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A class of Runge–Kutta methods for nonlinear Volterra integral equations of the second kind with singular kernels

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Abstract

This paper aims to obtain an approximate solution for fractional order Riccati differential equations (FRDEs). FRDEs are equivalent to nonlinear Volterra integral equations of the second kind. In order to solve nonlinear Volterra integral equations of the second kind, a class of Runge–Kutta methods has been applied. Runge–Kutta methods have been implemented to solve nonsingular integral equations. In this work Volterra integral equations are singular. The singularity by a suitable subtraction technique will be weakened; then, this method will be applied to gain an approximate solution. Fractional derivatives are defined in the Caputo form of order $0 < \alpha \le 1$.

Keywords: Fractional order Riccati differential equations; Runge–Kutta methods; Subtraction of the singularity; Nonlinear Volterra integral equations of the second kind; Caputo fractional derivative

1 Introduction

A generalization of the classical Newtonian calculus is called fractional calculus and appears in many natural phenomena such as physical, chemical, sociological, biological, and economical processes. Fractional differential equations are one of the most important branches of fractional calculus. Fractional differential equations are an essential tool in mathematical modeling for many engineering and scientific problems [1–9]. FRDEs are well known equations that find many applications in scientific phenomena. The general form of FRDEs is as follows:

$$D_s^{\alpha} x(s) = r(s)x^2(s) + q(s)x(s) + p(s), \quad s > 0, 0 < \alpha \le 1, \tag{1}$$

with the initial condition

$$x(0) = k, \tag{2}$$

where $p(s)$, $q(s)$, and $r(s)$ are known functions, D_s^{α} is the Caputo fractional derivative operator. For $\alpha = 1$, FRDEs are the same as classical Riccati differential equations.

There are numerous direct numerical approaches for solving such equations. Some of these methods are as follows: optimal homotopy asymptotic method [10], homotopy analysis [11–13], homotopy perturbation [14–18], variational iteration [19, 20], modified variational iteration [21], differential transform [22], shifted Jacobi spectral method [23], Taylor matrix method [24], Adomian decomposition [1, 25, 26], and B-spline operational matrix method [27]. In this work, first, FRDEs will be converted into nonlinear Volterra integral equations of the second kind and then we will look for the solution by a class of Runge–Kutta methods. However, we usually suppose that the kernel and driving terms are continuous functions in the interval of integration and the kernel satisfies a uniform Lipschitz condition in x [28]. We are aware that, when this assumption violated, the method will fail utterly, or at best converge slowly. For solving singular integral equations, the purpose is to achieve a method as a product integration which converges as fast as for smooth problems [28]. We know that obtained equivalent nonlinear Volterra integral equations of the second kind, which will be described in Sect. 2, are singular at the final point. So by an appropriate subtraction technique, the singularity will be weakened and then Runge–Kutta approach can be applied. However, at singular points, the modified method is slow, yet we see two advantages in this approach, namely the ability to use this method for special singular nonlinear Volterra integral equations of the second kind and getting relatively accurate results for the solution in comparison with other methods. The singularity of the nonlinear kernel, $k(s,t,x(t))$, implies that the construction of methods with high order accuracy will not be easy. Under best conditions, the rate of convergence decreases [29]. During recent years, various papers have been devoted to the solution of linear and nonlinear weakly singular integral equations. In [30], Chebyshev spectral collocation method has been implemented to solve multidimensional nonlinear Volterra integral equation with a weakly singular kernel. The local discrete collocation method, which does not need any meshes, is called the meshless local discrete collocation (MLDC) method. This approach has been utilized for solving weakly singular integral equations [31]. In [29], a strong approach based on Legendre multiwavelets is presented for obtaining the approximate solution of Fredholm weakly singular integro-differential equations. A kind of subtraction technique has been applied in this paper. The discrete Galerkin approach with thin-plate splines based on scattered points is used to calculate the solution of nonlinear weakly singular Fredholm integral equation in [32]. In [33], tau approximation method has been applied to solve weakly singular Volterra-Hammerstein integral equations. An interesting numerical method by combining the product integration and collocation methods based on the radial basis functions has been used for solving weakly singular Fredholm integral equations in [34]. Moreover, Newton product integration method [35], the piecewise polynomial collocation method [36, 37], and quadratic spline collocation method [38] have been utilized for solving weakly singular integral equations. An efficient approach based on combining the radial basis functions and discrete collocation method has been implemented to solve nonlinear Volterra integral equations of the second kind in [39]. A numerical scheme based on the moving least squares method has been applied to solve integral equations in [40]. This approach is meshless. Some other related works that can be useful to better understand the research are [41–47].

The rest of this paper is organized as follows: in the next section, we present a brief review of a class of Runge–Kutta methods for nonlinear Volterra integral equations of the second kind. In Sect. 3, we explain the subtraction of the singularity and application of the approach. In Sect. 4, we investigate two numerical examples. In Sect. 5, convergence analysis will be discussed. In the last section, we present the conclusions.

2 Preliminaries

The aim of this section is to recall some preliminaries about the objects used in our paper.

2.1 Definition

Definition 1 The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $x : (0, \infty) \to R$ is defined by

$$J^\alpha x(s) = \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} x(t)\, dt, \tag{3}$$

where Γ is the Gamma function. It must be noted that in this paper $0 < \alpha \leq 1$. See [7] for more details and examples.

Definition 2 The Caputo fractional derivative of order $\alpha > 0$ of a function $x : (0, \infty) \to R$ is defined as

$$D^\alpha x(s) = \frac{1}{\Gamma(m-\alpha)} \int_0^s (s-t)^{m-\alpha-1} x^{(m)}(t)\, dt, \quad m = \lceil \alpha \rceil. \tag{4}$$

See [7] for more details and examples.

2.2 Existence of solutions

Consider the initial value problem (IVP) with Caputo fractional derivative given by

$$D^\alpha x(s) = f\bigl(s, x(s)\bigr), \tag{5}$$

with initial conditions

$$D^k x(0) = x_0^{(k)}, \quad k = 0, 1, \dots, m-1. \tag{6}$$

We want to illustrate, by a theorem and a lemma, that every solution of the IVP given by (5) is also a solution of the following equation:

$$x(s) = \sum_{k=0}^{m-1} \frac{s^k}{k!} x_0^{(k)} + \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f\bigl(t, x(t)\bigr)\, dt, \quad m = \lceil \alpha \rceil. \tag{7}$$

Theorem 1 *Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $x_0^{(0)}, \dots, x_0^{(m-1)} \in R$, $L > 0$ and $h^* > 0$. Define $H := \{(s,x) : s \in [0, h^*], |x - \sum_{k=0}^{m-1} \frac{s^k}{k!} x_0^{(k)}| \leq L\}$. Moreover, suppose that the function $f : H \to R$ is continuous. Define $P := \sup_{(s,z) \in H} |f(s,z)|$ and*

$$h := \begin{cases} h^* & \text{if } P = 0, \\ \min\{h^*, (L\Gamma(\alpha+1)/P))^{1/n}\} & \text{else.} \end{cases}$$

Then there exists a function $x \in C[0, h]$, satisfying IVP (5) (see [7]).

Lemma 1 *Assume the hypotheses of Theorem 1. A function $x \in C[0, h]$ is a solution IVP (5) if and only if this function is a solution of the nonlinear Volterra integral equation of the second kind (7) (see [7]).*

Remark 1 As a direct result from Lemma 1, let us consider the IVP given by

$$D^{\alpha}x(s) = f\bigl(s, x(s)\bigr), \tag{8}$$

with the initial condition

$$x(0) = x_0, \tag{9}$$

where D^{α} is Caputo fractional derivative, and $f \in C([0,L] \times R, R)$, $0 < \alpha \leq 1$. Since f is presumed to be continuous, every solution of (8) is also a solution of the following nonlinear Volterra integral equation of the second kind:

$$x(s) = x_0 + \frac{1}{\Gamma(\alpha)} \int_0^s (s-t)^{\alpha-1} f\bigl(t, x(t)\bigr)\, dt, \quad t \in [0, L]. \tag{10}$$

Furthermore, every solution of the equation given by (10) is a solution of (8) (see [7, 48]).

2.3 Runge–Kutta methods

Consider nonsingular Volterra equations of the second kind of the general form given by

$$x(s) = y(s) + \int_a^s k\bigl(s, t, x(t)\bigr)\, dt, \quad a \leq s \leq b, \tag{11}$$

and suppose that the solution is defined over a finite interval $[a, b]$, y is continuous in the closed interval $[a, b]$, kernel is continuous in $a \leq t \leq s \leq b$ and it satisfies a uniform Lipschitz condition in x. In Eq. (11), these conditions guaranty the existence of a unique continuous solution. Rung–Kutta methods are efficient numerical methods to approximate the solution of (11). These methods are self-starting approaches which specify the approximate solution at the points $s_i = a + ih$, $i = 1, \ldots, N$; we generate approximations to the solution at some intermediate points in the closed interval $[s_i, s_{i+1}]$, $i = 0, \ldots, N-1$, where $s_i + \theta_r h$, $i = 0, \ldots, N-1$, $r = 1, \ldots, p-1$, and $0 = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_{p-1} \leq 1$. Then we apply the general p-stage Rung–Kutta approach to obtain an approximate solution of the initial value problem

$$x'(s) = f\bigl(s, x(s)\bigr), \tag{12}$$

$$x(a) = x_0, \tag{13}$$

given by

$$x_{i+1} = x_i + h \sum_{l=0}^{p-1} A_{pl} k_l^i, \tag{14}$$

where

$$k_0^i = f(a + ih, x_i), \tag{15}$$

$$k_r^i = f\left(a + (i + \theta_r)h, x_i + h \sum_{l=0}^{r-1} A_{rl} k_l^i\right), \quad r = 1, \ldots, p-1, \tag{16}$$

$$\sum_{l=0}^{r-1} A_{rl} = \begin{cases} \theta_r, & r = 1, 2, \ldots, p-1, \\ 1, & r = p, \end{cases} \tag{17}$$

and where x_l is an approximation to the solution at $s = s_l = a + lh$. We can rewrite Eq. (14) as follows:

$$x_{i+1} = x_i + h \sum_{l=0}^{p-1} A_{pl} f(s_i + \theta_l h, x_{i+\theta_l}). \tag{18}$$

It must be noted that A_{pl}, θ_l are chosen to obtain the final approximate solution of a special order. Equation (18), for a given pair (p, q), provides a set of nonlinear equations that may have no solutions, one solution, or a family of solutions; see [28] for more details and examples.

2.4 A class of Runge–Kutta methods

We can extend Eq. (18) to gain a class of Rung–Kutta methods to solve nonsingular Volterra equations of the second kind of the general form in (11). Substituting s_i into (11), results in

$$x(s_i) = y(s_i) + \int_a^{a+ih} k\bigl(a + ih, t, x(t)\bigr)\, dt, \quad i = 1, \ldots, N, \tag{19}$$

so

$$x(s_i) = y(s_i) + \sum_{j=0}^{i-1} \int_{a+jh}^{a+(j+1)h} k\bigl(a + ih, t, x(t)\bigr)\, dt, \quad i = 1, \ldots, N. \tag{20}$$

We can consider an approximation x_i to $x(s_i)$ from the following equation:

$$x_i = y(s_i) + h \sum_{j=0}^{i-1} \sum_{l=0}^{p-1} A_{pl} k\bigl(a + ih, a + (j + \theta_l)h, x_{j+\theta_l}\bigr). \tag{21}$$

For $s \in (s_i, s_{i+1})$, Eq. (11) can be written as the following form:

$$x(s) = y(s) + \sum_{j=0}^{i-1} \int_{s_j}^{s_{j+1}} k\bigl(s, t, x(t)\bigr)\, dt + \int_{s_i}^{s} k\bigl(s, t, x(t)\bigr)\, dt. \tag{22}$$

By setting $s = s_i + \theta_\vartheta h$, $\vartheta = 1, \ldots, p-1$, the last integral in (22) will be approximated as follows:

$$\int_{s_i}^{s_i+\theta_\vartheta h} k\bigl(s_i + \theta_\vartheta h, t, x(t)\bigr)\, dt \approx h \sum_{l=0}^{\vartheta-1} A_{\vartheta l} k(s_i + \theta_\vartheta h, s_i + \theta_l h, x_{i+\theta_l}). \tag{23}$$

According to (20), (21), (22), and (23), the Runge–Kutta method for (11) can be rewritten as the following form:

$$\begin{aligned} x_{i+\theta_\vartheta h} &= y(s_i + \theta_\vartheta h) + h \sum_{j=0}^{i-1} \sum_{l=0}^{p-1} A_{pl} k(s_i + \theta_\vartheta h, s_j + \theta_l h, x_{j+\theta_l}) \\ &\quad + h \sum_{l=0}^{\vartheta-1} A_{\vartheta l} k(s_i + \theta_\vartheta h, s_i + \theta_l h, x_{i+\theta_l}), \end{aligned} \tag{24}$$

$i = 0, 1, \ldots, N-1$, $\vartheta = 1, 2, \ldots, p-1$, where $x(a) = y(a)$, and A_{rj}, θ_j, $r = 1, 2, \ldots, p$, $j = 0, \ldots, p-1$, describe the particular method; see [28] for more details and examples.

3 Subtraction of the singularity

In this section, we want to apply a class of Runge–Kutta methods for solving nonlinear Volterra integral equations of the second kind with singular kernels given by (11). For this, we assume that $k(s,t,x(t)) = K(s,t)[\beta x^2(t) + \gamma x(t)]$, where

$$K(s,t) = \frac{K_0(s,t)}{(s-t)^{1-\alpha}}, \quad \beta, \gamma \in R, 0 < \alpha \leq 1, \tag{25}$$

and we suppose $K_0(s,t)$ is regular. Now we utilize the Runge–Kutta method given in Sect. 2.4 for the kernel provided in (25). According to Eq. (22), we have

$$\begin{aligned} x(s) = y(s) &+ \sum_{j=0}^{i-1} \int_{s_j}^{s_{j+1}} K(s,t)\bigl[\beta x^2(t) + \gamma x(t)\bigr]\,dt \\ &+ \int_{s_i}^{s} K(s,t)\bigl[\beta x^2(t) + \gamma x(t)\bigr]\,dt. \end{aligned} \tag{26}$$

We know that the singularity of the kernel is at $s = t$, in the last term of Eq. (26), so it is enough that we obtain the last term as follows:

$$\begin{aligned} &\int_{s_i}^{s} K(s,t)\bigl[\beta x^2(t) + \gamma x(t)\bigr]\,dt \\ &\quad = \int_{s_i}^{s} K(s,t)\bigl[\bigl(\beta x^2(t) + \gamma x(t)\bigr) - \bigl(\beta x^2(s) + \gamma x(s)\bigr) + \bigl(\beta x^2(s) + \gamma x(s)\bigr)\bigr]\,dt, \end{aligned} \tag{27}$$

and finally,

$$\begin{aligned} &\int_{s_i}^{s} K(s,t)\bigl[\beta x^2(t) + \gamma x(t)\bigr]\,dt \\ &\quad = \beta \int_{s_i}^{s} \frac{K_0(s,t)}{(s-t)^{1-\alpha}}\bigl(x^2(t) - x^2(s)\bigr)\,dt + \gamma \int_{s_i}^{s} \frac{K_0(s,t)}{(s-t)^{1-\alpha}}\bigl(x(t) - x(s)\bigr)\,dt \\ &\qquad + \bigl(\beta x^2(s) + \gamma x(s)\bigr)q(s), \end{aligned} \tag{28}$$

where $q(s) = \int_{s_i}^{s} \frac{K_0(s,t)}{(s-t)^{1-\alpha}}\,dt$ is known and can be computed easily. If the primal integral exists in the Riemann sense then the first and second terms of (28) are now regular at $s = t$. Since $x^2(t) - x^2(s) = 0$, and $x(t) - x(s) = 0$, at the singular point $s = t$, the singularity is weaker than in the previous case. So, we can now introduce Runge–Kutta method and apply it to (28). The singularity happens when $\theta_\vartheta = \theta_l$, so the term $\theta_\vartheta = \theta_l$, is omitted employing the identities $K(s,s)(x^2(s) - x^2(s)) = 0$ and $K(s,s)(x(s) - x(s)) = 0$. It must be noted that the singularity has been weakened, by this subtraction technique, but has not been removed completely. So by implementing the Runge–Kutta method, if $\theta_\vartheta = \theta_l$, we can write the numerical form of Eq. (28) as follows:

$$\begin{aligned} &\int_{s_i}^{s} K(s,t)\bigl[\beta x^2(t) + \gamma x(t)\bigr]\,dt \\ &\quad = \beta h \sum_{\substack{l=0 \\ \theta_\vartheta \neq \theta_l}}^{\vartheta-1} A_{\vartheta l}\bigl[k(s_i + \theta_\vartheta h, s_i + \theta_l h)\bigl(x_{i+\theta_l}^2 - x_{i+\theta_\vartheta}^2\bigr)\bigr] \end{aligned}$$

$$+\gamma h\sum_{\substack{l=0\\ \theta_\vartheta\neq\theta_l}}^{\vartheta-1}A_{\vartheta l}\big[k(s_i+\theta_\vartheta h,s_i+\theta_l h)\big](x_{i+\theta_l}-x_{i+\theta_\vartheta})\big]$$

$$+\big(\beta x^2(s_i+\theta_\vartheta h)+\gamma x(s_i+\theta_\vartheta h)\big)q(s_i+\theta_\vartheta h). \tag{29}$$

Finally, we write the numeric form of Eq. (24) as

$$\begin{aligned}
x_{i+\theta_\vartheta} &= y(s_i+\theta_\vartheta h)+h\sum_{j=0}^{i-1}\sum_{l=0}^{p-1}A_{pl}K(s_i+\theta_\vartheta h,s_j+\theta_{lh})\big[\beta x^2_{j+\theta_l}+\gamma x_{j+\theta_l}\big]\\
&\quad+\beta h\sum_{\substack{l=0\\ \theta_\vartheta\neq\theta_l}}^{\vartheta-1}A_{\vartheta l}\big[k(s_i+\theta_\vartheta h,s_i+\theta_l h)\big(x^2_{i+\theta_l}-x^2_{i+\theta_\vartheta}\big)\big]\\
&\quad+\gamma h\sum_{\substack{l=0\\ \theta_\vartheta\neq\theta_l}}^{\vartheta-1}A_{\vartheta l}\big[k(s_i+\theta_\vartheta h,s_i+\theta_l h)\big](x_{i+\theta_l}-x_{i+\theta_\vartheta})\big]\\
&\quad+\big(\beta x^2_{i+\theta_\vartheta}+\gamma x_{i+\theta_\vartheta}\big)q(s_i+\theta_\vartheta h),
\end{aligned} \tag{30}$$

where $\beta,\gamma\in R$, $\vartheta=1,2,\ldots,p-1$, $q(s)=\int_{s_i}^{s}\frac{K_0(s,t)}{(s-t)^{1-\alpha}}\,dt$, and $K(s,t)=\frac{K_0(s,t)}{(s-t)^{1-\alpha}}$, $i=0,1,\ldots,N-1$. Here $x_{i+\theta_\vartheta}$ is an approximation of $x(s_i+\theta_\vartheta h)$, $x(a)=y(a)$, and A_{rj}, θ_j, $r=1,2,\ldots,p$, $j=0,\ldots,p-1$, describe the particular method (see [28, 29]).

The proposed approach, with this subtraction technique, is called the new p-stage Runge–Kutta method (NRKp).

4 Examples

In this section, first, a new 4-stage Runge–Kutta method (NRK4) is obtained, and then the application of this approach in solving fractional Riccati differential equations is illustrated by two examples.

We set $p=4$, $\vartheta=1,2,3$, $i=0,1,\ldots,N-1$, $r=1,2,3,4$, $j=0,\ldots,3$ and $K_0(s,t)=c_0$, $c_0\in\mathbb{R}$, to gain a new 4-stage Runge–Kutta method from Eq. (30), and derive

$$\begin{aligned}
x_{i+\theta_\vartheta} &= y(s_i+\theta_\vartheta h)+h\sum_{j=0}^{i-1}\sum_{l=0}^{3}A_{4l}K(s_i+\theta_\vartheta h,s_j+\theta_l h)\big[\beta x^2_{j+\theta_l}+\gamma x_{j+\theta_l}\big]\\
&\quad+\beta h\sum_{\substack{l=0\\ \theta_\vartheta\neq\theta_l}}^{\vartheta-1}A_{\vartheta l}\big[k(s_i+\theta_\vartheta h,s_i+\theta_l h)\big(x^2_{i+\theta_l}-x^2_{i+\theta_\vartheta}\big)\big]\\
&\quad+\gamma h\sum_{\substack{l=0\\ \theta_\vartheta\neq\theta_l}}^{\vartheta-1}A_{\vartheta l}\big[k(s_i+\theta_\vartheta h,s_i+\theta_l h)\big](x_{i+\theta_l}-x_{i+\theta_\vartheta})\big]\\
&\quad+\big(\beta x^2_{i+\theta_\vartheta}+\gamma x_{i+\theta_\vartheta}\big)q(s_i+\theta_\vartheta h),
\end{aligned} \tag{31}$$

where $\theta_0=0$, $\theta_1=\theta_2=\frac{1}{2}$, $\theta_3=1$, $A_{10}=\frac{1}{2}$, $A_{20}=0$, $A_{21}=\frac{1}{2}$, $A_{30}=A_{31}=0$, $A_{32}=1$, $A_{40}=A_{43}=\frac{1}{6}$, and $A_{41}=A_{42}=\frac{1}{3}$; see [28] for more details about the values of θ_j, $j=0,1,2,3$.

Example 1 Consider the following fractional Riccati differential equation:

$$D^{\alpha}x(s) = 1 - x^2(s), \quad 0 \le s \le 1, 0 < \alpha \le 1, \tag{32}$$

with the initial condition

$$x(0) = 0. \tag{33}$$

The exact solution, for $\alpha = 1$, is

$$x(s) = \frac{e^{2s} - 1}{e^{2s} + 1}. \tag{34}$$

According to Remark 1, we can write (32) as follows:

$$x(s) = y(s) + \int_0^s k\bigl(s, t, x(t)\bigr)\,dt, \quad 0 \le s \le 1, \tag{35}$$

where $y(s) = \frac{s^{\alpha}}{\Gamma(\alpha+1)}$, $k(s,t,x(t)) = \beta k(s,t)x^2(t)$, $K(s,t) = \frac{1}{(s-t)^{1-\alpha}}$, $\beta = -\frac{1}{\Gamma(\alpha)}$, and $\gamma = 0$. By setting $i = 0$, $\vartheta = 1$ in (31), we have $x_{\frac{1}{2}} = \frac{h^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}$. In the next step, if $i = 0$, $\vartheta = 2$, the singularity happens at $s = \frac{h}{2}$, so the second $x_{\frac{1}{2}}$ can be achieved as

$$x_{\frac{1}{2}} = \frac{h^{\alpha}}{2^{\alpha}\Gamma(\alpha + 1)} + h\beta \sum_{\substack{l=0\\ l\neq 1}}^{1} A_{2l} k(s_0 + \theta_2 h, s_0 + \theta_l h)\bigl(x^2_{0+\theta_l} - x^2_{0+\theta_2}\bigr) + \beta x^2_{\frac{1}{2}} q\left(\frac{h}{2}\right), \tag{36}$$

where $q(\frac{h}{2}) = \int_0^{\frac{h}{2}} \frac{1}{(\frac{h}{2}-t)^{1-\alpha}}\,dt$, and $x_{\frac{1}{2}}$ is an approximation of $x(\frac{h}{2})$. The following quadratic polynomial will be obtained from (36):

$$x_{\frac{1}{2}} = \frac{h^{\alpha}}{2^{\alpha}\Gamma(\alpha + 1)}\bigl(1 - x^2_{\frac{1}{2}}\bigr). \tag{37}$$

To get $x_{\frac{1}{2}}$, we prefer to utilize the predictor–corrector method. For such a purpose, we use the first iteration of $x_{\frac{1}{2}} = \frac{h^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}$ on the right-hand side of (37) as a predictor. Let us consider $x_{\frac{1}{2}} = x^{(0)}_{\frac{1}{2}}$. Equation (37) can now be rewritten as follows:

$$x^{(1)}_{\frac{1}{2}} = \frac{h^{\alpha}}{2^{\alpha}\Gamma(\alpha + 1)}\bigl(1 - \bigl(x^{(0)}_{\frac{1}{2}}\bigr)^2\bigr), \tag{38}$$

i.e.,

$$x^{(1)}_{\frac{1}{2}} = \frac{h^{\alpha}}{2^{\alpha}\Gamma(\alpha + 1)}\left(1 - \left(\frac{h^{\alpha}}{2^{\alpha}\Gamma(\alpha + 1)}\right)^2\right). \tag{39}$$

By repeating this process, we obtain

$$x^{(2)}_{\frac{1}{2}} = \frac{h^{\alpha}}{2^{\alpha}\Gamma(\alpha + 1)}\bigl(1 - \bigl(x^{(1)}_{\frac{1}{2}}\bigr)^2\bigr). \tag{40}$$

Finally, after two iterations we get

$$x_{\frac{1}{2}}^{(2)} = \frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\left(1-\left(\frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\left(1-\left(\frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\right)^2\right)\right)^2\right). \tag{41}$$

With this technique given by (36), the singularity of $x_{\frac{1}{2}}$ will disappear, and with the predictor–corrector method, the approximation of $x(\frac{h}{2})$ will improve. In the following, we put $i = 0$, $\vartheta = 3$, so

$$x_1 = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\alpha} - \frac{(x_{\frac{1}{2}}^{(2)})^2}{2^{\alpha-1}}\right). \tag{42}$$

By substituting (41) into (42), we gain

$$x_1 = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\alpha} - \frac{(\frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}(1-(\frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}(1-(\frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)})^2))^2))^2}{2^{\alpha-1}}\right). \tag{43}$$

Similarly, for $i = 1$, $\vartheta = 1$,

$$x_{\frac{3}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\alpha}\frac{3^\alpha}{2^\alpha} - \frac{2}{3}\left(x_{\frac{1}{2}}^{(2)}\right)^2 - \frac{4}{3}\frac{x_1^2}{2^\alpha}\right). \tag{44}$$

In the following, if $i = 1$, $\vartheta = 2$, the singularity appears at $s = \frac{3h}{2}$, so we have

$$x_{\frac{3}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\left[\frac{1}{\alpha}\frac{3^\alpha}{2^\alpha} - \frac{2}{3}\left(x_{\frac{1}{2}}^{(2)}\right)^2 - \frac{1}{3}\frac{1}{2^\alpha}x_1^2 - \frac{1}{\alpha}\frac{3^\alpha}{2^\alpha}x_{\frac{3}{2}}^2\right]. \tag{45}$$

To obtain $x_{\frac{3}{2}}$, we prefer to use the predictor–corrector method. For such a purpose, we apply the first iteration of $x_{\frac{3}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}(\frac{2^\alpha}{\alpha 3^\alpha} - \frac{2}{3}(x_{\frac{1}{2}}^{(2)})^2 - \frac{4}{3}\frac{x_1^2}{2^\alpha})$ on the right-hand side of (45) as a predictor and consider $x_{\frac{3}{2}} = x_{\frac{3}{2}}^{(0)}$. Equation (45) can now be rewritten as follows:

$$x_{\frac{3}{2}}^{(1)} = \frac{h^\alpha}{\Gamma(\alpha)}\left[\frac{1}{\alpha}\frac{3^\alpha}{2^\alpha} - \frac{2}{3}\left(x_{\frac{1}{2}}^{(2)}\right)^2 - \frac{1}{3}\frac{1}{2^\alpha}x_1^2 - \frac{1}{\alpha}\frac{3^\alpha}{2^\alpha}\left(x_{\frac{3}{2}}^{(0)}\right)^2\right]. \tag{46}$$

By repeating this process, we improve the approximation of $x_{\frac{3}{2}}$ to

$$x_{\frac{3}{2}}^{(2)} = \frac{h^\alpha}{\Gamma(\alpha)}\left[\frac{1}{\alpha}\frac{3^\alpha}{2^\alpha} - \frac{2}{3}\left(x_{\frac{1}{2}}^{(2)}\right)^2 - \frac{1}{3}\frac{1}{2^\alpha}x_1^2 - \frac{1}{\alpha}\frac{3^\alpha}{2^\alpha}\left(x_{\frac{3}{2}}^{(1)}\right)^2\right]. \tag{47}$$

Let us consider $i = 1$, $\vartheta = 3$, and then

$$x_2 = \frac{h^\alpha}{\Gamma(\alpha)}\left[\frac{2^\alpha}{\alpha} - \frac{2}{3}\left(\frac{3}{2}\right)^{\alpha-1}\left(x_{\frac{1}{2}}^{(2)}\right)^2 - \frac{1}{6}x_1^2 - \frac{1}{2^{\alpha-1}}\left(x_{\frac{3}{2}}^{(2)}\right)^2\right]. \tag{48}$$

In the following, if $i = 2$, $\vartheta = 1$,

$$x_{\frac{5}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\left[\frac{1}{\alpha}\frac{5^\alpha}{2^\alpha} - \frac{2^\alpha}{3}\left(x_{\frac{1}{2}}^{(2)}\right)^2 - \frac{3^{\alpha-2}}{2^{\alpha-1}}x_1^2 - \frac{2}{3}\left(x_{\frac{3}{2}}^{(2)}\right)^2 - \frac{1}{3}\frac{x_2^2}{2^{\alpha-2}}\right]. \tag{49}$$

If $i = 2$, $\vartheta = 2$, the singularity is at $s = \frac{5h}{2}$, so we have

$$x_{\frac{5}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\left[\frac{1}{\alpha}\frac{5^\alpha}{2^\alpha} - \frac{2^\alpha}{3}\left(x^{(2)}_{\frac{1}{2}}\right)^2 - \frac{3^{\alpha-2}}{2^{\alpha-1}}x_1^2 - \frac{2}{3}\left(x^{(2)}_{\frac{3}{2}}\right)^2 - \frac{1}{3}\frac{x_2^2}{2^{\alpha-2}} - \frac{1}{\alpha 2^\alpha}x^2_{\frac{5}{2}}\right]. \tag{50}$$

To improve $x_{\frac{5}{2}}$, we prefer to use the predictor–corrector method. For such a purpose, we apply (49) on the right-hand side of (50) as a predictor, and so we consider $x_{\frac{5}{2}} = x^{(0)}_{\frac{5}{2}}$. After two steps we achieve

$$x^{(2)}_{\frac{5}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\left[\frac{1}{\alpha}\frac{5^\alpha}{2^\alpha} - \frac{2^\alpha}{3}\left(x^{(2)}_{\frac{1}{2}}\right)^2 - \frac{3^{\alpha-2}}{2^{\alpha-1}}x_1^2 - \frac{2}{3}\left(x^{(2)}_{\frac{3}{2}}\right)^2 - \frac{1}{3}\frac{x_2^2}{2^{\alpha-2}} - \frac{1}{\alpha 2^\alpha}\left(x^{(1)}_{\frac{5}{2}}\right)^2\right]. \tag{51}$$

Finally, for $i = 2$, $\vartheta = 3$,

$$x_3 = \frac{h^\alpha}{\Gamma(\alpha)}\left[\frac{3^\alpha}{\alpha} - \frac{2}{3}\left(\frac{5}{2}\right)^{\alpha-1}\left(x^{(2)}_{\frac{1}{2}}\right)^2 - \frac{2^\alpha x_1^2}{3} - \frac{3^{\alpha-2}}{2^{\alpha-2}}\left(x^{(2)}_{\frac{3}{2}}\right)^2 - \frac{x_2^2}{6} - \frac{1}{2^\alpha}\left(x^{(2)}_{\frac{5}{2}}\right)^2\right]. \tag{52}$$

In Example 1, the computed x_j, $j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$, form an approximation to the solution of the Eqs. (32)–(33) for $0 < \alpha \leq 1$, but x_3 is considered as the final approximation of the solution. In Tables 1 and 2, a p-stage Runge–Kutta method, for $p = 4$, which has led to an approximate solution, is called NRK4. Also, we present the solution achieved by three iterations from the modified variational iteration method (MVIM) [21] as well as the solution obtained by four terms from the modified homotopy perturbation method (HPM) [18]. A comparison of the results from NRK4 and HPM shows that the results obtained by NRK4 are more accurate in the closed interval $[0.6, 1]$, and have less variation in relative error. A comparison of the results of NRK4 and MVIM shows that the results of NRK4 are more accurate in the closed interval $[0.8, 1]$. To get any desired accuracy, we could proceed with this method and use more iterations; however, the relative errors are already small enough to be satisfied. It appears that the introduced modified Runge–Kutta method can be relatively accurate. The gained results are shown in Tables 1 and 2. It must be noted that $x(s)$ is an exact solution, for $\alpha = 1$.

Figure 1 shows a comparison between the exact solution and the numerical solution resulted from NRK4, for $\alpha = 1$. A comparison between the gained approximate solution by NRK4 and exact solution shows that the maximum of relative error happens at the last point and is less than or equal to 2.0523E–3. Moreover, the approximations of the solutions for various values of α are shown in Fig. 2. In (32), when α varies from 0 to 1, the approximate solution that is gained for a given α changes. For example, we suppose the arbitrary values of $\alpha = 0.25, 0.5, 0.75, 1$ in Fig. 2.

Table 1 The results of different methods for Example 1, $\alpha = 1$

s	$x(s)$	NRK4	HPM	MVIM
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.197375	0.197397	0.197375	0.197375
0.4	0.379949	0.380052	0.379944	0.379946
0.5	0.462117	0.462236	0.462078	0.462101
0.6	0.537050	0.537114	0.536857	0.537049
0.8	0.664037	0.663610	0.661706	0.663300
1.0	0.761594	0.760031	0.746032	0.757165

Table 2 Relative errors for Example 1, $\alpha = 1$

s	$x(s)$	NRK4	HPM	MIVM
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.197375	1.1146E–4	0.000000	0.000000
0.4	0.379949	2.7111E–4	1.3160E–5	7.8958E–6
0.5	0.462117	2.5751E–4	8.4394E–5	3.4623E–5
0.6	0.537050	1.1192E–4	3.5938E–4	1.8620E–6
0.8	0.664037	6.4304E–4	3.5103E–3	1.1099E–3
1.0	0.761594	2.0523E–3	2.0433E–2	5.8154E–3

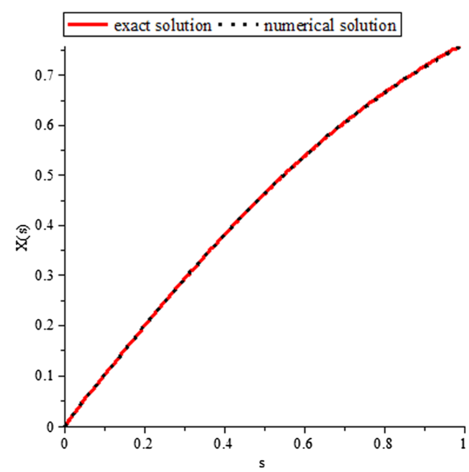

Figure 1 Exact and numerical solutions of Example 1

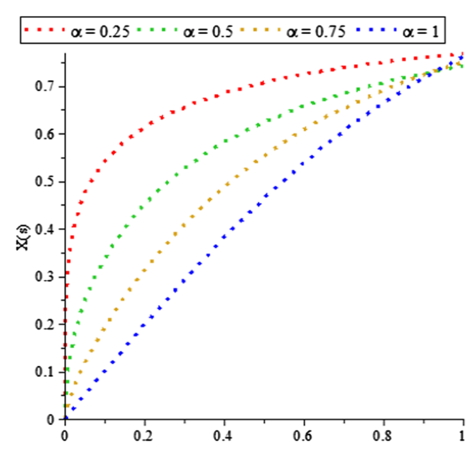

Figure 2 Numerical solutions of Example 1 for various values of $0 < \alpha \leq 1$

Example 2 Consider the following fractional Riccati differential equation:

$$D^{\alpha}x(s) = 1 + 2x(s) - x^2(s), \quad 0 \leq s \leq 1, 0 < \alpha \leq 1, \tag{53}$$

with the initial condition

$$x(0) = 0. \tag{54}$$

The exact solution, for $\alpha = 1$, is

$$x(s) = 1 + \sqrt{2}\tanh\left[\sqrt{2}s + \frac{1}{2}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)\right]. \tag{55}$$

According to Remark 1, (53) can be written in the following form:

$$x(s) = y(s) + \int_0^s k\bigl(s,t,x(t)\bigr)\,dt, \quad 0 \le s \le 1, \tag{56}$$

where $y(s) = \frac{s^\alpha}{\Gamma(\alpha+1)}$, $k(s,t,x(t)) = K(s,t)[\beta x^2(t) + \gamma x(t)]$, $K(s,t) = \frac{1}{(s-t)^{1-\alpha}}$, $\beta = -\frac{1}{\Gamma(\alpha)}$, and $\gamma = \frac{2}{\Gamma(\alpha)}$. We put $i = 0$, $\vartheta = 1$, in (31), and have $x_{\frac{1}{2}} = \frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}$. In the next step, if $i = 0$, $\vartheta = 2$, the singularity happens at $s = \frac{h}{2}$, so the second $x_{\frac{1}{2}}$ can be obtained as follows:

$$\begin{aligned} x_{\frac{1}{2}} = {}& \frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)} + h\beta\sum_{\substack{l=0\\l\neq 1}}^{1} A_{2l}k(s_0+\theta_2 h, s_0+\theta_l h)\bigl(x^2_{0+\theta_l} - x^2_{0+\theta_2}\bigr) \\ & + h\gamma\sum_{\substack{l=0\\l\neq 1}}^{1} A_{2l}k(s_0+\theta_2 h, s_0+\theta_l h)(x_{0+\theta_l} - x_{0+\theta_2}) \\ & + \bigl(\beta x^2_{0+\theta_2} + \gamma x_{0+\theta_2}\bigr)q(s_0+\theta_2 h), \end{aligned} \tag{57}$$

and hence the following quadratic polynomial will be obtained from (57):

$$x_{\frac{1}{2}} = \frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\bigl(1 + 2x_{\frac{1}{2}} - x^2_{\frac{1}{2}}\bigr). \tag{58}$$

To get $x_{\frac{1}{2}}$, we use the predictor–corrector method. We apply the first iteration of $x_{\frac{1}{2}} = \frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}$ on the right-hand side of (58) as a predictor. Let us consider $x_{\frac{1}{2}} = x^{(0)}_{\frac{1}{2}}$, so Eq. (58) can be rewritten as follows:

$$x^{(1)}_{\frac{1}{2}} = \frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\bigl(1 + 2x^{(0)}_{\frac{1}{2}} - \bigl(x^{(0)}_{\frac{1}{2}}\bigr)^2\bigr), \tag{59}$$

i.e.,

$$x^{(1)}_{\frac{1}{2}} = \frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\left(1 + \frac{2h^\alpha}{2^\alpha\Gamma(\alpha+1)} - \left(\frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\right)^2\right). \tag{60}$$

By repeating this process, we obtain

$$x^{(2)}_{\frac{1}{2}} = \frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\bigl(1 + 2x^{(1)}_{\frac{1}{2}} - \bigl(x^{(1)}_{\frac{1}{2}}\bigr)^2\bigr). \tag{61}$$

Finally, after two iterations, we get

$$\begin{aligned} x^{(2)}_{\frac{1}{2}} = {}& \frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\left(1 + \frac{2h^\alpha}{2^\alpha\Gamma(\alpha+1)}\left(1 + \frac{2h^\alpha}{2^\alpha\Gamma(\alpha+1)} - \left(\frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\right)^2\right)\right. \\ & \left. - \left(\frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\left(1 + \frac{2h^\alpha}{2^\alpha\Gamma(\alpha+1)} - \left(\frac{h^\alpha}{2^\alpha\Gamma(\alpha+1)}\right)^2\right)\right)^2\right). \end{aligned} \tag{62}$$

With the predictor–corrector method, the approximation of $x(\frac{h}{2})$ will improve, and with the proposed technique, given in (57), the singularity of $x_{\frac{1}{2}}$ will disappear. In the following, we put $i = 0$, $\vartheta = 3$, so

$$x_1 = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{1}{\alpha} + \frac{(2x^{(2)}_{\frac{1}{2}} - (x^{(2)}_{\frac{1}{2}})^2)}{2^{\alpha-1}}\right), \tag{63}$$

and, by substituting (62) into (63), x_1 will be obtained. Similarly, for $i = 1$, $\vartheta = 1$,

$$x_{\frac{3}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{3^\alpha}{\alpha 2^\alpha} + \frac{(2x^{(2)}_{\frac{1}{2}} - (x^{(2)}_{\frac{1}{2}})^2)}{3} + \frac{(2x_1 - x_1^2)}{32^\alpha} + \frac{(2x_1 - x_1^2)}{2^\alpha}\right). \tag{64}$$

In the following, if $i = 1$, $\vartheta = 2$, the singularity happens at $s = \frac{3h}{2}$, so by using (31), we have

$$x_{\frac{3}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{3^\alpha}{\alpha 2^\alpha} + \frac{2(2x^{(2)}_{\frac{1}{2}} - (x^{(2)}_{\frac{1}{2}})^2)}{3} + \frac{(2x_1 - x_1^2)}{32^\alpha} + \frac{(2x_{\frac{3}{2}} - x_{\frac{3}{2}}^2)}{\alpha 2^\alpha}\right), \tag{65}$$

where now the predictor–corrector method will be applied to get and improve $x_{\frac{3}{2}}$. For such a purpose, we apply the first iteration of $x_{\frac{3}{2}}$ on the right-hand side of (65) as a predictor and consider $x_{\frac{3}{2}} = x^{(0)}_{\frac{3}{2}}$. Equation (65) will be rewritten as follows:

$$x^{(1)}_{\frac{3}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{3^\alpha}{\alpha 2^\alpha} + \frac{2(2x^{(2)}_{\frac{1}{2}} - (x^{(2)}_{\frac{1}{2}})^2)}{3} + \frac{(2x_1 - x_1^2)}{32^\alpha} + \frac{(2x^{(0)}_{\frac{3}{2}} - (x^{(0)}_{\frac{3}{2}})^2)}{\alpha 2^\alpha}\right). \tag{66}$$

Finally, after two iterations, we get

$$x^{(2)}_{\frac{3}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{3^\alpha}{\alpha 2^\alpha} + \frac{2(2x^{(2)}_{\frac{1}{2}} - (x^{(2)}_{\frac{1}{2}})^2)}{3} + \frac{(2x_1 - x_1^2)}{32^\alpha} + \frac{(2x^{(1)}_{\frac{3}{2}} - (x^{(1)}_{\frac{3}{2}})^2)}{\alpha 2^\alpha}\right). \tag{67}$$

Let us consider $i = 1$, $\vartheta = 3$, and then

$$x_2 = \frac{h^\alpha}{\Gamma(\alpha)}\left(\frac{2^\alpha}{\alpha} + \frac{3^{\alpha-2}(2x^{(2)}_{\frac{1}{2}} - (x^{(2)}_{\frac{1}{2}})^2)}{2^{\alpha-2}} + \frac{(2x_1 - x_1^2)}{6} + \frac{(2x^{(2)}_{\frac{3}{2}} - (x^{(2)}_{\frac{3}{2}})^2)}{2^{\alpha-1}}\right). \tag{68}$$

In the following, if $i = 2$, $\vartheta = 1$, then

$$\begin{aligned} x_{\frac{5}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\Bigg(&\frac{5^\alpha}{\alpha 2^\alpha} + \frac{2^\alpha(2x^{(2)}_{\frac{1}{2}} - (x^{(2)}_{\frac{1}{2}})^2)}{3} + \frac{3^{\alpha-2}(2x_1 - x_1^2)}{2^{\alpha-1}} + \frac{2(2x^{(2)}_{\frac{3}{2}} - (x^{(2)}_{\frac{3}{2}})^2)}{3} \\ &+ \frac{(2x_2 - x_2^2)}{32^\alpha} + \frac{(2x_2 - x_2^2)}{2^\alpha}\Bigg). \end{aligned} \tag{69}$$

If $i = 2$, $\vartheta = 2$, the singularity happens at $s = \frac{5h}{2}$, so by using (31), we derive

$$\begin{aligned} x_{\frac{5}{2}} = \frac{h^\alpha}{\Gamma(\alpha)}\Bigg(&\frac{5^\alpha}{\alpha 2^\alpha} + \frac{2^\alpha(2x^{(2)}_{\frac{1}{2}} - (x^{(2)}_{\frac{1}{2}})^2)}{3} + \frac{3^{\alpha-2}(2x_1 - x_1^2)}{2^{\alpha-1}} + \frac{2(2x^{(2)}_{\frac{3}{2}} - (x^{(2)}_{\frac{3}{2}})^2)}{3} \\ &+ \frac{(2x_2 - x_2^2)}{32^\alpha} + \frac{(2x_{\frac{5}{2}} - x_{\frac{5}{2}}^2)}{\alpha 2^\alpha}\Bigg), \end{aligned} \tag{70}$$

where now the predictor–corrector method will be applied to get and improve $x_{\frac{5}{2}}$. For such a purpose, we apply the first iteration of $x_{\frac{5}{2}}$ on the right-hand side of (70) as a predictor and consider $x_{\frac{5}{2}} = x_{\frac{5}{2}}^{(0)}$. After two iterations, Eq. (70) will be transformed as follows:

$$\begin{aligned} x_{\frac{5}{2}}^{(2)} = \frac{h^{\alpha}}{\Gamma(\alpha)}\biggl(&\frac{5^{\alpha}}{\alpha 2^{\alpha}} + \frac{2^{\alpha}(2x_{\frac{1}{2}}^{(2)} - (x_{\frac{1}{2}}^{(2)})^2)}{3} + \frac{3^{\alpha-2}(2x_1 - x_1^2)}{2^{\alpha-1}} + \frac{2(2x_{\frac{3}{2}}^{(2)} - (x_{\frac{3}{2}}^{(2)})^2)}{3} \\ &+ \frac{(2x_2 - x_2^2)}{32^{\alpha}} + \frac{(2x_{\frac{5}{2}}^{(1)} - x_{\frac{5}{2}}^{(1)})}{\alpha 2^{\alpha}}\biggr). \end{aligned} \tag{71}$$

Finally, for $i = 2$, $\vartheta = 3$,

$$\begin{aligned} x_3 = \frac{h^{\alpha}}{\Gamma(\alpha)}\biggl(&\frac{3^{\alpha}}{\alpha} + \frac{5^{\alpha-1}(2x_{\frac{1}{2}}^{(2)} - (x_{\frac{1}{2}}^{(2)})^2)}{32^{\alpha-2}} + \frac{2^{\alpha-1}(2x_1 - x_1^2)}{3} + \frac{3^{\alpha-2}(2x_{\frac{3}{2}}^{(2)} - (x_{\frac{3}{2}}^{(2)})^2)}{2^{\alpha-2}} \\ &+ \frac{(2x_2 - x_2^2)}{6} + \frac{(2x_{\frac{5}{2}}^{(2)} - x_{\frac{5}{2}}^{(2)})}{2^{\alpha-1}}\biggr). \end{aligned} \tag{72}$$

In Example 2, the computed x_j, $j = \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3$, form an approximation to the solution of Eqs. (53)–(54) for $0 < \alpha \leq 1$, but x_3 is considered as the final approximation to the solution. The results of NRK4 have been compared with the results of the other two methods, HPM and MVIM. The solution gained by four terms is from HPM [18]. The solution achieved by three iterations comes from MVIM [21]. A comparison of the results of applying NRK4 and HPM shows that the results of NRK4 are almost as accurate as those of HPM in the closed interval $[0, 1]$. A comparison of the results of applying NRK4 and MVIM shows that the results of MVIM are more accurate in the closed interval $[0, 1]$. If we proceed with this method and use more iterations, we can get any desired accuracy. The obtained results have been shown in Tables 3 and 4. It must be noted that $x(s)$ is an exact solution, for $\alpha = 1$.

Table 3 The results of different methods for Example 2, $\alpha = 1$

s	$x(s)$	NRK4	HPM	MVIM
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.241977	0.240711	0.241965	0.241978
0.4	0.567812	0.563632	0.568115	0.567845
0.5	0.756014	0.749954	0.757564	0.756087
0.6	0.953566	0.945251	0.958259	0.953666
0.8	1.346364	1.332255	1.365240	1.346379
1.0	1.689498	1.668815	1.723810	1.686028

Table 4 Relative errors for Example 2, $\alpha = 1$

s	$x(s)$	NRK4	HPM	MIVM
0.0	0.000000	0.000000	0.000000	0.000000
0.2	0.241977	5.2319E–3	4.9592E–5	4.1326E–6
0.4	0.567812	7.3616E–3	5.3363E–4	5.8118E–5
0.5	0.756014	8.0157E–3	2.0502E–3	9.6559E–5
0.6	0.953566	8.7199E–3	4.9215E–3	1.0487E–4
0.8	1.346364	1.0479E–2	1.4020E–2	1.1141E–5
1.0	1.689498	1.2242E–2	2.0309E–2	2.0539E–3

Figure 3 Exact and numerical solutions of Example 2

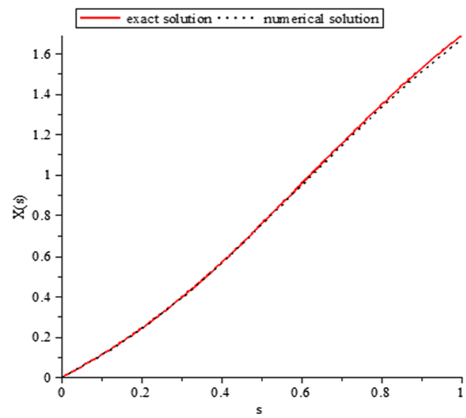

Figure 4 Numerical solutions of Example 2 for various values of $0 < \alpha \leq 1$

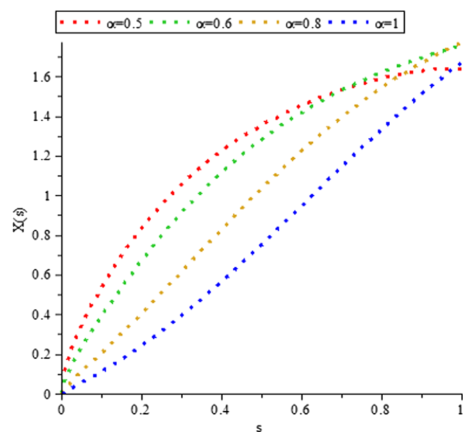

Figure 3 shows a comparison between the exact solution and the numerical solution resulted from NRK4, for $\alpha = 1$. A comparison between the gained approximate solution by NRK4 and the exact solution shows that the maximum of relative error happens at the last point and is less than or equal to 1.2242E–2. Moreover, the approximations of the solutions for various values of α are shown in Fig. 2. In (53), when α varies from 0 to 1, the approximate solution that was obtained for different α changes. For example, we suppose the arbitrary values of $\alpha = 0.5, 0.6, 0.8, 1$ in Fig. 4.

All calculations have been done using Maple on a computer with Intel Core i5-2430M CPU at 2.400 GHz, 4.00 GB of RAM and 64-bit operating system (Windows 7).

5 Convergence analysis

The convergence analysis of NRKp is the same as for a p-stage Runge–Kutta method; we refer the reader to [49], but, first, we need some preliminaries on convergence of the proposed approach. The solution of (11) with the special nonlinear kernel, $k(s,t,x(t)) = K(s,t)[\beta x^2(t) + \gamma x(t)]$, where $K(s,t) = \frac{K_0(s,t)}{(s-t)^{1-\alpha}}$, $\beta, \gamma \in R$, $0 < \alpha \leq 1$, is not differentiable when $s = t$; therefore, the rate of convergence and accuracy of proposed approach may be decreased, so when we utilize NRKp to solve nonlinear Volterra integral equations of the second kind, the predictor–corrector method needs to be applied at the last point $s = t$. In this section, to simplify the proof of convergence, we consider $k_0(s,t) = c_0$, $c_0 \in \mathbb{R}$.

Lemma 2 *Let*

$$|\beta|\left|x(t_1)+x(t_2)\right|+|\gamma|\leq\frac{LM}{|c_0|},\quad \beta,\gamma,c_0\in\mathbb{R}, \tag{73}$$

and $x(t_1)\neq x(t_2)$. *Then the nonlinear kernel*, $k(s,t,x(t))=(\beta x^2(t)+\gamma x(t))k(s,t)$, *satisfies a Lipschitz condition with respect to the dependent variable* x, *i.e.*,

$$\left|k\big(s,t_1,x(t_1)\big)-k\big(s,t_2,x(t_2)\big)\right|\leq L\left|x(t_1)-x(t_2)\right|, \tag{74}$$

where L *is Lipschitz constant*, $M=\frac{(s-t_2)^{1-\alpha}}{(b-a)^{1-\alpha}}$, $k(s,t)=\frac{c_0}{(s-t)^{1-\alpha}}$, *and* $a\leq t_1<t_2<s\leq b$.

Proof Write

$$\left|k\big(s,t_1,x(t_1)\big)-k\big(s,t_2,x(t_2)\big)\right|=|c_0|\left|\frac{\beta x^2(t_1)+\gamma x(t_1)}{(s-t_1)^{1-\alpha}}-\frac{\beta x^2(t_2)+\gamma x(t_2)}{(s-t_2)^{1-\alpha}}\right|, \tag{75}$$

then, according to the assumption, we have $b-a\geq s-t_1>s-t_2>0$, so $\frac{1}{s-t_2}>\frac{1}{s-t_1}\geq\frac{1}{b-a}$, and $\frac{1}{(s-t_2)^{1-\alpha}}>\frac{1}{(s-t_1)^{1-\alpha}}\geq\frac{1}{(b-a)^{1-\alpha}}$. In the following, we can write

$$\begin{aligned}
&\left|k\big(s,t_1,x(t_1)\big)-k\big(s,t_2,x(t_2)\big)\right| \\
&\quad\leq|c_0|\frac{|(s-t_2)^{1-\alpha}(\beta x^2(t_1)+\gamma x(t_1))-(s-t_1)^{1-\alpha}(\beta x^2(t_2)+\gamma x(t_2))|}{|(s-t_1)^{1-\alpha}(s-t_2)^{1-\alpha}|} \\
&\quad\leq|c_0|\frac{|(s-t_2)^{1-\alpha}(\beta x^2(t_1)+\gamma x(t_1))-(s-t_1)^{1-\alpha}(\beta x^2(t_2)+\gamma x(t_2))|}{|(s-t_2)^{2(1-\alpha)}|} \\
&\quad\leq|c_0|\frac{|(b-a)^{1-\alpha}|(|\beta(x^2(t_1)-x^2(t_2))|+|\gamma(x(t_1)-x(t_2))|)}{|(s-t_2)^{2(1-\alpha)}|} \\
&\quad\leq|c_0|\frac{|(b-a)^{1-\alpha}|(|\beta||(x(t_1)+x(t_2))|+|\gamma|)(|x(t_1)-x(t_2)|)}{|(s-t_2)^{2(1-\alpha)}|}.
\end{aligned} \tag{76}$$

Substituting (73) into (76), the lemma is proved. □

Theorem 2 *Let* $|\beta||x(t_1)+x(t_2)|+|\gamma|\leq\frac{LM}{|c_0|}$, β, γ, $c_0\in\mathbb{R}$, *then NRKp with the a special nonlinear kernel*, $k(s,t,x(t))=(\beta x^2(t)+\gamma x(t))k(s,t)$, *where* $k(s,t)=\frac{c_0}{(s-t)^{1-\alpha}}$, *is convergent; in other words*,

$$\lim_{h\to 0}\left|x(s_i+\theta_\vartheta h)-x_{i+\theta_\vartheta}\right|=0. \tag{77}$$

Proof Let $s=s_i+\theta_\vartheta h$, $i=0,1,\dots,N-1$, $\vartheta=1,2,\dots,p-1$, in (22). Then we have

$$\begin{aligned}
x(s_i+\theta_\vartheta h)={}&y(s_i+\theta_\vartheta h)+\sum_{j=0}^{i-1}\int_{s_j}^{s_{j+1}}k\big(s_i+\theta_\vartheta h,t,x(t)\big)\,dt \\
&+\int_{s_i}^{s_i+\theta_\vartheta h}k\big(s_i+\theta_\vartheta h,t,x(t)\big)\,dt.
\end{aligned} \tag{78}$$

Letting $h < \delta$, $e_{i+\theta_\vartheta} = x(s_i + \theta_\vartheta h) - x_{i+\theta_\vartheta}$, and subtracting (30) from (78), we obtain

$$\begin{aligned} e_{i+\theta_\vartheta} &= h\sum_{j=0}^{i-1}\sum_{l=0}^{p-1} A_{pl}\bigl\{k\bigl(s_i + \theta_\vartheta h, s_j + \theta_l h, x(s_j + \theta_l h)\bigr) - k(s_i + \theta_\vartheta h, s_j + \theta_l h, x_{j+\theta_l})\bigr\} \\ &\quad + h\sum_{l=0}^{\vartheta-1} A_{\vartheta l}\bigl\{k\bigl(s_i + \theta_\vartheta h, s_i + \theta_l h, x(s_i + \theta_l h)\bigr) - k(s_i + \theta_\vartheta h, s_i + \theta_l h, x_{i+\theta_l})\bigr\}, \end{aligned} \tag{79}$$

where

$$\begin{aligned} &k\bigl(s_i + \theta_\vartheta h, s_j + \theta_l h, x(s_j + \theta_l h)\bigr) = \frac{k_0(\beta x^2(s_j + \theta_l h) + \gamma x(s_j + \theta_l h))}{((i-j) + (\theta_\vartheta - \theta_l)h)^{1-\alpha}}, \\ &k(s_i + \theta_\vartheta h, s_j + \theta_l h, x_{j+\theta_l}) = \frac{k_0(\beta x^2_{j+\theta_l} + \gamma x_{j+\theta_l})}{((i-j) + (\theta_\vartheta - \theta_l)h)^{1-\alpha}}, \\ &k\bigl(s_i + \theta_\vartheta h, s_i + \theta_l h, x(s_i + \theta_l h)\bigr) = \frac{k_0(\beta x^2(s_i + \theta_l h) + \gamma x(s_i + \theta_l h))}{((\theta_\vartheta - \theta_l)h)^{1-\alpha}}, \\ &k(s_i + \theta_\vartheta h, s_i + \theta_l h, x_{i+\theta_l}) = \frac{k_0(\beta x^2_{i+\theta_l} + \gamma x_{i+\theta_l})}{((\theta_\vartheta - \theta_l)h)^{1-\alpha}}. \end{aligned} \tag{80}$$

Clearly, from Lemma 2 and [49], the theorem is proved. □

6 Conclusion

In this work, a class of Runge–Kutta methods has been successfully implemented for solving fractional Riccati differential equations. In the first step, we convert a fractional Riccati differential equation into a singular nonlinear Volterra integral equation of the second kind considering Remark 1. In the second step, we solve singular nonlinear Volterra integral equation of the second kind using the Runge–Kutta method, with some manipulation. We called proposed approach a new p-stage Runge–Kutta method (NRKp). We implemented NRKp, for $p = 4$, to get approximate solutions of two examples. In these two examples, x_3 is considered as an approximate solution, and obtained for different α. At singular points, to improve the accuracy of the approximate solution, we utilize the predictor–corrector method. The application results are compared with those of HPM and MVIM (Tables 1–4). A comparison of the approximate solutions shows that this method is accurate enough and can even be more accurate in many instances. We proved that the nonlinear kernel satisfies a Lipschitz condition, and then obtained that NRKp for a singular nonlinear Volterra integral equation of the second kind is convergent. As a direction for future research, we point that the final aim of presenting NRK4 is not only for finding approximate solutions to fractional Riccati differential equations but also for emphasizing that this approach can be applied for fractional differential equations, as well as singular Volterra integral equations. Moreover, this method can be used for solving those nonlinear Volterra integral equations of the second kind, which have a singular kernel, such as $k(s,t,x(t)) = K(s,t)[a_0x(t) + a_1x^2(t) + \cdots + a_{n-1}x^n(t)]$, where $k(s,t) = \frac{K_0(s,t)}{(s-t)^{1-\alpha}}$, $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$, $n \in \mathbb{N}$, $0 < \alpha \leq 1$.

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