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Nontrivial solutions for boundary value problems of a fourth order difference equation with sign-changing nonlinearity

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Abstract

In this paper, using the topological degree theory, we establish two existence theorems for nontrivial solutions for boundary value problems of a fourth order difference equation with a sign-changing nonlinearity.

Keywords: Difference equations boundary value problems; Sign-changing nonlinearity; Nontrivial solutions; Topological degree theory

1 Introduction

For $a, b \in \mathbb{Z}$, let $\mathbb{T}_a^b = \{a, a+1, a+2, \ldots, b\}$ with $a < b$. In this paper we consider the existence of nontrivial solutions for boundary value problems of the following fourth order difference equation with a sign-changing nonlinearity

$$\begin{cases} \Delta^4 u(t-2) = f(t, u(t)), \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{cases} \tag{1.1}$$

where T is an integer with $T \geq 5$, and $f : \mathbb{T}_2^T \times \mathbb{R} \to \mathbb{R}$ is a continuous function with $\mathbb{T}_2^T = \{2, 3, \ldots, T\}$ and $\mathbb{R} = (-\infty, +\infty)$ (it is assumed to be continuous from the topological space $\mathbb{T}_2^T \times \mathbb{R}$ into the topological space $\mathbb{R}$, the topology on $\mathbb{T}_2^T$ being the discrete topology).

Difference equations with discrete boundary value conditions have been widely studied in the literature; see, for example, [1–11] and the references therein. However, as mentioned in [6], very few results are available with sign-changing nonlinearities; see [6–11]. Other related work in this field can be found in [12–45] and the references therein. In [7], C.S. Goodrich used the Krasnosel'skiǐ fixed point theorem to obtain the existence of at least one positive solution to the following discrete fractional semipositone boundary value problem

$$\begin{cases} \Delta^\nu y(t) = \lambda f(t+\nu-1, y(t+\nu-1)), & t \in [0, T] \cap \mathbb{Z}, \\ y(\nu-1) = y(\nu+T) + \sum_{i=1}^N F(t_i, y(t_i)), \end{cases} \tag{1.2}$$

where Δ^ν is the νth fractional difference with $\nu \in (0,1)$, f is continuous, bounded below (i.e., $f + M \geq 0$ for some $M > 0$), and

$$\lim_{y\to+\infty} \frac{f(t,y)}{y} = 0 \quad \text{uniformly for } t \in [\nu-1, \nu+T]_{\mathbb{Z}_{\nu-1}}. \tag{1.3}$$

In [10], J. Xu and D. O'Regan used the fixed point index to obtain the existence of nontrivial solutions for (1.2) with weaker conditions than that of (1.3), and also in [11], J. Xu et al. considered the existence of positive solutions for system (1.2), with adopted convex and concave functions to depict the coupling behavior of nonlinearities. In [40], Y. Cui used the u_0-positive operator to study the uniqueness of solutions for the following nonlinear fractional boundary value problems:

$$\begin{cases} D^p x(t) + p(t)f(t,x(t)) + q(t) = 0, & t \in (0,1), \\ x(0) = x'(0) = 0, & x(1) = 0, \end{cases} \tag{1.4}$$

where D^p is the Riemann–Liouville fractional derivative, and f is a Lipschitz continuous function, with the Lipschitz constant associated with the first eigenvalue for the relevant operator. Using similar methods, the authors in [12, 39, 41] obtained some existence and nonexistence theorems for their problems.

Motivated by the works mentioned above, we consider the existence of nontrivial solutions for (1.1) involving sign-changing nonlinearities. Using the topological degree theory of a completely continuous field, and conditions concerning the first eigenvalue corresponding to the relevant linear problem, two existence theorems are obtained.

2 Preliminaries

For convenience, we let $\mathbb{T}_1^{T+1} = \{1,2,3,\dots,T,T+1\}$, $\mathbb{T}_0^{T+2} = \{0,1,2,3,\dots,T+1,T+2\}$, $\mathbb{T}_2^T = \{2,3,\dots,T\}$. Then we define our space E as the collection of all maps from $\mathbb{T}_0^{T+2}$ to $\mathbb{R}$ equipped with the norm $\|u\| = \max_{j\in\mathbb{T}_0^{T+2}} |u(j)|$. Consequently, E is a Banach space, and we let $P = \{u \in E : u(t) \geq 0, t \in \mathbb{T}_1^{T+1}\}$. Then P is a cone on E. Throughout our paper, we let $B_\rho = \{u \in E : \|u\| < \rho\}$ for $\rho > 0$. Now $\partial B_\rho = \{u \in E : \|u\| = \rho\}$ and $\overline{B}_\rho = \{u \in E : \|u\| \leq \rho\}$.

In what follows, we establish the Green's function for (1.1). As in [3, 4], we transform (1.1) into its equivalent sum equation

$$u(t) = \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j) f\bigl(j,u(j)\bigr), \quad t \in \mathbb{T}_1^{T+1}, \tag{2.1}$$

where

$$H(t,s) = \frac{1}{T} \begin{cases} (t-1)(T+1-s), & 1 \leq t \leq s \leq T, \\ (s-1)(T+1-t), & 2 \leq s \leq t \leq T+1. \end{cases} \tag{2.2}$$

Lemma 2.1 *Green's function H has the following properties*:

(i) $H(t,s) > 0$ *for* $(t,s) \in \mathbb{T}_2^T \times \mathbb{T}_2^T$,

(ii) $\frac{1}{T}H(t,t)H(s,s) \leq H(t,s) \leq H(s,s)$ *for* $(t,s) \in \mathbb{T}_2^T \times \mathbb{T}_1^{T+1}$.

Proof We only need to prove the first inequality of (ii). Indeed, for all $(t,s) \in \mathbb{T}_2^T \times \mathbb{T}_1^{T+1}$, from the definitions of $H(t,s)$ and $H(s,s)$ we have

$$\frac{H(t,s)}{H(s,s)} = \begin{cases} \frac{t-1}{s-1} \geq \frac{t-1}{T} \geq \frac{t-1}{T}\frac{T+1-t}{T} = \frac{1}{T}H(t,t), & 1 \leq t \leq s \leq T, \\ \frac{T+1-t}{T+1-s} \geq \frac{T+1-t}{T} \geq \frac{T+1-t}{T}\frac{t-1}{T} = \frac{1}{T}H(t,t), & 2 \leq s \leq t \leq T+1. \end{cases}$$

Then we have $H(t,s) \geq \frac{1}{T}H(t,t)H(s,s)$ for $(t,s) \in \mathbb{T}_2^T \times \mathbb{T}_1^{T+1}$. This completes the proof. □

We define an operator $A : E \to E$ as follows:

$$(Au)(t) = \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j) f\big(j, u(j)\big), \quad t \in \mathbb{T}_1^{T+1}. \tag{2.3}$$

The existence of solutions for (1.1) is equivalent to that of fixed points of A.

From [4], we know that $\sin \frac{\pi(t-1)}{T} := \varphi_0(t)$, $t \in \mathbb{T}_2^T$ is the eigenfunction related to the eigenvalue $\frac{1}{16}\sin^{-4}\frac{\pi}{2T}$ of the eigenproblem

$$\begin{cases} \Delta^4 u(t-2) = \lambda u(t), & t \in \mathbb{T}_2^T, \\ u(1) = u(T+1) = \Delta^2 u(0) = \Delta^2 u(T) = 0, \end{cases}$$

i.e., the following two equations hold:

$$\sum_{s=2}^{T}\sum_{j=2}^{T} H(t,s)H(s,j) \sin\frac{\pi(j-1)}{T} = \frac{1}{16}\sin^{-4}\frac{\pi}{2T}\sin\frac{\pi(t-1)}{T}, \quad t \in \mathbb{T}_2^T, \tag{2.4}$$

$$\sum_{s=2}^{T}\sum_{t=2}^{T} H(t,s)H(s,j) \sin\frac{\pi(t-1)}{T} = \frac{1}{16}\sin^{-4}\frac{\pi}{2T}\sin\frac{\pi(j-1)}{T}, \quad t \in \mathbb{T}_2^T. \tag{2.5}$$

Lemma 2.2 *Let* $e(t) = \frac{1}{T}H(t,t)$ *and* $P_0 = \{u \in P : u(t) \geq e(t)\|u\|, t \in \mathbb{T}_1^{T+1}\}$. *Then* $L(P) \subset P_0$, *where*

$$(Lu)(t) = \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j) u(j), \quad t \in \mathbb{T}_1^{T+1}. \tag{2.6}$$

This is a direct result from Lemma 2.1(ii), so we omit its proof.

Now, we offer two basic theorems from the topological degree theory; for details we refer the reader to [46].

Lemma 2.3 *Let* E *be a Banach space and* Ω *a bounded open set in* E. *Suppose that* $A : \Omega \to E$ *is a continuous compact operator. If there exists* $u_0 \in E \setminus \{0\}$ *such that*

$$u - Au \neq \mu u_0, \quad \forall u \in \partial\Omega, \mu \geq 0,$$

then the topological degree $\deg(I - A, \Omega, 0) = 0$.

Lemma 2.4 *Let E be a Banach space and Ω a bounded open set in E with $0 \in \Omega$. Suppose that $A : \Omega \to E$ is a continuous compact operator. If*

$$Au \neq \mu u, \quad \forall u \in \partial\Omega, \mu \geq 1,$$

then the topological degree $\deg(I - A, \Omega, 0) = 1$.

3 Nontrivial solutions for (1.1)

Now we present some assumptions for our nonlinearity f.

(H1) There exist two constants $a > 0$, $b > 0$ and a function $k \in C(\mathbb{R}, \mathbb{R}^+)$ such that

$$f(t,u) \geq -a - bk(u), \quad \forall u \in \mathbb{R}, t \in \mathbb{T}_2^T.$$

(H2) $\lim_{|u|\to+\infty} \frac{k(u)}{|u|} = 0$.

(H3) $\liminf_{|u|\to+\infty} \frac{f(t,u)}{|u|} > 16\sin^4\frac{\pi}{2T}$ uniformly on $t \in \mathbb{T}_2^T$,

(H4) $\limsup_{|u|\to 0} \frac{|f(t,u)|}{|u|} < 16\sin^4\frac{\pi}{2T}$ uniformly on $t \in \mathbb{T}_2^T$,

(H5) $\liminf_{u\to 0^+} \frac{f(t,u)}{u} > 16\sin^4\frac{\pi}{2T}$, $\limsup_{u\to 0^-} \frac{f(t,u)}{u} < 16\sin^4\frac{\pi}{2T}$, uniformly on $t \in \mathbb{T}_2^T$,

(H6) $\limsup_{|u|\to+\infty} \frac{|f(t,u)|}{|u|} < 16\sin^4\frac{\pi}{2T}$ uniformly on $t \in \mathbb{T}_2^T$.

Theorem 3.1 *Suppose that* (H1)–(H4) *hold. Then* (1.1) *has at least one nontrivial solution.*

Proof From (H3) there exist $\varepsilon_0 > 0$ and $X_0 > 0$ such that

$$f(t,u) \geq \left(16\sin^4\frac{\pi}{2T} + \varepsilon_0\right)|u|, \quad \forall t \in \mathbb{T}_2^T, |u| > X_0. \tag{3.1}$$

For any given ε with $\varepsilon_0 - b\varepsilon > 0$, and from (H2), there exists $X_1 > X_0$ such that

$$k(u) \leq \varepsilon|u|, \quad \forall |u| > X_1. \tag{3.2}$$

Now since $a > 0$, $b > 0$ and k is a nonnegative function, we have

$$\begin{aligned} f(t,u) &\geq \left(16\sin^4\frac{\pi}{2T} + \varepsilon_0\right)|u| - a - bk(u) \\ &\geq \left(16\sin^4\frac{\pi}{2T} + \varepsilon_0\right)|u| - a - b\varepsilon|u|, \quad \forall |u| > X_1. \end{aligned} \tag{3.3}$$

Now we choose $c_1 = (16\sin^4\frac{\pi}{2T} + \varepsilon_0 - b\varepsilon)X_1 + \max_{t\in\mathbb{T}_2^T, |u|\leq X_1} |f(t,u)|$ and $k^* = \max_{|u|\leq X_1} k(u)$. Then we have

$$\begin{aligned} f(t,u) &\geq \left(16\sin^4\frac{\pi}{2T} + \varepsilon_0 - b\varepsilon\right)|u| - a - c_1 \\ &= \left(16\sin^4\frac{\pi}{2T} + \varepsilon_0 - b\varepsilon\right)|u| - c_2, \quad \forall t \in \mathbb{T}_2^T, u \in \mathbb{R}, \end{aligned} \tag{3.4}$$

where $c_2 = c_1 + a$. Note that ε can be chosen arbitrarily small, and we let

$$R > \max\left\{\frac{(c_2+bk^*)[(\varepsilon_0-b\varepsilon)\sum_{s=2}^T H(s,s)\sum_{j=2}^T H(s,j) + (16\sin^4\frac{\pi}{2T}+\varepsilon_0-b\varepsilon)\sum_{s=2}^T\sum_{j=2}^T H(s,j)]}{\varepsilon_0-b\varepsilon-b\varepsilon[(\varepsilon_0-b\varepsilon)\sum_{s=2}^T H(s,s)\sum_{j=2}^T H(s,j)+(16\sin^4\frac{\pi}{2T}+\varepsilon_0-b\varepsilon)\sum_{s=2}^T\sum_{j=2}^T H(s,j)]},\right.$$
$$\left.\frac{\sum_{s=2}^T H(s,s)\sum_{j=2}^T H(s,j)(c_2+bk^*)}{1-b\varepsilon\sum_{s=2}^T H(s,s)\sum_{j=2}^T H(s,j)},0\right\}.$$

Now we prove that

$$u - Au \neq \mu\varphi_0, \quad \forall u \in \partial B_R, \mu \geq 0. \tag{3.5}$$

From (2.4) and Lemma 2.2, we have $\varphi_0 = 16\sin^4\frac{\pi}{2T}L\varphi_0 \in P_0$. Indeed, if (3.5) isn't true, then there exist $u_0 \in \partial B_R$ and $\mu_0 > 0$ such that

$$u_0 - Au_0 = \mu_0\varphi_0. \tag{3.6}$$

Let $\tilde{u}(t) = \sum_{s=2}^T H(t,s)\sum_{j=2}^T H(s,j)(a + bk(u_0) + c_1)$. Then

$$\begin{aligned}\tilde{u}(t) &\leq \sum_{s=2}^T H(t,s)\sum_{j=2}^T H(s,j)\bigl(c_2 + b\varepsilon|u_0| + bk^*\bigr)\\ &\leq \sum_{s=2}^T H(s,s)\sum_{j=2}^T H(s,j)\bigl(c_2 + b\varepsilon\|u_0\| + bk^*\bigr).\end{aligned}$$

Therefore,

$$\|\tilde{u}\| \leq \sum_{s=2}^T H(s,s)\sum_{j=2}^T H(s,j)\bigl(c_2 + b\varepsilon R + bk^*\bigr). \tag{3.7}$$

Then from $L(P) \subset P_0$, $\varphi_0 \in P_0$, and

$$\begin{aligned}u_0(t) + \tilde{u}(t) &= \tilde{u}(t) + (Au_0)(t) + \mu_0\varphi_0(t)\\ &= \sum_{s=2}^T H(t,s)\sum_{j=2}^T H(s,j)\bigl(f\bigl(j,u_0(j)\bigr) + bk\bigl(u_0(j)\bigr) + a + c_1\bigr) + \mu_0\varphi_0(t),\end{aligned}$$

we have

$$u_0 + \tilde{u} \in P_0.$$

As a result, we obtain

$$\begin{aligned}&(Au_0)(t) + \tilde{u}(t)\\ &\quad= \sum_{s=2}^T H(t,s)\sum_{j=2}^T H(s,j)\bigl(f\bigl(j,u_0(j)\bigr) + bk\bigl(u_0(j)\bigr) + c_2\bigr)\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)\left(\left(16\sin^4 \frac{\pi}{2T} + \varepsilon_0 - b\varepsilon\right)\bigl|u_0(j)\bigr| - c_2 + bk\bigl(u_0(j)\bigr) + c_2\right) \\
&\geq \left(16\sin^4 \frac{\pi}{2T} + \varepsilon_0 - b\varepsilon\right) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)\bigl|u_0(j)\bigr| \\
&\geq \left(16\sin^4 \frac{\pi}{2T} + \varepsilon_0 - b\varepsilon\right) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)u_0(j).
\end{aligned} \tag{3.8}$$

On the other hand, from the definition of L, we get

$$\begin{aligned}
&\left(16\sin^4 \frac{\pi}{2T} + \varepsilon_0 - b\varepsilon\right) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)u_0(j) \\
&\quad = 16\sin^4 \frac{\pi}{2T} \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)\bigl(u_0(j) + \tilde{u}(j)\bigr) \\
&\qquad - 16\sin^4 \frac{\pi}{2T} \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)\tilde{u}(j) \\
&\qquad + (\varepsilon_0 - b\varepsilon) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)u_0(j) \\
&\quad \geq 16\sin^4 \frac{\pi}{2T} \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)\bigl(u_0(j) + \tilde{u}(j)\bigr);
\end{aligned} \tag{3.9}$$

in order to obtain the above inequality, we prove that

$$\begin{aligned}
&-16\sin^4 \frac{\pi}{2T} \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)\tilde{u}(j) \\
&\quad + (\varepsilon_0 - b\varepsilon) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)u_0(j) \geq 0.
\end{aligned} \tag{3.10}$$

Indeed, since $u_0 + \tilde{u} \in P_0$, we have $u_0(t) + \tilde{u}(t) \geq e(t)\|u_0 + \tilde{u}\| \geq e(t)(\|u_0\| - \|\tilde{u}\|)$. Note that $H(t,s)$ vanishes at $t = 1$ and $t = T + 1$, $H(t,s)$ is symmetric on $\mathbb{T}_2^T$, i.e., $H(t,s) = H(s,t)$. Then

$$\begin{aligned}
&(\varepsilon_0 - b\varepsilon) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)\bigl(\tilde{u}(j) + u_0(j)\bigr) \\
&\qquad - \left(16\sin^4 \frac{\pi}{2T} + \varepsilon_0 - b\varepsilon\right) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)\tilde{u}(j) \\
&\quad \geq (\varepsilon_0 - b\varepsilon)\bigl(R - \|\tilde{u}\|\bigr) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)e(j) \\
&\qquad - \left(16\sin^4 \frac{\pi}{2T} + \varepsilon_0 - b\varepsilon\right) \sum_{s=2}^{T} H(t,s)
\end{aligned}$$

$$\times \sum_{j=2}^{T} H(s,j)e(j)\left(\sum_{s=2}^{T}\sum_{j=2}^{T} H(s,j)\big(c_2 + b\varepsilon R + bk^*\big)\right)$$
$$\geq 0.$$

Combining (3.8), (3.9) and (3.10), we have

$$\begin{aligned}(Au_0)(t) + \tilde{u}(t) &\geq 16\sin^4\frac{\pi}{2T}\sum_{s=2}^{T} H(t,s)\sum_{j=2}^{T} H(s,j)\big(u_0(j) + \tilde{u}(j)\big)\\ &= 16\sin^4\frac{\pi}{2T}\big(L(u_0 + \tilde{u})\big)(t).\end{aligned} \tag{3.11}$$

Using (3.6) we obtain

$$u_0 + \tilde{u} = Au_0 + \tilde{u} + \mu_0\varphi_0 \geq 16\sin^4\frac{\pi}{2T}L(u_0 + \tilde{u}) + \mu_0\varphi_0 \geq \mu_0\varphi_0. \tag{3.12}$$

Define

$$\mu^* = \sup\{\mu > 0 : u_0 + \tilde{u} \geq \mu\varphi_0\}.$$

Note that $\mu_0 \in \{\mu > 0 : u_0 + \tilde{u} \geq \mu\varphi_0\}$, and then $\mu^* \geq \mu_0$, $u_0 + \tilde{u} \geq \mu^*\varphi_0$. From (2.4) we have

$$16\sin^4\frac{\pi}{2T}L(u_0 + \tilde{u}) \geq \mu^* 16\sin^4\frac{\pi}{2T}L\varphi_0 = \mu^*\varphi_0,$$

and hence

$$u_0 + \tilde{u} \geq 16\sin^4\frac{\pi}{2T}L(u_0 + \tilde{u}) + \mu_0\varphi_0 \geq \big(\mu_0 + \mu^*\big)\varphi_0,$$

which contradicts the definition of μ^*. Therefore, (3.5) holds, and from Lemma 2.3 we obtain

$$\deg(I - A, B_R, 0) = 0. \tag{3.13}$$

On the other hand, from (H4), there exist $\varepsilon_1 \in (0, 16\sin^4\frac{\pi}{2T})$ and $r \in (0, R)$ such that

$$\big|f(t,u)\big| \leq \left(16\sin^4\frac{\pi}{2T} - \varepsilon_1\right)|u|, \quad \forall t \in \mathbb{T}_2^T, |u| < r. \tag{3.14}$$

Now for this r, we show that

$$Au \neq \mu u, \quad u \in \partial B_r, \mu \geq 1. \tag{3.15}$$

Otherwise, there would exist $u_1 \in \partial B_r$, $\mu_1 \geq 1$ such that

$$\begin{aligned}\big|u_1(t)\big| &= \frac{1}{\mu_1}\big|(Au_1)(t)\big| \leq \big|(Au_1)(t)\big|\\ &= \left|\sum_{s=2}^{T} H(t,s)\sum_{j=2}^{T} H(s,j)f\big(j,u_1(j)\big)\right|\end{aligned}$$

$$\le \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j) \big| f\big(j,u_1(j)\big)\big|$$
$$\le \left(16\sin^4\frac{\pi}{2T} - \varepsilon_1\right) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j)\big|u_1(j)\big|.$$

Multiplying both sides of the above inequality by $\sin\frac{\pi(t-1)}{T}$, then summing from 2 to T, and using (2.5), we obtain

$$\begin{aligned}
&\sum_{t=2}^{T}\big|u_1(t)\big|\sin\frac{\pi(t-1)}{T} \\
&\quad\le \left(16\sin^4\frac{\pi}{2T} - \varepsilon_1\right)\sum_{t=2}^{T}\left[\sum_{s=2}^{T} H(t,s)\sum_{j=2}^{T} H(s,j)\big|u_1(j)\big|\right]\sin\frac{\pi(t-1)}{T} \\
&\quad= \frac{16\sin^4\frac{\pi}{2T} - \varepsilon_1}{16\sin^4\frac{\pi}{2T}}\sum_{t=2}^{T}\big|u_1(t)\big|\sin\frac{\pi(t-1)}{T}.
\end{aligned}$$

This implies that $\sum_{t=2}^{T}|u_1(t)|\sin\frac{\pi(t-1)}{T} = 0$, and whence $u_1(t) \equiv 0$, which contradicts $u_1 \in \partial B_r$. Hence, (3.15) holds, and from Lemma 2.4 we obtain

$$\deg(I - A, B_r, 0) = 1. \tag{3.16}$$

This, together with (3.13), implies that

$$\deg(I - A, B_R \setminus \overline{B}_r, 0) = \deg(I - A, B_R, 0) - \deg(I - A, B_r, 0) = -1.$$

Therefore, the operator A has at least one fixed point in $B_R \setminus \overline{B}_r$, and (1.1) has at least one nontrivial solution. This completes the proof. □

Theorem 3.2 *Suppose that* (H5)–(H6) *hold. Then* (1.1) *has at least one nontrivial solution.*

Proof From (H5), there are $\varepsilon_2 \in (0, 16\sin^4\frac{\pi}{2T})$ and $r > 0$ such that

$$f(t,u) \ge \left(16\sin^4\frac{\pi}{2T} + \varepsilon_2\right)u, \quad \forall u \in [0,r], t \in \mathbb{T}_2^T,$$

and

$$f(t,u) \ge \left(16\sin^4\frac{\pi}{2T} - \varepsilon_2\right)u, \quad \forall u \in [-r,0], t \in \mathbb{T}_2^T.$$

The above two inequalities enable us to obtain

$$f(t,u) \ge \left(16\sin^4\frac{\pi}{2T} + \varepsilon_2\right)u, \quad \forall u \in [-r,r], t \in \mathbb{T}_2^T, \tag{3.17}$$
$$f(t,u) \ge \left(16\sin^4\frac{\pi}{2T} - \varepsilon_2\right)u, \quad \forall u \in [-r,r], t \in \mathbb{T}_2^T. \tag{3.18}$$

Define a cone P_1 as follows:

$$P_1 = \left\{ u \in P : \sum_{t=2}^{T} u(t) \sin \frac{\pi(t-1)}{T} \geq \delta \|u\| \right\},$$

where $\delta = \sum_{t=2}^{T} e(t) \sin \frac{\pi(t-1)}{T}$. Then we claim

$$L(P) \subset P_1. \tag{3.19}$$

Indeed, for $u \in P$, from Lemma 2.1 we have

$$\begin{aligned}\sum_{t=2}^{T} (Lu)(t) \sin \frac{\pi(t-1)}{T} &= \sum_{t=2}^{T} \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j) u(j) \sin \frac{\pi(t-1)}{T} \\ &\geq \sum_{t=2}^{T} \sum_{s=2}^{T} e(t) H(\tau, s) \sum_{j=2}^{T} H(s,j) u(j) \sin \frac{\pi(t-1)}{T} \\ &= \delta (Lu)(\tau), \quad \forall \tau \in \mathbb{T}_2^T,\end{aligned}$$

and thus

$$\sum_{t=2}^{T} (Lu)(t) \sin \frac{\pi(t-1)}{T} \geq \delta \|Lu\|.$$

Moreover, $\varphi_0 \in P_1$ since $\varphi_0 = 16 \sin^4 \frac{\pi}{2T} L\varphi_0 \in P_1$. Now we claim that

$$u - Au \neq \mu \varphi_0, \quad \forall u \in \partial B_r, \mu \geq 0. \tag{3.20}$$

If the claim is false, then there exist $u_2 \in \partial B_r$ and $\mu_2 \geq 0$ such that

$$u_2 - Au_2 = \mu_2 \varphi_0. \tag{3.21}$$

From (3.17) we have $Au_2 \geq (16 \sin^4 \frac{\pi}{2T} + \varepsilon_2) Lu_2$ and so $u_2 \geq (16 \sin^4 \frac{\pi}{2T} + \varepsilon_2) Lu_2$, i.e.,

$$u_2(t) \geq \left(16 \sin^4 \frac{\pi}{2T} + \varepsilon_2 \right) \sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j) u_2(j).$$

Multiplying both sides of the above inequality by $\sin \frac{\pi(t-1)}{T}$, then summing from 2 to T, and using (2.5), we obtain

$$\begin{aligned}&\sum_{t=2}^{T} u_2(t) \sin \frac{\pi(t-1)}{T} \\ &\quad \geq \left(16 \sin^4 \frac{\pi}{2T} + \varepsilon_2 \right) \sum_{t=2}^{T} \left[\sum_{s=2}^{T} H(t,s) \sum_{j=2}^{T} H(s,j) u_2(j) \right] \sin \frac{\pi(t-1)}{T} \\ &\quad = \frac{16 \sin^4 \frac{\pi}{2T} + \varepsilon_2}{16 \sin^4 \frac{\pi}{2T}} \sum_{t=2}^{T} u_2(t) \sin \frac{\pi(t-1)}{T},\end{aligned}$$

which implies that

$$\sum_{t=2}^{T} u_2(t)\sin\frac{\pi(t-1)}{T} \le 0. \tag{3.22}$$

On the other hand, from (3.21) we have

$$\begin{aligned} & u_2(t) - \left(16\sin^4\frac{\pi}{2T} - \varepsilon_2\right)(Lu_2)(t) \\ &\quad = (Au_2)(t) - \left(16\sin^4\frac{\pi}{2T} - \varepsilon_2\right)(Lu_2)(t) + \mu_2\varphi_0(t) \\ &\quad = \sum_{s=2}^{T} H(t,s)\sum_{j=2}^{T} H(s,j)\left[f\big(j,u_2(j)\big) - \left(16\sin^4\frac{\pi}{2T} - \varepsilon_2\right)u_2(j)\right] + \mu_2\varphi_0(t). \end{aligned}$$

Then (3.18), (3.19) and $\varphi_0 \in P_1$ enable us to find $u_2 - (16\sin^4\frac{\pi}{2T} - \varepsilon_2)Lu_2 \in P_1$, and thus

$$\begin{aligned} & \left\| u_2 - \left(16\sin^4\frac{\pi}{2T} - \varepsilon_2\right)Lu_2 \right\| \\ &\quad \le \frac{1}{\delta}\sum_{t=2}^{T}\left[u_2(t) - \left(16\sin^4\frac{\pi}{2T} - \varepsilon_2\right)(Lu_2)(t)\right]\sin\frac{\pi(t-1)}{T} \\ &\quad = \frac{\varepsilon_2}{\delta 16\sin^4\frac{\pi}{2T}}\sum_{t=2}^{T} u_2(t)\sin\frac{\pi(t-1)}{T} \le 0. \end{aligned}$$

Note that $(16\sin^4\frac{\pi}{2T} - \varepsilon_2)r(L) < 1$, where $r(L)$ is the spectral radius of L. Hence, we have $u_2 = 0$, contradicting $u_2 \in \partial B_r$. This implies that (3.20) holds, and from Lemma 2.3 we have

$$\deg(I - A, B_r, 0) = 0. \tag{3.23}$$

On the other hand, from (H6) there exist $\varepsilon_3 \in (0, 16\sin^4\frac{\pi}{2T})$ and $c_3 > 0$ such that

$$\big|f(t,u)\big| \le \left(16\sin^4\frac{\pi}{2T} - \varepsilon_3\right)|u| + c_3, \quad \forall t \in \mathbb{T}_2^T, u \in \mathbb{R}. \tag{3.24}$$

Let $\mathcal{M} = \{u \in E : u = \lambda Au, \lambda \in [0,1]\}$. Then we prove that $\mathcal{M}$ is bounded in E. If $u \in \mathcal{M}$, then from (3.24) we have

$$\begin{aligned} \big|u(t)\big| = \lambda\big|(Au)(t)\big| &\le \sum_{s=2}^{T} H(t,s)\sum_{j=2}^{T} H(s,j)\big|f\big(j,u(j)\big)\big| \\ &\le \sum_{s=2}^{T} H(t,s)\sum_{j=2}^{T} H(s,j)\left[\left(16\sin^4\frac{\pi}{2T} - \varepsilon_3\right)\big|u(j)\big| + c_3\right]. \end{aligned}$$

Multiplying both sides of the above inequality by $\sin\frac{\pi(t-1)}{T}$, then summing from 2 to T, and using (2.5), we obtain

$$\sum_{t=2}^{T}\big|u(t)\big|\sin\frac{\pi(t-1)}{T} \le \frac{1}{16\sin^4\frac{\pi}{2T}}\sum_{t=2}^{T}\left[\left(16\sin^4\frac{\pi}{2T} - \varepsilon_3\right)\big|u(t)\big| + c_3\right]\sin\frac{\pi(t-1)}{T},$$

and then

$$\sum_{t=2}^{T}\left|u(t)\right|\sin\frac{\pi(t-1)}{T}\le c_3\varepsilon_3^{-1}\sum_{t=2}^{T}\sin\frac{\pi(t-1)}{T}.$$

We know that there is a $t_0\in\mathbb{T}_2^T$ such that $\|u\|=|u(t_0)|$, and thus

$$\left|u(t_0)\right|\sin\frac{\pi(t_0-1)}{T}\le\sum_{t=2}^{T}\left|u(t)\right|\sin\frac{\pi(t-1)}{T}.$$

This implies that

$$\|u\|\le c_3\varepsilon_3^{-1}\sin^{-1}\frac{\pi(t_0-1)}{T}\sum_{t=2}^{T}\sin\frac{\pi(t-1)}{T},$$

proving the boundedness of $\mathcal{M}$. Choose $R>\max\{\sup_{u\in\mathcal{M}}\|u\|,r\}$ (r is defined by (3.17)), then

$$\lambda Au\neq u,\quad u\in\partial B_R,\lambda\in[0,1]. \tag{3.25}$$

Lemma 2.4 implies that

$$\deg(I-A,B_R,0)=1. \tag{3.26}$$

This, together with (3.23), implies that

$$\deg(I-A,B_R\setminus\overline{B}_r,0)=\deg(I-A,B_R,0)-\deg(I-A,B_r,0)=1.$$

Therefore, the operator A has at least one fixed point in $B_R\setminus\overline{B}_r$, and (1.1) has at least one nontrivial solution. This completes the proof. □

Example 3.3 Let $f(t,x)=a|x|-bk(x)$, $k(x)=\ln(|x|+1)$, $x\in\mathbb{R}$, where $a\in(16\sin^4\frac{\pi}{2T},+\infty)$ and $b\in(0,a+16\sin^4\frac{\pi}{2T})$. Then $\lim_{|x|\to+\infty}\frac{k(x)}{|x|}=0$, and $\lim_{|x|\to+\infty}\frac{a|x|-b\ln(|x|+1)}{|x|}=a>16\sin^4\frac{\pi}{2T}$, $\lim_{|x|\to0}\frac{|a|x|-b\ln(|x|+1)|}{|x|}=|a-b|<16\sin^4\frac{\pi}{2T}$. Therefore, (H1)–(H4) hold.

Example 3.4 Let $f(t,x)=\begin{cases}ax+b\sin x, & x\ge0,\\ ax-be^x+b, & x\le0,\end{cases}$ where $a,b>0$ with $a<16\sin^4\frac{\pi}{2T}$, $a+b>16\sin^4\frac{\pi}{2T}$ and $a-b<16\sin^4\frac{\pi}{2T}$. Then $\lim_{x\to0^+}\frac{ax+b\sin x}{x}=a+b$, $\lim_{x\to0^-}\frac{ax-be^x+b}{x}=a-b$, $\lim_{x\to+\infty}|\frac{ax+b\sin x}{x}|=a$, and $\lim_{x\to-\infty}|\frac{ax-be^x+b}{x}|=a$. Therefore, (H5)–(H6) hold.

4 Conclusions

In this paper, we established the existence of nontrivial solutions for the boundary value problems of the fourth order difference equation (1.1) with sign-changing nonlinearity using the topological degree theory. Under some conditions concerning the first eigenvalue corresponding to the relevant linear problem, the results here improve and generalize those obtained in [1–11].

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