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On the solutions of a max-type system of difference equations with period-two parameters

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Abstract

In this paper, we study the following max-type system of difference equations:

$$\begin{cases} x_n = \max\{A_n, \frac{y_{n-1}}{x_{n-2}}\}, \\ y_n = \max\{B_n, \frac{x_{n-1}}{y_{n-2}}\}, \end{cases} \quad n \in \{0, 1, 2, \dots\},$$

where $A_n, B_n \in (0, +\infty)$ are periodic sequences with period 2 and the initial values $x_{-1}, y_{-1}, x_{-2}, y_{-2} \in (0, +\infty)$. We show that every solution of the above system is eventually periodic.

Keywords: Max-type system of difference equations; Solution; Eventual periodicity

1 Introduction

Our purpose in this paper is to study eventual periodicity of the following max-type system of difference equations:

$$\begin{cases} x_n = \max\{A_n, \frac{y_{n-1}}{x_{n-2}}\}, \\ y_n = \max\{B_n, \frac{x_{n-1}}{y_{n-2}}\}, \end{cases} \quad n \in \mathbf{N}_0 \equiv \{0, 1, \dots\}, \tag{1.1}$$

where $A_n, B_n \in \mathbf{R}_+ \equiv (0, +\infty)$ are periodic sequences with period 2 and the initial values $x_{-2}, y_{-2}, x_{-1}, y_{-1} \in \mathbf{R}_+$.

In [1], Fotiades and Papaschinopoulos studied the following max-type system of difference equations:

$$\begin{cases} x_n = \max\{A, \frac{y_{n-1}}{x_{n-2}}\}, \\ y_n = \max\{B, \frac{x_{n-1}}{y_{n-2}}\}, \end{cases} \quad n \in \mathbf{N}_0 \tag{1.2}$$

with $A, B \in \mathbf{R}_+$ and showed that every positive solution of (1.2) is eventually periodic.

In [2], we studied the eventually periodic solutions of the following max-type system of difference equations:

$$\begin{cases} x_n = \max\{A, \frac{y_{n-k}}{x_{n-1}}\}, \\ y_n = \max\{B, \frac{x_{n-k}}{y_{n-1}}\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{1.3}$$

where $A, B \in \mathbf{R}_+$, $k \in \mathbf{N} \equiv \{1, 2, \ldots\}$ and the initial values $x_{-k}, y_{-k}, x_{-k+1}, y_{-k+1}, \ldots, x_{-1}, y_{-1} \in \mathbf{R}_+$.

Recently, there has been a great interest in studying max-type systems of difference equations. In 2012, Stević in [3] obtained in an elegant way the general solution to the following max-type system of difference equations:

$$\begin{cases} x_{n+1} = \max\{\frac{A}{x_n}, \frac{y_n}{x_n}\}, \\ y_{n+1} = \max\{\frac{A}{y_n}, \frac{x_n}{y_n}\}, \end{cases} \quad n \in \mathbf{N}_0 \tag{1.4}$$

for the case $x_0, y_0 \geq A > 0$ and $y_0/x_0 \geq \max\{A, 1/A\}$. The solvability of various systems of difference equations has reattracted some recent interest, see, e.g., [4–6] and the references therein.

In 2016, we in [7] studied the following max-type system of difference equations:

$$\begin{cases} x_n = \max\{\frac{1}{x_{n-m}}, \min\{1, \frac{A}{y_{n-r}}\}\}, \\ y_n = \max\{\frac{1}{y_{n-m}}, \min\{1, \frac{B}{x_{n-t}}\}\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{1.5}$$

where $A, B \in \mathbf{R}_+$, $m, r, t \in \mathbf{N}$ and the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \ldots, x_{-1}, y_{-1} \in R_+$ with $d = \max\{m, r, t\}$ and showed that every positive solution of (1.5) is eventually periodic with period $2m$.

When $m = r = t = 1$ and $A = B$, (1.5) reduces to the max-type system of difference equations

$$\begin{cases} x_n = \max\{\frac{1}{x_{n-1}}, \min\{1, \frac{A}{y_{n-1}}\}\}, \\ y_n = \max\{\frac{1}{y_{n-1}}, \min\{1, \frac{A}{x_{n-1}}\}\}, \end{cases} \quad n \in \mathbf{N}_0. \tag{1.6}$$

In 2015, the authors of [8] obtained the general solution of system (1.6).

Motivated by papers [9, 10], in 2014, Stević et al. in [11] investigated the following max-type system of difference equations:

$$\begin{cases} y_n^{(1)} = \max_{1 \leq i_1 \leq m_1}\{f_{1i_1}(y^{(1)}_{n-k^{(1)}_{i_1,1}}, y^{(2)}_{n-k^{(1)}_{i_1,2}}, \ldots, y^{(l)}_{n-k^{(1)}_{i_1,l}}, n), y^{(\sigma(1))}_{n-t_1 s}\}, \\ y_n^{(2)} = \max_{1 \leq i_2 \leq m_2}\{f_{2i_2}(y^{(1)}_{n-k^{(2)}_{i_2,1}}, y^{(2)}_{n-k^{(2)}_{i_2,2}}, \ldots, y^{(l)}_{n-k^{(2)}_{i_2,l}}, n), y^{(\sigma(2))}_{n-t_2 s}\}, \\ \ldots \\ y_n^{(l)} = \max_{1 \leq i_l \leq m_l}\{f_{li_l}(y^{(1)}_{n-k^{(l)}_{i_l,1}}, y^{(2)}_{n-k^{(l)}_{i_l,2}}, \ldots, y^{(l)}_{n-k^{(l)}_{i_l,l}}, n), y^{(\sigma(l))}_{n-t_l s}\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{1.7}$$

where $s, l, m_j, t_j, k^{(j)}_{i_j,h} \in \mathbf{N}$ $(j, h \in \{1, 2, \ldots, l\})$, $(\sigma(1), \ldots, \sigma(l))$ is a permutation of $(1, \ldots, l)$ and $f_{ji_j} : \mathbf{R}^l_+ \times \mathbf{N}_0 \longrightarrow \mathbf{R}_+$ $(j \in \{1, \ldots, l\}$ and $i_j \in \{1, \ldots, m_j\})$. They showed that every positive solution of (1.7) is eventually periodic with period sT for some $T \in \mathbf{N}$ if f_{ji_j} satisfy some conditions.

For some results of some properties of many max-type difference equations and systems, such as eventual periodicity, the boundedness character, and attractivity, see, e.g., [12–30] and the related references therein.

2 Main results and proofs

In this section, we study the eventual periodicity of positive solutions of system (1.1). Write $x_{2n} = p_n, x_{2n+1} = q_n, y_{2n} = s_n, y_{2n+1} = t_n$ for any $n \in \mathbf{N}_0$. Then system (1.1) reduces to the system

$$\begin{cases} p_n = \max\{A_0, \frac{t_{n-1}}{p_{n-1}}\}, \\ t_n = \max\{B_1, \frac{p_n}{t_{n-1}}\}, \\ q_n = \max\{A_1, \frac{s_n}{q_{n-1}}\}, \\ s_n = \max\{B_0, \frac{q_{n-1}}{s_{n-1}}\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{2.1}$$

where $A_0, A_1, B_0, B_1 \in \mathbf{R}_+$ and the initial values $s_{-1}, t_{-1}, p_{-1}, q_{-1} \in \mathbf{R}_+$.

The following lemma will be used in the proofs of our main results.

Lemma 2.1 *Let $\{x_n\}_{n\geq -1}$ be a solution of the following equation*:

$$x_n = \max\left\{A, \frac{B}{x_{n-1}}\right\}, \quad n \in \mathbf{N}_0 \tag{2.2}$$

with $A, B \in \mathbf{R}_+$ and the initial value $x_{-1} \in \mathbf{R}_+$. Then x_n is eventually periodic with period 2.

Proof By (2.2) we see $x_n x_{n-1} \geq B$ and $x_n \geq A$ for $n \in \mathbf{N}_0$ and for any $n \geq 2$,

$$\begin{aligned} A \leq x_n &= \max\left\{A, \frac{B}{x_{n-1}}\right\} \\ &= \max\left\{A, \frac{Bx_{n-2}}{x_{n-1}x_{n-2}}\right\} \\ &\leq \max\{A, x_{n-2}\} = x_{n-2}. \end{aligned} \tag{2.3}$$

Then, for every $i \in \{0, 1\}$, x_{2n+i} is eventually nonincreasing.

We claim that, for every $i \in \{0, 1\}$, x_{2n+i} is an eventually constant sequence. Assume on the contrary that for some $i \in \{0, 1\}$, x_{2n+i} is not an eventually constant sequence. Then there exists a sequence of positive integers $k_1 < k_2 < \cdots$ such that, for any $n \in \mathbf{N}$, we have

$$\begin{aligned} A < x_{2k_{n+1}+i} &= \frac{B}{x_{2k_{n+1}+i-1}} \\ &< x_{2k_n+i} = \frac{B}{x_{2k_n+i-1}}, \end{aligned}$$

which implies $x_{2k_{n+1}+i-1} > x_{2k_n+i-1}$ for any $n \in \mathbf{N}$. This is a contradiction. Thus x_n is eventually periodic with period 2. The proof is complete. □

From (2.1) we see that it suffices to consider the eventual periodicity of positive solutions of the following system:

$$\begin{cases} u_n = \max\{A, \frac{v_{n-1}}{u_{n-1}}\}, \\ v_n = \max\{B, \frac{u_n}{v_{n-1}}\}, \end{cases} \quad n \in \mathbf{N}_0, \tag{2.4}$$

where $A, B \in \mathbf{R}_+$ and the initial values $u_{-1}, v_{-1} \in \mathbf{R}_+$. Let $\{(u_n, v_n)\}_{n\geq -1}$ be a solution of (2.4). From (2.4) it immediately follows that, for any $n \in N_0$,

$$u_n \geq A \tag{2.5}$$

and

$$v_n \geq B \tag{2.6}$$

and

$$\begin{cases} u_n = \max\{A, \frac{B}{u_{n-1}}, \frac{1}{v_{n-2}}\}, \\ v_n = \max\{B, \frac{A}{v_{n-1}}, \frac{1}{u_{n-1}}\}. \end{cases} \quad n \in \mathbf{N}, \tag{2.7}$$

Lemma 2.2 *If there exist $k, p \in \mathbf{N}$ such that $u_{p+k} = u_k$ and $v_{p+k} = v_k$, then $u_{n+p} = u_n$ and $v_{n+p} = v_n$ for any $n \geq k$.*

Proof It is easy to see that

$$u_{k+p+1} = \max\left\{A, \frac{v_{k+p}}{u_{k+p}}\right\} = \max\left\{A, \frac{v_k}{u_k}\right\} = u_{k+1}$$

and

$$v_{k+p+1} = \max\left\{B, \frac{u_{k+p+1}}{v_{k+p}}\right\} = \max\left\{B, \frac{u_{k+1}}{v_k}\right\} = v_{k+1}.$$

Assume that, for some $N \in \mathbf{N}$, we have $u_{k+p+N} = u_{k+N}$ and $v_{k+p+N} = v_{k+N}$. Then

$$u_{k+p+N+1} = \max\left\{A, \frac{v_{k+p+N}}{u_{k+p+N}}\right\} = \max\left\{A, \frac{v_{k+N}}{u_{k+N}}\right\} = u_{k+N+1}$$

and

$$v_{k+p+N+1} = \max\left\{B, \frac{u_{k+p+N+1}}{v_{k+p+N}}\right\} = \max\left\{B, \frac{u_{k+N+1}}{v_{k+N}}\right\} = v_{k+N+1}.$$

By mathematical induction, we see that $u_{n+p} = u_n$ and $v_{n+p} = v_n$ for any $n \geq k$. The proof is complete. □

Proposition 2.1 *If $A > B \geq 1$, then $u_n = A$ eventually and v_n is eventually periodic with period 2. If $B \geq A \geq 1$, then $v_n = B$ eventually and u_n is eventually periodic with period 2.*

Proof If $A > B \geq 1$, then by (2.5)–(2.7) we see that, for $n \geq 2$,

$$
\begin{aligned}
A \leq u_n &= \max\left\{A, \frac{B}{u_{n-1}}, \frac{1}{v_{n-2}}\right\} \\
&\leq \max\left\{A, \frac{B}{A}, \frac{1}{B}\right\} = A.
\end{aligned}
$$

Thus, for $n \geq 2$, we have $u_n = A$ and

$$
v_n = \max\left\{B, \frac{A}{v_{n-1}}\right\}.
$$

By Lemma 2.1 we see that v_n is eventually periodic with period 2.

If $B \geq A \geq 1$, then by (2.5)–(2.7) we see that, for $n \geq 2$,

$$
\begin{aligned}
B \leq v_n &= \max\left\{B, \frac{A}{v_{n-1}}, \frac{1}{u_{n-1}}\right\} \\
&\leq \max\left\{B, \frac{A}{B}, \frac{1}{A}\right\} = B.
\end{aligned}
$$

Thus, for $n \geq 2$, we have $v_n = B$ and

$$
u_n = \max\left\{A, \frac{B}{u_{n-1}}\right\}.
$$

By Lemma 2.1 we see that u_n is eventually periodic with period 2. The proof is complete. □

Proposition 2.2 *If $B \geq 1 > A \geq 1/B$, then $v_n = B$ eventually and u_n is eventually periodic with period 2. If $1/A > B \geq 1 > A$, then $v_n = B$ eventually and u_n is eventually periodic with period 2 or u_n, v_n are eventually periodic with period 3.*

Proof Assume that $B \geq 1 > A \geq 1/B$. By (2.5)–(2.7) we see that, for $n \geq 1$,

$$
v_n = \max\left\{B, \frac{A}{v_{n-1}}, \frac{1}{u_{n-1}}\right\} = B
$$

since $A/v_{n-1} \leq 1$ and $1/u_{n-1} \leq B$. By Lemma 2.1 we see that u_n is eventually periodic with period 2.

Assume that $1/A > B \geq 1 > A$. Then by (2.5)–(2.7) we obtain

$$
\begin{aligned}
v_n &= \max\left\{B, \frac{A}{v_{n-1}}, \frac{1}{u_{n-1}}\right\} \\
&= \max\left\{B, \frac{1}{u_{n-1}}\right\} \quad (n \geq 1)
\end{aligned}
\tag{2.8}
$$

since $A/v_{n-1} \leq 1$.

If $v_n = 1/u_{n-1}$ eventually, then $v_n u_{n-1} = 1$ eventually and by (2.4) we have

$$
u_n = \max\left\{A, \frac{v_{n-1}}{u_{n-1}}\right\} = \max\{A, v_n v_{n-1}\}
$$

$$= v_n v_{n-1} \quad \text{eventually} \tag{2.9}$$

since $v_n v_{n-1} \geq B^2 > A$. Thus from (2.9) it follows that

$$\begin{aligned} u_{n+3} = v_{n+3}v_{n+2} &= \frac{v_{n+3}v_{n+2}v_{n+1}}{v_{n+1}} \\ &= \frac{v_{n+3}u_{n+2}v_{n+2}v_{n+1}}{u_{n+2}v_{n+1}u_n}u_n \\ &= \frac{1 \times u_{n+2}}{u_{n+2} \times 1}u_n = u_n \quad \text{eventually,} \end{aligned}$$

which implies that u_n, v_n are eventually periodic with period 3.

If $v_n = B$ eventually, then by (2.4) we have

$$u_n = \max\left\{A, \frac{B}{u_{n-1}}\right\} \quad \text{eventually.}$$

By Lemma 2.1 we see that u_n is eventually periodic with period 2.

If there exists some $k \in \mathbf{N}$ such that

$$v_k = B \geq \frac{1}{u_{k-1}} \quad \text{and} \quad v_{k+1} = \frac{1}{u_k} > B, \tag{2.10}$$

then by (2.4), (2.6), (2.8), and (2.10) it follows

$$\begin{aligned} u_{k+1} &= \max\left\{A, \frac{v_k}{u_k}\right\} = \max\{A, v_{k+1}v_k\} = v_{k+1}v_k, \\ v_{k+2} &= \max\left\{B, \frac{1}{u_{k+1}}\right\} = B, \\ u_{k+2} &= \max\left\{A, \frac{v_{k+1}}{u_{k+1}}\right\} = \max\left\{A, \frac{v_{k+1}}{v_{k+1}v_k}\right\} \\ &= \max\left\{A, \frac{1}{v_k}\right\} = \frac{1}{B}, \\ v_{k+3} &= \max\left\{B, \frac{1}{u_{k+2}}\right\} = B, \\ u_{k+3} &= \max\left\{A, \frac{v_{k+2}}{u_{k+2}}\right\} = B^2, \\ v_{k+4} &= \max\left\{B, \frac{1}{u_{k+3}}\right\} = B, \\ u_{k+4} &= \max\left\{A, \frac{v_{k+3}}{u_{k+3}}\right\} = \frac{1}{B}, \\ v_{k+5} &= \max\left\{B, \frac{1}{u_{k+4}}\right\} = B, \\ u_{k+5} &= \max\left\{A, \frac{v_{k+4}}{u_{k+4}}\right\} = \frac{1}{B}. \end{aligned}$$

By Lemma 2.2 we see that $v_n = B$ ($n \geq k+2$) and $u_{k+2n} = 1/B$ ($n \geq 1$) and $u_{k+2n+1} = B^2$ ($n \geq 1$), which implies that $v_n = B$ eventually and u_n is eventually periodic with period 2. The proof is complete. □

Proposition 2.3 *If $A \geq 1 > B$, then $u_n = A$ eventually and v_n is eventually periodic with period* 2.

Proof If $A \geq 1 > B \geq 1/A$, then by (2.5)–(2.7) we see that, for $n \geq 2$,

$$u_n = \max\left\{A, \frac{B}{u_{n-1}}, \frac{1}{v_{n-2}}\right\} = A$$

since $1/v_{n-2} \leq A$ and $B < u_{n-1}$. Thus from (2.4) it follows

$$v_n = \max\left\{B, \frac{A}{v_{n-1}}\right\} \quad \text{eventually.}$$

By Lemma 2.1 we see that v_n is eventually periodic with period 2.

Now assume that $1/B > A \geq 1 > B$. We claim that there exists a sequence of positive integers $n_1 < n_2 < \cdots$ such that $u_{n_k} = A$. Indeed, if $u_n = v_{n-1}/u_{n-1} > A$ eventually, then

$$\begin{aligned} A^2 < u_n u_{n-1} = v_{n-1} &= \max\left\{B, \frac{u_{n-1}}{v_{n-2}}\right\} \\ &= \frac{u_{n-1}}{v_{n-2}} = \max\left\{\frac{A}{v_{n-2}}, \frac{1}{u_{n-2}}\right\} \\ &= \frac{1}{u_{n-2}} \quad \text{eventually,} \end{aligned}$$

which implies $1 \leq A^3 < u_n u_{n-1} u_{n-2} = 1$, a contradiction.

If $u_n = A$ eventually, then by Lemma 2.1 we see that v_n is eventually periodic with period 2.

If there exists some $k \in \mathbf{N}$ such that

$$u_k = A \geq \frac{v_{k-1}}{u_{k-1}} \quad \text{and} \quad u_{k+1} = \frac{v_k}{u_k} = \frac{v_k}{A} > A, \tag{2.11}$$

then $v_k = u_{k+1} u_k > A^2$ and by (2.4) and (2.11) it follows

$$\begin{aligned} v_{k+1} &= \max\left\{B, \frac{u_{k+1}}{v_k}\right\} = \max\left\{B, \frac{1}{u_k}\right\} = \frac{1}{A}, \\ u_{k+2} &= \max\left\{A, \frac{v_{k+1}}{u_{k+1}}\right\} = \max\left\{A, \frac{1}{v_k}\right\} = A, \\ v_{k+2} &= \max\left\{B, \frac{u_{k+2}}{v_{k+1}}\right\} = A^2, \\ u_{k+3} &= \max\left\{A, \frac{v_{k+2}}{u_{k+2}}\right\} = A, \\ v_{k+3} &= \max\left\{B, \frac{u_{k+3}}{v_{k+2}}\right\} = \frac{1}{A}, \end{aligned}$$

$$u_{k+4} = \max\left\{A, \frac{v_{k+3}}{u_{k+3}}\right\} = A,$$
$$v_{k+4} = \max\left\{B, \frac{u_{k+4}}{v_{k+3}}\right\} = A^2.$$

By Lemma 2.2 we see that $u_n = A$ ($n \geq k+2$) and $v_{k+2n-1} = 1/A$ ($n \geq 1$) and $v_{k+2n} = A^2$ ($n \geq 1$), which implies that $u_n = A$ eventually and v_n is eventually periodic with period 2. The proof is complete. □

Proposition 2.4 *If $A < 1$ and $B < 1$, then $u_n = A$ eventually and v_n is eventually periodic with period 2 or $v_n = B$ eventually and u_n is eventually periodic with period 2 or u_n, v_n are eventually periodic with period 3.*

Proof Note

$$u_n = \max\left\{A, \frac{v_{n-1}}{u_{n-1}}\right\}.$$

There are three cases to consider.

Case 1. Assume that $u_n = A$. By Lemma 2.1 we see that v_n is eventually periodic with period 2.

Case 2. Assume that

$$u_n = v_{n-1}/u_{n-1} > A \quad \text{eventually}. \tag{2.12}$$

Then by (2.7) it follows

$$v_n = \max\left\{B, \frac{A}{v_{n-1}}, \frac{1}{u_{n-1}}\right\} = \max\left\{B, \frac{1}{u_{n-1}}\right\} \quad \text{eventually}. \tag{2.13}$$

If $v_n = B$ eventually, then by Lemma 2.1 we see that u_n is eventually periodic with period 2.

If $v_n = 1/u_{n-1} > B$ eventually, then by (2.12) we have

$$\begin{aligned} u_{n+3} &= \frac{v_{n+2}}{u_{n+2}} = \frac{1}{u_{n+2}u_{n+1}} \\ &= \frac{u_n}{u_{n+2}u_{n+1}u_n} \\ &= \frac{u_n}{v_{n+1}u_n} \\ &= u_n \quad \text{eventually}, \end{aligned}$$

which implies that u_n, v_n are eventually periodic with period 3.

In the following, we assume that there exists some $k \in \mathbf{N}$ such that, for every $n \geq k$,

$$u_n = \frac{v_{n-1}}{u_{n-1}}, \qquad v_k = B, \qquad v_{k+1} = \frac{1}{u_k} > B. \tag{2.14}$$

Thus by (2.13) and (2.14) it follows

$$u_{k+1} = \frac{B}{u_k},$$

$$u_{k+2} = \frac{v_{k+1}}{u_{k+1}} = \frac{1}{B}, \tag{2.15}$$

$$v_{k+2} = \max\left\{B, \frac{1}{u_{k+1}}\right\} = \max\left\{B, \frac{u_k}{B}\right\}. \tag{2.16}$$

If $v_{k+2} = B \geq u_k/B$, then by (2.13)–(2.16) we have

$$\begin{aligned}
u_{k+3} &= \frac{v_{k+2}}{u_{k+2}} = B^2,\\
v_{k+3} &= \max\left\{B, \frac{1}{u_{k+2}}\right\} = B,\\
u_{k+4} &= \frac{v_{k+3}}{u_{k+3}} = \frac{1}{B},\\
v_{k+4} &= \max\left\{B, \frac{1}{u_{k+3}}\right\} = \frac{1}{B^2},\\
u_{k+5} &= \frac{v_{k+4}}{u_{k+4}} = \frac{1}{B},\\
v_{k+5} &= \max\left\{B, \frac{1}{u_{k+4}}\right\} = B.
\end{aligned}$$

By Lemma 2.2 we see that $u_{n+3} = u_n$ and $v_{n+3} = v_n$ $(n \geq k+2)$, which implies that u_n, v_n are eventually periodic with period 3.

If $v_{k+2} = u_k/B > B$, then by (2.13)–(2.16) we have

$$\begin{aligned}
u_{k+3} &= \frac{v_{k+2}}{u_{k+2}} = u_k,\\
v_{k+3} &= \max\left\{B, \frac{1}{u_{k+2}}\right\} = B,\\
u_{k+4} &= \frac{v_{k+3}}{u_{k+3}} = \frac{B}{u_k},\\
v_{k+4} &= \max\left\{B, \frac{1}{u_{k+3}}\right\} = \frac{1}{u_k},\\
u_{k+5} &= \frac{v_{k+4}}{u_{k+4}} = \frac{1}{B},\\
v_{k+5} &= \max\left\{B, \frac{1}{u_{k+4}}\right\} = \frac{u_k}{B}.
\end{aligned}$$

By Lemma 2.2 we see that $u_{n+3} = u_n$ and $v_{n+3} = v_n$ $(n \geq k+2)$, which also implies that u_n, v_n are eventually periodic with period 3.

Case 3. Assume that there exists some $k \in \mathbf{N}$ such that

$$u_k = A \geq \frac{v_{k-1}}{u_{k-1}}, \qquad u_{k+1} = \frac{v_k}{u_k} = \frac{v_k}{A} > A. \tag{2.17}$$

Then $v_k = u_{k+1}u_k > A^2$ and by (2.7) and (2.17) we have

$$v_{k+1} = \max\left\{B, \frac{A}{v_k}, \frac{1}{u_k}\right\} = \max\left\{B, \frac{1}{A}\right\} = \frac{1}{A} \tag{2.18}$$

and

$$u_{k+2} = \max\left\{A, \frac{v_{k+1}}{u_{k+1}}\right\} = \max\left\{A, \frac{1}{v_k}\right\}. \tag{2.19}$$

If $u_{k+2} = A \geq 1/v_k$ and $\sqrt{B} \geq A \geq B^2$, then by (2.4), (2.18), and (2.19) it follows

$$\begin{aligned}
v_{k+2} &= \max\left\{B, \frac{u_{k+2}}{v_{k+1}}\right\} = B,\\
u_{k+3} &= \max\left\{A, \frac{v_{k+2}}{u_{k+2}}\right\} = \frac{B}{A},\\
v_{k+3} &= \max\left\{B, \frac{u_{k+3}}{v_{k+2}}\right\} = \frac{1}{A},\\
u_{k+4} &= \max\left\{A, \frac{v_{k+3}}{u_{k+3}}\right\} = \frac{1}{B},\\
v_{k+4} &= \max\left\{B, \frac{u_{k+4}}{v_{k+3}}\right\} = \frac{A}{B},\\
u_{k+5} &= \max\left\{A, \frac{v_{k+4}}{u_{k+4}}\right\} = A,\\
v_{k+5} &= \max\left\{B, \frac{u_{k+5}}{v_{k+4}}\right\} = B.
\end{aligned}$$

By Lemma 2.2 we see that $u_{n+3} = u_n$ and $v_{n+3} = v_n$ $(n \geq k+2)$, which implies that u_n, v_n are eventually periodic with period 3.

If $u_{k+2} = A \geq 1/v_k$ and $\sqrt{B} > B^2 > A$, then by (2.4), (2.18), and (2.19) it follows

$$\begin{aligned}
v_{k+2} &= B, \qquad u_{k+3} = \frac{B}{A},\\
v_{k+3} &= \frac{1}{A}, \qquad u_{k+4} = \frac{1}{B},\\
v_{k+4} &= \max\left\{B, \frac{u_{k+4}}{v_{k+3}}\right\} = B,\\
u_{k+5} &= \max\left\{A, \frac{v_{k+4}}{u_{k+4}}\right\} = B^2,\\
v_{k+5} &= \max\left\{B, \frac{u_{k+5}}{v_{k+4}}\right\} = B,\\
u_{k+6} &= \max\left\{A, \frac{v_{k+5}}{u_{k+5}}\right\} = \frac{1}{B},\\
v_{k+6} &= \max\left\{B, \frac{u_{k+6}}{v_{k+5}}\right\} = \frac{1}{B^2},\\
u_{k+7} &= \max\left\{A, \frac{v_{k+6}}{u_{k+6}}\right\} = \frac{1}{B},
\end{aligned}$$

$$v_{k+7} = \max\left\{B, \frac{u_{k+7}}{v_{k+6}}\right\} = B.$$

By Lemma 2.2 we see that $u_{n+3} = u_n$ and $v_{n+3} = v_n$ ($n \geq k+4$), which implies that u_n, v_n are eventually periodic with period 3.

If $u_{k+2} = A \geq 1/v_k$ and $\sqrt{B} < A$, then by (2.4), (2.18), and (2.19) it follows

$$v_{k+2} = \max\left\{B, \frac{u_{k+2}}{v_{k+1}}\right\} = A^2,$$

$$u_{k+3} = \max\left\{A, \frac{v_{k+2}}{u_{k+2}}\right\} = A,$$

$$v_{k+3} = \max\left\{B, \frac{u_{k+3}}{v_{k+2}}\right\} = \frac{1}{A},$$

$$u_{k+4} = \max\left\{A, \frac{v_{k+3}}{u_{k+3}}\right\} = \frac{1}{A^2},$$

$$v_{k+4} = \max\left\{B, \frac{u_{k+4}}{v_{k+3}}\right\} = \frac{1}{A},$$

$$u_{k+5} = \max\left\{A, \frac{v_{k+4}}{u_{k+4}}\right\} = A,$$

$$v_{k+5} = \max\left\{B, \frac{u_{k+5}}{v_{k+4}}\right\} = A^2.$$

By Lemma 2.2 we see that $u_{n+3} = u_n$ and $v_{n+3} = v_n$ ($n \geq k+2$), which implies that u_n, v_n are eventually periodic with period 3.

If $u_{k+2} = 1/v_k > A \geq Bv_k$, then by (2.4), (2.18), and (2.19) it follows

$$v_{k+2} = \max\left\{B, \frac{u_{k+2}}{v_{k+1}}\right\} = \frac{A}{v_k},$$

$$u_{k+3} = \max\left\{A, \frac{v_{k+2}}{u_{k+2}}\right\} = A,$$

$$v_{k+3} = \max\left\{B, \frac{u_{k+3}}{v_{k+2}}\right\} = v_k,$$

$$u_{k+4} = \max\left\{A, \frac{v_{k+3}}{u_{k+3}}\right\} = \frac{v_k}{A},$$

$$v_{k+4} = \max\left\{B, \frac{u_{k+4}}{v_{k+3}}\right\} = \frac{1}{A},$$

$$u_{k+5} = \max\left\{A, \frac{v_{k+4}}{u_{k+4}}\right\} = \frac{1}{v_k},$$

$$v_{k+5} = \max\left\{B, \frac{u_{k+5}}{v_{k+4}}\right\} = \frac{A}{v_k}.$$

By Lemma 2.2 we see that $u_{n+3} = u_n$ and $v_{n+3} = v_n$ ($n \geq k+2$), which implies that u_n, v_n are eventually periodic with period 3.

If $u_{k+2} = 1/v_k > A$ and $A/B < v_k \leq 1/B^2$, then by (2.4), (2.18), and (2.19) it follows

$$v_{k+2} = \max\left\{B, \frac{u_{k+2}}{v_{k+1}}\right\} = B,$$

$$u_{k+3} = \max\left\{A, \frac{v_{k+2}}{u_{k+2}}\right\} = Bv_k,$$
$$v_{k+3} = \max\left\{B, \frac{u_{k+3}}{v_{k+2}}\right\} = v_k,$$
$$u_{k+4} = \max\left\{A, \frac{v_{k+3}}{u_{k+3}}\right\} = \frac{1}{B},$$
$$v_{k+4} = \max\left\{B, \frac{u_{k+4}}{v_{k+3}}\right\} = \max\left\{B, \frac{1}{Bv_k}\right\} = \frac{1}{Bv_k},$$
$$u_{k+5} = \max\left\{A, \frac{v_{k+4}}{u_{k+4}}\right\} = \frac{1}{v_k},$$
$$v_{k+5} = \max\left\{B, \frac{u_{k+5}}{v_{k+4}}\right\} = B.$$

By Lemma 2.2 we see that $u_{n+3} = u_n$ and $v_{n+3} = v_n$ $(n \geq k + 2)$, which implies that u_n, v_n are eventually periodic with period 3.

If $u_{k+2} = 1/v_k > A$ and $v_k > 1/B^2$, then $A < B^2$ and by (2.4), (2.18), and (2.19) it follows

$$v_{k+2} = B, \qquad u_{k+3} = Bv_k,$$
$$v_{k+3} = v_k, \qquad u_{k+4} = \frac{1}{B},$$
$$v_{k+4} = \max\left\{B, \frac{u_{k+4}}{v_{k+3}}\right\} = \max\left\{B, \frac{1}{Bv_k}\right\} = B,$$
$$u_{k+5} = \max\left\{A, \frac{v_{k+4}}{u_{k+4}}\right\} = B^2,$$
$$v_{k+5} = \max\left\{B, \frac{u_{k+5}}{v_{k+4}}\right\} = B,$$
$$u_{k+6} = \max\left\{A, \frac{v_{k+5}}{u_{k+5}}\right\} = \frac{1}{B},$$
$$v_{k+6} = \max\left\{B, \frac{u_{k+6}}{v_{k+5}}\right\} = \frac{1}{B^2},$$
$$u_{k+7} = \max\left\{A, \frac{v_{k+6}}{u_{k+6}}\right\} = \frac{1}{B},$$
$$v_{k+7} = \max\left\{B, \frac{u_{k+7}}{v_{k+6}}\right\} = B.$$

By Lemma 2.2 we see that $u_{n+3} = u_n$ and $v_{n+3} = v_n$ $(n \geq k + 4)$, which implies that u_n, v_n are eventually periodic with period 3. The proof is complete. □

Combining (2.1) with (2.3), from Propositions 2.1–2.4 we obtain the following theorem.

Theorem 2.1 *Let $\{(x_n, y_n)\}_{n\geq -2}$ be a positive solution of (1.1). Then x_n and y_n are eventually periodic with periods T_x and T_y, respectively, and $T_x, T_y \in \{2, 4, 6, 12\}$.*

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Competing interests
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Authors' contributions
All authors participated in every phase of research conducted for this paper. All authors read and approved the final manuscript.

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