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Positive solutions of the periodic problems for quasilinear difference equation with sign-changing weight

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Abstract

We show the existence of positive solutions of the periodic problem of the quasilinear difference equation

$$\begin{cases} -\nabla[\phi(\Delta u_k)] + q_k u_k = \lambda g_k f(u_k), & k \in \mathbb{T}, \\ u_0 = u_T, \qquad u_1 = u_{T+1}, \end{cases}$$

where $\mathbb{T} = \{1, 2, \ldots, T\}$ with integer $T \geq 2$, $\phi(s) = s/\sqrt{1-s^2}$, $\boldsymbol{q} = (q_1, \ldots, q_T) \in \mathbb{R}^T$, $q_k \geq 0$ for all $k \in \mathbb{T}$ and $q_{k_0} > 0$ for some $k_0 \in \mathbb{T}$, $\boldsymbol{g} = (g_1, \ldots, g_T) \in \mathbb{R}^T$ changes the sign on $\mathbb{T}$, f is a continuous function, and $\lambda \in \mathbb{R}$ is a parameter. The proofs of the main results are based upon bifurcation techniques.

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1 Introduction

In this paper, we study the existence of positive solutions for boundary value problems of the type

$$\begin{cases} -\nabla[\phi(\Delta u_k)] + q_k u_k = \lambda g_k f(u_k), & k \in \mathbb{T}, \\ u_0 = u_T, \qquad u_1 = u_{T+1}, \end{cases} \tag{1.1}$$

where $\mathbb{T} = \{1, 2, \ldots, T\}$ with integer $T \geq 2$, $\lambda \in \mathbb{R}$ is a parameter, $\phi : (-1, 1) \to \mathbb{R}$ is given by

$$\phi(y) = y/\sqrt{1-y^2},$$

$\boldsymbol{q} = (q_1, \ldots, q_T) \in \mathbb{R}^T$, $q_k \geq 0$ for all $k \in \mathbb{T}$ and $q_{k_0} > 0$ for some $k_0 \in \mathbb{T}$, $f : [0, \infty) \to [0, \infty)$ is a continuous function with $f(s) > 0$ for $s > 0$, and $\boldsymbol{g} = (g_1, \ldots, g_T) \in \mathbb{R}^T$ satisfies the following assumption:

(A1) $\boldsymbol{g}$ changes sign on $\mathbb{T}$, that is, there exists a proper subset $\mathbb{T}^+$ of $\mathbb{T}$ such that $g_k > 0$ for $k \in \mathbb{T}^+$ and $g_k < 0$ for $k \in \mathbb{T} \backslash \mathbb{T}^+$.

Let i be the number of elements in $\mathbb{T}^+$. Then $T-i$ is the number of elements in $\mathbb{T}\backslash\mathbb{T}^+$. Let $\boldsymbol{u} \in \mathbb{R}^p$ be a vector $\boldsymbol{u} = (u_1,\dots,u_p)$. Define $\boldsymbol{u} \geq \boldsymbol{0}$ as $u_k \geq 0$ for $k = 1,\dots,p$; $\boldsymbol{u} > \boldsymbol{0}$ is defined as $\boldsymbol{u} \geq \boldsymbol{0}$ and $\boldsymbol{u} \neq \boldsymbol{0}$, that is, $u_k \geq 0$ for $k = 1,\dots,p$ and $u_{k_0} > 0$ for some $1 \leq k_0 \leq p$; $\boldsymbol{u} \succ \boldsymbol{0}$ is defined as $u_k > 0$ for $k = 1,\dots,p$.

For $\boldsymbol{u} \in \mathbb{R}^p$, set $|\boldsymbol{u}|_\infty = \max_{1\leq k\leq p} |u_k|$. For any $\boldsymbol{u} \in \mathbb{R}^p$, where $p \geq 4$ is a fixed integer, we define

$$\Delta\boldsymbol{u} = (\Delta u_1,\dots,\Delta u_{p-1}) \in \mathbb{R}^{p-1}$$

as follows:

$$\Delta u_k = u_{k+1} - u_k \quad (1 \leq k \leq p-1).$$

If $|\Delta\boldsymbol{u}|_\infty = \max_{1\leq k\leq p-1} |\Delta u_k| < 1$, then we define

$$\nabla\big[\phi(\Delta\boldsymbol{u})\big] = \big(\nabla\big[\phi(\Delta u_2)\big],\dots,\nabla\big[\phi(\Delta u_{p-1})\big]\big) \in \mathbb{R}^{p-2}$$

as follows:

$$\nabla\big[\phi(\Delta u_k)\big] = \phi(\Delta u_k) - \phi(\Delta u_{k-1}) \quad (2 \leq k \leq p-1).$$

A solution of problem (1.1) is a vector $\boldsymbol{u} = (u_0,\dots,u_{T+1}) \in \mathbb{R}^{T+2}$ satisfying (1.1) and such that $|\Delta\boldsymbol{u}|_\infty < 1$. A nontrivial solution of problem (1.1) is a solution of problem (1.1) such that $\boldsymbol{u} \neq \boldsymbol{0}$. A positive solution of problem (1.1) is a solution of problem (1.1) such that $\boldsymbol{u} > \boldsymbol{0}$. Further, it is said to be strictly positive if $\boldsymbol{u} \succ \boldsymbol{0}$.

The existence and multiplicity of solutions of

$$-\nabla\big[\phi(\Delta u_k)\big] = f_k(u_k, \Delta u_k), \quad k \in \mathbb{T}, \tag{1.2}$$

subject to diverse boundary conditions have been investigated by several authors; we refer the reader to [4–6] and the references therein. For example, Bereanu and Mawhin [4] proved some existence results of solutions for the periodic boundary value problem of Eq. (1.2) when the right-hand member $\boldsymbol{f} = (f_1,\dots,f_T)$ only satisfies some sign conditions. However, to the best of our knowledge, very little is known about the existence of positive solutions for the quasilinear periodic boundary value problem (1.1). Maybe the main reason is that the spectrum of the corresponding linear eigenvalue problem

$$\begin{cases} -\nabla(\Delta u_k) + q_k u_k = \lambda g_k u_k, & k \in \mathbb{T}, \\ u_0 = u_T, \qquad u_1 = u_{T+1}, \end{cases} \tag{1.3}$$

is incomplete when $\boldsymbol{g}$ changes its sign on $\mathbb{T}$.

It is worth pointing out that only partial information is known to the spectrum of the linear eigenvalue problem (1.3); see Gao and Ma [10] and Ji and Yang [11]. More precisely, from the results in [10] and [11] it follows that (1.3) has T real eigenvalues, including

i positive eigenvalues $\lambda_1^+,\dots,\lambda_i^+$ and $T-i$ negative eigenvalues $\lambda_1^-,\dots,\lambda_{T-i}^-$. From [10] it follows that these eigenvalues and the eigenvalues of the linear eigenvalue problem

$$\begin{cases} -\nabla[\phi(\Delta u_k)] + q_k u_k = \lambda g_k f(u_k), & k \in \mathbb{T}, \\ \Delta u_0 = 0, \qquad \Delta u_T = 0, \end{cases} \tag{1.4}$$

satisfy some order relation. Nevertheless, since in our study, we use bifurcation techniques, it may have some interest to determine the sign of the eigenfunctions corresponding to the eigenvalues.

However, [10] and [11] do not contain the sign of the eigenfunctions of (1.3), even of the eigenfunctions corresponding to the simple eigenvalues λ_1^+ and λ_1^-.

Recently, Ma et al. [16] used the infimum of *Rayleigh quotient* to characterize the principal eigenvalues of (1.3), proved the existence of the principal eigenvalues, and determined the sign of the corresponding eigenfunctions.

Motivated by papers [4, 10, 11, 16], we are concerned with the global structure of the positive solution set of (1.1) applying the spectrum structure of (1.3) under one of the conditions:

(H1) $f_0 \in (0,\infty)$;
(H2) $f_0 = 0$;
(H3) $f_0 = \infty$.

Here $f_0 := \lim_{s\to 0^+} \frac{f(s)}{s}$. Our main tools are the well-known Rabinowitz global bifurcation theorem [18] and the Crandall–Rabinowitz local bifurcation theorem [9, 12].

The rest of the paper is organized as follows. In Sect. 2, we state and prove some preliminary results. Section 3 is devoted to establish the existence result of positive solutions of problem (1.1) by the well-known Rabinowitz bifurcation theory. Finally, in Sect. 4, we give some further results in the cases that either $f_0 = 0$ or $f_0 = \infty$. Finally, we give an example to illustrate our main results.

For other results concerning the problem associated with prescribed mean curvature equations in the Minkowski space, we refer the reader to [2, 3, 7, 8, 15, 19].

2 Preliminary results

We first recall a fundamental result concerning the linear eigenvalue problem (1.3). To do this, let us make the following assumption:

(A2) $\boldsymbol{q} = (q_1,\dots,q_T) \in \mathbb{R}^T$, $q_k \geq 0$ for all $k \in \mathbb{T}$, and $q_{k_0} > 0$ for some $k_0 \in \mathbb{T}$.

We have the following:

Lemma 2.1 ([16, Theorem 2.1]) *Let* (A1) *and* (A2) *hold. Then problem* (1.3) *has exactly two principal eigenvalues* λ_1^- *and* λ_1^+ *such that*

(1) $\lambda_1^- < 0 < \lambda_1^+$;
(2) *the algebraic multiplicity of* λ_1^- *and* λ_1^+ *is* 1;
(3) *the eigenfunctions* $\boldsymbol{\varphi}_1^-$ *and* $\boldsymbol{\varphi}_1^+$ *corresponding to the eigenvalues* λ_1^- *and* λ_1^+ *are of one sign*.

Lemma 2.2 *Let*

$$h(y,z) = \begin{cases} \frac{\sqrt{1-y^2}\sqrt{1-z^2}[\sqrt{1-y^2}+\sqrt{1-z^2}]}{\sqrt{1-z^2}\sqrt{1-y^2}+1+zy} & \text{if } |y| < 1 \text{ and } |z| < 1, \\ 0 \quad \text{if } |y| \geq 1, \text{ or } |z| \geq 1, \text{ or } |y| \geq 1 \text{ and } |z| \geq 1. \end{cases} \tag{2.1}$$

Then

$$h(y,z) < 2 \tag{2.2}$$

and

$$\lim_{(y,z)\to(0,0)} \frac{h(y,z)-1}{\max\{|y|,|z|\}} = 0. \tag{2.3}$$

Proof It is easy to check that if $|y| < 1$ and $|z| < 1$, then

$$\begin{aligned}\frac{\sqrt{1-y^2}\sqrt{1-z^2}[\sqrt{1-y^2}+\sqrt{1-z^2}]}{\sqrt{1-z^2}\sqrt{1-y^2}+1+zy} &< \frac{\sqrt{1-y^2}\sqrt{1-z^2}[\sqrt{1-y^2}+\sqrt{1-z^2}]}{\sqrt{1-z^2}\sqrt{1-y^2}} \\ &= \sqrt{1-y^2}+\sqrt{1-z^2} \\ &\leq 2.\end{aligned}$$

On the other hand, since

$$\sqrt{1-x^2} = 1-x^2+o(x^2) \quad \text{as } x \to 0,$$

it follows from (2.1) that

$$\begin{aligned} h(y,z)-1 &= \frac{(1-y^2+o(y^2))(1-z^2+o(z^2))[(1-y^2+o(y^2))+(1-z^2+o(z^2))]}{(1-z^2+o(z^2))(1-y^2+o(y^2))+1+zy} - 1 \\ &= \frac{(1-y^2+o(y^2))(1-z^2+o(z^2))[(1-y^2+o(y^2))+(1-z^2+o(z^2))]}{2-z^2-y^2+zy+o(y^2)+o(z^2)} - 1 \\ &= \frac{2-3y^2-3z^2+o(y^2)+o(z^2)}{2-z^2-y^2+zy+o(y^2)+o(z^2)} - 1 \\ &= \frac{2-3y^2-3z^2-[2-z^2-y^2+zy]+o(y^2)+o(z^2)}{2-z^2-y^2+zy+o(y^2)+o(z^2)} \\ &= \frac{-2y^2-2z^2-zy+o(y^2)+o(z^2)}{2-z^2-y^2+zy+o(y^2)+o(z^2)}.\end{aligned}$$

Thus

$$\lim_{(y,z)\to(0,0)} \frac{h(y,z)-1}{\max\{|y|,|z|\}} = 0. \qquad \square$$

Lemma 2.3 *Let $\boldsymbol{u} = (u_1,\dots,u_T) \in \mathbb{R}^T$ be such that $|\Delta \boldsymbol{u}|_\infty < 1$. Then for any $k \in \mathbb{T}$, we have*

$$\begin{aligned}&\nabla\left(\frac{\Delta u_k}{\sqrt{1-(\Delta u_k)^2}}\right) \\ &\quad = \nabla(\Delta u_k)\left[\frac{\sqrt{1-(\Delta u_{k-1})^2}\sqrt{1-(\Delta u_k)^2}+1+\Delta u_{k-1}\Delta u_k}{\sqrt{1-(\Delta u_k)^2}\sqrt{1-(\Delta u_{k-1})^2}[\sqrt{1-(\Delta u_k)^2}+\sqrt{1-(\Delta u_{k-1})^2}]}\right].\end{aligned}$$

Proof Since $|\Delta \boldsymbol{u}|_\infty < 1$, by a simple calculation we get

$$\nabla\left(\frac{u_k}{v_k}\right) = \frac{\nabla u_k v_{k-1} - u_{k-1}\nabla v_k}{v_k v_{k-1}} \tag{2.4}$$

and

$$\nabla\left(\sqrt{1-(\Delta u_k)^2}\right) = \frac{-\nabla(\Delta u_k)(\Delta u_k + \Delta u_{k-1})}{\sqrt{1-(\Delta u_k)^2} + \sqrt{1-(\Delta u_{k-1})^2}}, \tag{2.5}$$

and, accordingly,

$$\begin{aligned}
&\nabla\left(\frac{\Delta u_k}{\sqrt{1-(\Delta u_k)^2}}\right) \\
&\quad = \frac{\nabla(\Delta u_k)\sqrt{1-(\Delta u_{k-1})^2} - \Delta u_{k-1}\nabla(\sqrt{1-(\Delta u_k)^2})}{\sqrt{1-(\Delta u_k)^2}\sqrt{1-(\Delta u_{k-1})^2}} \\
&\quad = \frac{\nabla(\Delta u_k)\sqrt{1-(\Delta u_{k-1})^2} - \Delta u_{k-1}\frac{-\nabla(\Delta u_k)(\Delta u_k+\Delta u_{k-1})}{\sqrt{1-(\Delta u_k)^2}+\sqrt{1-(\Delta u_{k-1})^2}}}{\sqrt{1-(\Delta u_k)^2}\sqrt{1-(\Delta u_{k-1})^2}} \\
&\quad = \nabla(\Delta u_k)\left[\frac{\sqrt{1-(\Delta u_{k-1})^2} + \Delta u_{k-1}\frac{(\Delta u_k+\Delta u_{k-1})}{\sqrt{1-(\Delta u_k)^2}+\sqrt{1-(\Delta u_{k-1})^2}}}{\sqrt{1-(\Delta u_k)^2}\sqrt{1-(\Delta u_{k-1})^2}}\right] \\
&\quad = \nabla(\Delta u_k)\left[\frac{\sqrt{1-(\Delta u_{k-1})^2}(\sqrt{1-(\Delta u_k)^2} + \sqrt{1-(\Delta u_{k-1})^2}) + \Delta u_{k-1}(\Delta u_k + \Delta u_{k-1})}{\sqrt{1-(\Delta u_k)^2}\sqrt{1-(\Delta u_{k-1})^2}[\sqrt{1-(\Delta u_k)^2} + \sqrt{1-(\Delta u_{k-1})^2}]}\right] \\
&\quad = \nabla(\Delta u_k)\left[\frac{\sqrt{1-(\Delta u_{k-1})^2}\sqrt{1-(\Delta u_k)^2} + 1 + \Delta u_{k-1}\Delta u_k}{\sqrt{1-(\Delta u_k)^2}\sqrt{1-(\Delta u_{k-1})^2}[\sqrt{1-(\Delta u_k)^2} + \sqrt{1-(\Delta u_{k-1})^2}]}\right]. \qquad \square
\end{aligned}$$

Let us introduce the vector space

$$\mathcal{D} = \left\{\boldsymbol{u} = (u_0, u_1, \ldots, u_T, u_{T+1}) \in \mathbb{R}^{T+2} : u_0 = u_T, u_1 = u_{T+1}, |\Delta \boldsymbol{u}|_\infty < 1\right\} \tag{2.6}$$

endowed with the usual norm $|\cdot|_\infty$.

To apply the global bifurcation theorem, we extend f to the whole $\mathbb{R}$ by setting

$$\tilde{f}(s) = \begin{cases} f(s) & \text{if } s \geq 0, \\ -f(-s) & \text{if } s < 0. \end{cases} \tag{2.7}$$

Clearly, $\tilde{f}$ is an odd continuous function. Note that since we consider the positive solutions, (1.1) is equivalent to the same problem with f replaced by $\tilde{f}$. We further use the same symbol f to denote the function f and the modified function $\tilde{f}$.

From Lemmas 2.2 and 2.3 we have the following:

Lemma 2.4 *$\boldsymbol{u} \in \mathcal{D}$ is a positive solution of problem* (1.1) *if and only if $\boldsymbol{u} \in \mathcal{D}$ is a positive solution of the following problem*:

$$\begin{cases} -\nabla(\Delta u_k) + q_k u_k = \lambda g_k f(u_k) h(\Delta u_k, \Delta u_{k-1}) - q_k u_k h(\Delta u_k, \Delta u_{k-1}) + q_k u_k, \\ \quad k \in \mathbb{T}, \\ u_0 = u_T, \qquad u_1 = u_{T+1}. \end{cases} \tag{2.8}$$

3 Existence of positive solutions

In this section, we consider the existence of positive solutions of the discrete nonlinear problem (1.1). To do this, we further assume that

(H4) $f_\infty := \lim_{s\to+\infty} \frac{f(s)}{s} = 0$.

Our main result is stated as follows.

Theorem 3.1 *Let* (A1), (A2), (H1), *and* (H4) *hold. Then there exist* $\lambda_* \in (0, \frac{\lambda_1^+}{f_0}]$ *and* $\lambda_{**} \in [\frac{\lambda_1^-}{f_0}, 0)$ *such that* (1.1) *has no positive solution if* $\lambda \in (\lambda_{**}, 0) \cup (0, \lambda_*)$ *and has at least one strictly positive solution if* $\lambda \in (-\infty, \frac{\lambda_1^-}{f_0}) \cup (\frac{\lambda_1^+}{f_0}, +\infty)$.

Proof We only prove the case $\lambda > 0$. The case $\lambda < 0$ is similar. We divide the proof into three steps.

Step 1. *A bifurcation result.*

Denote by $\mathcal{K} : \mathcal{D} \to \mathcal{D}$ the operator that sends any vector $\boldsymbol{v} \in \mathcal{D}$ onto the unique solution $\boldsymbol{w} \in \mathcal{D}$ of

$$\begin{cases} -\nabla(\Delta w_k) + q_k w_k = v_k, & k \in \mathbb{T}, \\ w_0 = w_T, \qquad w_1 = w_{T+1}. \end{cases} \tag{3.1}$$

Similarly, $\mathcal{L} : \mathcal{D} \to \mathcal{D}$ denotes the operator that sends any vector $\boldsymbol{z} \in \mathcal{D}$ onto the unique solution $\boldsymbol{r} \in \mathcal{D}$ of

$$\begin{cases} -\nabla(\Delta r_k) + q_k r_k = g_k z_k, & k \in \mathbb{T}, \\ r_0 = r_T, \qquad r_1 = r_{T+1}. \end{cases}$$

By [1, Lemma 3.1] both $\mathcal{K}$ and $\mathcal{L}$ are completely continuous, and (1.3) is equivalent to

$$\boldsymbol{u} = \lambda \mathcal{L}(\boldsymbol{u}), \tag{3.2}$$

so that the eigenvalues of (1.3) are precisely the characteristic values of $\mathcal{L}$.

Since $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and satisfies (H1), for any $s \in \mathbb{R}$, we can write

$$f(s) = \big(f_0 + l(s)\big)s,$$

where $l : \mathbb{R} \to \mathbb{R}$ is continuous, and

$$\lim_{s\to 0} l(s) = 0. \tag{3.3}$$

Let $b(y,z) = h(y,z) - 1$ for $(y,z) \in \mathbb{R}^2$. Then we have from Lemma 2.2 that

$$\lim_{|\Delta\boldsymbol{u}|_\infty \to 0} \frac{b(\Delta u_k, \Delta u_{k-1})}{|\Delta\boldsymbol{u}|_\infty} = 0, \quad k \in \mathbb{T}. \tag{3.4}$$

Let us consider

$$\begin{aligned} u_k = {} & \lambda f_0 \mathcal{L}(u_k) + \lambda \mathcal{L}\big(\big(f_0 + l(u_k)\big) b(\Delta u_k, \Delta u_{k-1}) + l(u_k)\big) \\ & - \mathcal{K}\big(q_k b(\Delta u_k, \Delta u_{k-1}) u_k\big), \quad k \in \mathbb{T}, \end{aligned} \tag{3.5}$$

as a bifurcation problem from the trivial solution axis.

Define the operator $\mathcal{H}:\mathbb{R}\times\mathcal{D}\to\mathcal{D}$ by

$$\mathcal{H}(\lambda,u_k)=\lambda\mathcal{L}\big(\big(f_0+l(u_k)\big)b(\Delta u_k,\Delta u_{k-1})+l(u_k)\big)-\mathcal{K}\big(q_k b(\Delta u_k,\Delta u_{k-1})u_k\big),\quad k\in\mathbb{T}.$$

Obviously, $\mathcal{H}$ is completely continuous.

Since for all $k\in\mathbb{T}$, $\Delta u_k=u_{k+1}-u_k$, as $|\boldsymbol{u}|_\infty\to 0$, we have that

$$|\Delta\boldsymbol{u}|_\infty=\max_{1\le k\le T-1}|\Delta u_k|\le\max_{1\le k\le T-1}\big(|u_{k+1}|+|u_k|\big)\to 0.\tag{3.6}$$

This fact, together with (3.3) and (3.4), yields that

$$\lim_{|\boldsymbol{u}|_\infty\to 0}\frac{\mathcal{H}(\lambda,\boldsymbol{u})}{|\boldsymbol{u}|_\infty}=0\tag{3.7}$$

uniformly on bounded λ intervals.

Note that, for any $\lambda>0$, the couple $(\lambda,\boldsymbol{u})\in\mathbb{R}\times\mathcal{D}$ with $\boldsymbol{u}>\mathbf{0}$ is a solution of the equation

$$\boldsymbol{u}=\lambda f_0\mathcal{L}(\boldsymbol{u})+\mathcal{H}(\lambda,\boldsymbol{u})\tag{3.8}$$

if and only if it is a positive solution of (1.1).

Denote

$$\mathcal{S}:=\overline{\big\{(\lambda,\boldsymbol{u}):\lambda>0,(\lambda,\boldsymbol{u})\text{ is a nontrivial solution of (3.8)}\big\}}^{\mathbb{R}^+\times\mathcal{D}}.$$

Then from Theorem 1.3 of [18] we have that there exists a connected component $\mathcal{C}$ in $\mathcal{S}$ such that $(\frac{\lambda_1^+}{f_0},\mathbf{0})\in\mathcal{C}$ and at least one of the following properties holds:

(i) $\mathcal{C}$ is unbounded in $\mathbb{R}\times\mathcal{D}$;

(ii) $\mathcal{C}$ contains $(\frac{\hat\lambda}{f_0},\mathbf{0})$, where $\hat\lambda$ is another characteristic value of $\mathcal{L}$.

Step 2. *Property* (i) *is valid.*

In what follows, we prove several properties, which will eventually lead to the conclusion.

Claim 1. *If* $(\frac{\tilde\lambda}{f_0},\mathbf{0})\in\mathcal{S}$, *then* $\tilde\lambda$ *is a characteristic value of* $\mathcal{L}$.

Suppose that there exists a sequence $\{(\lambda^{[n]},\boldsymbol{u}^{[n]})\}$ of nontrivial solutions of (3.8) converging to $(\frac{\tilde\lambda}{f_0},\mathbf{0})$ in $\mathbb{R}\times\mathcal{D}$.

Let $\boldsymbol{v}^{[n]}=\frac{\boldsymbol{u}^{[n]}}{|\boldsymbol{u}^{[n]}|_\infty}$ for all n. Then from (3.8) we have

$$\boldsymbol{v}^{[n]}=\lambda^{[n]}f_0\mathcal{L}\big(\boldsymbol{v}^{[n]}\big)+\frac{\mathcal{H}(\lambda^{[n]},\boldsymbol{u}^{[n]})}{|\boldsymbol{u}^{[n]}|_\infty}.\tag{3.9}$$

As $\mathcal{L}:\mathcal{D}\to\mathcal{D}$ is completely continuous and $\{\boldsymbol{v}^{[n]}\}$ is bounded in $\mathcal{D}$, after taking a subsequence if necessary, there exists $\boldsymbol{w}\in\mathcal{D}$ such that

$$\lim_{n\to+\infty}\mathcal{L}\big(\boldsymbol{v}^{[n]}\big)=\boldsymbol{w}.$$

From (3.7) and (3.9) it follows that

$$\lim_{n\to+\infty}\boldsymbol{v}^{[n]}=\tilde\lambda\boldsymbol{w}$$

and, accordingly,

$$\boldsymbol{w} = \tilde{\lambda}\mathcal{L}(\boldsymbol{w})$$

and

$$|\tilde{\lambda}\boldsymbol{w}|_\infty = 1.$$

In particular, $\boldsymbol{w} \neq \mathbf{0}$. Hence $\tilde{\lambda}$ is a characteristic value of $\mathcal{L}$.

Denote the nonnegative cone in $\mathcal{D}$ by P, that is,

$$P = \{\boldsymbol{u} \in \mathcal{D} : \boldsymbol{u} \geq \mathbf{0}\},$$

its interior by $\operatorname{int} P$, and its boundary by ∂P.

Claim 2. *Let $U \subset \mathbb{R} \times \mathcal{D}$ be a neighborhood of $(\frac{\lambda_1^+}{f_0}, \mathbf{0})$. Then, for any $(\lambda, \boldsymbol{u}) \in \mathcal{C} \cap U$, either $(\lambda, \boldsymbol{u}) = (\frac{\lambda_1^+}{f_0}, \mathbf{0})$, or $\boldsymbol{u} \in \operatorname{int} P$, or $-\boldsymbol{u} \in \operatorname{int} P$.*

This is a direct consequence of the well-known Crandall–Rabinowitz local bifurcation theorem [9, 12].

Claim 3. *Assume that $(\lambda, \boldsymbol{u}) \in \mathcal{C}$ and $\boldsymbol{u} \in \partial P$. Suppose further that $(\lambda, \boldsymbol{u})$ is the limit of a sequence $\{(\lambda^{[n]}, \boldsymbol{u}^{[n]})\}$ in $\mathcal{C}$ with $\boldsymbol{u}^{[n]} > \mathbf{0}$ for all n. Then $(\lambda, \boldsymbol{u}) = (\frac{\lambda_1^+}{f_0}, \mathbf{0})$.*

We first show that $\boldsymbol{u} = \mathbf{0}$. Assume on the contrary that $\boldsymbol{u} > \mathbf{0}$. From Lemma 2.2 it follows that there exists $c > 0$ such that

$$\lambda g_k\big(f_0 + l(u_k)\big)h(\Delta u_k, \Delta u_{k-1}) - q_k b(\Delta u_k, \Delta u_{k-1}) + c \geq 1, \quad k \in \mathbb{T}. \tag{3.10}$$

Therefore

$$-\nabla(\Delta u_k) + q_k u_k + c u_k = \big[\lambda g_k\big(f_0 + l(u_k)\big)h(\Delta u_k, \Delta u_{k-1}) - q_k b(\Delta u_k, \Delta u_{k-1}) + c\big]u_k > 0$$

for all $k \in \mathbb{T}$.

From this fact, since the Green's function $G(k,s)$ of the linear problem

$$-\nabla(\Delta u_k) + q_k u_k + c u_k = 0, \quad u_0 = u_T, u_1 = u_{T+1}, \tag{3.11}$$

satisfies $G(k,s) > 0$ for all $1 \leq k, s \leq T$ (see [17, Thms. 2.1 and 2.2]), we have $u_k > 0$ for all $k \in \mathbb{T}$, which contradicts with $\boldsymbol{u} \in \partial P$. Hence $\boldsymbol{u} = \mathbf{0}$.

By Claim 1, λ is a characteristic value of $\mathcal{L}$. Let $\boldsymbol{v}^{[n]} = \frac{\boldsymbol{u}^{[n]}}{|\boldsymbol{u}^{[n]}|_\infty}$. Arguing as in the proof of Claim 1, we conclude that, possibly passing to a subsequence,

$$\lim_{n\to\infty} \mathcal{L}\big(\boldsymbol{v}^{[n]}\big) = \boldsymbol{w}$$

in $\mathcal{D}$, where $\boldsymbol{w}$ is an eigenfunction of (1.3) associated with λ. Since $\boldsymbol{w} > \mathbf{0}$, we have $\lambda = \frac{\lambda_1^+}{f_0}$.

Claim 4. *For all $(\lambda, \boldsymbol{u}) \in \mathcal{C}$, either $(\lambda, \boldsymbol{u}) = (\frac{\lambda_1^+}{f_0}, \mathbf{0})$, or $\boldsymbol{u} \in \operatorname{int} P$, or $-\boldsymbol{u} \in \operatorname{int} P$.*

Let

$$\mathcal{E} = \left\{(\lambda, \boldsymbol{u}) \in \mathcal{C} : (\lambda, \boldsymbol{u}) \neq \left(\frac{\lambda_1^+}{f_0}, \mathbf{0}\right), \boldsymbol{u} \notin \operatorname{int} P, -\boldsymbol{u} \notin \operatorname{int} P\right\}. \tag{3.12}$$

From Claim 2 we have

$$\mathcal{E} = \big\{(\lambda, \boldsymbol{u}) \in (\mathcal{C} \setminus \mathcal{U})\big\},$$

and therefore $\mathcal{E} \subset \mathcal{C}$ is a closed subset of $\mathcal{C}$.

Let us prove that $\mathcal{E}$ is still open in $\mathcal{C}$. Suppose this is not the case. Then there exist $(\lambda, \boldsymbol{u}) \in \mathcal{E}$ and a sequence $\{(\lambda^{[n]}, \boldsymbol{u}^{[n]})\}$ in $\mathcal{C}/\mathcal{E}$ converging to $(\lambda, \boldsymbol{u})$. We may suppose that $\boldsymbol{u}^{[n]} \in \operatorname{int} P$ for all n, and hence by Claim 3 we obtain $(\lambda, \boldsymbol{u}) = (\frac{\lambda_1^+}{f_0}, \mathbf{0})$, contradicting the fact that $(\lambda, \boldsymbol{u}) \in \mathcal{E}$. Since $\mathcal{C}$ is connected and $(\frac{\lambda_1^+}{f_0}, \mathbf{0}) \in \mathcal{C} \setminus \mathcal{E}$, we conclude that $\mathcal{E} = \emptyset$.

By Claim 4 we have that if $(\frac{\hat{\lambda}}{f_0}, \mathbf{0}) \in \mathcal{C}$, then $\hat{\lambda} = \lambda_1^+$. Hence, property (ii) is not valid, whereas property (i) is valid.

Step 3. *The global behavior of* $\mathcal{C}$.

We first show that there exists $\epsilon > 0$ such that $\mathcal{S} \subset [\epsilon, +\infty) \times \mathcal{D}$.

Suppose on the contrary that there exists a sequence $\{(\lambda^{[n]}, \boldsymbol{u}^{[n]})\}$ of nontrivial solutions of (3.8) converging in $\mathbb{R} \times \mathcal{D}$ to some $(0, \boldsymbol{u}) \in \mathbb{R} \times \mathcal{D}$. Arguing as in the proof of Claim 1 (in Step 2) and setting $\boldsymbol{v}^{[n]} = \frac{\boldsymbol{u}^{[n]}}{|\boldsymbol{u}^{[n]}|_\infty}$, we have

$$\boldsymbol{v}^{[n]} = \lambda^{[n]} \mathcal{L}\big(\boldsymbol{v}^{[n]}\big) + \frac{\mathcal{H}(\lambda^{[n]}, \boldsymbol{u}^{[n]})}{|\boldsymbol{u}^{[n]}|_\infty}$$

and conclude that, possibly passing to a subsequence, $\lim_{n\to\infty} \boldsymbol{v}^{[n]} = \mathbf{0}$ in $\mathcal{D}$, which contradicts $|\boldsymbol{v}^{[n]}|_\infty = 1$.

Next, we will prove that $\operatorname{Proj}_{\mathbb{R}} \mathcal{C} = [\epsilon, +\infty)$, where ϵ is given by the previous fact.

Assume on the contrary that $\sup\{\lambda : (\lambda, \boldsymbol{y}) \in \mathcal{C}\} < \infty$. Then there exists a sequence $\{(\mu^{[n]}, \boldsymbol{u}^{[n]})\} \subset \mathcal{C}$ such that

$$\mu^{[n]} \le M_0, \qquad \big|\boldsymbol{u}^{[n]}\big|_\infty \to \infty, \quad n \to \infty, \tag{3.13}$$

where $M_0 = M_0(n)$ is a positive constant. Combining this fact with

$$u_k^{[n]} \ge \gamma \big|\boldsymbol{u}^{[n]}\big|_\infty \quad \text{for all } k \in \mathbb{T},$$

where $\gamma > 0$ satisfies $G(k,s) \ge \gamma G(s,s)$, we have

$$u_k^{[n]} \to \infty, \quad k \in \mathbb{T}, n \to \infty. \tag{3.14}$$

Since $\{(\mu^{[n]}, \boldsymbol{u}^{[n]})\} \subset \mathcal{C}$, we have

$$\begin{cases} -\nabla(\Delta u_k^{[n]}) + q_k u_k^{[n]} = \mu^{[n]} g_k f(u_k^{[n]}) h(\Delta u_k^{[n]}, \Delta u_{k-1}^{[n]}) - b(\Delta u_k^{[n]}, \Delta u_{k-1}^{[n]}) q_k u_k^{[n]}, \\ \quad k \in \mathbb{T}, \\ u_0^{[n]} = u_T^{[n]}, \qquad u_1^{[n]} = u_{T+1}^{[n]}. \end{cases} \tag{3.15}$$

We divide (3.15) by $|\boldsymbol{u}^{[n]}|_\infty$, and for all n, setting $\boldsymbol{v}_k^{[n]} = \frac{\boldsymbol{u}_k^{[n]}}{|\boldsymbol{u}^{[n]}|_\infty}$, we have

$$\begin{cases} -\nabla(\Delta v_k^{[n]}) + q_k v_k^{[n]} = \mu^{[n]} g_k \frac{f(u_k^{[n]})}{u_k^{[n]}} h(\Delta u_k^{[n]}, \Delta u_{k-1}^{[n]}) v_k^{[n]} - b(\Delta u_k^{[n]}, \Delta u_{k-1}^{[n]}) q_k v_k^{[n]}, \\ \quad k \in \mathbb{T}, \\ v_0^{[n]} = v_T^{[n]}, \qquad v_1^{[n]} = v_{T+1}^{[n]}. \end{cases} \tag{3.16}$$

Since $\{\boldsymbol{v}^{[n]}\}$ is bounded in $\mathcal{D}$, after taking a subsequence and relabeling if necessary, we have that $\boldsymbol{v}^{[n]} \to \overline{\boldsymbol{v}}$ for all $n \in \mathbb{N}$, where $\overline{\boldsymbol{v}} \in \mathcal{D}$ with $|\overline{\boldsymbol{v}}|_\infty = 1$. However, by (3.4), (3.6), (3.13), (3.14), and (H4) we have

$$\overline{\boldsymbol{v}} = \boldsymbol{0},$$

which is a contradiction.

We are now in position of getting the conclusions of the theorem. By Step 3 we infer that (1.1) has at least one nontrivial solution for all $\lambda > \frac{\lambda_1^+}{f_0}$. From the fact that f, and hence $\mathcal{H}$, is odd with respect to the second variable it follows that $(\lambda, \boldsymbol{u}) \in \mathcal{C}$ if and only if $(\lambda, -\boldsymbol{u}) \in \mathcal{C}$. This fact, together with Claim 4, also implies that at least one of those solutions belongs to $\operatorname{int} P$, that is, (1.1) has at least one strictly positive solution for all $\lambda > \frac{\lambda_1^+}{f_0}$.

Let

$$\Lambda = \left\{\lambda : (\lambda, \boldsymbol{u}) \text{ is a positive solution of (1.1)}, \lambda > 0\right\},$$

and let

$$\lambda_* = \inf \Lambda.$$

From Step 3 we have $\lambda_* > 0$. Therefore we conclude that, for all $\lambda \in (0, \lambda_*)$, (1.1) has no positive solution. □

4 Some further results

In this section, we deal with the cases $f_0 = 0$ and $f_0 = \infty$.

Theorem 4.1 *Let* (A1), (A2), (H2), *and* (H4) *hold. Then there exist* $0 < \lambda_* \le \lambda^*$ *and* $\lambda_{**} \le \lambda^{**} < 0$ *such that* (1.1) *has at least two strictly positive solutions if* $\lambda \in (-\infty, \lambda_{**}) \cup (\lambda^*, +\infty)$ *and has no positive solution if* $\lambda \in (\lambda^{**}, 0) \cup (0, \lambda_*)$.

Proof (*Sketched*) We will use a similar argument as in [13] and [14] to get the desired results. For each $n \in \mathbb{N}$, let us define function $f^{[n]} : [0, \infty) \to \mathbb{R}$ by

$$f^{[n]}(s) = \begin{cases} f(s) & \text{if } s \in (\frac{1}{n}, \infty), \\ n f(\frac{1}{n}) s & \text{if } s \in [0, \frac{1}{n}]. \end{cases} \tag{4.1}$$

Then for each $n \in \mathbb{N}$, $f^{[n]}$ is a continuous function such that

$$\limsup_{n\to\infty} \bigl[f^{[n]}(s) - f(s)\bigr] = 0 \quad \text{uniformly for } s \in [0, \infty) \tag{4.2}$$

and

$$\left(f^{[n]}\right)_0 = \lim_{s\to 0}\frac{f^{[n]}(s)}{s} = nf\left(\frac{1}{n}\right). \tag{4.3}$$

To apply the global bifurcation theorem, we extend $f^{[n]}$ to an odd function $\tilde{f}^{[n]}:\mathbb{R}\to\mathbb{R}$ by

$$\tilde{f}^{[n]}(s) = \begin{cases} f^{[n]}(s) & \text{if } s \geq 0, \\ -f^{[n]}(-s) & \text{if } s < 0. \end{cases}$$

By the same method, to prove Theorem 3.1, with obvious changes, we get that for each $n \in \mathbb{N}$, the positive solution set of the auxiliary problem

$$\begin{cases} -\nabla[\phi(\Delta u_k)] + q_k u_k = \lambda g_k \tilde{f}^{[n]}(u_k), & k \in \mathbb{T}, \\ u_0 = u_T, \qquad u_1 = u_{T+1}, \end{cases} \tag{4.4}$$

possesses a connected component $\mathcal{C}^{[n]}$ that joins $(\frac{\lambda_1^+}{nf(1/n)}, \mathbf{0})$ with (∞, ∞).

From (H2) we have

$$\lim_{n\to\infty}\left(f^{[n]}\right)_0 = \lim_{n\to\infty} nf\left(\frac{1}{n}\right) = 0 \tag{4.5}$$

and, accordingly,

$$\lim_{n\to\infty}\frac{\lambda_1^+}{nf(1/n)} = \infty. \tag{4.6}$$

According to [13, Lemma 2.4] and [14, Lemma 2.2], the set $\limsup \mathcal{C}^{[n]}$ contains a connected component of positive solutions that joins $(\infty, \mathbf{0})$ to (∞, ∞).

Therefore there exists $0 < \lambda_* \leq \lambda^*$ such that problem (1.1) has at least two strictly positive solutions for $\lambda > \lambda^*$ and has no positive solution for $\lambda \in (0, \lambda_*)$. Similarly, there exist $\lambda_{**} \leq \lambda^{**} < 0$ such that (1.1) has at least two strictly positive solutions if $\lambda < \lambda_{**}$ and has no positive solution if $\lambda \in (\lambda^{**}, 0)$. □

Theorem 4.2 *Let* (A1), (A2), (H3), *and* (H4) *hold. Then* (1.1) *has at least one strictly positive solution if* $\lambda \in (-\infty, 0) \cup (0, +\infty)$.

Proof (*Sketched*) From (H3) it follows that

$$\lim_{n\to\infty}\left(f^{[n]}\right)_0 = \infty \tag{4.7}$$

and

$$\lim_{n\to\infty}\frac{\lambda_1^+}{nf(1/n)} = 0. \tag{4.8}$$

Similarly to the proof of Theorem 4.1, we get that, for each $n \in \mathbb{N}$, the positive solution set of the auxiliary problem (4.4) possesses a connected component $\mathcal{C}^{[n]}$ that joins

$(0, \mathbf{0})$ with (∞, ∞). According to [13, Lemma 2.4] and [14, Lemma 2.2], the set $\limsup \mathcal{C}^{[n]}$ contains an unbounded connected component $\mathcal{C}$,

$$(0, \mathbf{0}) \in \mathcal{C} \subset \limsup_{n\to\infty} \mathcal{C}^{[n]}, \tag{4.9}$$

which joins $(0, \mathbf{0})$ with (∞, ∞).

Therefore problem (1.1) has at least one strictly positive solution for all $\lambda > 0$. With obvious changes, we may obtain the existence of (1.1) in the case $\lambda < 0$. $\square$

Example 4.1 Let us consider the periodic problem of the quasilinear difference equation

$$\begin{cases} -\nabla[\phi(\Delta u_k)] + \frac{1}{4}u_k = \lambda f(k, u_k), & k \in \mathbb{T} := \{1, 2, 3\}, \\ u_0 = u_3, \qquad u_1 = u_4, \end{cases} \tag{4.10}$$

with

$$f(k, s) = \alpha^*(s) + \gamma(k, s)s,$$

where

$$\alpha^*(s) = \begin{cases} 0, & s = 0, \\ \frac{1}{4}s, & s \in (0, 1], \\ \frac{1}{4}, & s \in (1, \infty), \end{cases}$$

and

$$\gamma(k, s) = \begin{cases} \frac{1}{2}, & k \in \{1, 2\}, s \in [0, 1), \\ \frac{1}{2s}, & k \in \{1, 2\}, s \in [1, \infty), \\ -\frac{1}{2}, & k = 3, s \in [0, 1], \\ \frac{1}{4}s - \frac{3}{4}, & k = 3, s \in (1, 2), \\ -\frac{1}{2s}, & k = 3, s \in [2, \infty). \end{cases}$$

Clearly,

$$\lim_{s\to\infty} \frac{f(k, s)}{s} = 0, \qquad \lim_{s\to 0} \frac{f(k, s)}{s} = g_k \quad \text{for all } k \in \mathbb{T},$$

and

$$g_k = \begin{cases} \frac{3}{4}, & k \in \{1, 2\}, \\ -\frac{1}{4}, & k = 3. \end{cases}$$

By a direct calculation we have

$$\lambda_1^- \doteq -7.883, \qquad \lambda_1^+ \doteq 0.550.$$

Therefore by Theorem 3.1 there exist $\lambda_* \in (0, 0.550]$ and $\lambda_{**} \in [-7.883, 0)$ such that (4.10) has no positive solution if $\lambda \in (0, \lambda_*) \cup (\lambda_{**}, 0)$ and has at least one strictly positive solution if $\lambda > 0.550$ or $\lambda < -7.883$.

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MX and RM completed the main study and drafted the paper together. ZH checked the proofs and verified the calculation. All authors read and approved the final manuscript.

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