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On the oscillation of Hadamard fractional differential equations

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Abstract

Hadamard fractional derivatives are nonlocal fractional derivatives with singular logarithmic kernel with memory, and hence they are suitable to describe complex systems. In this paper, sufficient conditions are established for the oscillation of solutions fractional differential equations in the frame of left Hadamard fractional derivatives of order $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) \geq 0$. The results are also obtained for fractional Hadamard derivatives in the Caputo setting. Examples are provided to illustrate the applicability of the main results.

MSC: 34A08; 34C10; 26A33

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1 Introduction

The oscillation theory for fractional differential and difference equations was studied by some authors (see [1–8]). The study of oscillation and other qualitative properties of fractional dynamical systems such as stability, existence, and uniqueness of solutions is necessary to analyze the systems under consideration [9–13]. The analysis of the memory of certain fractional dynamical systems has gained a high impact in the last few years [14, 15]. Several definitions of fractional derivatives and fractional integral operators with different kernels exist in the literature. The Hadamard definition, which depends on a logarithmic type kernel, was introduced in [16], and later on some authors contributed to the development of its theory (see [17–19]). In this paper, we study the oscillation of Hadamard fractional differential equation of the form

$$\begin{cases} \mathcal{D}_a^{\alpha} x(t) + f_1(t,x) = r(t) + f_2(t,x), & t > a \\ \lim_{t\to a^+} \mathcal{D}_a^{\alpha-j} x(t) = b_j \quad (j = 1,2,\dots,n), \end{cases} \tag{1}$$

where $n = \lceil \alpha \rceil$, $\mathcal{D}_a^{\alpha}$ is the left-fractional Hadamard derivative of order $\alpha \in \mathbb{C}, \mathrm{Re}(\alpha) \geq 0$ in the Riemann–Liouville setting.

The objective of this paper is to study the oscillation of Hadamard fractional differential equations of the form (1) besides their Caputo setting.

This paper is organized as follows. Section 2 introduces some notations and provides the definitions of Hadamard fractional integral and differential operators together with some basic properties and lemmas that are needed in the proofs of the main theorems.

In Sect. 3, the main theorems are presented. Section 4 is devoted to the results obtained for Hadamard fractional operators in the Caputo setting. Finally, examples are provided in Sect. 5 to explain the effectiveness of the main results.

2 Notations and preliminary assertions

We start this section by introducing the definition of the Hadamard type fractional integrals and derivatives (see [20]).

Let (a,b) be a finite or infinite interval of $\mathbb{R}^+$ starting from a.The left-sided Hadamard fractional integral of order $\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0$ is defined by

$$\left(\mathcal{I}_a^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t}\, dt \quad (a < x < b). \tag{2}$$

The right-sided Hadamard fractional integral of order $\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0$ is defined by

$$\left(\mathcal{I}_b^\alpha f\right)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t}\, dt \quad (a < x < b). \tag{3}$$

The delta derivative is defined by $\delta = xD$, where $D = \frac{d}{dx}$. The left-sided Hadamard fractional derivative of order $\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, n = \lceil \operatorname{Re}(\alpha) \rceil$ is defined by

$$\begin{aligned} \left(\mathcal{D}_a^\alpha y\right)(x) &:= \delta^n \left(\mathcal{I}_a^{n-\alpha} y\right)(x) \\ &= \left(x \frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{n-\alpha-1} \frac{y(t)}{t}\, dt \quad (a < x < b). \end{aligned} \tag{4}$$

The right-sided Hadamard fractional derivative of order $\alpha \in \mathbb{C}, \operatorname{Re}(\alpha) > 0, n = \lceil \operatorname{Re}(\alpha) \rceil$ is defined by

$$\begin{aligned} \left(\mathcal{D}_b^\alpha y\right)(x) &:= (-\delta)^n \left(\mathcal{I}_b^{n-\alpha} y\right)(x) \\ &= \left(-x \frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{n-\alpha-1} \frac{y(t)}{t}\, dt \quad (a < x < b). \end{aligned} \tag{5}$$

Likewise Riemann–Liouville and Caputo fractional operators, left and right Hadamard type fractional operators are dual to each other via the Q-operator [21, 22] ($Qf(t) = f(a + b - t)$).

Now, we recall the definitions of Caputo type Hadamard fractional derivatives (see [17]).

Definition 2.1 ([17]) The left- and right-sided Hadamard fractional derivatives of Caputo type respectively are defined by

$${}^C\mathcal{D}_a^\alpha y(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{n-\alpha-1} \frac{\delta^n y(t)}{t}\, dt = \mathcal{I}_a^{n-\alpha} \delta^n y(x) \tag{6}$$

and

$${}^C\mathcal{D}_b^\alpha y(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{n-\alpha-1} \frac{\delta^n y(t)}{t}\, dt = (-1)^n \mathcal{I}_b^{n-\alpha} \delta^n y(x), \tag{7}$$

where $\operatorname{Re}(\alpha) > 0, n = \lceil \operatorname{Re}(\alpha) \rceil$, and $y(x) \in AC_\delta^n[a,b]$.

The following properties are useful in the sequel.

Property 2.1 ([17, 20]) Let $n = \lceil\alpha\rceil, \operatorname{Re}(\alpha) \geq 0$, and $\operatorname{Re}(\beta) > 0$. Then

1. $\mathcal{D}_a^\alpha(\ln\frac{x}{a})^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln\frac{x}{a})^{\beta-\alpha-1}$,
2. $\mathcal{D}_b^\alpha(\ln\frac{b}{x})^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln\frac{b}{x})^{\beta-\alpha-1}$,
3. $\mathcal{D}_a^\alpha(\ln\frac{x}{a})^{\alpha-n} = 0$ and $\mathcal{D}_b^\alpha(\ln\frac{b}{x})^{\alpha-n} = 0$,
4. ${}^C\mathcal{D}_a^\alpha(\ln\frac{x}{a})^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln\frac{x}{a})^{\beta-\alpha-1}, \operatorname{Re}(\beta) > n$,
5. ${}^C\mathcal{D}_b^\alpha(\ln\frac{b}{x})^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\ln\frac{b}{x})^{\beta-\alpha-1}, \operatorname{Re}(\beta) > n$,
6. ${}^C\mathcal{D}_a^\alpha(\ln\frac{x}{a})^k = 0$ and ${}^C\mathcal{D}_b^\alpha(\ln\frac{b}{x})^k = 0$ for $k = 0, 1, \ldots, n-1$,
7. $\mathcal{D}_a^\alpha 1 = \frac{1}{\Gamma(1-\alpha)}(\ln\frac{x}{a})^{-\alpha}$ and ${}^C\mathcal{D}_a^\alpha 1 = 0$.

Lemma 2.1 ([23] Young's inequality)

(i) *Let* $X, Y \geq 0, u > 1$, *and* $\frac{1}{u} + \frac{1}{v} = 1$, *then* $XY \leq \frac{1}{u}X^u + \frac{1}{v}Y^v$.

(ii) *Let* $X \geq 0, Y > 0, 0 < u < 1$, *and* $\frac{1}{u} + \frac{1}{v} = 1$, *then* $XY \geq \frac{1}{u}X^u + \frac{1}{v}Y^v$, *where equalities hold if and only if* $Y = X^{u-1}$.

3 Oscillation of Hadamard fractional differential equations in the frame of Riemann

In this section we study the oscillation theory for equation (1).

Lemma 3.1 ([20]) *Let* $\operatorname{Re}(\alpha) > 0, n = -[-\operatorname{Re}(\alpha)], y(x) \in L(a,b)$, *and* $(\mathcal{I}_a^{n-\alpha}y)(x) \in AC_\delta^n[a,b]$. *Then*

$$\left(\mathcal{I}_a^\alpha\mathcal{D}_a^\alpha y\right)(x) = y(x) - \sum_{j=1}^n \frac{(\delta^{n-j}\mathcal{I}_a^{n-\alpha}y)(a)}{\Gamma(\alpha-j+1)}\left(\ln\frac{x}{a}\right)^{\alpha-j}. \tag{8}$$

Using Lemma 3.1, the solution representation of (1) can be written as

$$x(t) = \sum_{j=1}^n \frac{(\mathcal{D}_a^{\alpha-j}x)(a)}{\Gamma(\alpha-j+1)}\left(\ln\frac{t}{a}\right)^{\alpha-j} + \mathcal{I}_a^\alpha F(t,x), \tag{9}$$

where $F(t,x) = r(t) + f_2(t,x) - f_1(t,x)$.

A solution of (1) is said to be oscillatory if it has arbitrarily large zeros on $(0,\infty)$; otherwise, it is called nonoscillatory. An equation is said to be oscillatory if all of its solutions are oscillatory.

We prove our results under the following assumptions:

$$xf_i(t,x) > 0 \quad (i = 1,2), x \neq 0, t \geq 0, \tag{10}$$

$$\left|f_1(t,x)\right| \geq p_1(t)|x|^\beta \quad \text{and} \quad \left|f_2(t,x)\right| \leq p_2(t)|x|^\gamma, \quad x \neq 0, t \geq 0, \tag{11}$$

$$\left|f_1(t,x)\right| \leq p_1(t)|x|^\beta \quad \text{and} \quad \left|f_2(t,x)\right| \geq p_2(t)|x|^\gamma, \quad x \neq 0, t \geq 0, \tag{12}$$

where $p_1, p_2 \in C([0,\infty),(0,\infty))$, and β, γ are positive constants.

Define

$$\Phi(t) = \Gamma(\alpha)\sum_{j=1}^n \frac{(\mathcal{D}_a^{\alpha-j}x)(a)}{\Gamma(\alpha-j+1)}\left(\ln\frac{t}{a}\right)^{\alpha-j} \tag{13}$$

and

$$\Psi(t,T_1)=\int_a^{T_1}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{F(s,x(s))}{s}\,ds. \tag{14}$$

Theorem 3.2 *Let $f_2=0$ in (1) and condition (10) hold. If*

$$\liminf_{t\to\infty}(\ln t)^{1-\alpha}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds=-\infty \tag{15}$$

and

$$\limsup_{t\to\infty}(\ln t)^{1-\alpha}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds=\infty \tag{16}$$

for every sufficiently large T, then every solution of (1) is oscillatory.

Proof Let $x(t)$ be a nonoscillatory solution of equation (1) with $f_2=0$. Suppose that $T_1>a$ is large enough so that $x(t)>0$ for $t\geq T_1$. Hence, (10) implies that $f_1(t,x)>0$ for $t\geq T_1$. Using (2), we get from (9)

$$\begin{aligned}\Gamma(\alpha)x(t)&=\Gamma(\alpha)\sum_{j=1}^{n}\frac{(\mathcal{D}^{\alpha-j}x)(a)}{\Gamma(\alpha-j+1)}\left(\ln\frac{t}{a}\right)^{\alpha-j}\\&\quad+\int_a^{T_1}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{F(s,x(s))}{s}\,ds\\&\quad+\int_{T_1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)-f_1(s,x(s))]}{s}\,ds\\&\leq\Phi(t)+\Psi(t,T_1)+\int_{T_1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds,\end{aligned} \tag{17}$$

where Φ and Ψ are defined in (13) and (14), respectively.

Multiplying (17) by $(\ln t)^{1-\alpha}$, we get

$$\begin{aligned}0&<(\ln t)^{1-\alpha}\Gamma(\alpha)x(t)\leq(\ln t)^{1-\alpha}\Phi(t)+(\ln t)^{1-\alpha}\Psi(t,T_1)\\&\quad+(\ln t)^{1-\alpha}\int_{T_1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds.\end{aligned} \tag{18}$$

Take $T_2>T_1$. We consider two cases as follows.

Case (1): Let $0<\alpha\leq1$. Then $n=1$ and $|(\ln t)^{1-\alpha}\Phi(t)|=|b_1(\ln t)^{1-\alpha}(\ln\frac{t}{a})^{\alpha-1}|$.

Since $h_1(t)=(\frac{\ln t-\ln a}{\ln t})^{\alpha-1}$ is decreasing for $a>1$ and $t>T_2$, we get

$$\left|(\ln t)^{1-\alpha}\Phi(t)\right|=\left|b_1\left(\frac{\ln t-\ln a}{\ln t}\right)^{\alpha-1}\right|\leq|b_1|\left(\frac{\ln T_2-\ln a}{\ln T_2}\right)^{\alpha-1}=c_1(T_2). \tag{19}$$

We also have

$$\left|(\ln t)^{1-\alpha}\Psi(t,T_1)\right|$$

$$\begin{aligned}
&= \left|(\ln t)^{1-\alpha} \int_a^{T_1} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{[r(s) + f_2(s,x(s)) - f_1(s,x(s))]}{s}\,ds\right| \\
&\le \int_a^{T_1} \left(\frac{\ln t - \ln s}{\ln t}\right)^{\alpha-1} \frac{|r(s) + f_2(s,x(s)) - f_1(s,x(s))|}{s}\,ds \\
&\le \int_a^{T_1} \left(\frac{\ln T_2 - \ln s}{\ln T_2}\right)^{\alpha-1} \frac{|r(s) + f_2(s,x(s)) - f_1(s,x(s))|}{s}\,ds \\
&:= c_2(T_1, T_2).
\end{aligned} \tag{20}$$

Then from equation (18) we get

$$(\ln t)^{1-\alpha} \int_{T_1}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{r(s)}{s}\,ds \ge -\big[c_1(T_1) + c_2(T_1,T_2)\big],$$

hence

$$\liminf_{t\to\infty} (\ln t)^{1-\alpha} \int_{T_1}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{r(s)}{s}\,ds \ge -\big[c_1(T_1) + c_2(T_1,T_2)\big] > -\infty,$$

which contradicts condition (15).

Case (2): Let $\alpha > 1$. Then $n \ge 2$. Since the function $h_2(t) = (\ln t - \ln a)^{1-j}$ is decreasing and $(\frac{\ln t - \ln a}{\ln t})^{\alpha-1} \le 1$ for $\alpha > 1$, we get, for $t \ge T_2$,

$$\begin{aligned}
\big|(\ln t)^{1-\alpha}\Phi(t)\big| &= \left|(\ln t)^{1-\alpha}\Gamma(\alpha)\sum_{j=1}^{n} \frac{b_j(\ln \frac{t}{a})^{\alpha-j}}{\Gamma(\alpha-j+1)}\right| \\
&\le \Gamma(\alpha)\left(\frac{\ln t - \ln a}{\ln t}\right)^{\alpha-1} \sum_{j=1}^{n} |b_j| \frac{(\ln t - \ln a)^{1-j}}{\Gamma(\alpha-j+1)} \\
&\le \Gamma(\alpha)\sum_{j=1}^{n} |b_j| \frac{(\ln t - \ln a)^{1-j}}{\Gamma(\alpha-j+1)} \\
&\le \Gamma(\alpha)\sum_{j=1}^{n} |b_j| \frac{(\ln T_2 - \ln a)^{1-j}}{\Gamma(\alpha-j+1)} := c_3(T_2).
\end{aligned} \tag{21}$$

We also have

$$\begin{aligned}
\big|(\ln t)^{1-\alpha}\Psi(t,T_1)\big| &= \left|(\ln t)^{1-\alpha} \int_a^{T_1} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{[r(s) + f_2(s,x(s)) - f_1(s,x(s))]}{s}\,ds\right| \\
&\le \int_a^{T_1} \left(\frac{\ln t - \ln s}{\ln t}\right)^{\alpha-1} \frac{|r(s) + f_2(s,x(s)) - f_1(s,x(s))|}{s}\,ds \\
&\le \int_a^{T_1} \frac{|r(s) + f_2(s,x(s)) - f_1(s,x(s))|}{s}\,ds \\
&:= c_4(T_1).
\end{aligned} \tag{22}$$

So, we conclude that

$$(\ln t)^{1-\alpha} \int_{T_1}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{r(s)}{s}\,ds \ge -\big[c_3(T_2) + c_4(T_1)\big],$$

hence

$$\liminf_{t\to\infty}(\ln t)^{1-\alpha}\int_{T_1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds\geq-\bigl[c_3(T_2)+c_4(T_1)\bigr]>-\infty,$$

which contradicts condition (15).

Therefore, we conclude that $x(t)$ is oscillatory. In case $x(t)$ is eventually negative, similar arguments lead to a contradiction with condition (16). □

Theorem 3.3 *Let conditions* (10) *and* (11) *hold with* $\beta>\gamma$. *If*

$$\liminf_{t\to\infty}(\ln t)^{1-\alpha}\int_{T}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+H(s)]}{s}\,ds=-\infty \tag{23}$$

and

$$\limsup_{t\to\infty}(\ln t)^{1-\alpha}\int_{T}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)-H(s)]}{s}\,ds=\infty \tag{24}$$

for every sufficiently large T, *where*

$$H(s)=\frac{\beta-\gamma}{\gamma}\bigl[p_1(s)\bigr]^{\frac{\gamma}{\gamma-\beta}}\left[\frac{\gamma p_2(s)}{\beta}\right]^{\frac{\beta}{\beta-\gamma}}, \tag{25}$$

then every solution of (1) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of equation (1), say, $x(t)>0$ for $t\geq T_1>a$. Let $s\geq T_1$. Using conditions (10) and (11), we get

$$f_2(s,x)-f_1(s,x)\leq p_2(s)x^{\gamma}(s)-p_1(s)x^{\beta}(s).$$

Let $X=x^{\gamma}(s)$, $Y=\frac{\gamma p_2(s)}{\beta p_1(s)}$, $u=\frac{\beta}{\gamma}$, and $v=\frac{\beta}{\beta-\gamma}$, then from part (i) of Lemma 2.1 we get

$$\begin{aligned}p_2(s)x^{\gamma}(s)-p_1(s)x^{\beta}(s)&=\frac{\beta p_1(s)}{\gamma}\left[x^{\gamma}(s)\frac{\gamma p_2(s)}{\beta p_1(s)}-\frac{\gamma}{\beta}\bigl(x^{\gamma}(s)\bigr)^{\frac{\beta}{\gamma}}\right]\\&=\frac{\beta p_1(s)}{\gamma}\left[XY-\frac{1}{u}X^{u}\right]\leq\frac{\beta p_1(s)}{\gamma}\frac{1}{v}Y^{v}=H(s),\end{aligned} \tag{26}$$

where H is defined by (25). Then from equation (9) we obtain

$$\begin{aligned}\Gamma(\alpha)x(t)&=\Phi(t)+\Psi(t,T_1)+\int_{T_1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+f_2(s,x(s))-f_1(s,x(s))]}{s}\,ds\\&\leq\Phi(t)+\Psi(t,T_1)+\int_{T_1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+p_2(s)x^{\gamma}(s)-p_1(s)x^{\beta}(s)]}{s}\,ds\\&\leq\Phi(t)+\Psi(t,T_1)+\int_{T_1}^{t}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+H(s)]}{s}\,ds.\end{aligned} \tag{27}$$

The rest of the proof is the same as that of Theorem 3.2 and hence is omitted. □

Theorem 3.4 *Let $\alpha \geq 1$ and suppose that* (10) *and* (12) *hold with $\beta < \gamma$. If*

$$\limsup_{t\to\infty}(\ln t)^{1-\alpha}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+H(s)]}{s}\,ds=\infty \tag{28}$$

and

$$\liminf_{t\to\infty}(\ln t)^{1-\alpha}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)-H(s)]}{s}\,ds=-\infty \tag{29}$$

for every sufficiently large T, where H is defined by (25)*, then every bounded solution of* (1) *is oscillatory.*

Proof Let $x(t)$ be a bounded nonoscillatory solution of equation (1). Then there exist constants M_1 and M_2 such that

$$M_1 \leq x(t) \leq M_2 \quad \text{for } t \geq a. \tag{30}$$

Assume that x is a bounded eventually positive solution of (1). Then there exists $T_1 > a$ such that $x(t) > 0$ for $t \geq T_1 > a$. Using conditions (10) and (12), we get $f_2(s,x) - f_1(s,x) \geq p_2(s)x^{\gamma}(s) - p_1(s)x^{\beta}(s)$. Using (ii) of Lemma 2.1, and similar to the proof of (26), we find

$$p_2(s)x^{\gamma}(s) - p_1(s)x^{\beta}(s) \geq H(s) \quad \text{for } s \geq T_1.$$

From (9) and similar to (27), we obtain

$$\Gamma(\alpha)x(t) \geq \Phi(t) + \Psi(t,T_1) + \int_{T_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+H(s)]}{s}\,ds.$$

Multiplying by $(\ln t)^{1-\alpha}$, we get

$$\begin{aligned}(\ln t)^{1-\alpha}\Gamma(\alpha)x(t) &\geq (\ln t)^{1-\alpha}\Phi(t) + (\ln t)^{1-\alpha}\Psi(t,T_1)\\ &\quad + (\ln t)^{1-\alpha}\int_{T_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+H(s)]}{s}\,ds.\end{aligned} \tag{31}$$

Take $T_2 > T_1$. We consider two cases as follows.

Case (1): Let $\alpha = 1$. Then (19) and (20) are still correct. Hence, from (31) and using (30), we find that

$$\begin{aligned}M_2\Gamma(\alpha) \geq (\ln t)^{1-\alpha}\Gamma(\alpha)x(t) &\geq -c_1(T_2) - c_2(T_1,T_2)\\ &\quad + (\ln t)^{1-\alpha}\int_{T_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+H(s)]}{s}\,ds\end{aligned}$$

for $t \geq T_2$. Thus, we get

$$\limsup_{t\to\infty}(\ln t)^{1-\alpha}\int_{T_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+H(s)]}{s}\,ds \leq c_1(T_2) + c_2(T_1,T_2) + M_2\Gamma(\alpha) < \infty,$$

which contradicts condition (28).

Case (2): Let $\alpha > 1$. Then (21) and (22) are still true. Hence, from (31) and using (30), we find that

$$\begin{aligned} M_2\Gamma(\alpha)(\ln t)^{1-\alpha} &\geq -c_3(T_2) - c_4(T_1) \\ &\quad + (\ln t)^{1-\alpha} \int_{T_1}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{[r(s) + H(s)]}{s}\, ds \end{aligned}$$

for $t \geq T_2$. Since $\lim_{t\to\infty}(\ln t)^{1-\alpha} = 0$ for $\alpha > 1$, we conclude that

$$\limsup_{t\to\infty}(\ln t)^{1-\alpha} \int_{T_1}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{[r(s) + H(s)]}{s}\, ds \leq c_3(T_2) + c_4(T_1) < \infty,$$

which contradicts condition (28). Therefore, we conclude that $x(t)$ is oscillatory. In case $x(t)$ is eventually bounded negative, similar arguments lead to a contradiction with condition (29). □

4 Oscillation of Hadamard fractional differential equations in the frame of Caputo

In this section, we study the oscillation of the Hadamard fractional differential equations in the Caputo setting of the form

$$\begin{cases} {}^{C}\mathcal{D}_a^{\alpha}x(t) + f_1(t,x) = r(t) + f_2(t,x), & t > a \\ \delta^k x(a) = b_k \quad (k = 0, 1, \ldots, n-1), \end{cases} \tag{32}$$

where $n = \lceil\alpha\rceil$ and ${}^{C}\mathcal{D}_a^{\alpha}$ is defined by (6).

Lemma 4.1 ([17]) *Let $y \in AC_\delta^n[a,b]$ or $C_\delta^n[a,b]$ and $\alpha \in \mathbb{C}$. Then*

$$\mathcal{I}_a^{\alpha}\big({}^{C}\mathcal{D}_a^{\alpha}\big)y(x)) = y(x) - \sum_{k=0}^{n-1} \frac{\delta^k y(a)}{k!} \left(\ln \frac{x}{a}\right)^{k}. \tag{33}$$

Using Lemma 4.1, the solution representation of (32) can be written as

$$x(t) = \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \left(\ln \frac{t}{a}\right)^{k} + \mathcal{I}_a^{\alpha} F(t,x), \tag{34}$$

where $F(t,x) = r(t) + f_2(t,x) - f_1(t,x)$.

Define

$$\chi(t) = \Gamma(\alpha) \sum_{k=0}^{n-1} \frac{\delta^k x(a)}{k!} \left(\ln \frac{t}{a}\right)^{k}. \tag{35}$$

Theorem 4.2 *Let $f_2 = 0$ in (32) and condition (10) hold. If*

$$\liminf_{t\to\infty}(\ln t)^{1-n} \int_{T}^{t} \left(\ln \frac{t}{s}\right)^{\alpha-1} \frac{r(s)}{s}\, ds = -\infty \tag{36}$$

and

$$\limsup_{t\to\infty}(\ln t)^{1-n}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds=\infty \tag{37}$$

for every sufficiently large T, then every solution of (32) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of equation (32) with $f_2 = 0$. Suppose that $T_1 > a$ is large enough so that $x(t) > 0$ for $t \geq T_1$. Hence (10) implies that $f_1(t,x) > 0$ for $t \geq T_1$. Using (2), we get from (34)

$$\begin{aligned}\Gamma(\alpha)x(t) &= \Gamma(\alpha)\sum_{k=0}^{n-1}\frac{\delta^k x(a)}{k!}\left(\ln\frac{t}{a}\right)^k \\ &\quad+\int_a^{T_1}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{F(s,x(s))}{s}\,ds \\ &\quad+\int_{T_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)-f_1(s,x(s))]}{s}\,ds \\ &\leq \Phi(t)+\Psi(t,T_1)+\int_{T_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds,\end{aligned} \tag{38}$$

where χ and Ψ are defined in (35) and (14), respectively.

Multiplying (38) by $(\ln t)^{1-n}$, we get

$$\begin{aligned}0 &< (\ln t)^{1-n}\Gamma(\alpha)x(t) \leq (\ln t)^{1-n}\chi(t)+(\ln t)^{1-n}\Psi(t,T_1) \\ &\quad+(\ln t)^{1-n}\int_{T_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds.\end{aligned} \tag{39}$$

Take $T_2 > T_1$. We consider two cases as follows.

Case (1): Let $0 < \alpha \leq 1$. Then $n = 1$ and $(\ln t)^{1-n}\chi(t) = \Gamma(\alpha)$.

Since $h_3(t) = (\ln\frac{t}{s})^{\alpha-1}$ is decreasing for $t > s$ and $t > T_2$, we get

$$\begin{aligned}\left|(\ln t)^{1-n}\Psi(t,T_1)\right| &= \left|\int_a^{T_1}\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+f_2(s,x(s))-f_1(s,x(s))]}{s}\,ds\right| \\ &\leq \int_a^{T_1}\left(\ln\frac{T_2}{s}\right)^{\alpha-1}\frac{|r(s)+f_2(s,x(s))-f_1(s,x(s))|}{s}\,ds \\ &:= c_5(T_1,T_2).\end{aligned}$$

Then, from equation (39) and for $t \geq T_2$, we get

$$(\ln t)^{1-n}\int_{T_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds \geq -\left[\Gamma(\alpha)+c_5(T_1,T_2)\right],$$

hence

$$\liminf_{t\to\infty}(\ln t)^{1-n}\int_{T_1}^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds \geq -\left[\Gamma(\alpha)+c_5(T_1,T_2)\right] > -\infty,$$

which contradicts condition (36).

Case (2): Let $\alpha > 1$. Then $n \geq 2$. Also, $(\frac{\ln t - \ln a}{\ln t})^{n-1} < 1$ for $n \geq 2$. The function $h_4(t) = (\ln \frac{t}{a})^{k-n+1}$ is decreasing for $k < n-1$. Thus, for $t \geq T_2$, we have

$$\begin{aligned}
\left|(\ln t)^{1-n}\chi(t)\right| &= \left|(\ln t)^{1-n}\Gamma(\alpha)\sum_{k=0}^{n-1}\frac{\delta^k x(a)}{k!}\left(\ln\frac{t}{a}\right)^k\right| \\
&= \left|\left(\frac{\ln t - \ln a}{\ln t}\right)^{n-1}\Gamma(\alpha)\sum_{k=0}^{n-1}\frac{\delta^k x(a)}{k!}\left(\ln\frac{t}{a}\right)^{k-n+1}\right| \\
&\leq \Gamma(\alpha)\sum_{k=0}^{n-1}\frac{|\delta^k x(a)|}{k!}\left(\ln\frac{t}{a}\right)^{k-n+1} \\
&\leq \Gamma(\alpha)\sum_{k=0}^{n-1}\frac{|\delta^k x(a)|}{k!}\left(\ln\frac{T_2}{a}\right)^{k-n+1} := c_6(T_2).
\end{aligned} \tag{40}$$

Also, since $(\frac{\ln t - \ln s}{\ln t})^{n-1} < 1$ for $n \geq 2$ and similar to (22), we get

$$\left|(\ln t)^{1-\alpha}\Psi(t, T_1)\right| \leq c_4(T_1).$$

Using the last inequality and (40), from (39) we get a contradiction with condition (36). Therefore, we conclude that $x(t)$ is oscillatory. In case $x(t)$ is eventually negative, similar arguments lead to a contradiction with condition (37). □

We state the following two theorems without proof.

Theorem 4.3 *Let conditions* (10) *and* (11) *hold with* $\beta > \gamma$. *If*

$$\liminf_{t\to\infty}(\ln t)^{1-n}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+H(s)]}{s}\,ds = -\infty \tag{41}$$

and

$$\limsup_{t\to\infty}(\ln t)^{1-n}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)-H(s)]}{s}\,ds = \infty \tag{42}$$

for every sufficiently large T, *where* H *is defined by* (25), *then every solution of* (32) *is oscillatory.*

Theorem 4.4 *Let* $\alpha \geq 1$ *and suppose that* (10) *and* (12) *hold with* $\beta < \gamma$. *If*

$$\limsup_{t\to\infty}(\ln t)^{1-n}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)+H(s)]}{s}\,ds = \infty \tag{43}$$

and

$$\liminf_{t\to\infty}(\ln t)^{1-n}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{[r(s)-H(s)]}{s}\,ds = -\infty \tag{44}$$

for every sufficiently large T, *where* H *is defined by* (25), *then every bounded solution of* (32) *is oscillatory.*

5 Examples

In this section, we construct examples to illustrate the effectiveness of our theoretical results.

Example 5.1 Consider the following Hadamard fractional differential equation in the Riemann setting:

$$\begin{cases} \mathcal{D}_a^{\alpha} x(t) + x^5(t)\ln(t+e) \\ \quad = \frac{2}{\Gamma(3-\alpha)}(\ln\frac{x}{a})^{2-\alpha} + [(\ln\frac{x}{a})^{10} - (\ln\frac{x}{a})^{\frac{2}{3}}]\ln(t+e) + x^{\frac{1}{3}}(t)\ln(t+e), \\ \lim_{t\to a^+} \mathcal{D}_a^{\alpha-j} x(t) = 0 \quad (j=1,2), 1<\alpha<2, \end{cases} \tag{45}$$

where $n = 2$, $f_1(t,x) = x^5(t)\ln(t+e)$, $f_2(t,x) = x^{\frac{1}{3}}(t)\ln(t+e)$, and $r(t) = \frac{2}{\Gamma(3-\alpha)}(\ln\frac{x}{a})^{2-\alpha} + [(\ln\frac{x}{a})^{10} - (\ln\frac{x}{a})^{\frac{2}{3}}]\ln(t+e)$. It is easy to verify that conditions (10) and (11) are satisfied for $\beta = 5$, $\gamma = \frac{1}{3}$, and $p_1(t) = p_2(t) = \ln(t+e)$. However, we show in the following that condition (23) does not hold. For every sufficiently large $T \ge 1$ and all $t \ge T$, we have $r(t) > 0$. Calculating $H(s)$ as defined by (25), we find that $H(s) = 14(15)^{-\frac{15}{14}}\ln(s+e) \ge 0.77$. Then, using (property 2.1) for $\beta = 1$ and the fact that $\mathcal{D}^{\alpha} = \mathcal{I}^{-\alpha}$, we get

$$\begin{aligned} &\liminf_{t\to\infty} (\ln t)^{1-\alpha} \int_T^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{[r(s)+H(s)]}{s}\,ds \\ &\quad \ge \liminf_{t\to\infty} (\ln t)^{1-\alpha} \int_T^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{H(s)}{s}\,ds \\ &\quad \ge \liminf_{t\to\infty} 0.77 (\ln t)^{1-\alpha} \int_T^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \left(\ln\frac{s}{a}\right)^0 \frac{ds}{s} \\ &\quad = \liminf_{t\to\infty} 0.77 (\ln t)^{1-\alpha} \Gamma(\alpha) \left(\mathcal{I}_a^{\alpha} \left(\ln\frac{s}{a}\right)^0\right)(t) \\ &\quad = \liminf_{t\to\infty} 0.77 \frac{\ln t}{\alpha} \left(1 - \frac{\ln a}{\ln t}\right)^{\alpha} = \infty. \end{aligned}$$

However, one can easily verify that $x(t) = (\ln\frac{t}{a})^2$ is a nonoscillatory solution of (45). The initial condition is also satisfied because

$$\mathcal{D}_a^{\alpha-j} \left(\ln\frac{t}{a}\right)^2 = \mathcal{I}_a^{j-\alpha} \left(\ln\frac{t}{a}\right)^2 = \frac{2}{\Gamma(3+j-\alpha)} \left(\ln\frac{t}{a}\right)^{2+j-\alpha}, \quad j = 1,2.$$

Example 5.2 Consider the Hadamard fractional differential equation

$$\begin{cases} \mathcal{D}_a^{\alpha} x(t) + x^3(t) = \sin t, \\ \lim_{t\to a^+} \mathcal{D}_a^{\alpha-1} x(t) = 0, \quad 0<\alpha<1, \end{cases} \tag{46}$$

where $f_1(t,x) = x^3(t)$, $r(t) = \sin t$, and $f_2(t,x) = 0$. Then condition (10) holds. Furthermore, one can easily check that

$$\liminf_{t\to\infty} (\ln t)^{1-\alpha} \int_T^t \left(\ln\frac{t}{s}\right)^{\alpha-1} \frac{\sin s}{s}\,ds = -\infty$$

and

$$\limsup_{t\to\infty}(\ln t)^{1-\alpha}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{\sin s}{s}\,ds=\infty.$$

This shows that conditions (15) and (16) of Theorem 3.2 hold. Hence, every solution of (46) is oscillatory.

Example 5.3 Consider the following Hadamard fractional differential equation of Caputo type:

$$\begin{cases} {}^{C}\mathcal{D}_a^{\alpha}x(t)+e^tx^3(t)=\frac{1}{\Gamma(2-\alpha)}(\ln\frac{t}{a})^{1-\alpha}+(\ln\frac{t}{a})^3e^t, \\ x(a)=0, \qquad \delta x(a)=1, \quad 1<\alpha<2, \end{cases} \tag{47}$$

where $n=2$, $f_1(t,x)=e^tx^3(t)$, $r(t)=\frac{1}{\Gamma(2-\alpha)}(\ln\frac{t}{a})^{1-\alpha}+(\ln\frac{t}{a})^3e^t$, and $f_2(t,x)=0$. Then condition (10) is satisfied. Since $r(s)>0$ and using (2), we get

$$\begin{aligned}
&\liminf_{t\to\infty}(\ln t)^{1-n}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\frac{r(s)}{s}\,ds\\
&\quad\geq\liminf_{t\to\infty}(\ln t)^{1-n}\int_T^t\left(\ln\frac{t}{s}\right)^{\alpha-1}\left(\ln\frac{s}{a}\right)^{0}\frac{ds}{s}\\
&\quad=\liminf_{t\to\infty}(\ln t)^{1-m}\Gamma(\alpha)\left(\mathcal{I}_a^{\alpha}\left(\ln\frac{s}{a}\right)^{0}\right)(t)\\
&\quad=\liminf_{t\to\infty}(\ln t)^{1-n}\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)}\left(\ln\frac{t}{a}\right)^{\alpha}\\
&\quad=\liminf_{t\to\infty}\frac{\Gamma(\alpha)}{\Gamma(1+\alpha)}\left(\ln\frac{t}{a}\right)^{\alpha-n+1}\left(1-\frac{\ln a}{\ln t}\right)^{n-1}\\
&\quad=\infty,
\end{aligned}$$

which means that condition (36) does not hold. However, it is forward to check that $x(t)=\ln\frac{t}{a}$ is a nonoscillatory solution of (47).

6 Conclusion

In this article, the oscillation theory for Hadamard fractional differential equations was studied. Sufficient conditions for the oscillation of solutions of Hadamard fractional differential equations in the Riemann setting (1) were given in three theorems in Sect. 3. The main approach is based on applying Young's inequality which will help us in obtaining sharper conditions. The oscillation for the Hadamard fractional differential equations in the Caputo setting has been investigated as well. Numerical examples are presented to demonstrate the effectiveness of the obtained results. We finally remark that dual oscillation properties for fractional systems in the frame of right Hadamard fractional derivatives given in (5) can be concluded from the left cases through applying the Q-operator. It would be of interest, in future works, to study the oscillation of other types of fractional systems in the frame of fractional derivatives with more generalized kernels or nonsingular kernels [24].

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