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Positive solutions to one-dimensional quasilinear impulsive indefinite boundary value problems

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Abstract

Consider the one-dimensional quasilinear impulsive boundary value problem involving the p-Laplace operator

$$\begin{cases} -(\phi_p(u'))' = \lambda\omega(t)f(u), & 0 < t < 1, \\ -\Delta u|_{t=t_k} = \mu I_k(u(t_k)), & k = 1, 2, \ldots, n, \\ \Delta u'|_{t=t_k} = 0, & k = 1, 2, \ldots, n, \\ u'(0) = 0, & u(1) = \int_0^1 g(t)u(t)\,dt, \end{cases}$$

where $\lambda, \mu > 0$ are two positive parameters, $\phi_p(s)$ is the p-Laplace operator, i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $\omega(t)$ changes sign on $[0, 1]$. Several new results are obtained for the above quasilinear indefinite problem.

Keywords: Multiplicity of positive solutions; Indefinite weight function; p-Laplace operator; Quasilinear impulsive differential equation

1 Introduction

Impulsive differential equation is regarded as a critical mathematical tool to provide a natural description of observed evolution processes (see [1–4]). So the consideration of impulsive differential equations has gained prominence and many authors have begun to take a great interest in the subject of impulsive differential equations, for example, see [5–22] and the references cited therein.

Meanwhile, the p-Laplace operator equation is a typical quasilinear operator equation, which comes naturally from glaciology, nonlinear flow laws, and non-Newtonian mechanics (see [23, 24]). Recently, various existence, multiplicity, and uniqueness results of positive solutions for differential equations with one-dimensional p-Laplace operator have been considered [25–33]. Specially, Zhang and Ge [34] investigated the following second order one-dimensional p-Laplace operator equation

$$\begin{cases} -(\phi_p(u'(t)))' = f(t, u(t)), & t \neq t_k, t \in (0, 1), \\ \Delta u|_{t=t_k} = I_k(u(t_k)), & k = 1, 2, \ldots, n, \\ u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), & u'(1) = 0, \end{cases} \tag{1.1}$$

where $\phi_p(s)$ is p-Laplace operator, i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, t_k ($k = 1, 2, \dots, n$, where n is a fixed positive integer) are fixed points with $0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n < 1$, ξ_i ($i = 1, 2, \dots, m-2$) $\in (0,1)$ is given $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $\xi_i \neq t_k$, $i = 1, 2, \dots, m-2$, $k = 1, 2, \dots, n$, $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e.,

$$\Delta u|_{t=t_k} = u\bigl(t_k^+\bigr) - u\bigl(t_k^-\bigr),$$

where $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand limit and left-hand limit of $u(t)$ at $t = t_k$, respectively. Applying the classical fixed-point index theorem for compact maps, the authors got several new multiplicity results of positive solutions.

On the other hand, we observe that many authors (see [35–49]) have paid more attention to a class of boundary value problems involving integral boundary conditions, which contains two-point, three-point, and general multi-point boundary value problems as exceptional cases, see [50–58] and the references cited therein.

However, in literature there are almost no papers on multiple positive solutions for second order impulsive nonlocal indefinite boundary value problems with one-dimensional p-Laplace operator and multiple parameters. More precisely, the study of $\lambda > 0$, $\mu > 0$, $p \not\equiv 2$, $I_k \neq 0$ ($k = 1, 2, \dots, n$) and ω changes sign is still open for the second order nonlocal boundary value problem

$$\begin{cases} -(\phi_p(u'))' = \lambda\omega(t)f(u), & 0 < t < 1, \\ -\Delta u|_{t=t_k} = \mu I_k(u(t_k)), & k = 1, 2, \dots, n, \\ \Delta u'|_{t=t_k} = 0, & k = 1, 2, \dots, n, \\ u'(0) = 0, & u(1) = \int_0^1 g(t)u(t)\,dt, \end{cases} \tag{1.2}$$

where $\lambda > 0$ and $\mu > 0$ are two parameters, $\omega(t)$ may change sign, $\phi_p(s)$ is a p-Laplace operator, i.e., $\phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. t_k ($k = 1, 2, \dots, n$) (where n is a fixed positive integer) are fixed points with $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n < t_{n+1} = 1$, $\Delta u|_{t=t_k}$ denotes the jump of $u(t)$ at $t = t_k$, i.e., $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ represent the right-hand limit and left-hand limit of $u(t)$ at $t = t_k$, respectively.

In addition, set $J = [0,1]$, $R_+ = [0, +\infty)$, $R = (-\infty, +\infty)$, and let ω, f, I_k, and g satisfy the following conditions:

(H_1) $\omega : J \to R$ is continuous, and there exists a constant $\xi \in (0,1)$ such that

$$\omega(t) \geq 0, \quad t \in [0, \xi], \qquad \omega(t) \leq 0, \quad t \in [\xi, 1].$$

Moreover, $\omega(t)$ does not vanish identically on any subinterval of J.

(H_2) $f : R_+ \to R_+$ is continuous, and $f(u) > 0$ for all $u > 0$, there exists $0 < c \leq 1$ such that

$$f(x) \geq c\psi(x), \quad x \in R_+,$$

where $\psi(x) = \max\{f(y) : 0 \leq y \leq x\}$;

(H_3) $I_k \in C(R_+, R_+)$, and $I_k(u) > 0$ for all $u > 0$.

(H_4) $g \in L^1[0,1]$ is nonnegative and $\eta \in [0,1)$, where

$$\eta = \int_0^1 g(s)\,ds. \tag{1.3}$$

(H_5) There exist $0 < \theta_1 \le +\infty$, $\theta_1 \neq p-1$, $0 < \theta_2 \le +\infty$, $\theta_2 \neq 1$, and $k_1, k_2, k_3, k_4 > 0$ such that

$$k_1 u^{\theta_1} \le f(u) \le k_2 u^{\theta_1}, \qquad k_3 u^{\theta_2} \le I_k(u) \le k_4 u^{\theta_2}.$$

(H_6) There exists a number $0 < \sigma < \xi$ such that

$$c^2 k_1 \sigma^{\theta_1} \int_\sigma^\xi \omega^+(t)\,dt \ge k_2 \xi^{\theta_1} \int_\xi^1 \omega^-(t)\,dt.$$

We define $\omega^+(t) = \max\{\omega(t), 0\}$, $\omega^-(t) = -\min\{\omega(t), 0\}$. Then $\omega(t) = \omega^+(t) - \omega^-(t)$.

It is well accepted that the fixed point theorem in a cone is crucial in showing the existence of positive solutions of various boundary value problems for second order differential equations.

Lemma 1.1 (Theorem 2.3.4 of [59]) *Let Ω_1 and Ω_2 be two bounded open sets in a real Banach space E such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let the operator $T : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be completely continuous, where P is a cone in E. Suppose that one of the two conditions*

(i) $\|Tx\| \le \|x\|$, $\forall x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \ge \|x\|$, $\forall x \in P \cap \partial\Omega_2$,

or

(ii) $\|Tx\| \ge \|x\|$, $\forall x \in P \cap \partial\Omega_1$, *and* $\|Tx\| \le \|x\|$, $\forall x \in P \cap \partial\Omega_2$,

is satisfied. Then T has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

This paper is organized in the following fashion. In Sect. 2, we present some lemmas to be used in the subsequent sections. Section 3 is devoted to proving the multiplicity of positive solutions for problem (1.2), and we give an example to illustrate the main results in the final section.

2 Preliminaries

Let $J' = J \backslash \{t_1, t_2, \ldots, t_n\}$. The basic space used in this paper $PC[0,1] = \{u | u : J \to R$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $u(t_k^+)$ exists, $k = 1, 2, \ldots, n\}$. Then $PC[0,1]$ is a real Banach space with the norm $\|\cdot\|_{PC}$ defined by $\|u\|_{PC} = \sup_{t \in J} |u(t)|$. By a solution of (1.2), we mean that a function $u \in PC[0,1] \cap C^2(J')$ which satisfies (1.2).

In these main results, we will make use of the following lemmas.

Lemma 2.1 *Assume that (H_1)–(H_4) hold. Then $u \in PC[0,1] \cap C^2(J')$ is a solution of problem (1.2) if and only if $u \in PC[0,1]$ is a solution of the following impulsive integral equation*:

$$\begin{aligned} u(t) = {} & \frac{1}{1-\eta}\biggl[\int_0^1 g(t) \int_t^1 \phi_q\biggl(\int_0^s \lambda\omega(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds\,dt \\ & + \mu \int_0^1 g(t)\biggl(\sum_{t \le t_k} I_k\bigl(u(t_k)\bigr)\biggr)\,dt\biggr] \\ & + \int_t^1 \phi_q\biggl(\int_0^s \lambda\omega(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds + \mu \sum_{t \le t_k} I_k\bigl(u(t_k)\bigr). \end{aligned} \tag{2.1}$$

Proof The proof is similar to that of Lemma 3.1 in [38]. □

To establish the existence of multiple positive solutions in $PC[0,1]\cap C^2(J')$ of problem (1.2), we denote

$$PC^+[0,1] = \Big\{u \in PC[0,1] : \min_{t\in J} u(t) \geq 0\Big\},$$

and a cone K in $PC[0,1]$ by

$$K = \big\{u \in PC^+[0,1] : u \text{ is concave on } [0,\xi], \text{and } u \text{ is convex on } [\xi,1]\big\}. \tag{2.2}$$

Let $R > r > 0$, define $K_r = \{u \in K : \|u\| < r\}$, $K_{R,r} = \{u \in K : r < \|u\| < R\}$. Note that $\partial K_r = \{u \in K : \|u\| = r\}$, $\overline{K}_{R,r} = \{u \in K : r \leq \|u\| \leq R\}$.

We define a map $T : K \to PC[0,1]$ by

$$\begin{aligned}(Tu)(t) = {}& \frac{1}{1-\eta}\bigg[\int_0^1 g(t)\int_t^1 \phi_q\bigg(\int_0^s \lambda\omega(\tau)f\big(u(\tau)\big)\,d\tau\bigg)\,ds\,dt \\ &+ \mu\int_0^1 g(t)\bigg(\sum_{t\leq t_k} I_k\big(u(t_k)\big)\bigg)\,dt\bigg] \\ &+ \int_t^1 \phi_q\bigg(\int_0^s \lambda\omega(\tau)f\big(u(\tau)\big)\,d\tau\bigg)\,ds + \mu\sum_{t\leq t_k} I_k\big(u(t_k)\big),\end{aligned} \tag{2.3}$$

where η is defined in (1.3).

Lemma 2.2 *From (2.1), we know that $u \in PC[0,1]$ is a solution of problem (1.2) if and only if u is a fixed point of the map T.*

Lemma 2.3 *Assume that (H_1)–(H_6) hold. Then we have $T(K) \subset K$, and $T : K \to K$ is completely continuous.*

Proof From (2.3), we know that

$$(Tu)'(t) = -\phi_q\bigg(\int_0^t \lambda\omega(s)f\big(u(s)\big)\,ds\bigg). \tag{2.4}$$

Define $q(t) : J \to J$ as follows:

$$q(t) = \min\bigg\{\frac{t}{\xi}, \frac{1-t}{1-\xi}\bigg\},$$

and $\min_{\sigma\leq t\leq\xi} q(t) = \frac{\sigma}{\xi}$, $\max_{\xi\leq t\leq 1} q(t) = 1$.

Firstly, for any $u \in K$, we have

$$\int_0^1 \omega(s)f\big(u(s)\big)\,ds \geq \int_0^\sigma \omega^+(s)f\big(u(s)\big)\,ds. \tag{2.5}$$

In fact, by (2.2), we know that $u(t) \geq 0$. Since $u \in K$, $u(0) \geq 0$, and $u(1) \geq 0$, we have

$$\begin{aligned}&\frac{u(t)-u(0)}{t-0} \geq \frac{u(\xi)-u(0)}{\xi-0}, \quad t\in[0,\xi] \quad\Rightarrow\quad u(t) \geq q(t)u(\xi), \quad t\in[0,\xi], \\ &\frac{u(t)-u(1)}{t-1} \geq \frac{u(\xi)-u(1)}{\xi-1}, \quad t\in[\xi,1] \quad\Rightarrow\quad u(t) \leq q(t)u(\xi), \quad t\in[\xi,1].\end{aligned}$$

As we all know, ψ is nondecreasing on J, so we have

$$\psi\big(u(t)\big) \ge \psi\big(q(t)u(\xi)\big), \quad t \in [0,\xi], \qquad \psi\big(u(t)\big) \le \psi\big(q(t)u(\xi)\big), \quad t \in [\xi,1].$$

So, it follows from (H_5) and (H_6) that

$$\begin{aligned}
&\int_0^1 \omega(s)f\big(u(s)\big)\,ds - \int_0^\sigma \omega^+(s)f\big(u(s)\big)\,ds \\
&\quad = \int_\sigma^\xi \omega^+(s)f\big(u(s)\big)\,ds - \int_\xi^1 \omega^-(s)f\big(u(s)\big)\,ds \\
&\quad \ge c\int_\sigma^\xi \omega^+(s)\psi\big(u(s)\big)\,ds - \int_\xi^1 \omega^-(s)\psi\big(u(s)\big)\,ds \\
&\quad \ge c\int_\sigma^\xi \omega^+(s)\psi\big(q(s)u(\xi)\big)\,ds - \int_\xi^1 \omega^-(s)\psi\big(q(s)u(\xi)\big)\,ds \\
&\quad \ge c\int_\sigma^\xi \omega^+(s)f\big(q(s)u(\xi)\big)\,ds - \frac{1}{c}\int_\xi^1 \omega^-(s)f\big(q(s)u(\xi)\big)\,ds \\
&\quad \ge ck_1u^\theta(\xi)\frac{\sigma^\theta}{\xi^\theta}\int_\sigma^\xi \omega^+(s)\,ds - \frac{1}{c}k_2u^\theta(\xi)\int_\xi^1 \omega^-(s)\,ds \\
&\quad \ge u^\theta(\xi)\bigg(ck_1\frac{\sigma^\theta}{\xi^\theta}\int_\sigma^\xi \omega^+(s)\,ds - \frac{1}{c}k_2\int_\xi^1 \omega^-(s)\,ds\bigg) \\
&\quad \ge 0.
\end{aligned}$$

Secondly, if $t \in [0,\xi]$, we have

$$\int_0^t \omega(s)f\big(u(s)\big)\,ds = \int_0^t \omega^+(s)f\big(u(s)\big)\,ds \ge 0.$$

Since $p,q > 1$, we get

$$\begin{aligned}
(Tu)''(t) &= \bigg(-\phi_q\bigg(\int_0^t \lambda\omega(s)f\big(u(s)\big)\,ds\bigg)\bigg)' \\
&= \bigg(-\bigg(\int_0^t \lambda\omega^+(s)f\big(u(s)\big)\,ds\bigg)^{q-1}\bigg)' \\
&= -(q-1)\bigg(\int_0^t \lambda\omega^+(s)f\big(u(s)\big)\,ds\bigg)^{q-2}\lambda\omega^+(t)f\big(u(t)\big) \\
&\le 0.
\end{aligned}$$

If $t \in [\xi,1]$, then we have

$$\begin{aligned}
\int_0^t \omega(s)f\big(u(s)\big)\,ds &= \int_0^\xi \omega^+(s)f\big(u(s)\big)\,ds - \int_\xi^t \omega^-(s)f\big(u(s)\big)\,ds \\
&\ge \int_0^\xi \omega^+(s)f\big(u(s)\big)\,ds - \int_\xi^1 \omega^-(s)f\big(u(s)\big)\,ds \\
&= \int_0^1 \omega(s)f\big(u(s)\big)\,ds
\end{aligned}$$

$$\begin{aligned} &\geq \int_0^{\sigma} \omega^+(s) f\bigl(u(s)\bigr)\, ds \\ &\geq 0. \end{aligned}$$

And then, for $t \in [\xi, 1]$, it follows from $p, q > 1$ that

$$\begin{aligned} (Tu)''(t) &= \biggl(-\phi_q\biggl(\int_0^t \lambda\omega(s) f\bigl(u(s)\bigr)\, ds\biggr)\biggr)' \\ &= \biggl(-\phi_q\biggl(\int_0^{\xi} \lambda\omega^+(s) f\bigl(u(s)\bigr)\, ds - \int_{\xi}^t \lambda\omega^-(s) f\bigl(u(s)\bigr)\biggr)\biggr)' \\ &= \biggl(-\biggl(\int_0^{\xi} \lambda\omega^+(s) f\bigl(u(s)\bigr)\, ds - \int_{\xi}^t \lambda\omega^-(s) f\bigl(u(s)\bigr)\biggr)^{q-1}\biggr)' \\ &= -(q-1)\biggl(\int_0^{\xi} \lambda\omega^+(s) f\bigl(u(s)\bigr)\, ds - \int_{\xi}^t \lambda\omega^-(s) f\bigl(u(s)\bigr)\biggr)^{q-2} \bigl(-\lambda\omega^-(t) f\bigl(u(t)\bigr)\bigr) \\ &\geq 0. \end{aligned}$$

Moreover, by direct calculating, we get $(Tu)(t) \geq 0$ for $t \in J$, $(Tu)''(t) \leq 0$ for $t \in [0, \xi]$, and $(Tu)''(t) \geq 0$ for $t \in [\xi, 1]$. Thus, $T(K) \subset K$.

Then it finally follows from the Arzelà–Ascoli theorem that the operator T is completely continuous. □

From Lemma 2.3, since $(Tu)'(t) \leq 0$, then T is nonincreasing for $u \in K$. It is not difficult to see that

$$\begin{aligned} \|Tu\|_{PC} &= (Tu)(0) \\ &= \frac{1}{1-\eta}\biggl[\int_0^1 g(0) \int_0^1 \phi_q\biggl(\int_0^s \lambda\omega(\tau) f\bigl(u(\tau)\bigr)\, d\tau\biggr)\, ds\, dt \\ &\quad + \mu \int_0^1 g(0)\biggl(\sum_{t \leq t_k} I_k\bigl(u(t_k)\bigr)\biggr)\, dt\biggr] \\ &\quad + \int_0^1 \phi_q\biggl(\int_0^s \lambda\omega(\tau) f\bigl(u(\tau)\bigr)\, d\tau\biggr)\, ds + \mu \sum_{k=1}^n I_k\bigl(u(t_k)\bigr). \end{aligned} \tag{2.6}$$

Lemma 2.4 *If* (H_1)–(H_4) *hold, then for* $u \in K$ *we get*

$$\|Tu\|_{PC} \leq \frac{1}{1-\eta}\phi_q\biggl(\int_0^{\xi} \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\, d\tau\biggr) + \mu \frac{1}{1-\eta} \sum_{k=1}^n I_k\bigl(u(t_k)\bigr), \tag{2.7}$$

$$\|Tu\|_{PC} \geq \frac{(1 - \int_{\xi}^1 g(t)\, dt)(1-\xi)}{1-\eta}\phi_q\biggl(\int_{\frac{\sigma}{2}}^{\sigma} \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\, d\tau\biggr) + \mu \sum_{k=1}^n I_k\bigl(u(t_k)\bigr). \tag{2.8}$$

Proof By (2.6), for $u \in K$, we have

$$\begin{aligned} \|Tu\|_{PC} &= \frac{1}{1-\eta}\biggl[\int_0^1 g(t) \int_t^1 \phi_q\biggl(\int_0^s \lambda\omega(\tau) f\bigl(u(\tau)\bigr)\, d\tau\biggr)\, ds\, dt \\ &\quad + \mu \int_0^1 g(t)\biggl(\sum_{t \leq t_k} I_k\bigl(u(t_k)\bigr)\biggr)\, dt\biggr] \end{aligned}$$

$$
\begin{aligned}
&+\int_0^1 \phi_q\left(\int_0^s \lambda\omega(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds+\mu\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\\
&\le \frac{1}{1-\eta}\left[\int_0^1 g(t)\int_0^1 \phi_q\left(\int_0^s \lambda\omega(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds\,dt\right.\\
&\quad\left.+\mu\int_0^1 g(t)\left(\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\right)dt\right]\\
&\quad+\int_0^1 \phi_q\left(\int_0^s \lambda\omega(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds+\mu\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\\
&=\frac{1}{1-\eta}\int_0^1 \phi_q\left(\int_0^s \lambda\omega(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds+\mu\frac{1}{1-\eta}\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\\
&=\frac{1}{1-\eta}\left[\int_0^\xi \phi_q\left(\int_0^s \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds+\int_\xi^1 \phi_q\left(\int_0^\xi \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right.\right.\\
&\quad\left.\left.-\int_\xi^s \lambda\omega^-(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)\right]ds+\mu\frac{1}{1-\eta}\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\\
&\le \frac{1}{1-\eta}\left[\int_0^\xi \phi_q\left(\int_0^\xi \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds\right.\\
&\quad\left.+\int_\xi^1 \phi_q\left(\int_0^\xi \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds\right]\\
&\quad+\mu\frac{1}{1-\eta}\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\\
&=\frac{1}{1-\eta}\int_0^1 \phi_q\left(\int_0^\xi \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds+\mu\frac{1}{1-\eta}\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\\
&=\frac{1}{1-\eta}\phi_q\left(\int_0^\xi \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)+\mu\frac{1}{1-\eta}\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr).
\end{aligned}
$$

Then (2.7) holds.

From (2.5) and (2.6), we have

$$
\begin{aligned}
\|Tu\|_{PC}&=\frac{1}{1-\eta}\left[\int_0^1 g(t)\int_t^1 \phi_q\left(\int_0^s \lambda\omega(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds\,dt\right.\\
&\quad\left.+\mu\int_0^1 g(t)\left(\sum_{t\le t_k} I_k\bigl(u(t_k)\bigr)\right)dt\right]\\
&\quad+\int_0^1 \phi_q\left(\int_0^s \lambda\omega(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds+\mu\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr)\\
&=\frac{1}{1-\eta}\left\{\int_0^\xi g(t)\left[\int_t^\xi \phi_q\left(\int_0^s \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right)ds\right.\right.\\
&\quad+\int_\xi^1 \phi_q\left(\int_0^\xi \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right.
\end{aligned}
$$

$$
\begin{aligned}
&- \int_\xi^s \lambda\omega^-(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds\biggr]\,dt + \int_\xi^1 g(t)\int_t^1 \phi_q\biggl(\int_0^\xi \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau \\
&- \int_\xi^s \lambda\omega^-(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds\,dt\biggr\} + \int_0^\xi \phi_q\biggl(\int_0^s \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&+ \int_\xi^1 \phi_q\biggl(\int_0^\xi \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau \\
&- \int_\xi^s \lambda\omega^-(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds + \mu\frac{1}{1-\eta}\int_0^1 g(t)\sum_{t\le t_k} I_k\bigl(u(t_k)\bigr)\,dt \\
&+ \mu\sum_{k=1}^n I_k\bigl(u(t_k)\bigr) \\
\ge{}& \frac{1}{1-\eta}\int_0^\xi g(t)\int_\xi^1 \phi_q\biggl(\int_0^\xi \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau - \int_\xi^s \lambda\omega^-(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds\,dt \\
&+ \int_\xi^1 \phi_q\biggl(\int_0^\xi \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau - \int_\xi^s \lambda\omega^-(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&+ \mu\sum_{k=1}^n I_k\bigl(u(t_k)\bigr) \\
={}& \frac{1-\int_\xi^1 g(t)\,dt}{1-\eta}\int_\xi^1 \phi_q\biggl(\int_0^\xi \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau - \int_\xi^s \lambda\omega^-(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&+ \mu\sum_{k=1}^n I_k\bigl(u(t_k)\bigr) \\
\ge{}& \frac{1-\int_\xi^1 g(t)\,dt}{1-\eta}\int_\xi^1 \phi_q\biggl(\int_0^\xi \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau - \int_\xi^1 \lambda\omega^-(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr)\,ds \\
&+ \mu\sum_{k=1}^n I_k\bigl(u(t_k)\bigr) \\
={}& \frac{(1-\int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\biggl(\int_0^1 \lambda\omega(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr) + \mu\sum_{k=1}^n I_k\bigl(u(t_k)\bigr) \\
\ge{}& \frac{(1-\int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\biggl(\int_0^\sigma \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr) + \mu\sum_{k=1}^n I_k\bigl(u(t_k)\bigr) \\
\ge{}& \frac{(1-\int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\biggl(\int_{\frac{\sigma}{2}}^\sigma \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau\biggr) + \mu\sum_{k=1}^n I_k\bigl(u(t_k)\bigr).
\end{aligned}
$$

Then (2.8) holds. □

3 Main results

Based on the lemmas mentioned above, we give the following theorems and their proofs.

Theorem 3.1 *Assume that* (H_1)–(H_6) *hold. If* $\theta_1 > p-1$ *and* $\theta_2 > 1$, *there exist* $\lambda_0 > 0$ *and* $\mu_0 > 0$ *such that problem* (1.2) *admits two positive solutions for* $\lambda \in [\lambda_0, +\infty)$, $\mu \in [\mu_0, +\infty)$.

Proof Denote

$$A_1 = \frac{1}{\int_0^{\xi} \lambda\omega^+(\tau)\,d\tau}\phi_p\left(\frac{1-\eta}{2}\right), \qquad A_2 = \frac{1-\eta}{2\mu n},$$
$$B_1 = \frac{1}{\int_{\frac{\sigma}{2}}^{\sigma} \lambda\omega^+(\tau)\,d\tau}\phi_p\left(\frac{1-\eta}{2\alpha(1-\xi)(1-\int_{\xi}^1 g(t)\,dt)}\right), \qquad B_2 = \frac{1}{2n\mu\alpha}.$$

On the one hand, since $\theta_1 > p-1$ and $\theta_2 > 1$, by (H_5), we get

$$\lim_{u\to 0}\frac{f(u)}{\phi_p(u)} \le \lim_{u\to 0}\frac{k_2 u^{\theta_1}}{u^{p-1}} = 0, \qquad \lim_{u\to 0}\frac{I_k(u)}{u} \le \lim_{u\to 0}\frac{k_4 u^{\theta_2}}{u} = 0.$$

Hence, there exists $r > 0$ such that

$$f(u) < A_1\phi_p(u), \qquad I_k(u) < A_2 u, \quad u \in [0, r].$$

Then from (2.7), for $u \in \partial K_r$, then $\|u\|_{PC} = r$ and $0 \le u(t) \le \|u\| = r$ for all $t \in J$. It is clear that $f(u(t)) < A_1\phi_p(u(t))$ and $I_k(u(t)) < A_2 u(t)$ for all $t \in J$. Then from (2.7), for $u \in \partial K_r$, we get

$$\begin{aligned}
\|Tu\|_{PC} &\le \frac{1}{1-\eta}\phi_q\left(\int_0^{\xi} \lambda\omega^+(\tau)f\big(u(\tau)\big)\,d\tau\right) + \mu\frac{1}{1-\eta}\sum_{k=1}^n I_k\big(u(t_k)\big) \\
&< \frac{1}{1-\eta}\phi_q\left(\int_0^{\xi} \lambda\omega^+(\tau)A_1\phi_p\big(u(\tau)\big)\,d\tau\right) + \mu\frac{1}{1-\eta}\sum_{k=1}^n A_2 u(t_k) \\
&\le \frac{1}{1-\eta}\phi_q\left(\int_0^{\xi} \lambda\omega^+(\tau)A_1\phi_p\big(\|u\|_{PC}\big)\,d\tau\right) + \mu\frac{1}{1-\eta}\sum_{k=1}^n A_2\|u\|_{PC} \\
&= \frac{\|u\|_{PC}}{2} + \frac{\|u\|_{PC}}{2} \\
&= \|u\|_{PC}.
\end{aligned}$$

Consequently,

$$\|Tu\|_{PC} < \|u\|_{PC}, \quad \forall u \in \partial K_r. \tag{3.1}$$

On the other hand, we denote $\delta(t) = \min\{\frac{t}{\xi}, \frac{\xi-t}{\xi}\}$, $t \in [0, \xi]$. If $u \in K$, then u is a nonnegative function on $[0, \xi]$. So we get

$$u(t) \ge \delta(t)\|u\|_{PC}, \quad t \in [0, \xi].$$

It follows that $u(t) \ge \alpha\|u\|_{PC}$, $t \in [\frac{\sigma}{2}, \sigma]$, where $\alpha = \min_{\frac{\sigma}{2}\le t\le\sigma}\delta(t)$.

Since $\theta_1 > p-1$ and $\theta_2 > 1$, by (H_5), we have

$$\lim_{u\to+\infty}\frac{f(u)}{\phi_p(u)} \ge \lim_{u\to+\infty}\frac{k_1 u^{\theta_1}}{u^{p-1}} = +\infty, \qquad \lim_{u\to+\infty}\frac{I_k(u)}{u} \ge \lim_{u\to+\infty}\frac{k_3 u^{\theta_2}}{u} = +\infty.$$

Furthermore, there exists $0 < r < R'$ such that

$$f(u) \geq B_1\phi_p(u), \qquad I_k(u) \geq B_2 u, \quad u \in [R', +\infty),$$

Choose $R \geq \frac{R'}{\alpha}$. Then, for any $u \in \partial K_R$, we have $\min_{\frac{\sigma}{2} \leq t \leq \sigma} u(t) \geq \min_{\frac{\sigma}{2} \leq t \leq \sigma} \delta(t)\|u\|_{PC} = \alpha R \geq R'$, and $f(u(t)) \geq B_1 u^{p-1}(t)$, $I_k(u(t)) \geq B_2 u(t)$, $t \in [\frac{\sigma}{2}, \sigma]$.

Then by (2.8), for $u \in \partial K_R$, we have

$$\begin{aligned}
\|Tu\|_{PC} &\geq \frac{(1 - \int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\left(\int_{\frac{\sigma}{2}}^{\sigma} \lambda\omega^+(\tau)f\big(u(\tau)\big)\,d\tau\right) + \mu\sum_{k=1}^{n} I_k\big(u(t_k)\big) \\
&\geq \frac{(1 - \int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\left(\int_{\frac{\sigma}{2}}^{\sigma} \lambda\omega^+(\tau)B_1\phi_p\big(u(\tau)\big)\,d\tau\right) + \mu\sum_{k=1}^{n} B_2 u(t_k) \\
&\geq \frac{(1 - \int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\left(\int_{\frac{\sigma}{2}}^{\sigma} \lambda\omega^+(\tau)B_1\phi_p\big(\alpha\|u\|_{PC}\big)\,d\tau\right) + \mu\sum_{k=1}^{n} B_2\alpha\|u\|_{PC} \\
&= \frac{\alpha(1 - \int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\|u\|_{PC}\phi_q\left(\int_{\frac{\sigma}{2}}^{\sigma} \lambda\omega^+(\tau)B_1\,d\tau\right) + \mu n B_2\alpha\|u\|_{PC} \\
&\geq \frac{1}{2}\|u\|_{PC} + \frac{1}{2}\|u\|_{PC} \\
&= \|u\|_{PC}.
\end{aligned}$$

Consequently,

$$\|Tu\|_{PC} \geq \|u\|_{PC}, \quad \forall u \in \partial K_R. \tag{3.2}$$

In addition, choose a number $r' \in (0, r)$. Noticing that $f(u) > 0$ for all $u > 0$ and $I_k(u) > 0$ for all $u > 0$, we can define

$$\begin{aligned}
&f_{r'} = \min\{f(u) : \alpha r' \leq u \leq r'\}, \qquad I_{kr'} = \min\{I_k : \alpha r' \leq u \leq r'\}, \\
&I_{r'} = \min\{I_{kr'} : k = 1, 2, \ldots, n\}.
\end{aligned}$$

Let $\lambda_0 = \frac{1}{\int_{\frac{\sigma}{2}}^{\sigma} \omega^+(\tau)f_{r'}\,d\tau}\phi_p(\frac{r'(1-\eta)}{2(1-\int_\xi^1 g(t)\,dt)(1-\xi)})$, $\mu_0 = \frac{r'}{2nI_{r'}}$. Thus we have

$$\begin{aligned}
&\frac{(1 - \int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\left(\int_{\frac{\sigma}{2}}^{\sigma} \lambda_0\omega^+(\tau)f_{r'}\,d\tau\right) = \frac{1}{2}r', \\
&\mu_0 n I_{r'} = \frac{1}{2}r'.
\end{aligned}$$

If $u \in \partial K_{r'}$, then $\|u\|_{PC} = r'$ and $\alpha r' = \min_{\frac{\sigma}{2} \leq t \leq \sigma} \delta(t)\|u\|_{PC} \leq u(t) \leq \|u\|_{PC} = r'$, $t \in [\frac{\sigma}{2}, \sigma]$. It is clear that $f(u(t)) \geq f_{r'}$ and $I_k(u(t)) \geq I_{r'}$, $t \in [\frac{\sigma}{2}, \sigma]$.

Then from (2.8), for $u \in \partial K_{r'}$, we have

$$
\begin{aligned}
\|Tu\|_{PC} &\geq \frac{(1-\int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\left(\int_{\frac{\sigma}{2}}^{\sigma} \lambda\omega^+(\tau) f\bigl(u(\tau)\bigr)\,d\tau\right) + \mu\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr) \\
&\geq \frac{(1-\int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\left(\int_{\frac{\sigma}{2}}^{\sigma} \lambda\omega^+(\tau) f_{r'}\,d\tau\right) + \mu\sum_{k=1}^{n} I_{r'} \\
&\geq \frac{(1-\int_\xi^1 g(t)\,dt)(1-\xi)}{1-\eta}\phi_q\left(\int_{\frac{\sigma}{2}}^{\sigma} \lambda_0\omega^+(\tau) f_{r'}\,d\tau\right) + \mu_0 n I_{r'} \\
&= \frac{1}{2}r' + \frac{1}{2}r' \\
&= r' = \|u\|_{PC}.
\end{aligned}
$$

Consequently,

$$
\|Tu\|_{PC} \geq \|u\|_{PC}, \quad \forall u \in \partial K_{r'}. \tag{3.3}
$$

Therefore, applying Lemma 1.1 to (3.1), (3.2), and (3.3) yields that T has two fixed points $u_1 \in \overline{K}_R \setminus \overline{K}_r$ and $u_2 \in K_r \setminus K_{r'}$. Thus, if $\theta_1 > p-1$ and $\theta_2 > 1$, there exist $\lambda_0 > 0$ and $\mu_0 > 0$ such that problem (1.2) admits two positive solutions for $\lambda \in [\lambda_0, +\infty)$ and $\mu \in [\mu_0, +\infty)$. The proof of Theorem 3.1 is completed. □

Theorem 3.2 *Assume that* (H_1)–(H_6) *hold. If* $0 < \theta_1 < p-1$ *and* $0 < \theta_2 < 1$, *there exist* $\lambda^0 > 0$ *and* $\mu^0 > 0$ *such that problem* (1.2) *admits two positive solutions for* $\lambda \in (0, \lambda^0]$ *and* $\mu \in (0, \mu^0]$.

Proof On the one hand, since $0 < \theta_1 < p-1$ and $0 < \theta_2 < 1$, by (H_5), we get

$$
\lim_{u\to 0}\frac{f(u)}{\phi_p(u)} \geq \lim_{u\to 0}\frac{k_1 u^{\theta_1}}{u^{p-1}} = +\infty, \qquad \lim_{u\to 0}\frac{I_k(u)}{u} \geq \lim_{u\to 0}\frac{k_3 u^{\theta_2}}{u} = +\infty.
$$

Hence, there exists $r_1 > 0$ such that

$$
f(u) > B_1\phi_p(u), \qquad I_k(u) > B_2 u, \quad u \in [0, r_1].
$$

Then we have $\min\{f(u) : \alpha r_1 \leq u \leq r_1\} > B_1\phi_p(u)$ and $\min\{I_k(u) : \alpha r_1 \leq u \leq r_1\} > B_2 u$.

If $u \in \partial K_{r_1}$, then $\|u\|_{PC} = r_1$ and $\alpha r_1 = \min_{\frac{\sigma}{2}\leq t\leq\sigma}\delta(t)\|u\|_{PC} \leq u(t) \leq \|u\|_{PC} = r_1$, $t \in [\frac{\sigma}{2}, \sigma]$. It is easy to see that $f(u(t)) > B_3\phi_p(u(t))$, $I_k(u(t)) > B_4 u(t)$, $t \in [\frac{\sigma}{2}, \sigma]$. Then from (2.8), for $u \in \partial K_{r_1}$, similar to (3.2), we have

$$
\|Tu\|_{PC} > \|u\|_{PC}, \quad \forall u \in \partial K_{r_1}. \tag{3.4}
$$

On the other hand, since $0 < \theta_1 < p-1$ and $0 < \theta_2 < 1$, by (H_5), we have

$$
\lim_{u\to +\infty}\frac{f(u)}{\phi_p(u)} \leq \lim_{u\to +\infty}\frac{k_2 u^{\theta}}{u^{p-1}} = 0, \qquad \lim_{u\to +\infty}\frac{I_k(u)}{u} \leq \lim_{u\to +\infty}\frac{k_4 u^{\theta}}{u} = 0.
$$

Furthermore, there exists $0 < r_1 < R_1' < +\infty$ such that

$$f(u) \le \frac{A_1}{2}\phi_p(u), \qquad I_k(u) \le \frac{A_2}{2}u, \quad u \in \bigl[R_1', +\infty\bigr).$$

Let $M_1 = \max\{f(u) : 0 \le u \le R_1'\}$ and $M_2 = \max\{I_k : 0 \le u \le R_1', k = 1, 2, \dots, n\}$. It implies that

$$f(u) \le \frac{A_1}{2}\phi_p(u) + M_1, \qquad I_k(u) \le \frac{A_2}{2}u + M_2, \quad u \in [0, +\infty).$$

Choose $R_1 \ge \{R_1', \frac{2\phi_q(2\int_0^\xi \lambda\omega^+(\tau)M_1\,d\tau)}{1-\eta}, 4\mu n M_2\}$. If $u \in \partial K_{R_1}$, then $\|u\| = R_1$ and $0 \le u(t) \le R_1$, $t \in J$. It is easy to see that $f(u(t)) \le \frac{A_1}{2}\phi_p(u(t)) + M_1$, $I_k(u(t)) \le \frac{A_2}{2}u(t) + M_2$, $t \in J$. Then from (2.7), for $u \in \partial K_{R_1}$, we have

$$\begin{aligned}
\|Tu\|_{PC} &\le \frac{1}{1-\eta}\phi_q\biggl(\int_0^\xi \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\biggr) + \mu\sum_{k=1}^n I_k\bigl(u(t_k)\bigr) \\
&\le \frac{1}{1-\eta}\phi_q\biggl(\int_0^\xi \lambda\omega^+(\tau)\biggl(\frac{A_1}{2}\phi_p\bigl(u(\tau)\bigr) + M_1\biggr)\,d\tau\biggr) + \mu\sum_{k=1}^n\biggl(\frac{A_2}{2}u(t_k) + M_2\biggr) \\
&\le \frac{1}{1-\eta}\phi_q\biggl(\int_0^\xi \lambda\omega^+(\tau)\frac{A_1}{2}\phi_p\bigl(\|u\|_{PC}\bigr)\,d\tau + \int_0^\xi \lambda\omega^+(\tau)M_1\,d\tau\biggr) \\
&\quad + \mu\sum_{k=1}^n \frac{A_2}{2}\|u\|_{PC} + \mu n M_2 \\
&\le \frac{1}{1-\eta}\phi_q\biggl(\frac{1}{2}\phi_p\biggl(\frac{\|u\|_{PC}(1-\eta)}{2}\biggr) + \frac{1}{2}\phi_p\biggl(\frac{R_1(1-\eta)}{2}\biggr)\biggr) + \frac{\|u\|_{PC}}{4} + \frac{R_1}{4} \\
&= \frac{1}{2}R_1 + \frac{1}{2}R_1 \\
&= R_1 = \|u\|_{PC}.
\end{aligned}$$

Consequently,

$$\|Tu\|_{PC} \le \|u\|_{PC}, \quad \forall u \in \partial K_{R_1}. \tag{3.5}$$

In addition, choosing a number $r_1' \in (0, r_1)$, we can define

$$\begin{aligned}
&f^{r_1'} = \max\bigl\{f(u) : 0 < u \le r_1'\bigr\}, \qquad I_k^{r_1'} = \max\bigl\{I_k(u) : 0 < u \le r_1'\bigr\}, \\
&I^{r_1'} = \max\bigl\{I_k^{r_1'} : k = 1, 2, \dots, n\bigr\}.
\end{aligned}$$

Let $\lambda^0 = \frac{1}{\int_0^\xi \omega^+(\tau)f^{r_1'}\,d\tau}\phi_p(\frac{r_1'(1-\eta)}{2})$ and $\mu^0 = \frac{r_1'}{2nI^{r_1'}}$. It is clear that

$$\frac{1}{1-\eta}\phi_q\biggl(\int_0^\xi \lambda^0\omega^+(\tau)f^{r_1'}\,d\tau\biggr) \le \frac{1}{2}r_1', \qquad \mu^0 n I^{r_1'} \le \frac{1}{2}r_1'.$$

If $u \in \partial K_{r_1'}$, then $\|u\|_{PC} = r_1'$ and $0 \leq u(t) \leq \|u\|_{PC} = r_1'$, $t \in J$. It is clear that $f(u(t)) \leq f^{r_1'}$, $I_k(u(t)) \leq I^{r_1'}$, $t \in J$. Then from (2.7), for $u \in \partial K_{r_1'}$, we have

$$\begin{aligned}
\|Tu\|_{PC} &\leq \frac{1}{1-\eta}\phi_q\left(\int_0^{\xi} \lambda\omega^+(\tau)f\bigl(u(\tau)\bigr)\,d\tau\right) + \mu\sum_{k=1}^{n} I_k\bigl(u(t_k)\bigr) \\
&\leq \frac{1}{1-\eta}\phi_q\left(\int_0^{\xi} \lambda\omega^+(\tau)f^{r_1'}\,d\tau\right) + \mu\sum_{k=1}^{n} I^{r_1'} \\
&\leq \frac{1}{1-\eta}\phi_q\left(\int_0^{\xi} \lambda^0\omega^+(\tau)f^{r_1'}\,d\tau\right) + \mu^0 n I^{r_1'} \\
&= \frac{1}{2}r_1' + \frac{1}{2}r_1' \\
&= r_1' = \|u\|_{PC}.
\end{aligned}$$

Consequently,

$$\|Tu\|_{PC} \leq \|u\|_{PC}, \quad \forall u \in \partial K_{r_1'}. \tag{3.6}$$

Therefore, applying Lemma 1.1 to (3.4), (3.5), and (3.6) yields that T has two fixed points $u_1' \in \overline{K}_{R_1} \setminus \overline{K}_{r_1}$ and $u_2' \in K_{r_1} \setminus K_{r_1'}$. Thus, if $0 < \theta_1 < p-1$ and $0 < \theta_2 < 1$, there exist $\lambda^0 > 0$ and $\mu^0 > 0$ such that problem (1.2) admits two positive solutions for $\lambda \in (0, \lambda^0]$ and $\mu \in (0, \mu^0]$. The proof of Theorem 3.2 is finished. □

Remark 3.1 If $I_k = 0$ ($k = 1, 2, \ldots, n$), even for the case $g(t) \equiv 0$ on J, the results of the present paper are still novel.

Remark 3.2 Comparing with Li, Feng, and Qin [60], the main features of this paper are as follows:

(i) $p > 1$ is considered, not only $p \equiv 2$.
(ii) $I_k \not\equiv 0$ ($k = 1, 2, \ldots, n$) is considered.
(iii) The basic space $PC[0,1]$ is available, not $C[0,1]$.

4 An example

We give an example to illustrate our main conclusions.

Example 4.1 Let $p = \frac{3}{2}$, $n = 1$, $t_1 = \frac{1}{2}$. Consider the following problem:

$$\begin{cases} -(\phi_p(u'))' = \lambda\omega(t)(u + \sin u), & 0 < t < 1, \\ -\Delta u|_{t=t_1} = \mu I_1(u(t_1)), \\ \Delta u'|_{t=t_1} = 0, \\ u'(0) = 0, \quad u(1) = \int_0^1 g(t)u(t)\,dt, \end{cases} \tag{4.1}$$

where

$$\omega(t) = \begin{cases} 12(\frac{2}{3} - t), & t \in [0, \frac{2}{3}], \\ \frac{2}{3} - t, & t \in [\frac{2}{3}, 1], \end{cases} \qquad I_1(u) = u^2, \qquad g(t) = t.$$

From the definition of $\omega(t)$ and $g(t)$, we know that $\xi = \frac{1}{2}$ and $\eta = \int_0^1 t\,dt = \frac{1}{2}$. From $p = \frac{3}{2}$, we can get that $q = 3$.

Since f is nondecreasing, then $c = 1$. For fixed $k_1 = 1$, $k_2 = 2$, $\theta_1 = 1$, $k_3 = k_4 = 1$, $\theta_2 = 2$, $\sigma = \frac{1}{4}$, we can prove that (H_5) holds.

In fact,

$$\begin{aligned}\frac{1}{2}\int_{\frac{1}{2}}^{\frac{2}{3}} 12\left(\frac{2}{3}-\tau\right)d\tau &= 6\int_{\frac{1}{2}}^{\frac{2}{3}}\left(\frac{2}{3}-\tau\right)d\tau \\ &= 6\left(\frac{2}{3}\tau-\frac{\tau^2}{2}\right)\Big|_{\frac{1}{2}}^{\frac{2}{3}} d\tau \\ &= \frac{1}{12},\end{aligned}$$

and

$$2\times\frac{2}{3}\int_{\frac{2}{3}}^{1}\left(\tau-\frac{2}{3}\right)d\tau = \frac{4}{3}\left(\frac{\tau^2}{2}-\frac{2}{3}\tau\right)\Big|_{\frac{2}{3}}^{1} = \frac{2}{27}.$$

Obviously, $\frac{1}{12} > \frac{2}{27}$. Thus

$$\frac{1}{2}\int_{\frac{1}{2}}^{\frac{2}{3}} 12\left(\frac{2}{3}-\tau\right)d\tau \geq \frac{4}{3}\int_{\frac{2}{3}}^{1}\left(\tau-\frac{2}{3}\right)d\tau.$$

This shows that (H_6) holds.

Let $\lambda_0 = \frac{12}{7}\sqrt{\frac{3}{13}}(\frac{1}{8}+\sin\frac{1}{8})^{-1}$, $\mu_0 = 16$. Then it follows from Theorem 3.1 that problem (4.1) admits two positive solutions for $\lambda \in [\frac{12}{7}\sqrt{\frac{3}{13}}(\frac{1}{8}+\sin\frac{1}{8})^{-1}, +\infty)$, $\mu \in [16,\infty)$.

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