ &\leq \frac{1}{T}\int_{iT}^{(i+1)T}\left|\left[Au'\right](s)\right|ds \\ &\leq T^{\frac{1-q}{q}}\left(\int_{iT}^{(i+1)T}\left|\left[Au'\right](s)\right|^p ds\right)^{\frac{1}{p}} \\ &\leq T^{\frac{1-q}{q}}\left(\int_{-kT}^{kT}\left|\left[Au'\right](s)\right|^p ds\right)^{\frac{1}{p}} \\ &\leq T^{\frac{1-q}{q}}\max\{M_1,M_2\}.\end{aligned} \tag{3.18}$$

By (3.16)–(3.18) we have

$$\begin{aligned}&\left|\varphi_p\left(\left[Au'\right](t)\right) + \lambda\nabla F\left(u(t)\right)\right| \\ &\quad\leq \left|\int_{t_i}^{t}\frac{d}{ds}\left[\varphi_p\left(\left[Au'\right](s)\right) + \lambda\nabla F\left(u(s)\right)\right]ds + \varphi_p\left(\left[Au'\right](t_i)\right) + \lambda\nabla F\left(u(t_i)\right)\right| \\ &\quad\leq \int_{iT}^{(i+1)T}\left|\left[\varphi_p\left(\left[Au'\right](s)\right) + \lambda\nabla F\left(u(s)\right)\right]\right|ds + \left|\varphi_p\left(\left[Au'\right](t_i)\right)\right| + F_{\rho_0} \\ &\quad\leq \tilde{\rho}T + \left[T^{\frac{1-q}{q}}\max\{M_1,M_2\}\right]^{p-1} + F_{\rho_0} := \rho,\end{aligned}$$

which yields

$$\left|\left[Au'\right](t)\right| \leq [\rho + F_{\rho_0}]^{\frac{1}{p-1}}. \tag{3.19}$$

It follows from Lemma 2.6 and (3.19) that

$$\left|u'\right|_0 = \left\|A^{-1}Au'\right\| \leq \left(\sum_{i=1}^{n}\frac{1}{|1-|c_i||}\right)\left\|Au'\right\| \leq \left(\sum_{i=1}^{n}\frac{1}{|1-|c_i||}\right)[\rho + F_{\rho_0}]^{\frac{1}{p-1}} := \rho_1.$$

Note that ρ_1 is independent of λ and k. The proof of Theorem 3.1 is completed. □

Theorem 3.2 *If the conditions of Theorem 3.1 are satisfied, then Eq. (3.2) has at least one $2kT$-periodic solution $u_k(t)$ for each $k \in \mathbb{N}$ such that*

$$\|u_k\|_p \leq A_0, \qquad \|u_k'\|_p \leq A_1, \qquad |u_k|_0 \leq \rho_0, \qquad |u_k'|_0 \leq \rho_1.$$

Proof To apply Lemma 2.5, we study the p-Laplacian neutral systems

$$\bigl(\varphi_p\bigl(u(t) - Cu(t-\tau)\bigr)'\bigr)' + \lambda\frac{d}{dt}\nabla F\bigl(u(t)\bigr) + \lambda G\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) = \lambda e_k(t), \quad \lambda \in (0,1). \tag{3.20}$$

Let $\Omega_1 \subset C^1_{2kT}$ be the set of all $2kT$-periodic of Eq. (3.20). From Theorem 3.1, assuming that $u \in \Omega_1 \subset \Sigma$ by $(0,1) \subset (0,1]$, we get

$$|u|_0 \leq \rho_0, \qquad |u'|_0 \leq \rho_1.$$

Set $\Omega_2 = \{x : x \in \operatorname{Ker} L, QNx = 0\}$,

$$\begin{aligned}
&L : D(L) \subset C_{2kT} \to C_{2kT}, \qquad Lu = \bigl(\varphi_p(Au)'\bigr)', \\
&N : C_{2kT} \to C^1_{2kT}, \qquad Nu = -\frac{d}{dt}\nabla F\bigl(u(t)\bigr) - G\bigl(u\bigl(t-\gamma(t)\bigr)\bigr) + e_k(t), \\
&Q : C_{2kT} \to C_{2kT}/\operatorname{Im} L, \quad Qy = \frac{1}{2kT}\int_{-kT}^{kT} y(s)\,ds.
\end{aligned}$$

Obviously, $x = a \in \mathbb{R}^n$ when $x \in \Omega_2$. Meanwhile, it follows from $[H_1]$ that

$$2kTm_0|a|^p \leq \int_{-kT}^{kT} \bigl|\bigl\langle (E-C)a, e_k(t)\bigr\rangle\bigr|\,dt \leq B|a|\bigl(1+|c_M|\bigr)(2kT)^{\frac{1}{p}},$$

that is,

$$|a| \leq m_0^{\frac{1}{1-p}} B^{\frac{1}{p-1}} T^{\frac{-1}{p}} \bigl(1+|c_M|\bigr)^{\frac{1}{p-1}} := B_0,$$

where $|c_M| = \max|c_i|$, $i = 1,2,\ldots,n$.

Let $\Omega = \{x : x \in C^1_{2kT}, |x|_0 < \rho_0 + B_0, |x'|_0 < \rho_1 + 1\}$. Then $\Omega \supset \Omega_1 \cup \Omega_2$. Thus assumptions $[A_1]$ and $[A_2]$ of Lemma 2.5 are satisfied. Next, we can prove that $[A_3]$ of Lemma 2.5 is also satisfied. Let

$$H(x,\mu) : \bigl(\Omega \cap \mathbb{R}^n\bigr) \times [0,1] \longrightarrow \mathbb{R}^n : H(x,\mu) = -\mu x + (1-\mu)\Delta(x),$$

where $\Delta(x) = \frac{1}{2kT}\int_{-kT}^{kT}[G(x) - e_k(t)]\,dt$ is determined by Lemma 2.5. By $[H_1]$ we get

$$H(x,\mu) \neq 0, \quad \forall (x,\mu) \in \bigl[\partial\bigl(\Omega \cap \mathbb{R}^n\bigr)\bigr] \times [0,1].$$

Thus

$$\begin{aligned}
&\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \\
&\quad = \deg\bigl\{H(x,0), \Omega \cap \operatorname{Ker} L, 0\bigr\}
\end{aligned}$$

$$
\begin{aligned}
&= \deg\{H(x,1), \Omega \cap \operatorname{Ker} L, 0\} \\
&\neq 0.
\end{aligned}
$$

So, A_3 of Lemma 2.5 holds. By Lemma 2.5, $u_k \in \bar{\Omega}$ is a $2kT$-periodic solution for Eq. (1.2) when $\lambda = 1$. Therefore, by means of Theorem 3.1 we have

$$
\|u_k\|_p \le A_0, \qquad \|u_k'\|_p \le A_1, \qquad |u_k|_0 \le \rho_0, \qquad |u_k'|_0 \le \rho_1. \tag{3.21}
$$

□

Theorem 3.3 *Assume that the conditions in Theorem 3.1 are satisfied. Then Eq. (1.1) has a nontrivial homoclinic solution.*

Proof By Theorem 3.2, Eq. (1.5) has a $2kT$-periodic solution $u_k(t)$ for each $k \in \mathbb{N}$. Thus $u_k(t)$ satisfies

$$
\bigl(\varphi_p\bigl(u_k(t) - Cu_k(t-\tau)\bigr)'\bigr)' = -\frac{d}{dt}\nabla F\bigl(u_k(t)\bigr) - G\bigl(u_k\bigl(t-\gamma(t)\bigr)\bigr) + e_k(t). \tag{3.22}
$$

Set $y_k = \varphi_p(Au_k')$ for $k > k_0$. From (3.19) and (3.22) we see that

$$
|y_k|_0 \le \rho + F_{\rho_0}
$$

and

$$
|y_k'|_0 \le \max_{|x|\le\rho_0}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{\partial^2 F(x)}{\partial x_i \partial x_j}\right|^2\right)^{\frac{1}{2}} |u_k'|_0 + G_{\rho_0} + \sup_{t\in R}|e(t)| := \rho_2.
$$

By the method similar to that of Lemma 2.4 in [12] we can get that there is $u_0 \in C^1(\mathbb{R}, \mathbb{R}^n)$ such that $u_{k_j}'(t) \to u_0'(t)$ uniformly on $[c,d] \subset \mathbb{R}$, where $\{u_{k_j}\}$ is a subsequence of $\{u_k\}$.

There exists $j_0 > 0$ such that $[a - |\gamma|_0, b + |\gamma|_0] \subset [-k_jT, k_jT - \varepsilon_0]$ with $j > j_0$ and $a < b \in \mathbb{R}$. Therefore, by (1.5) and (3.15), for $t \in [a - |\gamma|_0, b + |\gamma|_0]$, we get

$$
\bigl(\varphi_p\bigl(u_{k_j}(t) - Cu_{k_j}(t-\tau)\bigr)'\bigr)' = -\frac{d}{dt}\nabla F\bigl(u_{k_j}(t)\bigr) - G\bigl(u_{k_j}\bigl(t-\gamma(t)\bigr)\bigr) + e(t). \tag{3.23}
$$

From (3.23) we get

$$
\begin{aligned}
y_k' &= \bigl(\varphi_p\bigl(Au_{k_j}'\bigr)\bigr)' \\
&= -\frac{d}{dt}\nabla F\bigl(u_{k_j}(t)\bigr) - G\bigl(u_{k_j}\bigl(t-\gamma(t)\bigr)\bigr) + e(t) \\
&\to -\frac{d}{dt}\nabla F\bigl(u_0(t)\bigr) - G\bigl(u_0\bigl(t-\gamma(t)\bigr)\bigr) + e(t) \\
&:= \chi(t), \quad \text{uniformly on } [a,b],
\end{aligned}
$$

because $y_{k_j}'(t)$ is continuously differentiable on (a,b) for $j > j_0$ and $y_{k_j}'(t) \to \chi(t)$ uniformly on $[a,b]$. We know that $\chi(t) = (\varphi_p(u_0(t) - Cu_0(t-\tau))')'$, $t \in \mathbb{R}$. Since $a, b \in \mathbb{R}$ are arbitrary, $u_0(t)$ is a solution of (1.1).

Next, we prove that $u_0(t) \to 0$ and $u_0'(t) \to 0$ as $|t| \to +\infty$. Since

$$\begin{aligned}\int_{-\infty}^{+\infty} \left(\left|u_0(t)\right|^p + \left|u_0'(t)\right|^p\right) dt &= \lim_{i\to+\infty} \int_{-iT}^{iT} \left(\left|u_0(t)\right|^p + \left|u_0'(t)\right|^p\right) dt \\ &= \lim_{i\to+\infty} \lim_{j\to+\infty} \int_{-iT}^{iT} \left(\left|u_{k_j}(t)\right|^p + \left|u_{k_j}'(t)\right|^p\right) dt,\end{aligned}$$

if $k_j > i$, $i \in \mathbb{N}$, then it follows from (3.14) and (3.15) that

$$\int_{-iT}^{iT} \left(\left|u_{k_j}(t)\right|^p + \left|u_{k_j}'(t)\right|^p\right) dt \le \int_{-k_jT}^{k_jT} \left(\left|u_{k_j}(t)\right|^p + \left|u_{k_j}'(t)\right|^p\right) dt \le A_0^p + A_1^p.$$

Letting $i \to +\infty$ and $j \to +\infty$, we have

$$\int_{-\infty}^{+\infty} \left(\left|u_0(t)\right|^p + \left|u_0'(t)\right|^p\right) dt \le A_0^p + A_1^p \tag{3.24}$$

and

$$\int_{|t|\ge r} \left(\left|u_0(t)\right|^p + \left|u_0'(t)\right|^p\right) dt \to 0, \quad r \to +\infty. \tag{3.25}$$

From (3.13), similarly to the previous method, we get

$$\int_{-\infty}^{+\infty} \left|u_0'(t) - Cu_0'(t-\tau)\right|^p dt \le M^p. \tag{3.26}$$

From Lemma 2.1 we can see that

$$\begin{aligned}\left|u_0(t)\right| &\le (2T)^{-\frac{1}{p}} \left(\int_{t-T}^{t+T} \left|u_0(s)\right|^p ds\right)^{\frac{1}{p}} + T(2T)^{-\frac{1}{p}} \left(\int_{t-T}^{t+T} \left|u_0'(s)\right|^p ds\right)^{\frac{1}{p}} \\ &\le \max\left\{(2T)^{-\frac{1}{p}}, T(2T)^{-\frac{1}{p}}\right\} \int_{t-T}^{t+T} \left(\left|u_0(t)\right|^p + \left|u_0'(t)\right|^p\right) dt \to 0, \quad |t| \to +\infty.\end{aligned}$$

Finally, we will prove that $|u_0'(t)| \to 0$ as $|t| \to +\infty$ if the following condition holds:

$$\left|\left[\tilde{A}u_0'\right](t)\right| := \left|u_0'(t) - Cu_0'(t-\tau)\right| \to 0, \quad |t| \to +\infty. \tag{3.27}$$

On the one hand, from (3.16) we have $|u_0| \le \rho_0$, and applying (1.1) yields

$$\begin{aligned}&\left|\frac{d}{dt}\left(\left|\left[\tilde{A}u_0'\right](t)\right|^{p-2}\left[\tilde{A}u_0'\right](t)\right)\right| \\ &\quad \le \left|\frac{d}{dt}\nabla F\left(u_0(t)\right)\right| + \left|G\left(u_0\left(t-\gamma(t)\right)\right)\right| + \sup_{t\in\mathbb{R}}\left|e(t)\right| \\ &\quad \le \sup_{|u|\le\rho_0}\left|\frac{d}{dt}\nabla F(u)\right| + \sup_{|u|\le\rho_0}\left|G(u)\right| + \sup_{t\in\mathbb{R}}\left|e(t)\right| := \tilde{M} \quad \text{for } t \in \mathbb{R}.\end{aligned}$$

If (3.27) does not hold, then there exist a parameter $\varepsilon_0 \in (0, \frac{1}{2})$ and a sequence $\{t_k\}$ such that

$$|t_1| < |t_2| < |t_3| < \cdots, \qquad |t_k| + 1 < |t_{k+1}|, \quad k = 1, 2, \ldots,$$

and

$$\left|\tilde{A}u_0'(t_k)\right| \geq (2\varepsilon_0)^{\frac{1}{p-1}}, \quad k = 1, 2, \ldots.$$

So, for $t \in [t_k, t_k + \varepsilon_0/(1+\tilde{M})]$, we have

$$\begin{aligned}
\left|[\tilde{A}u_0'](t)\right|^{p-1} &= \left|\left|[\tilde{A}u_0'](t_k)\right|^{p-2}[\tilde{A}u_0'](t_k) + \int_{t_k}^{t} \frac{d}{ds}\left(\left|[\tilde{A}u_0'](s)\right|^{p-2}[\tilde{A}u_0'](s)\right) ds\right| \\
&\geq \left|[\tilde{A}u_0'](t_k)\right|^{p-1} - \int_{t_k}^{t}\left|\frac{d}{ds}\left|\left([\tilde{A}u_0'](s)\right|^{p-2}[\tilde{A}u_0'](s)\right)\right| ds \\
&\geq \varepsilon_0.
\end{aligned}$$

Note that

$$\int_{-\infty}^{+\infty}\left|[\tilde{A}u_0'](t_k)\right|^{p} dt \geq \sum_{k=1}^{\infty}\int_{t_k}^{t_k+\varepsilon_0/(1+\tilde{M})}\left|[\tilde{A}u_0'](t_k)\right|^{p} dt = \infty,$$

which contradicts (3.26), and thus (3.27) holds.

On the other hand, let $u_0'(t) = (u_{0_1}'(t), u_{0_2}'(t), \ldots, u_{0_n}'(t))$. From (3.21) we know that $|Au_k'| < (1 + \sqrt{\sum_{i=1}^{n}|c_i|^2})\rho_1 := B_1$. For all $\varepsilon > 0$, let $N = [\log_{|c_i|}^{\frac{\varepsilon(1-|c_i|)}{2B_1}}] > 0$. Then $\sum_{h=N+1}^{\infty}|c_i|^h < \frac{\varepsilon}{2B_1}$ ($|c_i| < 1$). According to (3.27), it is easy to find that there exists a constant $G > 0$ such that $|u_{0_i}'(t) - c_i u_{0_i}'(t-\tau)| < \frac{\varepsilon}{2(N+1)}$ for $t > G$. Set $P_T = \{x | x \in C(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t)\}$ and $A_0 : P_T \to P_T$, $[A_0 x](t) = x(t) - cx(t-\tau)$ with $|c| \neq 1$. Then applying Lemma 2.3 in [13], we obtain

$$[A_0^{-1}f](t) = \begin{cases} \sum_{j\geq 0} c^j f(t-j\tau), & |c| < 1\ \forall f \in P_T, \\ -\sum_{j\geq 0} c^{-j} f(t+j\tau), & |c| > 1\ \forall f \in P_T. \end{cases}$$

When $|c_i| < 1$, this yields

$$\begin{aligned}
&\left|u_{0_i}'(t)\right| \\
&\quad = \lim_{j\to+\infty}\left|[A^{-1}Au_{k_{j0_i}}'](t)\right| \\
&\quad \leq \left|\lim_{j\to\infty}\sum_{h\geq 0}^{N} c_i^h [Au_{k_{j0_i}}'](t-h\tau) + \sum_{h=N+1}^{\infty} c_i^h [Au_{k_{j0_i}}'](t-h\tau)\right| \\
&\quad \leq \left|\lim_{j\to\infty}\sum_{h\geq 0}^{N} c_i^h [Au_{k_{j0_i}}'](t-h\tau)\right| + \left|\lim_{j\to\infty}\sum_{h=N+1}^{\infty} c_i^h [Au_{k_{j0_i}}'](t-h\tau)\right| \\
&\quad \leq \lim_{j\to\infty}\sum_{h\geq 0}^{N}|c_i|^h\left|[Au_{k_{j0_i}}'](t-h\tau)\right| + B_1\sum_{h=N+1}^{\infty}|c_i|^h \\
&\quad = \sum_{h\geq 0}^{N}|c_i|^h\left|\left(u_{0_i}'(t-h\tau) - c_i u_{0_i}'(t-(h+1)\tau)\right)\right| + B_1\sum_{h=N+1}^{\infty}|c_i|^h. \qquad (3.28)
\end{aligned}$$

By (3.28), for arbitrary $\varepsilon > 0$, there exists $\bar{N} = G + N$ such that, for $t > \bar{N}$,

$$\begin{aligned} \left|u'_{0_i}(t)\right| &\le \sum_{h\ge 0}^{N} |c_i|^h \left|\left(u'_{0_i}(t-h\tau) - c_i u'_{0_i}\left(t-(h+1)\tau\right)\right)\right| + \left|B_1 \sum_{h=N+1}^{\infty} c_i^h\right| \\ &< (N+1)\frac{\varepsilon}{2(N+1)} + B_1 \frac{\varepsilon}{2B_1} \\ &= \varepsilon. \end{aligned}$$

So, $|u'_{0_i}(t)| \to 0$ as $|t| \to +\infty$. Similarly to the previous method, when $|c_i| > 1$, $|u'_{0_i}(t)| \to 0$ also holds as $|t| \to +\infty$. Thus $|u'_0(t)| \to 0$ as $|t| \to +\infty$. Obviously, $u_0(t) \not\equiv 0$; otherwise, $e(t) = 0$, which contradicts condition $[H_2]$. This completes the proof. □

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Competing interests
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Authors' contributions
Both authors have equally contributed to obtaining new results in this paper and also read and approved the final manuscript.

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