Agarwal et al. *Advances in Difference Equations* (2018) 2018:342
https://doi.org/10.1186/s13662-018-1802-9

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence results for multi-term fractional differential equations with nonlocal multi-point and multi-strip boundary conditions

Ravi P. Agarwal[1,2,3], Ahmed Alsaedi[3], Najla Alghamdi[3,4], Sotiris K. Ntouyas[3,5] and Bashir Ahmad[3*]

*Correspondence:
bashirahmad_qau@yahoo.com
[3]Nonlinear Analysis and Applied Mathematics (NAAM)—Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia
Full list of author information is available at the end of the article

Abstract
In this paper, we discuss the existence and uniqueness of solutions for a new class of multi-point and multi-strip boundary value problems of multi-term fractional differential equations by using standard fixed point theorems. We demonstrate the application of the obtained results with the aid of examples. Some new results are also deduced by fixing the parameters involved in the problem at hand.

MSC: 34A08; 34B10

Keywords: Caputo fractional derivative; Multi-term fractional differential equations; Multi-point and multi-strip boundary conditions; Existence; Fixed point

1 Introduction

Multi-term fractional differential equations involve more than one fractional order differential operators and appear in the mathematical models of many real world problems. Bagley–Torvik [1] and Basset equations [2] are important examples of this class of equations.

Fractional differential equations find useful applications in several disciplines of science and engineering such as blood flow phenomena, virology, bio-engineering, image processing, control theory, etc. For details and examples, see [3–7].

The literature on initial and boundary value problems of differential equations and inclusions containing a single fractional order operator is now much enriched and one can find useful results in a series of articles [8–19] and the references cited therein. However, the topic of boundary value problems of differential equations and inclusions containing more than two fractional order operators needs to be investigated. For some works on differential equations and inclusions involving two fractional order operators (sequential fractional differential equations) can be found in [20–23].

In this paper, we introduce and investigate a new boundary value problem of multi-term fractional differential equations supplemented with nonlocal multi-point and multi-strip boundary conditions given by

$$\left(\delta_2 {}^cD^{\alpha+2} + \delta_1 {}^cD^{\alpha+1} + \delta_0 {}^cD^{\alpha}\right)x(t) = f\left(t, x(t)\right), \quad 0 < \alpha < 1, 0 < t < 1, \tag{1.1}$$

$$x(0) = 0, \qquad x(\xi) = \sum_{i=1}^{n} j_i x(\eta_i), \qquad x(1) = \sum_{i=1}^{k} \lambda_i \int_{\upsilon_i}^{\sigma_i} x(s)\,ds, \tag{1.2}$$

where ${}^{c}D^{\alpha}$ denotes the Caputo fractional derivative of order α, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a given continuous function, $0 < \xi < \eta_1 < \eta_2 < \cdots < \eta_n < \upsilon_1 < \sigma_1 < \upsilon_2 < \sigma_2 < \cdots < \upsilon_k < \sigma_k < 1$, $j_i \in \mathbb{R}$, $i = 1,\dots,n$, $\lambda_i \in \mathbb{R}$, $i = 1,\dots,k$, δ_i are real numbers $\{i = 0,1,2\}$, with $\delta_2 \neq 0$.

The rest of the paper is organized as follows. In Sect. 2, we recall some preliminary ideas of fractional calculus and prove some important lemmas. Section 3 contains existence and uniqueness results for the problem (1.1)–(1.2) with $\delta_1^2 - 4\delta_0\delta_2 > 0$, which are obtained by applying some well-known theorems of the fixed point theory. Though the tools of the fixed point theory are standard, their exposition helps to develop the existence theory for the given problem. In Sects. 4 and 5, we outline the idea for dealing with the problem (1.1)–(1.2) involving the cases $\delta_1^2 - 4\delta_0\delta_2 = 0$ and $\delta_1^2 - 4\delta_0\delta_2 < 0$, respectively. The last section describes the importance and the scope of the obtained work.

2 Basic results

Before presenting some auxiliary results, let us recall some preliminary concepts of fractional calculus [24, 25].

Definition 2.1 Let g be a locally integrable real-valued function on $-\infty \leq a < t < b \leq +\infty$. The Riemann–Liouville fractional integral I_a^q of order $q \in \mathbb{R}$ $(q > 0)$ is defined as

$$I_a^q g(t) = (g * K_q)(t) = \frac{1}{\Gamma(q)} \int_a^t (t-s)^{q-1} g(s)\,ds,$$

where $K_q(t) = \frac{t^{q-1}}{\Gamma(q)}$, Γ denotes the Euler gamma function.

Definition 2.2 Let $g \in L^1[a,b]$, $-\infty \leq a < t < b \leq +\infty$ and $g * K_{m-q} \in W^{m,1}[a,b]$, $m = [q]+1$, $q > 0$, where $W^{m,1}[a,b]$ is the Sobolev space defined as

$$W^{m,1}[a,b] = \left\{ g \in L^1[a,b] : \frac{d^m}{dt^m} g \in L^1[a,b] \right\}.$$

The Riemann–Liouville fractional derivative D_a^q of order $q > 0$ $(m-1 < q < m,\ m \in \mathbb{N})$ is defined as

$$D_a^q g(t) = \frac{d^m}{dt^m} I_a^{1-q} g(t) = \frac{1}{\Gamma(m-q)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-q} g(s)\,ds.$$

Definition 2.3 Let $g \in L^1[a,b]$, $-\infty \leq a < t < b \leq +\infty$ and $g * K_{m-q} \in W^{m,1}[a,b]$, $m = [q]$, $q > 0$. The Caputo fractional derivative ${}^{c}D_a^q$ of order $q \in \mathbb{R}$ $(m-1 < q < m,\ m \in \mathbb{N})$ is defined as

$${}^{c}D_a^q g(t) = D_a^q \left[g(t) - g(a) - g'(a)\frac{(t-a)}{1!} - \cdots - g^{(m-1)}(a)\frac{(t-a)^{m-1}}{(m-1)!} \right].$$

If $g \in C^m[a,b]$, then the Caputo fractional derivative ${}^{c}D_a^q$ of order $q \in \mathbb{R}$ $(m-1 < q < m,\ m \in \mathbb{N})$ is defined as

$${}^{c}D_a^q[g](t) = I_a^{1-q} f^{(m)}(t) = \frac{1}{\Gamma(m-q)} \int_a^t (t-s)^{m-1-q} g^{(m)}(s)\,ds.$$

In the sequel, the Riemann–Liouville fractional integral I_a^q and the Caputo fractional derivative ${}^cD_a^q$ with $a = 0$ are respectively denoted by I^q and ${}^cD^q$.

Property 2.4 ([24]) With the given notations, the following equality holds:

$$I^q\big({}^cD^q\varphi(t)\big) = \varphi(t) - c_0 - c_1 t - \cdots - c_{n-1}t^{n-1}, \quad t > 0, n-1 < q < n, \tag{2.1}$$

where c_i ($i = 1, \dots, n-1$) are arbitrary constants.

Definition 2.5 A function $x \in C^3[0,1]$ satisfying (1.1)–(1.2) is called a solution of this problem on $[0,1]$.

The following lemma associated with the linear variant of problem (1.1)–(1.2) plays an important role in the sequel.

Lemma 2.6 *For any $y \in C([0,1],\mathbb{R})$ and $\delta_1^2 - 4\delta_0\delta_2 > 0$, the solution of linear multi-term fractional differential equation*

$$\big(\delta_2 {}^cD^{\alpha+2} + \delta_1 {}^cD^{\alpha+1} + \delta_0 {}^cD^{\alpha}\big)x(t) = y(t), \quad 0 < \alpha < 1, 0 < t < 1, \tag{2.2}$$

supplemented with the boundary conditions (1.2) is given by

$$\begin{aligned} x(t) = \frac{1}{\hat{\delta}}\Bigg\{ & \int_0^t \int_0^s \big(e^{m_2(t-s)} - e^{m_1(t-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds \\ & + \rho_1(t)\Bigg[\int_0^{\xi} \int_0^s \big(e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds \\ & - \sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s \big(e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds\Bigg] \\ & + \rho_2(t)\Bigg[\int_0^1 \int_0^s \big(e^{m_2(1-s)} - e^{m_1(1-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds \\ & - \sum_{i=1}^k \lambda_i \int_{\upsilon_i}^{\sigma_i} \int_0^s \bigg(\frac{(e^{m_2(\sigma_i-s)} - 1)}{m_2} - \frac{(e^{m_1(\sigma_i-s)} - 1)}{m_1}\bigg)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds\Bigg]\Bigg\}, \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} & m_1 = \frac{-\delta_1 - \sqrt{\delta_1^2 - 4\delta_0\delta_2}}{2\delta_2}, \qquad m_2 = \frac{-\delta_1 + \sqrt{\delta_1^2 - 4\delta_0\delta_2}}{2\delta_2}, \\ & \delta_2(m_2 - m_1) = \frac{\sqrt{\delta_1^2 - 4\delta_0\delta_2}}{2} = \hat{\delta}, \\ & \rho_1(t) = \frac{\omega_4\varrho_1(t) - \omega_3\varrho_2(t)}{\mu_1}, \qquad \rho_2(t) = \frac{\omega_1\varrho_2(t) - \omega_2\varrho_1(t)}{\mu_1}, \\ & \varrho_1(t) = \frac{\delta_2[m_1(1 - e^{m_2 t}) - m_2(1 - e^{m_1 t})]}{\delta_0}, \\ & \varrho_2(t) = \hat{\delta}\big(e^{m_2 t} - e^{m_1 t}\big), \qquad m_1 m_2 = \frac{\delta_0}{\delta_2}, \qquad \mu_1 = \omega_1\omega_4 - \omega_2\omega_3 \neq 0, \end{aligned}$$

$$\begin{aligned}
\omega_1 &= \frac{\delta_2}{\delta_0}\Bigg[m_2\Bigg(1-\sum_{i=1}^{n} j_i - e^{m_1\xi} + \sum_{i=1}^{n} j_i e^{m_1\eta_i}\Bigg) \\
&\quad - m_1\Bigg(1-\sum_{i=1}^{n} j_i - e^{m_2\xi} + \sum_{i=1}^{n} j_i e^{m_2\eta_i}\Bigg)\Bigg], \\
\omega_2 &= \hat{\delta}\Bigg(e^{m_1\xi} - e^{m_2\xi} - \sum_{i=1}^{n} j_i e^{m_1\eta_i} + \sum_{i=1}^{n} j_i e^{m_2\eta_i}\Bigg), \\
\omega_3 &= \frac{\delta_2}{\delta_0}\Bigg[m_2\Bigg(1-e^{m_1} - \sum_{i=1}^{k}\lambda_i(\sigma_i-\upsilon_i) + \sum_{i=1}^{k}\lambda_i/m_1\big(e^{m_1\sigma_i}-e^{m_1\upsilon_i}\big)\Bigg) \\
&\quad - m_1\Bigg(1-e^{m_2} - \sum_{i=1}^{k}\lambda_i(\sigma_i-\upsilon_i) + \sum_{i=1}^{k}\lambda_i/m_2\big(e^{m_2\sigma_i}-e^{m_2\upsilon_i}\big)\Bigg)\Bigg], \\
\omega_4 &= \hat{\delta}\Bigg(e^{m_1} - e^{m_2} - \sum_{i=1}^{k}\lambda_i/m_1\big(e^{m_1\sigma_i}-e^{m_1\upsilon_i}\big) + \sum_{i=1}^{k}\lambda_i/m_2\big(e^{m_2\sigma_i}-e^{m_2\upsilon_i}\big)\Bigg).
\end{aligned} \tag{2.4}$$

Proof Applying the operator I^α on (2.2) and using (2.1), we get

$$\big(\delta_2 D^2 + \delta_1 D + \delta_0\big)x(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,ds + c_1, \tag{2.5}$$

where c_1 is an arbitrary constant. By the method of variation of parameters, the solution of (2.5) can be written as

$$\begin{aligned}
x(t) &= c_2 e^{m_1 t} + c_3 e^{m_2 t} - \frac{1}{\hat{\delta}}\Bigg[\int_0^t e^{m_1(t-s)}\bigg(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du + c_1\bigg)\,ds \\
&\quad + \int_0^t e^{m_2(t-s)}\bigg(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du + c_1\bigg)\,ds\Bigg],
\end{aligned} \tag{2.6}$$

where m_1 and m_2 are given by (2.4). Using $x(0) = 0$ in (2.6) and simplifying the coefficient of c_1, we get

$$\begin{aligned}
x(t) &= c_1\bigg(\frac{\delta_2[m_2(1-e^{m_1 t}) - m_1(1-e^{m_2 t})]}{\delta_0\hat{\delta}}\bigg) + c_2\big(e^{m_1 t} - e^{m_2 t}\big) \\
&\quad + \frac{1}{\hat{\delta}}\Bigg[\int_0^t \big(e^{m_2(t-s)} - e^{m_1(t-s)}\big)\bigg(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du\bigg)\,ds\Bigg],
\end{aligned} \tag{2.7}$$

which, together with the conditions $x(\xi) = \sum_{i=1}^{n} j_i x(\eta_i)$ and $x(1) = \sum_{i=1}^{k}\lambda_i \int_{\upsilon_i}^{\sigma_i} x(s)\,ds$, yields the following system of equations in the unknown constants c_1 and c_2:

$$c_1\omega_1 + c_2\omega_2 = V_1, \tag{2.8}$$

$$c_1\omega_3 + c_2\omega_4 = V_2, \tag{2.9}$$

where $\hat{\delta}$ and ω_i ($i = 1,2,3,4$) are given by (2.4), and

$$\begin{aligned}
V_1 &= -\int_0^\xi \int_0^s \big(e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du\,ds \\
&\quad + \sum_{i=1}^{n} j_i \int_0^{\eta_i}\int_0^s \big(e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du\,ds,
\end{aligned}$$

$$V_2 = -\int_0^1 \int_0^s \left(e^{m_2(1-s)} - e^{m_1(1-s)}\right) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du\,ds$$
$$+ \sum_{i=1}^k \lambda_i \int_{\upsilon_i}^{\sigma_i} \int_0^s \left[\frac{(e^{m_1(\sigma-s)}-1)}{m_1} - \frac{(e^{m_2(\sigma-s)}-1)}{m_2} \right] \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du\,ds.$$

Solving the system (2.8)–(2.9), we find that

$$c_1 = \frac{V_1\omega_4 - V_2\omega_2}{\mu_1}, \qquad c_2 = \frac{V_2\omega_1 - V_1\omega_3}{\mu_1}.$$

Substituting the value of c_1 and c_2 in (2.7), we obtain the solution (2.3). This completes the proof. □

Lemma 2.7 *For any $y \in C([0,1],\mathbb{R})$ and $\delta_1^2 - 4\delta_0\delta_2 = 0$, the solution of linear multi-term fractional differential equation*

$$\left(\delta_2{}^c D^{\alpha+2} + \delta_1{}^c D^{\alpha+1} + \delta_0{}^c D^{\alpha}\right) x(t) = y(t), \quad 0 < \alpha < 1, 0 < t < 1, \tag{2.10}$$

supplemented with the boundary conditions (1.2) is given by

$$x(t) = \frac{1}{\delta_2} \left\{ \int_0^t \int_0^s (t-s) e^{m(t-s)} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du\,ds \right.$$
$$+ \chi_1(t) \left[\int_0^\xi \int_0^s (\xi - s) e^{m(\xi-s)} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du\,ds \right.$$
$$\left. - \sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s (\eta_i - s) e^{m(\eta_i - s)} \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,du\,ds \right]$$
$$+ \chi_2(t) \left[\int_0^1 \int_0^s (1-s) e^{m(1-s)} \frac{(s-u)^{\alpha-1}}{\Gamma(q)} y(u)\,du\,ds \right.$$
$$\left.\left. - \sum_{i=1}^k \lambda_i \int_{\upsilon_i}^{\sigma_i} \int_0^s \left(\frac{m(\sigma_i - s) e^{m(\sigma_i - s)} - e^{m(\sigma_i - s)} + 1}{m^2} \right) \frac{(s-u)^{q-1}}{\Gamma(q)} y(u)\,du\,ds \right] \right\}, \tag{2.11}$$

where

$$m = \frac{-\delta_1}{2\delta_2}, \qquad \chi_1(t) = \frac{\varpi_3 z_2(t) - \varpi_4 z_1(t)}{\mu_2}, \qquad \chi_2(t) = \frac{\varpi_2 z_1(t) - \varpi_1 z_2(t)}{\mu_2},$$
$$z_1(t) = \frac{mte^{mt} - e^{mt} + 1}{m^2}, \qquad z_2(t) = \delta_2 t e^{mt},$$
$$\varpi_1 = \frac{m\xi e^{m\xi} - e^{m\xi} + 1 - \sum_{i=1}^n j_i (m\eta_i e^{m\eta_i} - e^{m\eta_i} + 1)}{m^2},$$
$$\varpi_2 = \delta_2 \left(\xi e^{m\xi} - \sum_{i=1}^n j_i \eta_i e^{m\eta_i} \right), \tag{2.12}$$
$$\varpi_3 = \frac{1}{m^3} \left[m^2 e^m - m e^m + m - m \sum_{i=1}^k \lambda_i \left(\sigma_i e^{m\sigma_i} - \upsilon_i e^{m\upsilon_i} \right) \right.$$

$$
\begin{aligned}
&+2\sum_{i=1}^{k}\lambda_i\left(e^{m\sigma_i}-e^{m\upsilon_i}\right)-m\sum_{i=1}^{k}\lambda_i(\sigma_i-\upsilon_i)\Bigg],\\
\varpi_4 &= \delta_2\left(\frac{m^2e^m-m\sum_{i=1}^{k}\lambda_i(\sigma_i e^{m\sigma_i}-\upsilon_i e^{m\upsilon_i})+\sum_{i=1}^{k}\lambda_i(e^{m\sigma_i}-e^{m\upsilon_i})}{m^2}\right),\\
\mu_2 &= \varpi_1\varpi_4-\varpi_2\varpi_3\neq 0.
\end{aligned}
$$

Proof Since the proof is similar to that of Lemma 2.6, we omit it. □

Lemma 2.8 *For any $y\in C([0,1],\mathbb{R})$ and $\delta_1^2-4\delta_0\delta_2<0$, the solution of linear multi-term fractional differential equation*

$$
\left(\delta_2{}^{c}D^{\alpha+2}+\delta_1{}^{c}D^{\alpha+1}+\delta_0{}^{c}D^{\alpha}\right)x(t)=y(t),\quad 0<\alpha<1, 0<t<1, \tag{2.13}
$$

supplemented with the boundary conditions (1.2) is given by

$$
\begin{aligned}
x(t) &= \frac{1}{\delta_2 b}\Bigg\{\int_0^t\int_0^s e^{-a(t-s)}\sin b(t-s)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds\\
&\quad+\tau_1(t)\Bigg[\int_0^{\xi}\int_0^s e^{-a(\xi-s)}\sin b(\xi-s)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds\\
&\quad-\sum_{i=1}^{n}j_i\int_0^{\eta_i}\int_0^s e^{-a(\eta_i-s)}\sin b(\eta_i-s)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds\Bigg]\\
&\quad+\tau_2(t)\Bigg[\int_0^1\int_0^s e^{-a(1-s)}\sin b(1-s)\frac{(s-u)^{q-1}}{\Gamma(q)}y(u)\,du\,ds\\
&\quad-\frac{\sum_{i=1}^{k}\lambda_i}{a^2+b^2}\int_{\upsilon_i}^{\sigma_i}\int_0^s\Big(b-be^{-a(\sigma_i-s)}\cos b(\sigma_i-s)\\
&\quad-ae^{-a(\sigma_i-s)}\sin b(\sigma_i-s)\Big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,du\,ds\Bigg]\Bigg\},
\end{aligned}
\tag{2.14}
$$

where

$$
\begin{aligned}
m_{1,2} &= -a\pm bi,\qquad a=\frac{\delta_1}{2\delta_2},\qquad b=\frac{\sqrt{4\delta_0\delta_2-\delta_1{}^2}}{2\delta_2},\\
\tau_1(t) &= \frac{q_3\nu_2(t)-q_4\nu_1(t)}{\mu_3},\qquad \tau_2(t)=\frac{q_2\nu_1(t)-q_1\nu_2(t)}{\mu_3},\\
\nu_1(t) &= \frac{b+be^{-at}\cos bt-ae^{-at}\sin bt}{a^2+b^2},\qquad \nu_2(t)=\delta_2 be^{-at}\sin bt,\\
q_1 &= \frac{b-be^{-a\xi}\cos b\xi-ae^{-a\xi}\sin b\xi-\sum_{i=1}^{n}j_i(b-be^{-a\eta_i}\cos b\eta_i-ae^{-a\eta_i}\sin b\eta_i)}{a^2+b^2},\\
q_2 &= \delta_2 b\left(e^{-a\xi}\sin b\xi-\sum_{i=1}^{n}j_ie^{-a\eta_i}\sin b\eta_i\right),\\
q_3 &= \frac{1}{a^2+b^2}\Bigg[b-be^{-a}\cos b-ae^{-a}\sin b-b\sum_{i=1}^{k}\lambda_i(\sigma_i-\upsilon_i)
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
&-\frac{a^2-b^2}{a^2+b^2}\sum_{i=1}^{k}\lambda_i\bigl(e^{-a\sigma_i}\sin b\sigma_i - e^{-a\upsilon_i}\sin b\upsilon_i\bigr)\\
&-\frac{2ab}{a^2+b^2}\sum_{i=1}^{k}\lambda_i\bigl(e^{-a\sigma_i}\cos b\sigma_i - e^{-a\upsilon_i}\cos b\upsilon_i\bigr)\Biggr],
\end{aligned}
$$

$$
\begin{aligned}
q_4 = \delta_2 b\Biggl[&e^{-a}\sin b + \frac{\sum_{i=1}^{k}\lambda_i}{a^2+b^2}\bigl(be^{-a\sigma_i}\cos b\sigma_i - be^{-a\upsilon_i}\cos b\upsilon_i\\
&+ ae^{-a\sigma_i}\sin b\sigma_i - ae^{-a\upsilon_i}\sin b\upsilon_i\bigr)\Biggr],
\end{aligned}
$$

$$
\mu_3 = q_1q_4 - q_2q_3 \neq 0.
$$

Proof We do not provide the proof as it is similar to that of Lemma 2.6. □

3 Existence and uniqueness results for the case $\delta_1{}^2 - 4\delta_0\delta_2 > 0$

Denote by $\mathcal{C} = C([0,1],\mathbb{R})$ the Banach space of all continuous functions from $[0,1]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$.

By Lemma 2.6, we transform the problem (1.1)-(1.2) with $\delta_1{}^2 - 4\delta_0\delta_2 > 0$ into a fixed point problem as

$$
x = \mathcal{J}x, \tag{3.1}
$$

where the operator $\mathcal{J} : \mathcal{C} \to \mathcal{C}$ is defined by

$$
\begin{aligned}
(\mathcal{J}x)(t) = \frac{1}{\hat{\delta}}\Biggl\{&\int_0^t\int_0^s\bigl(e^{m_2(t-s)} - e^{m_1(t-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds\\
&+ \rho_1(t)\Biggl[\int_0^{\xi}\int_0^s\bigl(e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds\\
&- \sum_{i=1}^{n}j_i\int_0^{\eta_i}\int_0^s\bigl(e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds\Biggr]\\
&+ \rho_2(t)\Biggl[\int_0^1\int_0^s\bigl(e^{m_2(1-s)} - e^{m_1(1-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds\\
&- \sum_{i=1}^{k}\lambda_i\int_{\upsilon_i}^{\sigma_i}\int_0^s\left(\frac{(e^{m_2(\sigma-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma-s)}-1)}{m_1}\right)\\
&\times\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds\Biggr]\Biggr\},
\end{aligned}
$$

with $\rho_1(t)$ and $\rho_2(t)$ given by (2.4).

In the sequel, for the sake of computational convenience, we set

$$
\begin{aligned}
&\widehat{\rho}_1 = \max_{t\in[0,1]}\bigl|\rho_1(t)\bigr|, \qquad \widehat{\rho}_2 = \max_{t\in[0,1]}\bigl|\rho_2(t)\bigr|,\\
&\varepsilon = \max_{t\in[0,1]}\bigl|m_2\bigl(1-e^{m_1t}\bigr) - m_1\bigl(1-e^{m_2t}\bigr)\bigr|,
\end{aligned}
$$

$$\begin{aligned}\phi &= \frac{|\delta_2|}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\Bigg\{\varepsilon+\widehat{\rho}_1\Bigg[\xi^{\alpha}\big|m_2\big(1-e^{m_1\xi}\big)-m_1\big(1-e^{m_2\xi}\big)\big| \\ &\quad+\sum_{i=1}^{n}|j_i|\eta_i^{\alpha}\big|m_2\big(1-e^{m_1\eta_i}\big)-m_1\big(1-e^{m_2\eta_i}\big)\big|\Bigg] \\ &\quad+\widehat{\rho}_2\Bigg[\big|m_2\big(1-e^{m_1}\big)-m_1\big(1-e^{m_2}\big)\big| \\ &\quad+\frac{|\delta_2|}{|\delta_0|}\sum_{i=1}^{k}|\lambda_i|\sigma_i^{\alpha}\big|m_2^2\big(m_1(\sigma_i-\upsilon_i)-e^{m_1(\sigma_i-\upsilon_i)}+1\big) \\ &\quad-m_1^2\big(m_2(\sigma_i-\upsilon_i)-e^{m_2(\sigma_i-\upsilon_i)}+1\big)\big|\Bigg]\Bigg\}, \\ \phi_1 &= \phi-\frac{|\delta_2|\varepsilon}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}.\end{aligned} \tag{3.2}$$

Now the stage is set to present our main results. In the first result, we use Krasnoselskii's fixed point theorem to prove the existence of solutions for the problem (1.1)–(1.2) with $\delta_1^2-4\delta_0\delta_2>0$.

Theorem 3.1 (Krasnoselskii's fixed point theorem [26]) *Let Y be a bounded, closed, convex, and nonempty subset of a Banach space X. Let F_1 and F_2 be operators satisfying the conditions*: (i) $F_1y_1+F_2y_2\in Y$ *whenever* $y_1,y_2\in Y$; (ii) F_1 *is compact and continuous*; (iii) F_2 *is a contraction mapping. Then there exists a $y\in Y$ such that $y=F_1y+F_2y$.*

In the forthcoming analysis, we need the following assumptions:

(A_1) $|f(t,x)-f(t,y)|\le\ell\|x-y\|$, for all $t\in[0,1]$, $x,y\in\mathbb{R}$, $\ell>0$.

(A_2) $|f(t,x)|\le\vartheta(t)$, for all $(t,x)\in[0,1]\times\mathbb{R}$ and $\vartheta\in C([0,1],\mathbb{R}^+)$.

Theorem 3.2 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying conditions* (A_1) *and* (A_2). *Then the problem* (1.1)–(1.2), *with* $\delta_1^2-4\delta_0\delta_2>0$, *has at least one solution on* $[0,1]$ *if*

$$\ell\phi_1<1, \tag{3.3}$$

where ϕ_1 is given by (3.2).

Proof Setting $\sup_{t\in[0,1]}|\vartheta(t)|=\|\vartheta\|$, we can fix

$$\begin{aligned}r &\ge \frac{|\delta_2|\|\vartheta\|}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\Bigg\{\varepsilon+\widehat{\rho}_1\Bigg[\xi^{\alpha}\big|m_2\big(1-e^{m_1\xi}\big)-m_1\big(1-e^{m_2\xi}\big)\big| \\ &\quad+\sum_{i=1}^{n}|j_i|\eta_i^{\alpha}\big|m_2\big(1-e^{m_1\eta_i}\big)-m_1\big(1-e^{m_2\eta_i}\big)\big|\Bigg]+\widehat{\rho}_2\Bigg[\big|m_2\big(1-e^{m_1}\big)-m_1\big(1-e^{m_2}\big)\big| \\ &\quad+\frac{\delta_2}{\delta_0}\sum_{i=1}^{k}|\lambda_i|\big|m_2^2\big(m_1(\sigma_i-\upsilon_i)-e^{m_1(\sigma_i-\upsilon_i)}+1\big)-m_1^2\big(m_2(\sigma_i-\upsilon_i)-e^{m_2(\sigma_i-\upsilon_i)}+1\big)\big|\Bigg]\Bigg\},\end{aligned}$$

and define $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. Introduce the operators $\mathcal{J}_1$ and $\mathcal{J}_2$ on B_r as follows:

$$(\mathcal{J}_1 x)(t) = \frac{1}{\hat{\delta}} \int_0^t \int_0^s \big(e^{m_2(t-s)} - e^{m_1(t-s)}\big) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\big(u, x(u)\big)\, du\, ds \tag{3.4}$$

and

$$\begin{aligned}(\mathcal{J}_2 x)(t) = \frac{1}{\hat{\delta}} \Bigg\{ & \rho_1(t) \Bigg[\int_0^\xi \int_0^s \big(e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\big) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\big(u, x(u)\big)\, du\, ds \\ & - \sum_{i=1}^n j_i \int_0^{\eta_i} \int_0^s \big(e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\big) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\big(u, x(u)\big)\, du\, ds \Bigg] \\ & + \rho_2(t) \Bigg[\int_0^1 \int_0^s \big(e^{m_2(1-s)} - e^{m_1(1-s)}\big) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\big(u, x(u)\big)\, du\, ds \\ & - \sum_{i=1}^k \lambda_i \int_{\upsilon_i}^{\sigma_i} \int_0^s \bigg(\frac{(e^{m_2(\sigma_i-s)} - 1)}{m_2} - \frac{(e^{m_1(\sigma_i-s)} - 1)}{m_1} \bigg) \\ & \times \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\big(u, x(u)\big)\, du\, ds \Bigg] \Bigg\}. \end{aligned} \tag{3.5}$$

Observe that $\mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2$. For $x, y \in B_r$, we have

$$\begin{aligned} & \|\mathcal{J}_1 x + \mathcal{J}_2 y\| \\ & \quad = \sup_{t\in[0,1]} \big|(\mathcal{J}_1 x)(t) + (\mathcal{J}_2 y)(t)\big| \\ & \quad \leq \frac{1}{|\hat{\delta}|} \sup_{t\in[0,1]} \Bigg\{ \int_0^t \int_0^s \big(e^{m_2(t-s)} - e^{m_1(t-s)}\big) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(u, x(u)\big)\big|\, du\, ds \\ & \qquad + \big|\rho_1(t)\big| \Bigg[\int_0^\xi \int_0^s \big(e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\big) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(u, y(u)\big)\big|\, du\, ds \\ & \qquad + \sum_{i=1}^n |j_i| \int_0^{\eta_i} \int_0^s \big(e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\big) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(u, y(u)\big)\big|\, du\, ds \Bigg] \\ & \qquad + \big|\rho_2(t)\big| \Bigg[\int_0^1 \int_0^s \big(e^{m_2(1-s)} - e^{m_1(1-s)}\big) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(u, y(u)\big)\big|\, du\, ds \\ & \qquad + \sum_{i=1}^k |\lambda_i| \int_{\upsilon_i}^{\sigma_i} \int_0^s \bigg(\frac{(e^{m_2(\sigma_i-s)} - 1)}{m_2} - \frac{(e^{m_1(\sigma_i-s)} - 1)}{m_1} \bigg) \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \big|f\big(u, y(u)\big)\big|\, du\, ds \Bigg] \Bigg\} \\ & \quad \leq \frac{\|\vartheta\|}{|\hat{\delta}|\Gamma(\alpha+1)} \sup_{t\in[0,1]} \Bigg\{ t^\alpha \int_0^t \big|e^{m_2(t-s)} - e^{m_1(t-s)}\big|\, ds \\ & \qquad + \big|\rho_1(t)\big| \Bigg[\xi^\alpha \int_0^\xi \big|e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\big|\, ds + \sum_{i=1}^n |j_i| \eta_i^\alpha \int_0^{\eta_i} \big|e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\big|\, ds \Bigg] \\ & \qquad + \big|\rho_2(t)\big| \Bigg[\int_0^1 \big|e^{m_2(1-s)} - e^{m_1(1-s)}\big|\, ds \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}|\lambda_i|\sigma_i^{\alpha}\int_{\upsilon_i}^{\sigma_i}\left|\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\right|ds\Bigg]\Bigg\}\\
\le{}&\frac{|\delta_2|\|\vartheta\|}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\Bigg\{\varepsilon+\widehat{\rho_1}\Bigg[\xi^{\alpha}\big|m_2\big(1-e^{m_1\xi}\big)-m_1\big(1-e^{m_2\xi}\big)\big|\\
&+\sum_{i=1}^{n}|j_i|\eta_i^{\alpha}\big|m_2\big(1-e^{m_1\eta_i}\big)-m_1\big(1-e^{m_2\eta_i}\big)\big|\Bigg]+\widehat{\rho_2}\Bigg[\big|m_2\big(1-e^{m_1}\big)-m_1\big(1-e^{m_2}\big)\big|\\
&+\frac{|\delta_2|}{|\delta_0|}\sum_{i=1}^{k}|\lambda_i|\sigma_i^{\alpha}\big|m_2^2\big(m_1(\sigma_i-\upsilon_i)-e^{m_1(\sigma_i-\upsilon_i)}+1\big)\\
&-m_1^2\big(m_2(\sigma_i-\upsilon_i)-e^{m_2(\sigma_i-\upsilon_i)}+1\big)\big|\Bigg]\Bigg\}\\
\le{}& r.
\end{aligned}
$$

Thus $\mathcal{J}_1x+\mathcal{J}_2y\in B_r$. Using assumption ($A_1$) together with (3.3), we show that $\mathcal{J}_2$ is a contraction as follows:

$$
\begin{aligned}
&\|\mathcal{J}_2x-\mathcal{J}_2y\|\\
&=\sup_{t\in[0,1]}\big|(\mathcal{J}_2x)(t)-(\mathcal{J}_2y)(t)\big|\\
&\le\frac{1}{|\hat{\delta}|}\sup_{t\in[0,1]}\Bigg\{\big|\rho_1(t)\big|\Bigg[\int_0^{\xi}\int_0^{s}\big(e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\\
&\quad\times\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|\,du\,ds\\
&\quad+\sum_{i=1}^{n}|j_i|\int_0^{\eta_i}\int_0^{s}\big(e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|\,du\,ds\Bigg]\\
&\quad+\big|\rho_2(t)\big|\Bigg[\int_0^{1}\int_0^{s}\big(e^{m_2(1-s)}-e^{m_1(1-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|\,du\,ds\\
&\quad+\sum_{i=1}^{k}|\lambda_i|\int_{\upsilon_i}^{\sigma_i}\int_0^{s}\left(\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\right)\\
&\quad\times\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|\,du\,ds\Bigg]\Bigg\}\\
&\le\frac{\ell}{|\hat{\delta}|}\sup_{t\in[0,1]}\Bigg\{\big|\rho_1(t)\big|\Bigg[\xi^{\alpha}\int_0^{\xi}\big|e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\big|\,ds\\
&\quad+\sum_{i=1}^{n}|j_i|\eta_i^{\alpha}\int_0^{\eta_i}\big|e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\big|\,ds\Bigg]\\
&\quad+\big|\rho_2(t)\big|\Bigg[\int_0^{1}\int_0^{s}\big|e^{m_2(1-s)}-e^{m_1(1-s)}\big|\,du\,ds\\
&\quad+\sum_{i=1}^{k}|\lambda_i|\sigma_i^{\alpha}\int_{\upsilon_i}^{\sigma_i}\left|\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\right|ds\Bigg]\Bigg\}\|x-y\|
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{|\delta_2|\ell}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\biggl\{\widehat{\rho}_1\biggl[\xi^{\alpha}\bigl|m_2\bigl(1-e^{m_1\xi}\bigr)-m_1\bigl(1-e^{m_2\xi}\bigr)\bigr| \\
&\quad + \sum_{i=1}^{n}|j_i|\eta_i^{\alpha}\bigl|m_2\bigl(1-e^{m_1\eta_i}\bigr)-m_1\bigl(1-e^{m_2\eta_i}\bigr)\bigr|\biggr] + \widehat{\rho}_2\biggl[\bigl|m_2\bigl(1-e^{m_1}\bigr)-m_1\bigl(1-e^{m_2}\bigr)\bigr| \\
&\quad + \frac{|\delta_2|}{|\delta_0|}\sum_{i=1}^{k}|\lambda_i|\sigma_i^{\alpha}\bigl|m_2^2\bigl(m_1(\sigma_i-\upsilon_i)-e^{m_1(\sigma_i-\upsilon_i)}+1\bigr) \\
&\quad - m_1^2\bigl(m_2(\sigma_i-\upsilon_i)-e^{m_2(\sigma_i-\upsilon_i)}+1\bigr)\bigr|\biggr]\biggr\}\|x-y\| \\
&= \ell\phi_1\|x-y\|.
\end{aligned}$$

Note that continuity of f implies that operator $\mathcal{J}_1$ is continuous. Also, $\mathcal{J}_1$ is uniformly bounded on B_r as

$$\|\mathcal{J}_1x\| = \sup_{t\in[0,1]}\bigl|(\mathcal{J}_1x)(t)\bigr| \leq \frac{|\delta_2|\|\vartheta\|\varepsilon}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}.$$

Now we prove the compactness of operator $\mathcal{J}_1$. We define $\sup_{(t,x)\in[0,1]\times B_r}|f(t,x)| = \overline{f}$. Thus, for $0 < t_1 < t_2 < 1$, we have

$$\begin{aligned}
&\bigl|(\mathcal{J}_1x)(t_2)-(\mathcal{J}_1x)(t_1)\bigr| \\
&\quad = \frac{1}{|\hat{\delta}|}\biggl|\int_0^{t_1}\int_0^{s}\bigl[\bigl(e^{m_2(t_2-s)}-e^{m_1(t_2-s)}\bigr)-\bigl(e^{m_2(t_1-s)}-e^{m_1(t_1-s)}\bigr)\bigr] \\
&\qquad \times \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds \\
&\qquad + \int_{t_1}^{t_2}\int_0^{s}\bigl(e^{m_2(t_2-s)}-e^{m_1(t_2-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds\biggr| \\
&\quad \leq \frac{|\delta_2|\overline{f}}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\bigl\{\bigl(t_1^{\alpha}-t_2^{\alpha}\bigr)\bigl|m_1\bigl(1-e^{m_2(t_2-t_1)}\bigr)-m_2\bigl(1-e^{m_1(t_2-t_1)}\bigr)\bigr| \\
&\qquad + t_1^{\alpha}\bigl|m_1\bigl(e^{m_2t_2}-e^{m_2t_1}\bigr)-m_2\bigl(e^{m_1t_2}-e^{m_1t_1}\bigr)\bigr|\bigr\} \to 0, \quad \text{as } t_1 \to t_2,
\end{aligned}$$

independent of $x \in B_r$. Thus, $\mathcal{J}_1$ is relatively compact on B_r. Hence, by the Arzelá–Ascoli Theorem, $\mathcal{J}_1$ is compact on B_r. Thus all the assumptions of Theorem 3.1 are satisfied. So, by the conclusion of Theorem 3.1, the problem (1.1)–(1.2) with ${\delta_1}^2 - 4\delta_0\delta_2 > 0$ has at least one solution on $[0,1]$. The proof is completed. □

Remark 3.3 In the above theorem we can interchange the roles of operators $\mathcal{J}_1$ and $\mathcal{J}_2$ to obtain a second result by replacing (3.3) with the following condition:

$$\frac{|\delta_2|\ell\varepsilon}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)} < 1.$$

In the next result, we prove the uniqueness of solutions for the problem (1.1)–(1.2) with ${\delta_1}^2 - 4\delta_0\delta_2 > 0$ by applying Banach contraction mapping principle.

Theorem 3.4 *Assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that* (A_1) *is satisfied. Then there exists a unique solution for the problem* (1.1)–(1.2), *with* ${\delta_1}^2-4\delta_0\delta_2>0$, *on* $[0,1]$ *if* $\ell<1/\phi$, *where* ϕ *is given by* (3.2).

Proof Let us define $\sup_{t\in[0,1]}|f(t,0)|=M$ and select $\bar{r}\geq\frac{\phi M}{1-\ell\phi}$ to show that $\mathcal{J}B_{\bar{r}}\subset B_{\bar{r}}$, where $B_{\bar{r}}=\{x\in\mathcal{C}:\|x\|\leq\bar{r}\}$ and $\mathcal{J}$ is defined by (3.1). Using condition (A_1), we have

$$\begin{aligned}\bigl|f(t,x)\bigr| &= \bigl|f(t,x)-f(t,0)+f(t,0)\bigr|\leq\bigl|f(t,x)-f(t,0)\bigr|+\bigl|f(t,0)\bigr|\\ &\leq\ell\|x\|+M\leq\ell\bar{r}+M.\end{aligned}\tag{3.6}$$

Then, for $x\in B_{\bar{r}}$, we obtain

$$\begin{aligned}&\bigl\|\mathcal{J}(x)\bigr\|\\
&\quad=\sup_{t\in[0,1]}\bigl|\mathcal{J}(x)(t)\bigr|\\
&\quad\leq\frac{1}{|\hat{\delta}|}\sup_{t\in[0,1]}\biggl\{\int_0^t\int_0^s\bigl(e^{m_2(t-s)}-e^{m_1(t-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\\
&\qquad+\bigl|\rho_1(t)\bigr|\biggl[\int_0^\xi\int_0^s\bigl(e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\\
&\qquad+\sum_{i=1}^n|j_i|\int_0^{\eta_i}\int_0^s\bigl(e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\biggr]\\
&\qquad+\bigl|\rho_2(t)\bigr|\biggl[\int_0^1\int_0^s\bigl(e^{m_2(1-s)}-e^{m_1(1-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\\
&\qquad+\sum_{i=1}^k|\lambda_i|\int_{\upsilon_i}^{\sigma_i}\int_0^s\biggl(\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\biggr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\biggr]\biggr\}\\
&\quad\leq\frac{(\ell\bar{r}+M)}{|\hat{\delta}|}\sup_{t\in[0,1]}\biggl\{\int_0^t\bigl|e^{m_2(t-s)}-e^{m_1(t-s)}\bigr|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\\
&\qquad+\bigl|\rho_1(t)\bigr|\biggl[\int_0^\xi\bigl|e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\bigr|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\\
&\qquad+\sum_{i=1}^n|j_i|\int_0^{\eta_i}\bigl|e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\bigr|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\biggr]\\
&\qquad+\bigl|\rho_2(t)\bigr|\biggl[\int_0^1\bigl|e^{m_2(1-s)}-e^{m_1(1-s)}\bigr|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\\
&\qquad+\sum_{i=1}^k|\lambda_i|\int_{\upsilon_i}^{\sigma_i}\biggl|\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\biggr|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\biggr]\biggr\}\\
&\quad\leq\frac{|\delta_2|(\ell\bar{r}+M)}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\biggl\{\varepsilon+\widehat{\rho}_1\biggl[\xi^\alpha\bigl|m_2\bigl(1-e^{m_1\xi}\bigr)-m_1\bigl(1-e^{m_2\xi}\bigr)\bigr|\\
&\qquad+\sum_{i=1}^n|j_i|\eta_i^\alpha\bigl|m_2\bigl(1-e^{m_1\eta_i}\bigr)-m_1\bigl(1-e^{m_2\eta_i}\bigr)\bigr|\biggr]+\widehat{\rho}_2\biggl[\bigl|m_2\bigl(1-e^{m_1}\bigr)-m_1\bigl(1-e^{m_2}\bigr)\bigr|\end{aligned}$$

$$\begin{aligned}
&+\frac{|\delta_2|}{|\delta_0|}\sum_{i=1}^{k}|\lambda_i|\sigma_i^{\alpha}\big|m_2^2\big(m_1(\sigma_i-\upsilon_i)-e^{m_1(\sigma_i-\upsilon_i)}+1\big)\\
&-m_1^2\big(m_2(\sigma_i-\upsilon_i)-e^{m_2(\sigma_i-\upsilon_i)}+1\big)\big|\bigg]\bigg]\bigg\}\\
&=(\ell\bar{r}+M)\phi\le\bar{r},
\end{aligned}$$

which clearly shows that $\mathcal{J}x\in B_{\bar{r}}$ for any $x\in B_{\bar{r}}$. Thus $\mathcal{J}B_{\bar{r}}\subset B_{\bar{r}}$. Now, for $x,y\in\mathcal{C}$ and for each $t\in[0,1]$, we have

$$\begin{aligned}
&\big\|(\mathcal{J}x)-(\mathcal{J}y)\big\|\\
&\le\frac{1}{|\hat{\delta}|}\sup_{t\in[0,1]}\bigg\{\int_0^t\int_0^s\big(e^{m_2(t-s)}-e^{m_1(t-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|\,du\,ds\\
&+\big|\rho_1(t)\big|\bigg[\int_0^{\xi}\int_0^s\big(e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|\,du\,ds\\
&+\sum_{i=1}^{n}|j_i|\int_0^{\eta_i}\int_0^s\big(e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|\,du\,ds\bigg]\\
&+\big|\rho_2(t)\big|\bigg[\int_0^{1}\int_0^s\big(e^{m_2(1-s)}-e^{m_1(1-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|\,du\,ds\\
&+\sum_{i=1}^{k}|\lambda_i|\int_{\upsilon_i}^{\sigma_i}\int_0^s\bigg(\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\bigg)\\
&\times\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f\big(u,x(u)\big)-f\big(u,y(u)\big)\big|\,du\,ds\bigg]\bigg\}\\
&\le\frac{\ell}{|\hat{\delta}|}\sup_{t\in[0,1]}\bigg\{\int_0^t\big|e^{m_2(t-s)}-e^{m_1(t-s)}\big|\frac{s^{\alpha}}{\Gamma(\alpha+1)}\,ds\\
&+\big|\rho_1(t)\big|\bigg[\int_0^{\xi}\big|e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\big|\frac{s^{\alpha}}{\Gamma(\alpha+1)}\,ds\\
&+\sum_{i=1}^{n}|j_i|\int_0^{\eta_i}\big|e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\big|\frac{s^{\alpha}}{\Gamma(\alpha+1)}\,ds\bigg]\\
&+\big|\rho_2(t)\big|\bigg[\int_0^{1}\big|e^{m_2(1-s)}-e^{m_1(1-s)}\big|\frac{s^{\alpha}}{\Gamma(\alpha+1)}\,ds\\
&+\sum_{i=1}^{k}|\lambda_i|\int_{\upsilon_i}^{\sigma_i}\bigg|\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\bigg|\frac{s^{\alpha}}{\Gamma(\alpha+1)}\,ds\bigg]\bigg\}\|x-y\|\\
&\le\frac{|\delta_2|\ell}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\bigg\{\varepsilon+\widehat{\rho}_1\bigg[\xi^{\alpha}\big|m_2\big(1-e^{m_1\xi}\big)-m_1\big(1-e^{m_2\xi}\big)\big|\\
&+\sum_{i=1}^{n}|j_i|\eta_i^{\alpha}\big|m_2\big(1-e^{m_1\eta_i}\big)-m_1\big(1-e^{m_2\eta_i}\big)\big|\bigg]+\widehat{\rho}_2\bigg[\big|m_2\big(1-e^{m_1}\big)-m_1\big(1-e^{m_2}\big)\big|\\
&+\frac{|\delta_2|}{|\delta_0|}\sum_{i=1}^{k}|\lambda_i|\sigma_i^{\alpha}\big|m_2^2\big(m_1(\sigma_i-\upsilon_i)-e^{m_1(\sigma_i-\upsilon_i)}+1\big)
\end{aligned}$$

$$- m_1^2\bigl(m_2(\sigma_i - \upsilon_i) - e^{m_2(\sigma_i-\upsilon_i)} + 1\bigr)\bigr|\Bigr]\Bigr]\Bigr\}\|x-y\|$$
$$= \ell\phi\|x-y\|,$$

where ϕ is given by (3.2) and depends only on the parameters involved in the problem. In view of the condition $\ell < 1/\phi$, it follows that $\mathcal{J}$ is a contraction. Thus, by the contraction mapping principle (Banach fixed point theorem), the problem (1.1)–(1.2) with ${\delta_1}^2 - 4\delta_0\delta_2 > 0$ has a unique solution on $[0,1]$. This completes the proof. □

The next existence result is based on Leray–Schauder nonlinear alternative.

Theorem 3.5 (Nonlinear alternative for single valued maps [27]) *Let C be a closed, convex subset of be a Banach space E and U an open subset of C with $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*
(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\epsilon \in (0,1)$ with $u = \epsilon F(u)$.*

We need the following assumptions:

(H_1) There exist a function $g \in C([0,1],\mathbb{R}^+)$, and a nondecreasing function $Q : \mathbb{R}^+ \to \mathbb{R}^+$ such that $|f(t,y)| \le g(t)Q(\|y\|)$, $\forall(t,y) \in [0,1]\times\mathbb{R}$.
(H_2) There exists a constant $K > 0$ such that
$$\frac{K}{\|g\|Q(K)\phi} > 1.$$

Theorem 3.6 *Let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ be a continuous function and suppose assumptions (H_1) and (H_2) are satisfied. Then the problem (1.1)–(1.2), with ${\delta_1}^2 - 4\delta_0\delta_2 > 0$, has at least one solution on $[0,1]$.*

Proof Consider the operator $\mathcal{J} : \mathcal{C} \to \mathcal{C}$ defined by (3.1). We show that $\mathcal{J}$ maps bounded sets into bounded sets in $\mathcal{C} = C([0,1],\mathbb{R})$. For a positive number ζ, let $\mathcal{B}_\zeta = \{x \in \mathcal{C} : \|x\| \le \zeta\}$ be a bounded set in $\mathcal{C}$. Then we have

$$\begin{aligned}
\bigl\|\mathcal{J}(x)\bigr\| &= \sup_{t\in[0,1]}\bigl|\mathcal{J}(x)(t)\bigr| \\
&\le \frac{1}{|\hat{\delta}|}\sup_{t\in[0,1]}\Biggl\{\int_0^t\int_0^s\bigl(e^{m_2(t-s)} - e^{m_1(t-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds \\
&\quad + \bigl|\rho_1(t)\bigr|\Biggl[\int_0^\xi\int_0^s\bigl(e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds \\
&\quad + \sum_{i=1}^n|j_i|\int_0^{\eta_i}\int_0^s\bigl(e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\Biggr] \\
&\quad + \bigl|\rho_2(t)\bigr|\Biggl[\int_0^1\int_0^s\bigl(e^{m_2(1-s)} - e^{m_1(1-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}|\lambda_i|\int_{\upsilon_i}^{\sigma_i}\int_0^s\left(\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\right) \\
&\times\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\Biggr]\Biggr\} \\
\leq{}& \frac{\|g\|Q(\zeta)}{|\hat{\delta}|}\sup_{t\in[0,1]}\Biggl\{\int_0^t\bigl|e^{m_2(t-s)}-e^{m_1(t-s)}\bigr|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds \\
&+\bigl|\rho_1(t)\bigr|\Biggl[\int_0^\xi\bigl|e^{m_2(\xi-s)}-e^{m_1(\xi-s)}\bigr|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds \\
&+\sum_{i=1}^{n}j_i\int_0^{\eta_i}\bigl|e^{m_2(\eta_i-s)}-e^{m_1(\eta_i-s)}\bigr|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\Biggr] \\
&+\bigl|\rho_2(t)\bigr|\Biggl[\int_0^1\bigl|e^{m_2(1-s)}-e^{m_1(1-s)}\bigr|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds \\
&+\sum_{i=1}^{k}|\lambda_i|\int_{\upsilon_i}^{\sigma_i}\left|\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2}-\frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\right|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\Biggr]\Biggr\} \\
\leq{}& \frac{|\delta_2|\|g\|Q(\zeta)}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\Biggl\{\varepsilon+\widehat{\rho}_1\Biggl[\xi^\alpha\bigl|m_2\bigl(1-e^{m_1\xi}\bigr)-m_1\bigl(1-e^{m_2\xi}\bigr)\bigr| \\
&+\sum_{i=1}^{n}|j_i|\eta_i^\alpha\bigl|m_2\bigl(1-e^{m_1\eta_i}\bigr)-m_1\bigl(1-e^{m_2\eta_i}\bigr)\bigr|\Biggr] \\
&+\widehat{\rho}_2\Biggl[\bigl|m_2\bigl(1-e^{m_1}\bigr)-m_1\bigl(1-e^{m_2}\bigr)\bigr| \\
&+\frac{|\delta_2|}{|\delta_0|}\sum_{i=1}^{k}|\lambda_i|\sigma_i^\alpha\bigl|m_2^2\bigl(m_1(\sigma_i-\upsilon_i)-e^{m_1(\sigma_i-\upsilon_i)}+1\bigr) \\
&-m_1^2\bigl(m_2(\sigma_i-\upsilon_i)-e^{m_2(\sigma_i-\upsilon_i)}+1\bigr)\bigr|\Biggr]\Biggr\},
\end{aligned}
$$

which yields

$$
\begin{aligned}
\|\mathcal{J}x\|\leq{}& \frac{|\delta_2|\|g\|Q(\zeta)}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\Biggl\{\varepsilon+\widehat{\rho}_1\xi^\alpha\bigl|m_2\bigl(1-e^{m_1\xi}\bigr)-m_1\bigl(1-e^{m_2\xi}\bigr)\bigr| \\
&+\sum_{i=1}^{n}|j_i|\eta_i^\alpha\bigl|m_2\bigl(1-e^{m_1\eta_i}\bigr)-m_1\bigl(1-e^{m_2\eta_i}\bigr)\bigr| \\
&+\widehat{\rho}_2\Biggl[\bigl|m_2\bigl(1-e^{m_1}\bigr)-m_1\bigl(1-e^{m_2}\bigr)\bigr| \\
&+\frac{|\delta_2|}{|\delta_0|}\sum_{i=1}^{k}|\lambda_i|\sigma_i^\alpha\bigl|m_2^2\bigl(m_1(\sigma_i-\upsilon_i)-e^{m_1(\sigma_i-\upsilon_i)}+1\bigr) \\
&-m_1^2\bigl(m_2(\sigma_i-\upsilon_i)-e^{m_2(\sigma_i-\upsilon_i)}+1\bigr)\bigr|\Biggr]\Biggr\}.
\end{aligned}
$$

Next we show that $\mathcal{J}$ maps bounded sets into equicontinuous sets of $\mathcal{C}$. Let $t_1, t_2 \in [0,1]$ with $t_1 < t_2$ and $y \in \mathcal{B}_\zeta$, where $\mathcal{B}_\zeta$ is a bounded set of $\mathcal{C}$. Then we obtain

$$
\begin{aligned}
&\bigl|(\mathcal{J}x)(t_2) - (\mathcal{J}x)(t_1)\bigr| \\
&\quad \leq \frac{1}{|\hat{\delta}|}\Biggl\{\biggl|\int_0^{t_1}\int_0^s \bigl[\bigl(e^{m_2(t_2-s)} - e^{m_1(t_2-s)}\bigr) - \bigl(e^{m_2(t_1-s)} - e^{m_1(t_1-s)}\bigr)\bigr] \\
&\qquad \times \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(u,x(u)\bigr)\,du\,ds \\
&\qquad + \int_{t_1}^{t_2}\int_0^s \bigl(e^{m_2(t_2-s)} - e^{m_1(t_2-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(u,x(u)\bigr)\,du\,ds\biggr| \\
&\qquad + \bigl|\rho_1(t_2) - \rho_1(t_1)\bigr| \int_0^{\xi}\int_0^s \bigl(e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds \\
&\qquad + \sum_{i=1}^{n} |j_i| \int_0^{\eta_i}\int_0^s \biggl[\bigl(e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\biggr] \\
&\qquad + \bigl|\rho_2(t_2) - \rho_2(t_1)\bigr| \biggl[\int_0^{1}\int_0^s \bigl(e^{m_2(1-s)} - e^{m_1(1-s)}\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds \\
&\qquad + \sum_{i=1}^{k} |\lambda_i| \int_{\upsilon_i}^{\sigma_i}\int_0^s \biggl(\frac{(e^{m_2(\sigma_i-s)} - 1)}{m_2} - \frac{(e^{m_1(\sigma_i-s)} - 1)}{m_1}\biggr) \\
&\qquad \times \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} \bigl|f\bigl(u,x(u)\bigr)\bigr|\,du\,ds\biggr]\Biggr\} \\
&\quad \leq \frac{|\delta_2|\|g\|Q(\zeta)}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\biggl\{\bigl(t_1^\alpha - t_2^\alpha\bigr)\bigl|m_1\bigl(1 - e^{m_2(t_2-t_1)}\bigr) - m_2\bigl(1 - e^{m_1(t_2-t_1)}\bigr|\bigr) \\
&\qquad + t_1^\alpha\bigl|m_1\bigl(e^{m_2t_2} - e^{m_2t_1}\bigr) - m_2\bigl(e^{m_1t_2} - e^{m_1t_1}\bigr|\bigr) \\
&\qquad + \bigl|\rho_1(t_2) - \rho_1(t_1)\bigr| \biggl[\xi^\alpha\bigl|m_2\bigl(1 - e^{m_1\xi}\bigr) - m_1\bigl(1 - e^{m_2\xi}\bigr)\bigr| \\
&\qquad + \sum_{i=1}^{n} |j_i|\eta_i^\alpha\bigl|m_2\bigl(1 - e^{m_1\eta_i}\bigr) - m_1\bigl(1 - e^{m_2\eta_i}\bigr)\bigr|\biggr] \\
&\qquad + \bigl|\rho_2(t_2) - \rho_2(t_1)\bigr| \biggl[\bigl|m_2\bigl(1 - e^{m_1}\bigr) - m_1\bigl(1 - e^{m_2}\bigr)\bigr| \\
&\qquad + \frac{|\delta_2|}{|\delta_0|}\sum_{i=1}^{k} |\lambda_i|\sigma_i^\alpha\bigl|m_1^2\bigl(m_2(\sigma_i - \upsilon_i) - e^{m_2(\sigma_i-\upsilon_i)} + 1\bigr) \\
&\qquad - m_2^2\bigl(m_1(\sigma_i - \upsilon_i) - e^{m_1(\sigma_i-\upsilon_i)} + 1\bigr)\bigr|\biggr]\biggr\},
\end{aligned}
$$

which tends to zero independently of $x \in \mathcal{B}_\zeta$ as $t_2 - t_1 \to 0$. As $\mathcal{J}$ satisfies the above assumptions, it follows by the Arzelá–Ascoli theorem that $\mathcal{J} : \mathcal{C} \to \mathcal{C}$ is completely continuous.

The result will follow from the Leray–Schauder nonlinear alternative once it is shown that the set of all solutions to the equation $x = \theta\mathcal{J}x$ is bounded for $\theta \in [0,1]$. For that, let

x be a solution of $x = \theta\mathcal{J}x$ for $\theta \in [0,1]$. Then, for $t \in [0,1]$, we have

$$
\begin{aligned}
|x(t)| &= |\theta\mathcal{J}x(t)| \\
&\le \frac{1}{|\hat{\delta}|}\sup_{t\in[0,1]}\Bigg\{\int_0^t\int_0^s \big(e^{m_2(t-s)} - e^{m_1(t-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f(u,x(u))\big|\,du\,ds \\
&\quad + |\rho_1(t)|\bigg[\int_0^\xi\int_0^s \big(e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f(u,x(u))\big|\,du\,ds \\
&\quad + \sum_{i=1}^n |j_i|\int_0^{\eta_i}\int_0^s \big(e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f(u,x(u))\big|\,du\,ds\bigg] \\
&\quad + |\rho_2(t)|\bigg[\int_0^1\int_0^s \big(e^{m_2(1-s)} - e^{m_1(1-s)}\big)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f(u,x(u))\big|\,du\,ds \\
&\quad + \sum_{i=1}^k |\lambda_i|\int_{\upsilon_i}^{\sigma_i}\int_0^s \left(\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\right) \\
&\quad \times \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\big|f(u,x(u))\big|\,du\,ds\bigg]\Bigg\} \\
&\le \frac{\|g\|Q(\|x\|)}{|\hat{\delta}|}\sup_{t\in[0,1]}\Bigg\{\int_0^t \big|e^{m_2(t-s)} - e^{m_1(t-s)}\big|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds \\
&\quad + |\rho_1(t)|\bigg[\int_0^\xi \big|e^{m_2(\xi-s)} - e^{m_1(\xi-s)}\big|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds \\
&\quad + \sum_{i=1}^n |j_i|\int_0^{\eta_i}\big|e^{m_2(\eta_i-s)} - e^{m_1(\eta_i-s)}\big|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\bigg] \\
&\quad + |\rho_2(t)|\bigg[\int_0^1 \big|e^{m_2(1-s)} - e^{m_1(1-s)}\big|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds \\
&\quad + \sum_{i=1}^k |\lambda_i|\int_{\upsilon_i}^{\sigma_i}\left|\frac{(e^{m_2(\sigma_i-s)}-1)}{m_2} - \frac{(e^{m_1(\sigma_i-s)}-1)}{m_1}\right|\frac{s^\alpha}{\Gamma(\alpha+1)}\,ds\bigg]\Bigg\} \\
&\le \frac{|\delta_2|\|g\|Q(\|x\|)}{|\delta_0\hat{\delta}|\Gamma(\alpha+1)}\Bigg\{\varepsilon + \widehat{\rho_1}\bigg[\xi^\alpha\big|m_2(1-e^{m_1\xi}) - m_1(1-e^{m_2\xi})\big| \\
&\quad + \sum_{i=1}^n |j_i|\eta_i^\alpha\big|m_2(1-e^{m_1\eta_i}) - m_1(1-e^{m_2\eta_i})\big|\bigg] \\
&\quad + \widehat{\rho_2}\bigg[\big|m_2(1-e^{m_1}) - m_1(1-e^{m_2})\big| \\
&\quad + \frac{|\delta_2|}{|\delta_0|}\sum_{i=1}^k |\lambda_i|\sigma_i^\alpha\big|m_1^2\big(m_2(\sigma_i-\upsilon_i) - e^{m_2(\sigma_i-\upsilon_i)} + 1\big) \\
&\quad - m_2^2\big(m_1(\sigma_i-\upsilon_i) - e^{m_1(\sigma_i-\upsilon_i)} + 1\big)\big|\bigg]\Bigg\} \\
&= \|g\|Q(\|x\|)\phi,
\end{aligned}
$$

which implies that

$$\frac{\|x\|}{\|g\|Q(\|x\|)\phi} \leq 1.$$

In view of (H_2), there is no solution x such that $\|x\| \neq K$. Let us set

$$U = \{x \in \mathcal{C} : \|x\| < K\}.$$

The operator $\mathcal{J} : \overline{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u = \theta \mathcal{J}(u)$ for some $\theta \in (0,1)$. Consequently, by the nonlinear alternative of Leray–Schauder type [27], we deduce that $\mathcal{J}$ has a fixed point $u \in \overline{U}$ which is a solution of the problem (1.1)–(1.2) with ${\delta_1}^2 - 4\delta_0\delta_2 > 0$. The proof is completed. □

Example 3.7 Consider the following multi-term fractional differential equation

$$\left(2^{c}D^{7/3} + 3^{c}D^{4/3} + {}^{c}D^{1/3}\right)x(t) = \frac{A}{4(1+t)^2}\tan^{-1}x + \sin(t+3), \quad 0 < t < 1, \tag{3.7}$$

subject to the boundary conditions

$$x(0) = 0, \qquad x(1/6) = 2x(1/5) + x(2/5), \qquad x(1) = \int_{1/4}^{3/5} x(s)\,ds + 3\int_{2/3}^{4/5} x(s)\,ds. \tag{3.8}$$

Here $\alpha = 1/3$, $\xi = 1/6$, $\eta_1 = 1/5$, $\eta_2 = 2/5$, $\upsilon_1 = 1/4$, $\upsilon_2 = 2/3$, $\sigma_1 = 3/5$, $\sigma_2 = 4/5$ $j_1 = 2$, $j_2 = 1$, $\lambda_1 = 1$, $\lambda_2 = 3$ and

$$f(t,x) = \frac{A}{4(1+t)^2}\tan^{-1}x + \sin(t+3),$$

A is positive number. Clearly, $\delta_1^2 - 4\delta_0\delta_2 = 1 > 0$, $|f(t,x) - f(t,y)| \leq \ell|x-y|$ with $\ell = A/4$. Using the given values, we find that $\phi \approx 0.66348$ and $\phi_1 \approx 0.49011$. Further, we have that $|f(t,x)| \leq \frac{\pi A}{8(1+t)^2} + \sin(t+3) = \vartheta(t)$ and $\ell\phi_1 < 1$ when $A < 8.16143$. As all the conditions of Theorem 3.2 are satisfied, the conclusion of Theorem 3.2 applies to the problem (3.7)–(3.8). On the other hand, as $\ell\phi < 1$ for $A < 6.02882$, there exists a unique solution for the problem (3.7)–(3.8) on $[0,1]$ by Theorem 3.4.

Example 3.8 Consider the multi-term fractional differential equation:

$$\left(2^{c}D^{7/3} + 3^{c}D^{4/3} + {}^{c}D^{1/3}\right)x(t) = \frac{2}{\sqrt{t^2+64}}\left(|x|\left(\frac{|x|}{|x|+1}\right) + \frac{1}{5}\right), \quad 0 < t < 1, \tag{3.9}$$

supplemented with the boundary conditions (3.8).

Observe that $\delta_1^2 - 4\delta_2\delta_0 = 1 > 0$ and $|f(t,x)| \leq g(t)Q(\|x\|)$ with $g(t) = \frac{2}{\sqrt{t^2+64}}$ and $Q(\|x\|) = \|x\| + \frac{1}{5}$. Due to condition ($H_2$), using $\phi \approx 0.66348$, we find that $K > 0.15908$. Thus, by the conclusion of Theorem 3.6, there exists at least one solution for the equation (3.9) with the boundary conditions (3.8).

4 Existence results for problem (1.1)–(1.2) with $\delta_1^2 - 4\delta_0\delta_2 = 0$

In view of Lemma 2.7, we can transform problem (1.1)–(1.2) with $\delta_1^2 - 4\delta_0\delta_2 = 0$ into an equivalent fixed point problem as

$$x = \mathcal{H}x, \tag{4.1}$$

where the operator $\mathcal{H} : \mathcal{C} \to \mathcal{C}$ is defined by

$$\begin{aligned}
(\mathcal{H}x)(t) = {} & \frac{1}{\delta_2}\Biggl\{\int_0^t \int_0^s (t-s)e^{m(t-s)}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(u,x(u)\bigr)\,du\,ds \\
& + \chi_1(t)\Biggl[\int_0^{\xi} \int_0^s (\xi-s)e^{m(\xi-s)}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(u,x(u)\bigr)\,du\,ds \\
& - \sum_{i=1}^{n} j_i \int_0^{\eta_i} \int_0^s (\eta_i-s)e^{m(\eta_i-s)}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(u,x(u)\bigr)\,du\,ds\Biggr] \\
& + \chi_2(t)\Biggl[\int_0^{1} \int_0^s (1-s)e^{m(1-s)}\frac{(s-u)^{q-1}}{\Gamma(q)} f\bigl(u,x(u)\bigr)\,du\,ds \\
& - \sum_{i=1}^{k} \lambda_i \int_{\upsilon_i}^{\sigma_i} \int_0^s \biggl(\frac{m(\sigma-s)e^{m(\sigma-s)} - e^{m(\sigma-s)} + 1}{m^2}\biggr) \\
& \times \frac{(s-u)^{q-1}}{\Gamma(q)} f\bigl(u,x(u)\bigr)\,du\,ds\Biggr]\Biggr\},
\end{aligned}$$

$\chi_1(t)$ and $\chi_2(t)$ are defined by (2.12). Moreover, we set

$$\begin{aligned}
& \widehat{\chi}_1 = \max_{t\in[0,1]} \bigl|\chi_1(t)\bigr|, \qquad \widehat{\chi}_2 = \max_{t\in[0,1]} \bigl|\chi_2(t)\bigr|, \\
& \beta = \frac{1}{|\delta_2|m^2\Gamma(\alpha+1)}\Biggl\{(1+\widehat{\chi}_2)\bigl|(m-1)e^m + 1\bigr| \\
& \quad + \widehat{\chi}_1\Biggl[\xi^{\alpha}\bigl|(m\xi-1)e^{m\xi}+1\bigr| + \sum_{i=1}^{n}|j_i|\eta_i^{\alpha}\bigl|(m\eta_i-1)e^{m\eta_i}+1\bigr|\Biggr] \\
& \quad + \widehat{\chi}_2 \frac{\sum_{i=1}^{k}|\lambda_i|\sigma_i^{\alpha}}{|m|}\bigl|m(\sigma_i-\upsilon_i)\bigl(e^{m(\sigma_i-\upsilon_i)}+1\bigr) + 2\bigl(1-e^{m(\sigma_i-\upsilon_i)}\bigr)\bigr|\Biggr\}, \\
& \beta_1 = \beta - \frac{|(m-1)e^m+1|}{|\delta_2|m^2\Gamma(\alpha+1)}.
\end{aligned} \tag{4.2}$$

Now we present existence results for the problem (1.1)–(1.2) with $\delta_1^2 - 4\delta_0\delta_2 = 0$ without proof. One can complete the proofs for these results following the arguments used in the previous section.

Theorem 4.1 *Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying conditions* (A_1) *and* (A_2). *Then the problem* (1.1)–(1.2), *with* $\delta_1^2 - 4\delta_0\delta_2 = 0$, *has at least one solution on* $[0,1]$ *if* $\ell\beta_1 < 1$, *where* β_1 *is given by* (4.2).

Theorem 4.2 *Assume that $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function and condition* (A_1) *is satisfied. Then there exists a unique solution for problem* (1.1)–(1.2), *with $\delta_1^2-4\delta_0\delta_2=0$, on $[0,1]$ if $\ell<1/\beta$, where β is given by* (4.2).

Theorem 4.3 *Let $f:[0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. In addition, suppose that* (H_1) *and the following condition hold*:

(H_2') *There exists a constant $K_1>0$ such that $\frac{K_1}{\|g\|Q(K_1)\beta}>1$, where β is defined by* (4.2).

Then the problem (1.1)–(1.2), *with $\delta_1^2-4\delta_0\delta_2=0$, has at least one solution on* $[0,1]$.

Example 4.4 Let us consider the multi-term fractional differential equation

$$\left({}^{c}D^{7/3}+2{}^{c}D^{4/3}+{}^{c}D^{1/3}\right)x(t)=\left(\frac{|x|}{1+|x|}+t\right)\frac{B}{2\sqrt{t^2+4}}+\cos t,\quad 0<t<1, \tag{4.3}$$

supplemented with the boundary conditions (3.8), where

$$f(t,x)=\left(\frac{|x|}{1+|x|}+t\right)\frac{B}{2\sqrt{t^2+4}}+\cos t$$

and B is positive number.

Obviously, $\delta_1^2-4\delta_0\delta_2=0$, and $|f(t,x)-f(t,y)|\le\ell|x-y|$ with $\ell=B/4$. Using the given values, we find that $\beta\approx 0.39636$ and $\beta_1\approx 0.10045$. It is easy to check that $|f(t,x)|\le\frac{B(1+t)}{2\sqrt{t^2+4}}+\cos t=\vartheta(t)$ and $\ell\beta_1<1$ when $B<39.82081$. As all the condition of Theorem 4.1 are satisfied, equation (4.3) with the boundary data (3.8) has at least one solution on $[0,1]$. On the other hand, $\ell\beta<1$ whenever $B<10.091836$, so there exists a unique solution for equation (4.3) with the boundary data (3.8) on $[0,1]$ by Theorem 4.2.

5 Existence results for problem (1.1)–(1.2) with $\delta_1^2-4\delta_0\delta_2<0$

By Lemma 2.8, the fixed point problem equivalent to the problem (1.1)–(1.2) with $\delta_1^2-4\delta_0\delta_2<0$ can be written as

$$x=\mathcal{K}x, \tag{5.1}$$

where the operator $\mathcal{K}:\mathcal{C}\to\mathcal{C}$ is defined by

$$\begin{aligned}(\mathcal{K}x)(t)=\frac{1}{\delta_2 b}\Biggl\{&\int_0^t\int_0^s e^{-a(t-s)}\sin b(t-s)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds\\&+\tau_1(t)\Biggl[\int_0^{\xi}\int_0^s e^{-a(\xi-s)}\sin b(\xi-s)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds\\&-\sum_{i=1}^n j_i\int_0^{\eta_i}\int_0^s e^{-a(\eta_i-s)}\sin b(\eta_i-s)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}f\bigl(u,x(u)\bigr)\,du\,ds\Biggr]\\&+\tau_2(t)\Biggl[\int_0^1\int_0^s e^{-a(1-s)}\sin b(1-s)\frac{(s-u)^{q-1}}{\Gamma(q)}f\bigl(u,x(u)\bigr)\,du\,ds\end{aligned}$$

$$
\begin{aligned}
&- \frac{\sum_{i=1}^{k}\lambda_i}{a^2+b^2}\int_{\upsilon_i}^{\sigma_i}\int_0^s \bigl(b - be^{-a(\sigma-s)}\cos b(\sigma-s) \\
&- ae^{-a(\sigma-s)}\sin b(\sigma-s)\bigr)\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} f\bigl(u,x(u)\bigr)\,du\,ds\biggr]\biggr\},
\end{aligned}
$$

$\tau_1(t)$ and $\tau_2(t)$ are defined by (2.15).

Further, we set

$$
\begin{aligned}
&\widehat{\tau}_1 = \max_{t\in[0,1]}\bigl|\tau_1(t)\bigr|, \qquad \widehat{\tau}_2 = \max_{t\in[0,1]}\bigl|\tau_2(t)\bigr| \\
&\gamma = \frac{1}{|\delta_0 b|\Gamma(\alpha+1)}\biggl\{(1+\widehat{\tau}_2)\bigl|b - be^{-a}\cos b - ae^{-a}\sin b\bigr| \\
&\quad + \widehat{\tau}_1\biggl[\xi^{\alpha}\bigl|b - be^{-a\xi}\cos b\xi - ae^{-a\xi}\sin b\xi\bigr| \\
&\quad + \sum_{i=1}^{n}|j_i|\eta_i^{\alpha}\bigl|b - be^{-a\eta_i}\cos b\eta_i - ae^{-a\eta_i}\sin b\eta_i\bigr|\biggr] \\
&\quad + \widehat{\tau}_2\sum_{i=1}^{k}|\lambda_i|\sigma_i^{\alpha}\bigl|b(\sigma_i-\upsilon_i) - e^{-a(\sigma_i-\upsilon_i)}\sin b(\sigma_i-\upsilon_i)\bigr|\biggr\}, \\
&\gamma_1 = \gamma - \frac{|b - be^{-a}\cos b - ae^{-a}\sin b|}{|\delta_0 b|\Gamma(\alpha+1)}, \qquad a = \frac{\delta_1}{2\delta_2}, \qquad b = \frac{\sqrt{4\delta_0\delta_2 - \delta_1{}^2}}{2\delta_2}.
\end{aligned}
\tag{5.2}
$$

As before, we can formulate existence results for the problem (1.1)–(1.2) with $\delta_1^2 - 4\delta_0\delta_2 < 0$ as follows.

Theorem 5.1 *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying conditions* (A_1) *and* (A_2). *Then the problem* (1.1)–(1.2), *with $\delta_1^2 - 4\delta_0\delta_2 < 0$, has at least one solution on* $[0,1]$ *provided that $\ell\gamma_1 < 1$, where γ is given by* (5.2).

Theorem 5.2 *Assume that $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that* (A_1) *is satisfied. Then there exists a unique solution for the problem* (1.1)–(1.2), *with $\delta_1^2 - 4\delta_0\delta_2 < 0$, on* $[0,1]$ *if $\ell < 1/\gamma$, where γ is given by* (5.2).

Theorem 5.3 *Let $f : [0,1]\times\mathbb{R}\to\mathbb{R}$ be a continuous function. Further, suppose that* (H_1) *and the following condition hold*:

(H_2'') *There exists a constant $K_2 > 0$ such that $\frac{K_2}{\|g\|Q(K_2)\gamma} > 1$, where γ is defined by* (5.2).

Then the problem (1.1)–(1.2), *with $\delta_1^2 - 4\delta_0\delta_2 < 0$, has at least one solution on* $[0,1]$.

Example 5.4 Consider the following multi-term fractional differential equation

$$
\bigl(2{}^{c}D^{7/3} + 2{}^{c}D^{4/3} + {}^{c}D^{1/3}\bigr)x(t) = f(t,x), \quad 0 < t < 1, \tag{5.3}
$$

equipped with the boundary conditions (3.8), where

$$
f(t,x) = \frac{L}{3\sqrt{t^3+9}}\bigl(\cos x + e^{-2t}\bigr), \quad L > 0.
$$

Clearly, $\delta_1^2 - 4\delta_0\delta_2 = -4 < 0$ and $|f(t,x) - f(t,y)| \leq \ell|x - y|$ with $\ell = L/9$. Using the given values, it is found that $\gamma \approx 0.57912$ and $\gamma_1 \approx 0.38098$. Further, it is easy to check that $|f(t,x)| \leq \frac{L(1+e^{-2t})}{3\sqrt{t^3+9}} = \vartheta(t)$ and $\ell\gamma_1 < 1$ when $L < 23.62329$. As all the conditions of Theorem 5.1 are satisfied, equation (5.3) with the boundary conditions (3.8) has at least one solution on $[0,1]$. On the other hand, since $\ell\gamma < 1$ for $L < 15.54082$, there exists a unique solution for equation (5.3) with the boundary conditions (3.8) on $[0,1]$ by Theorem 5.2.

6 Conclusions

We have derived existence results for a multi-term fractional differential equation associated with different combinations ($\delta_1^2 - 4\delta_0\delta_2 > 0$, $\delta_1^2 - 4\delta_0\delta_2 = 0$, and $\delta_1^2 - 4\delta_0\delta_2 < 0$) of the constants involved in the equation equipped with nonlocal multi-point and multi-strip boundary conditions. Our results are not only new in the given context, but also yield some interesting new results as special cases of the obtained work. For instance, by taking $\lambda_i = 0$, $i = i, \dots, k$ in the results of this paper, we obtain new results for the multi-term fractional differential equation (1.1) associated with the boundary condition of the form: $x(0) = 0$, $x(\xi) = \sum_{i=1}^{n} j_i x(\eta_i)$, $x(1) = 0$. Our results correspond to those for (1.1) with the nonlocal multi-strip boundary condition: $x(0) = 0$, $x(\xi) = 0$, $x(1) = \sum_{i=1}^{k} \lambda_i \int_{\upsilon_i}^{\sigma_i} x(s)\,ds$ if we fix $j_i = 0$, $i = 1, \dots, n$ in the obtained results.

Acknowledgements
This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PhD-11-130-39). The authors, therefore, acknowledge with thanks DSR technical and financial support. The authors also thank the reviewers for their constructive remarks on this paper.

Funding
This project was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, Saudi Arabia under grant no. (KEP-PhD-11-130-39).

Abbreviations
Not applicable.

Availability of data and materials
Not applicable.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, RPA, AA, NA, SKN and BA, contributed equally to each part of this work. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Texas A&M University, Kingsville, USA. [2]Florida Institute of Technology, Melbourne, USA. [3]Nonlinear Analysis and Applied Mathematics (NAAM)—Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia. [4]Department of Mathematics, Faculty of Science, University of Jeddah, Jeddah, Saudi Arabia. [5]Department of Mathematics, University of Ioannina, Ioannina, Greece.

Publisher's Note

Received: 11 July 2018 Accepted: 10 September 2018 Published online: 25 September 2018

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