Guo et al. *Advances in Difference Equations* (2018) 2018:379
https://doi.org/10.1186/s13662-018-1796-3

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH

Open Access

Existence and uniqueness results to positive solutions of integral boundary value problem for fractional q-derivatives

Furi Guo[1], Shugui Kang[1] and Fu Chen[1*]

*Correspondence:
chenfu-88@163.com
[1]School of Mathematics and Computer Science, Shanxi Datong University, Datong, P.R. China

Abstract

In this paper,we are interested in the existence and uniqueness of positive solutions for integral boundary value problem with fractional q-derivative:

$$D_q^\alpha u(t) + f(t,u(t),u(t)) + g(t,u(t)) = 0, \quad 0 < t < 1,$$

$$u(0) = D_q u(0) = 0, \qquad u(1) = \mu \int_0^1 u(s)\,d_q s,$$

where D_q^α is the fractional q-derivative of Riemann–Liouville type, $0 < q < 1$, $2 < \alpha \leq 3$, and μ is a parameter with $0 < \mu < [\alpha]_q$. By virtue of fixed point theorems for mixed monotone operators, we obtain some results on the existence and uniqueness of positive solutions.

Keywords: Positive solution; Mixed monotone operator; Fractional q-difference equation; Existence and uniqueness

1 Introduction

The theory that fractional differential equations arise in the fields of science and engineering such as physics, chemistry, mechanics, economics, and biological sciences, etc.; see, for example, [1–6]. The q-difference calculus or quantum calculus is an old subject that was put forward by Jackson [7, 8]. The essential definitions and properties of q-difference calculus can be found in [9, 10]. Early development for q-fractional calculus can be seen in the papers by Al-Salam [11] and Agarwal [12] on the existence theory of fractional q-difference. These days the fractional q-difference equation have given fire to increasing scholars' imaginations. Some works considered the existence of positive solutions for nonlinear q-fractional boundary value problem [13–32]. For example, Ferreira [13] studied the existence of positive solutions to the fractional q-difference equation

$$\begin{cases} D_q^\alpha u(t) + f(t,u(t)) = 0, & 0 < t < 1, 1 < \alpha \leq 2, \\ u(0) = u(1) = 0. \end{cases} \tag{1.1}$$

Ferreira [14] also considered the existence of positive solutions to the nonlinear q-difference boundary value problem

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, 1 < \alpha \leq 3, \\ u(0) = D_q u(0) = 0, & D_q u(1) = \beta \geq 0. \end{cases} \tag{1.2}$$

EI-Shahed and AI-Askar [15] studied the existence of a positive solution to the fractional q-difference equation

$$\begin{cases} {}_cD_q^\alpha u(t) + a(t)f(t) = 0, & 0 \leq t \leq 1, 2 < \alpha \leq 3, \\ u(0) = D_q^2(0) = 0, & \gamma D_q u(1) + \beta D_q^2 u(1) = 0, \end{cases} \tag{1.3}$$

where $\gamma, \beta \leq 0$, and ${}_cD_q^\alpha$ is the fractional q-derivative of Caputo type.

Darzi and Agheli [16] studied the existence of a positive solution to the fractional q-difference equation

$$\begin{cases} D_q^\alpha u(t) + a(t)f(t) = 0, & 0 \leq t \leq 1, 3 < \alpha \leq 4, \\ u(0) = D_q u(0) = D_q^2 u(0) = 0, & D_q^2 u(1) = \beta D_q^2 u(\eta), \end{cases} \tag{1.4}$$

where $0 < \eta < 1$ and $1 - \beta\eta^{\alpha-3} > 0$.

The methods used in the papers mentioned are mainly the Krasnoselskii fixed point theorem, the Schauder fixed point theorem, the Leggett–Williams fixed point theorem, and so on. Differently from methods used in the literature mentioned, on the basis of the enlightenment of the works [17, 18, 26], we will use fixed point theorems for mixed monotone operators to demonstrate the existence and uniqueness of positive solutions for integral boundary value problems of the form

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t), u(t)) + g(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = D_q u(0) = 0, & u(1) = \mu \int_0^1 u(s)\, d_q s, \end{cases} \tag{1.5}$$

where D_q^α is the fractional q-derivative of Riemann–Liouville type, $0 < q < 1$, $2 < \alpha \leq 3$, $0 < \mu < [\alpha]_q$. Our results ensure the existence of a unique positive solution. Moreover, an iterative scheme is constructed for approximating the solution. As far as we know, there are still very few works utilizing the fixed point results for mixed monotone operators to study the existence and uniqueness of a positive solution for fractional q-derivative integral boundary value problems.

The plan of the paper is as follows. In Sect. 2, we give not only basic definitions of q-fractional integral, but also some properties of certain Green's functions, which play a fundamental role in the process of proofs. In Sect. 3, in light of some sufficient conditions, we obtained some results on the existence and uniqueness of positive solutions to problem (1.5). At the closing part, two examples are given to demonstrate the serviceability of our main results in Sect. 4.

2 Preliminaries

For convenience of the reader, on one hand, we recall some well-known facts on q-calculus and, on the other hand, some notations and lemmas that will be used in the proofs of our theorems.

A nonempty closed convex set $P \subset E$ is a cone if (1) $x \in P, r \geq 0 \Rightarrow rx \in P$ and (2) $x \in P, -x \in P \Rightarrow x = \theta$ (θ is the zero element of E), where $(E, \|\cdot\|)$ is a real Banach space. For all $x, y \in E$, if there exist $\mu, \nu > 0$ such that $\mu x \leq y \leq \nu x$, then we write $x \sim y$. Obviously, $\sim$ is an equivalence relation. Let $P_h = \{x \in E | x \sim h, h > \theta\}$.

Let $q \in (0,1)$. Then the q-number is given by

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in R.$$

The q-analogue of the power function $(a-b)^{(n)}$ with $n \in N_0$ is

$$(a-b)^{(0)} = 1, \qquad (a-b)^{(n)} = \prod_{k=0}^{n-1}\bigl(a - bq^k\bigr), \quad n \in N, a, b \in R.$$

More generally, if $\alpha \in R$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \frac{a - bq^k}{a - bq^{\alpha+k}}, \quad \alpha \neq 0.$$

Note that if $b = 0$, then $a^{(\alpha)} = a^\alpha$. The q-gamma function is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{1-q^{x-1}}, \quad x \in R^+,$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

The q-derivative of a function f is defined by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \qquad (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),$$

and q-derivatives of higher order by

$$\bigl(D_q^0 f\bigr)(x) = f(x), \qquad \bigl(D_q^n f\bigr)(x) = D_q\bigl(D_q^{n-1} f\bigr)(x), \quad n \in N.$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(s)\, d_q s = x(1-q) \sum_{k=0}^{\infty} f\bigl(xq^k\bigr) q^k, \quad x \in [0, b].$$

If $a \in [0, b]$ and f is defined in the interval $[0, b]$, then its integral from a to b is defined by

$$\int_a^b f(s)\, d_q s = \int_0^b f(s)\, d_q s - \int_0^a f(s)\, d_q s.$$

Similarly to the derivatives, the operator I_q^n is given by

$$\big(I_q^0 f\big)(x) = f(x), \qquad \big(I_q^n f\big)(x) = I_q\big(I_q^{n-1} f\big)(x), \quad n \in N.$$

The fundamental theorem of calculus applies to the operators I_q and D_q, that is,

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

The following formulas will be used later (${}_tD_q$ denotes the derivative with respect to variable t):

$$\begin{aligned}
&{}_tD_q(t-s)^{(\alpha)} = [\alpha]_q(t-s)^{(\alpha-1)},\\
&\left({}_xD_q \int_0^x f(x,t)\,d_qt\right)(x) = \int_0^x {}_xD_q f(x,t)\,d_qt + f(qx,x).
\end{aligned}$$

Definition 2.1 (see [4]) Let $\alpha \geq 0$, and let f be a function defined on $[0,1]$. The fractional q-integral of the Riemann–Liouville type is defined by $(I_q^0 f)(x) = f(x)$ and

$$\big(I_q^\alpha f\big)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t)\,d_qt, \quad \alpha > 0, x \in [0,1].$$

Definition 2.2 (see [10]) The fractional q-derivative of the Riemann–Liouville type is defined by

$$\big(D_q^0 f\big)(x) = f(x), \qquad \big(D_q^\alpha f\big)(x) = \big(D_q^p I_q^{p-\alpha} f\big)(x), \quad \alpha > 0,$$

where p is the smallest integer greater than or equal to α.

Lemma 2.1 (see [10]) *Let $\alpha, \beta \geq 0$, and let f be a function defined on $[0,1]$. Then the following formulas hold*:

(i) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\beta+\alpha} f)(x)$,
(ii) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2.2 (see [10]) *Let $\alpha > 0$, and let be p be a positive integer. Then the following equality holds*:

$$\big(I_q^\alpha D_q^p f\big)(x) = \big(D_q^p I_q^\alpha f\big)(x) - \sum_{k=0}^{p-1} \frac{x^{\alpha-p+k}}{\Gamma_q(\alpha+k-p+1)} \big(D_q^k f\big)(0).$$

Lemma 2.3 (see [27]) *Let $2 < \alpha \leq 3$ and $0 < \mu < [\alpha]_q$. Let $x \in C[0,1]$. Then the boundary value problem*

$$D_q^\alpha u(t) + x(t) = 0, \quad 0 < t < 1, \tag{2.1}$$

$$u(0) = D_q u(0) = 0, \qquad u(1) = \mu \int_0^1 u(s)\,d_qs, \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,qs)x(s)\,d_qs,$$

where

$$G(t,s) = \begin{cases} \frac{t^{\alpha-1}(1-s)^{(\alpha-1)}([\alpha]_q-\mu+\mu q^{\alpha-1}s)-([\alpha]_q-\mu)(t-s)^{\alpha-1}}{([\alpha]_q-\mu)\Gamma_q(\alpha)}, & 0 \le s \le t \le 1, \\ \frac{t^{\alpha-1}(1-s)^{(\alpha-1)}([\alpha]_q-\mu+\mu q^{\alpha-1}s)}{([\alpha]_q-\mu)\Gamma_q(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.3}$$

Lemma 2.4 (see [27]) *The function $G(t,qs)$ defined by* (2.3) *has the following properties*:

(i) *$G(t,qs)$ is a continuous function and $G(t,qs) \ge 0$*;

(ii) $\frac{\mu q^{\alpha}t^{\alpha-1}(1-qs)^{(\alpha-1)}s}{([\alpha]_q-\mu)\Gamma_q(\alpha)} \le G(t,qs) \le \frac{M_0 t^{\alpha-1}}{([\alpha]_q-\mu)\Gamma_q(\alpha)}$, $t,s \in [0,1]$,

where $M_0 = \max\{[\alpha-1]_q([\alpha]_q-\mu) + \mu q^{\alpha}, q^{\alpha-1}[\alpha]_q\}$.

Definition 2.3 (see [17]) An operator $A : P \times P \to P$ is said to be a mixed monotone operator if $A(x,y)$ is increasing in x and decreasing in y, that is, $x_i, y_i \in P$ $(i = 1,2)$, $x_1 \le x_2$, $y_1 \ge y_2$ imply $A(x_1,y_1) \le A(x_2,y_2)$. An element $x \in P$ is called a fixed point of A if $A(x,x) = x$.

Definition 2.4 (see [18]) An operator $A : P \to P$ is said to be subhomogeneous if

$$A(tx) \ge tA(x) \quad \text{for any } t \in (0,1), x \in P. \tag{2.4}$$

Definition 2.5 (see [18]) Let $D = P$, and let γ be a real number with $0 \le \gamma < 1$. An operator $A : D \to D$ is said to be γ-concave if it satisfies

$$A(tx) \ge t^{\gamma}A(x) \quad \text{for any } t \in (0,1), x \in D. \tag{2.5}$$

Lemma 2.5 (see [17]) *Let $h > \theta$ and $\gamma \in (0,1)$.*

Let $A : P \times P \to P$ be a mixed monotone operator satisfying

$$A\bigl(tx, t^{-1}y\bigr) \ge t^{\gamma}A(x,y) \quad \textit{for any } t \in (0,1), x,y \in P, \tag{2.6}$$

and let $B : P \to P$ be an increasing subhomogeneous operator. Assume that

(i) *there is $h_0 \in P_h$ such that $A(h_0,h_0) \in P_h$ and $Bh_0 \in P_h$*;

(ii) *there exists a constant δ_0 such that $A(x,y) \ge \delta_0 Bx$ for any $x,y \in P$.*

Then:

(1) *$A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$*;

(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \le u_0 < v_0, \qquad u_0 \le A(u_0,v_0) + Bu_0 \le A(v_0,u_0) + Bv_0 \le v_0;$$

(3) *the operator equation $A(x,x) + Bx = x$ has a unique solution $x^* \in P_h$*;

(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1},y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1},x_{n-1}) + By_{n-1}, \quad n = 1,2,\ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Remark 2.1 When $B = \theta$ in Lemma 2.5, then the corresponding conclusion still holds.

Lemma 2.6 (see [18]) *Let $h > \theta$ and $\gamma \in (0,1)$.*
Let $A : P \times P \to P$ be a mixed monotone operator satisfying

$$A\left(tx, t^{-1}y\right) \geq tA(x,y), \quad \text{for any } t \in (0,1), x, y \in P, \tag{2.7}$$

and let $B : P \to P$ be an increasing γ-concave operator. Assume that

(i) *there is $h_0 \in P_h$ such that $A(h_0, h_0) \in P_h$ and $Bh_0 \in P_h$;*
(ii) *there exists a constant δ_0 such that $A(x,y) \leq \delta_0 Bx$ for any $x, y \in P$.*

Then:

(1) *$A : P_h \times P_h \to P_h$ and $B : P_h \to P_h$;*
(2) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that*

$$rv_0 \leq u_0 < v_0, \qquad u_0 \leq A(u_0, v_0) + Bu_0 \leq A(v_0, u_0) + Bv_0 \leq v_0;$$

(3) *the operator equation $A(x,x) + Bx = x$ has a unique solution $x^* \in P_h$;*
(4) *for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = A(x_{n-1}, y_{n-1}) + Bx_{n-1}, \qquad y_n = A(y_{n-1}, x_{n-1}) + By_{n-1}, \quad n = 1, 2, \ldots,$$

we have $x_n \to x^*$ and $y_n \to x^*$ as $n \to \infty$.

Remark 2.2 When $A = \theta$ in Lemma 2.6, then the corresponding conclusion still holds.

3 Main results

In this section, we give and prove our main results by applying Lemmas 2.5 and 2.6. We consider the Banach space $X = C[0,1]$ endowed with standard norm $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Clearly, this space can be equipped with a partial order given by

$$x, y \in C[0,1], \quad x \leq y \quad \Leftrightarrow \quad x(t) \leq y(t) \quad \text{for } t \in [0,1].$$

We define the cone $P = \{x \in X : x(t) \geq 0, t \in [0,1]\}$. Notice that P is a normal cone in $C[0,1]$ and the normality constant is 1.

Theorem 3.1 *Suppose that*

(F_1) *a function $f(t,x,y) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is continuous, increasing with respect to the second variable, and decreasing with respect to the third variable*;
(F_2) *a function $g(t,x) : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and increasing with respect to the second variable*;
(F_3) *there exists a constant $\gamma \in (0,1)$ such that $f(t, \lambda x, \lambda^{-1}y) \geq \lambda^{\gamma} f(t,x,y)$ for any $t \in [0,1]$, $\lambda \in (0,1)$, $x, y \in [0,+\infty)$, and $g(t, \lambda x) \geq \lambda g(t,x)$ for $\lambda \in (0,1)$, $t \in [0,1]$, $u \in [0,+\infty)$, and $g(t,0) \not\equiv 0$;*
(F_4) *there exists a constant $\delta_0 > 0$ such that $f(t,x,y) \geq \delta_0 g(t,x)$, $t \in [0,1]$, $x, y \geq 0$.*

Then:

(1) *there exist* $x_0, y_0 \in P_h$ *and* $r \in (0,1)$ *such that* $ry_0 \le x_0 < y_0$ *and*

$$x_0 \le \int_0^1 G(t,qs)\big[f\big(s,x_0(s),y_0(s)\big)+g\big(s,x_0(s)\big)\big]\,d_qs, \quad t\in[0,1],$$
$$y_0 \ge \int_0^1 G(t,qs)\big[f\big(s,y_0(s),y_0(s)\big)+g\big(s,y_0(s)\big)\big]\,d_qs, \quad t\in[0,1],$$

where $G(t,qs)$ *is defined by* (2.3), *and* $h(t)=t^{\alpha-1}$, $t\in[0,1]$;

(2) *the boundary value problem* (1.5) *has a unique positive solution* u^* *in* P_h, *and for any* $x_0, y_0 \in P_h$, *constructing successively the sequences*

$$x_{n+1} = \int_0^1 G(t,qs)\big[f\big(s,x_n(s),y_n(s)\big)+g\big(s,x_n(s)\big)\big]\,d_qs, \quad n=0,1,2,\ldots,$$
$$y_{n+1} = \int_0^1 G(t,qs)\big[f\big(s,y_n(s),x_n(s)\big)+g\big(s,y_n(s)\big)\big]\,d_qs, \quad n=0,1,2,\ldots,$$

we have $\|x_n-u^*\|\to 0$ *and* $\|y_n-u^*\|\to 0$ *as* $n\to\infty$.

Proof We note that if u is a solution of boundary value problem (1.5), then

$$u(t) = \int_0^1 G(t,qs)\big[f\big(s,u(s),u(s)\big)+g\big(s,u(s)\big)\big]\,d_qs, \quad 0\le t\le 1. \tag{3.1}$$

Define two operators $T_1 : P\times P\to E$ and $T_2 : P\to E$ by

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 G(t,qs)f\big(s,u(s),v(s)\big)\,d_qs, \\ (T_2u)(t) &= \int_0^1 G(t,qs)g\big(s,u(s)\big)\,d_qs. \end{aligned} \tag{3.2}$$

We transform the boundary value problem (1.5) into a fixed point problem $u=T_1(u,u)+T_2u$. From (F_1), (F_2), and Lemma 2.4 it is easy to see that $T_1 : P\times P\to P$ and $T_2 : P\to P$. Next, we want to prove that T_1 and T_2 satisfy the conditions of Lemma 2.5.

To begin with, we prove that T_1 is a mixed monotone operator. In fact, for $u_1,u_2,v_1,v_2\in P$ with $u_1\ge u_2$ and $v_1\le v_2$, it is easy to see that $u_1(t)\ge u_2(t)$, $v_1(t)\le v_2(t)$, $t\in[0,1]$, and by Lemma 2.4 and (F_1),

$$\begin{aligned} T_1(u_1,v_1)(t) &= \int_0^1 G(t,qs)f\big(s,u_1(s),v_1(s)\big)\,d_qs \\ &\ge \int_0^1 G(t,qs)f\big(s,u_2(s),v_2(s)\big)\,d_qs = T_1(u_2,v_2)(t). \end{aligned} \tag{3.3}$$

For any $\lambda\in(0,1)$ and $u,v\in P$, by (F_3) we have

$$\begin{aligned} T_1\big(\lambda u,\lambda^{-1}v\big)(t) &= \int_0^1 G(t,qs)f\big(s,\lambda u(s),\lambda^{-1}v(s)\big)\,d_qs \\ &\ge \lambda^\gamma \int_0^1 G(t,qs)f\big(s,u(s),v(s)\big)\,d_qs \ge \lambda^\gamma T_1(u,v)(t). \end{aligned} \tag{3.4}$$

So, the operator T_1 satisfies (2.6).

For any $u_1(t) \geq u_2(t)$, $t \in [0,1]$, from $G(t,qs) \geq 0$ and (F_2) we know that

$$T_2u_1(t) = \int_0^1 G(t,qs)g\bigl(s,u_1(s)\bigr)\,d_qs \geq \int_0^1 G(t,qs)g\bigl(s,u_2(s)\bigr)\,d_qs = T_2u_2(t).$$

So T_2 is increasing. Further, for any $\lambda \in (0,1)$ and $u \in P$, from hypothesis (F_3) we get

$$T_2(\lambda u)(t) = \int_0^1 G(t,qs)g\bigl(s,\lambda u(s)\bigr)\,d_qs \geq \lambda \int_0^1 G(t,qs)g\bigl(s,u(s)\bigr)\,d_qs = \lambda T_2u(t), \tag{3.5}$$

that is, the operator T_2 is subhomogeneous. By (F_1) and Lemma 2.4, for any $t \in [0,1]$, we have

$$\begin{aligned} T_1(h,h)(t) &= \int_0^1 G(t,qs)f\bigl(s,h(s),h(s)\bigr)\,d_qs \\ &= \int_0^1 G(t,qs)f\bigl(s,s^{\alpha-1},s^{\alpha-1}\bigr)\,d_qs \\ &\leq \frac{M_0}{\Gamma_q(\alpha)([\alpha]_q-\mu)}h(t)\int_0^1 f(s,1,0)\,d_qs \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} T_1(h,h)(t) &= \int_0^1 G(t,qs)f\bigl(s,h(s),h(s)\bigr)\,d_qs \\ &= \int_0^1 G(t,qs)f\bigl(s,s^{\alpha-1},s^{\alpha-1}\bigr)\,d_qs \\ &\geq \frac{\mu q^\alpha}{\Gamma_q(\alpha)([\alpha]_q-\mu)}h(t)\int_0^1 s(1-qs)^{(\alpha-1)}f(s,0,1)\,d_qs. \end{aligned} \tag{3.7}$$

From (F_2) and (F_4) we have the inequality

$$f(s,1,0) \geq f(s,0,1) \geq \delta_0 g(s,0) \geq 0.$$

Since $g(t,0) \not\equiv 0$, we also obtain

$$\int_0^1 f(s,1,0)\,d_qs \geq \int_0^1 f(s,0,1)\,d_qs \geq \delta_0\int_0^1 g(s,0)\,d_qs > 0. \tag{3.8}$$

Let

$$\begin{aligned} M_1 &= \frac{M_0}{\Gamma_q(\alpha)([\alpha]_q-\mu)}\int_0^1 f(s,1,0)\,d_qs, \\ M_2 &= \frac{\mu q^\alpha}{\Gamma_q(\alpha)([\alpha]_q-\mu)}\int_0^1 s(1-qs)^{(\alpha-1)}f(s,0,1)\,d_qs, \\ M_3 &= \frac{\mu q^\alpha}{\Gamma_q(\alpha)([\alpha]_q-\mu)}\int_0^1 s(1-qs)^{(\alpha-1)}g(s,0)\,d_qs, \\ M_4 &= \frac{M_0}{\Gamma_q(\alpha)([\alpha]_q-\mu)}\int_0^1 g(s,1)\,d_qs. \end{aligned}$$

Thus we have $M_2h(t) \le T_1(h,h) \le M_1h(t), M_3h(t) \le T_2h \le M_4h(t), t \in [0,1]$. So, $T_1(h,h) \in P_h$. From $g(t,0) \not\equiv 0$ it is easy to see that $T_2h \in P_h$. So, there is $h(t) = t^{\alpha-1} \in P_h$ such that $T_1(h,h) \in P_h$ and $T_2h \in P_h$.

Next, we prove that the operators T_1 and T_2 satisfy condition (ii) of Lemma 2.5. In fact, for $u, v \in P$ and any $t \in [0,1]$, by (F_4) we have

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 G(t,qs)f\big(s,u(s),v(s)\big)\,d_qs \\ &\ge \delta_0 \int_0^1 G(t,qs)g\big(s,u(s)\big)\,d_qs = \delta_0(T_2u)(t). \end{aligned} \tag{3.9}$$

Then we have $T_1(u,v) \ge \delta_0T_2u$ for $u, v \in P$. By Lemma 2.5 we can deduce: there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \le u_0 \le v_0$, $u_0 \le T_1(u_0,v_0) + T_2u_0 \le T_1(v_0,u_0) + T_2v_0 \le v_0$; the operator equation $T_1(u,u) + T_2u = u$ has a unique solution $u^* \in P_h$; and for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences

$$x_n = T_1(x_{n-1},y_{n-1}) + T_2x_{n-1}, \qquad y_n = T_1(y_{n-1},x_{n-1}) + T_2y_{n-1}, \quad n = 1,2,\ldots,$$

we get $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$. We have the following two inequalities:

$$\begin{aligned} u_0(t) &\le \int_0^1 G(t,qs)\big[f\big(s,u_0(s),v_0(s)\big) + g\big(s,u_0(s)\big)\big]\,d_qs, \quad t \in [0,1], \\ v_0(t) &\ge \int_0^1 G(t,qs)\big[f\big(s,v_0(s),u_0(s)\big) + g\big(s,v_0(s)\big)\big]\,d_qs, \quad t \in [0,1]. \end{aligned}$$

Thus problem (1.5) has a unique positive solution $u^* \in P_h$; for any $u_0, v_0 \in P_h$, constructing successively the sequences

$$\begin{aligned} x_{n+1}(t) &= \int_0^1 G(t,qs)\big[f\big(s,x_n(s),y_n(s)\big) + g\big(s,x_n(s)\big)\big]\,d_qs, \quad n = 0,1,2,\ldots, \\ y_{n+1}(t) &= \int_0^1 G(t,qs)\big[f\big(s,y_n(s),x_n(s)\big) + g\big(s,y_n(s)\big)\big]\,d_qs, \quad n = 0,1,2,\ldots, \end{aligned}$$

we have $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$. □

Corollary 3.1 *Suppose that f satisfies the conditions of Theorem 3.1 and $g \equiv 0$, $f(t,0,1) \not\equiv 0$. Then:*

(i) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \le u_0 < v_0$, and*

$$\begin{aligned} u_0(t) &\le \int_0^1 G(t,qs)\big[f\big(s,u_0(s),v_0(s)\big)\big]\,d_qs, \quad t \in [0,1], \\ v_0(t) &\ge \int_0^1 G(t,qs)\big[f\big(s,v_0(s),u_0(s)\big)\big]\,d_qs, \quad t \in [0,1], \end{aligned}$$

where $G(t,qs)$ is defined by (2.3), and $h(t) = t^{\alpha-1}$, $t \in [0,1]$;

(ii) *the BVP*

$$\begin{cases} D_q^\alpha u(t) + f(t,u(t),u(t)) = 0, & 0 < t < 1, 2 < \alpha \le 3, \\ u(0) = D_q u(0) = 0, & u(1) = \mu \int_0^1 u(s)\, d_q s, \end{cases} \tag{3.10}$$

has a unique positive solution u^* *in* P_h;

(iii) *for any* $x_0, y_0 \in P_h$, *the sequences*

$$x_{n+1} = \int_0^1 G(t,qs)\big[f\big(s,x_n(s),y_n(s)\big)\big]\, d_q s, \quad n = 0,1,2,\ldots,$$

$$y_{n+1} = \int_0^1 G(t,qs)\big[f\big(s,y_n(s),x_n(s)\big)\big]\, d_q s, \quad n = 0,1,2,\ldots,$$

satisfy $\|x_n - u^*\| \to 0$ *and* $\|y_n - u^*\| \to 0$ *as* $n \to \infty$.

Theorem 3.2 *Suppose that* (F_1)–(F_2) *hold. In addition, suppose that* f, g *satisfy the following conditions*:

(F_5) *there exists a constant* $\gamma \in (0,1)$ *such that* $g(t,\lambda u) \ge \lambda^\gamma g(t,u)$ *for any* $t \in [0,1]$, $\lambda \in (0,1)$, $u \in [0,+\infty)$, *and* $f(t,\lambda u, \lambda^{-1} v) \ge \lambda f(t,u,v)$ *for* $\lambda \in (0,1)$, $t \in [0,1]$, $u, v \in [0,+\infty)$;

(F_6) $f(t,0,1) \not\equiv 0$ *for* $t \in [0,1]$, *and there exists a constant* $\delta_0 > 0$ *such that* $f(t,u,v) \le \delta_0 g(t,u)$, $t \in [0,1]$, $u, v \ge 0$.

Then:

(1) *there exist* $u_0, v_0 \in P_h$ *and* $r \in (0,1)$ *such that* $rv_0 \le u_0 < v_0$ *and*

$$u_0 \le \int_0^1 G(t,qs)\big[f\big(s,u_0(s),v_0(s)\big) + g\big(s,u_0(s)\big)\big]\, d_q s, \quad t \in [0,1],$$

$$v_0 \ge \int_0^1 G(t,qs)\big[f\big(s,v_0(s),u_0(s)\big) + g\big(s,v_0(s)\big)\big]\, d_q s, \quad t \in [0,1],$$

where $G(t,qs)$ *is defined by* (2.3), *and* $h(t) = t^{\alpha-1}$, $t \in [0,1]$;

(2) *the boundary value problem* (1.5) *has a unique positive solution* u^* *in* P_h; *and for any* $x_0, y_0 \in P_h$, *the sequences*

$$x_{n+1} = \int_0^1 G(t,qs)\big[f\big(s,x_n(s),y_n(s)\big) + g\big(s,x_n(s)\big)\big]\, d_q s, \quad n = 0,1,2,\ldots,$$

$$y_{n+1} = \int_0^1 G(t,qs)\big[f\big(s,y_n(s),x_n(s)\big) + g\big(s,y_n(s)\big)\big]\, d_q s, \quad n = 0,1,2,\ldots,$$

satisfy $\|x_n - u^*\| \to 0$ *and* $\|y_n - u^*\| \to 0$ *as* $n \to \infty$.

Proof Similarly to the proof of Theorem 3.1, T_1 and T_2 are given in (3.2). From (F_1) and (F_2) we know that $T_1 : P \times P \to P$ is a mixed monotone operator and $T_2 : P \to P$ is increasing. By (F_5) we obtain

$$T_1\big(\lambda u, \lambda^{-1} v\big) \ge \lambda T_1(u,v), \qquad T_2(\lambda u) \ge \lambda^\gamma T_2 u, \quad \text{for } \lambda \in (0,1), u, v \in P.$$

According to (F_2) and (F_6), we have

$$f(s,0,1) \leq \delta_0 g(s,0), \qquad f(s,0,1) \leq f(s,1,0), \quad s \in [0,1].$$

From $f(t,0,1) \not\equiv 0$ we get

$$\begin{aligned} &0 < \int_0^1 f(s,0,1)\,d_qs \leq \int_0^1 f(s,1,0)\,d_qs, \\ &0 < \frac{1}{\delta_0}\int_0^1 f(s,0,1)\,d_qs \leq \int_0^1 g(s,0)\,d_qs \leq \int_0^1 g(s,1)\,d_qs, \end{aligned}$$

and the following inequalities hold:

$$\begin{aligned} 0 &< \frac{\mu q^\alpha}{\Gamma_q(\alpha)([\alpha]_q - \mu)} \int_0^1 s(1-qs)^{(\alpha-1)} f(s,0,1)\,d_qs \\ &\leq \frac{M_0}{\Gamma_q(\alpha)([\alpha]_q - \mu)} \int_0^1 f(s,1,0)\,d_qs, \qquad (3.11) \\ 0 &< \frac{\mu q^\alpha}{\Gamma_q(\alpha)([\alpha]_q - \mu)} \int_0^1 s(1-qs)^{(\alpha-1)} g(s,0)\,d_qs \\ &\leq \frac{M_0}{\Gamma_q(\alpha)([\alpha]_q - \mu)} \int_0^1 g(s,1)\,d_qs. \qquad (3.12) \end{aligned}$$

Hence we can easily check that $T_1(h,h) \in P$, $T_2h \in P$, $t \in [0,1]$, and, by using (F_6), we have

$$\begin{aligned} T_1(u,v)(t) &= \int_0^1 G(t,s) f\bigl(s,u(s),v(s)\bigr)\,d_qs \\ &\leq \delta_0 \int_0^1 G(t,s) g\bigl(s,u(s)\bigr)\,d_qs = \delta_0 T_2u(t). \qquad (3.13) \end{aligned}$$

Then we have $T_1(u,v) \leq \delta_0 T_2u$ for $u,v \in P$. Thus, from Lemma 2.6 we get that there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leq u_0 \leq v_0, u_0 \leq T_1(u_0,v_0) + T_2u_0 \leq T_1(v_0,u_0) + T_2v_0 \leq v_0$; the operator equation $T_1(u,u) + T_2u = u$ has a unique solution $u^* \in P_h$; and for any initial values $x_0, y_0 \in P_h$, the sequences

$$x_n = T_1(x_{n-1}, y_{n-1}) + T_2x_{n-1}, \qquad y_n = T_1(y_{n-1}, x_{n-1}) + T_2y_{n-1}, \quad n = 1,2,\ldots,$$

satisfy $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$. That is,

$$\begin{aligned} &u_0(t) \leq \int_0^1 G(t,qs)\bigl[f\bigl(s,u_0(s),v_0(s)\bigr) + g\bigl(s,u_0(s)\bigr)\bigr]\,d_qs, \quad t \in [0,1], \\ &v_0(t) \geq \int_0^1 G(t,qs)\bigl[f\bigl(s,v_0(s),u_0(s)\bigr) + g\bigl(s,v_0(s)\bigr)\bigr]\,d_qs, \quad t \in [0,1], \end{aligned}$$

The boundary value problem (1.5) has a unique positive solution $u^* \in P_h$; for $u_0, v_0 \in P_h$, the sequences

$$x_{n+1}(t) = \int_0^1 G(t,qs)\big[f\big(s,x_n(s),y_n(s)\big) + g\big(s,x_n(s)\big)\big]\,d_qs, \quad n = 0,1,2,\ldots,$$
$$y_{n+1}(t) = \int_0^1 G(t,qs)\big[f\big(s,y_n(s),x_n(s)\big) + g\big(s,y_n(s)\big)\big]\,d_qs, \quad n = 0,1,2,\ldots,$$

satisfy $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$. □

Corollary 3.2 *Suppose that g satisfies the conditions of Theorem 3.2, $f \equiv 0$, and $g(t,0) \not\equiv 0$ for $t \in [0,1]$. Then:*

(i) *there exist $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \le u_0 < v_0$, and*

$$u_0(t) \le \int_0^1 G(t,qs)\big[g\big(s,u_0(s)\big)\big]\,d_qs,$$
$$v_0(t) \ge \int_0^1 G(t,qs)\big[g\big(s,v_0(s)\big)\big]\,d_qs, \quad t \in [0,1],$$

where $G(t,qs)$ is defined by (2.3), and $h(t) = t^{\alpha-1}$, $t \in [0,1]$;

(ii) *the BVP*

$$\begin{cases} D_q^\alpha u(t) + g(t,u(t)) = 0, & 0 < t < 1, 2 < \alpha \le 3, \\ u(0) = D_qu(0) = 0, & u(1) = \mu \int_0^1 u(s)\,d_qs, \end{cases} \tag{3.14}$$

has a unique positive solution u^* in P_h; and for any $x_0, y_0 \in P_h$, the sequences

$$x_{n+1} = \int_0^1 G(t,qs)g\big(s,x_n(s)\big)\,d_qs, \quad n = 0,1,2,\ldots,$$
$$y_{n+1} = \int_0^1 G(t,qs)g\big(s,y_n(s)\big)\,d_qs, \quad n = 0,1,2,\ldots,$$

satisfy $\|x_n - u^*\| \to 0$ and $\|y_n - u^*\| \to 0$ as $n \to \infty$.

4 Example

Now, we give two examples to illustrate our results.

Example 4.1 Consider the following boundary value problem:

$$\begin{cases} -D_{\frac{1}{2}}^{\frac{5}{2}}u(t) = u(t)^{\frac{1}{3}} + [u(t)+1]^{-\frac{1}{2}} + \frac{u(t)}{1+u(t)}t^3 + t^2 + 4, & 0 < t < 1, \\ u(0) = D_{\frac{1}{2}}u(0) = 0, & u(1) = \mu \int_0^1 u(s)\,d_{\frac{1}{2}}s. \end{cases} \tag{4.1}$$

In this example, we let

$$f(t,u,v) = u^{\frac{1}{3}} + [v+1]^{-\frac{1}{2}} + t^2 + 2, \qquad g(t,u) = \frac{u}{1+u}t^3 + 2,$$
$$\gamma = \frac{1}{2}, \qquad \mu = \frac{1}{2}.$$

It is not difficult to find that $f(t,x,y):[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous, increasing with respect to the second variable, and decreasing with respect to the third variable and that $g(t,x):[0,1]\times[0,+\infty)\to[0,+\infty)$ is continuous with $g(t,0)=2>0$ and increasing with respect to the second variable. We also have

$$\begin{aligned} g(t,\lambda u) &= \frac{\lambda u}{1+\lambda u}t^3+2\geq\frac{\lambda u}{1+u}t^3+2\lambda=\lambda g(t,u),\quad \lambda\in(0,1),\\ f\bigl(t,\lambda u,\lambda^{-1}v\bigr) &= \lambda^{\frac{1}{3}}u^{\frac{1}{3}}+\lambda^{\frac{1}{2}}[v+\lambda]^{-\frac{1}{2}}+t^2+2\\ &\geq \lambda^{\frac{1}{2}}\bigl\{u^{\frac{1}{3}}+[v+1]^{-\frac{1}{2}}+t^2+2\bigr\}\\ &= \lambda^{\gamma}f(t,u,v). \end{aligned}$$

Further, if we take $\delta_0\in(0,\frac{2}{3}]$, then we easily get

$$\begin{aligned} f(t,u,v) &= u^{\frac{1}{3}}+[v+1]^{-\frac{1}{2}}+t^2+2\geq 2=\frac{2}{3}\cdot 3\\ &\geq \delta_0\biggl[\frac{u}{1+u}t^3+2\biggr]=\delta_0 g(t,u). \end{aligned}$$

So f and g satisfy the conditions of Theorem 3.1. Thus by Theorem 3.1 the boundary value problem (4.1) has a unique positive solution in P_h, where $h(t)=t^{\alpha-1}=t^{\frac{3}{2}}$, $t\in[0,1]$.

Example 4.2 Consider the following boundary value problem:

$$\begin{cases} -D_{\frac{1}{2}}^{\frac{5}{2}}u(t)=(\frac{u(t)}{1+u(t)})^{\frac{1}{4}}+[u(t)+1]^{-\frac{1}{3}}+t^3+u(t)^{\frac{1}{3}}+t^2+1, & 0<t<1,\\ u(0)=D_{\frac{1}{2}}u(0)=0, \qquad u(1)=\mu\int_0^1 u(s)\,d_{\frac{1}{2}}s. \end{cases} \tag{4.2}$$

We let

$$f(t,u,v)=\biggl(\frac{u}{1+u}\biggr)^{\frac{1}{4}}+[v+1]^{-\frac{1}{3}}+t^3,\qquad g(t,u)=u^{\frac{1}{3}}+t^2+1,\qquad \gamma=\frac{1}{3},\qquad \mu=\frac{1}{2}.$$

It is not difficult to find that $f(t,x,y):[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ is continuous, increasing with respect to the second variable, and decreasing with respect to the third variable and that $g(t,x):[0,1]\times[0,+\infty)\to[0,+\infty)$ and increasing with respect to the second variable. We also have

$$\begin{aligned} g(t,\lambda u) &= \lambda^{\frac{1}{3}}u^{\frac{1}{3}}+t^2+1\geq\lambda^{\frac{1}{3}}\bigl[u^{\frac{1}{3}}+t^2+1\bigr]=\lambda^{\gamma}g(t,u),\quad \lambda\in(0,1),\\ f\bigl(t,\lambda u,\lambda^{-1}v\bigr) &= \biggl(\frac{\lambda u}{1+\lambda u}\biggr)^{\frac{1}{4}}+\bigl[\lambda^{-1}v+1\bigr]^{-\frac{1}{3}}+t^3\\ &\geq \lambda^{\frac{1}{3}}\biggl\{\biggl(\frac{u}{1+u}\biggr)^{\frac{1}{4}}+[v+\lambda]^{-\frac{1}{3}}+t^3\biggr\}\\ &\geq \lambda\biggl\{\biggl(\frac{u}{1+u}\biggr)^{\frac{1}{4}}+[v+1]^{-\frac{1}{3}}+t^3\biggr\}\\ &= \lambda f(t,u,v). \end{aligned}$$

If we take $\delta_0 = 1 > 0$, then we have

$$f(t,u,v) = \left(\frac{u}{1+u}\right)^{\frac{1}{4}} + [v+1]^{-\frac{1}{3}} + +t^3 \leq u^{\frac{1}{4}} + t^2 + 1 \leq u^{\frac{1}{3}} + t^2 + 1 = \delta_0 g(u,t).$$

So f and g satisfy the conditions of Theorem 3.2. Thus by Theorem 3.2 the boundary value problem (4.2) has a unique positive solution in P_h, where $h(t) = t^{\alpha-1} = t^{\frac{3}{2}}$, $t \in [0,1]$.

Acknowledgements
The authors are very grateful to the reviewers for their valuable suggestions and useful comments, which led to an improvement of this paper.

Funding
This project was supported by the National Natural Science Foundation of China (Grant No. 11271235).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
FG carried out the molecular genetic studies, participated in the sequence alignment, and drafted the manuscript. SK conceived the study and participated in its design and coordination. FC helped to draft the manuscript. All authors read and approved the final manuscript.

Publisher's Note

Received: 23 April 2018 Accepted: 4 September 2018 Published online: 22 October 2018

References

1. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)
2. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equation. Wiley, New York (1993)
3. Glockle, W.G., Nonnenmacher, T.F.: A fractional calculus approach of self-similar protein dynamics. Biophys. J. **68**, 46–53 (1995)
4. Podlubny, I.: Fractional Differential Equations. Mathematics in Science and Engineering. Academic Press, New York (1999)
5. Field, C., Joshi, N., Nijhoff, F.: q-Difference equations of KdV type and Chazy-type second-degree difference equations. J. Phys. A, Math. Theor. **41**, 1–13 (2008)
6. Abreu, L.: Sampling theory associated with q-difference equations of the Sturm–Liouville type. J. Phys. A **38**(48), 10311–10319 (2005)
7. Jackson, F.: On q-functions and a certain difference operator. Trans. R. Soc. Edinb. **46**, 253–281 (1908)
8. Jackson, F.: On q-definite integrals. Pure Appl. Math. Q. **41**, 193–203 (1910)
9. Rajković, P., Marinković, S., Stanković, M.: Fractional integrals and derivatives in q-calculus. Appl. Anal. Discrete Math. **1**(1), 311–323 (2007)
10. Annaby, M.H., Mansour, Z.S.: q-Fractional Calculus and Equations. Lecture Notes in Mathematics, vol. 2056. Springer, Berlin (2012)
11. Al-Salam, W.A.: Some fractional q-integrals and q-derivatives. Proc. Edinb. Math. Soc. **15**, 135–140 (1966)
12. Agarwal, R.P.: Certain fractional q-integrals and q-derivatives. Proc. Camb. Philos. Soc. **66**, 365–370 (1969)
13. Ferreira, R.A.C.: Nontrivial solutions for fractional q-difference boundary value problems. Electron. J. Qual. Theory Differ. Equ., **2010**, 70 (2010)
14. Ferreira, R.A.C.: Positive solutions for a class of boundary value problems with fractional q-differences. Comput. Math. Appl. **61**(2), 367–373 (2011)
15. El-Shahed, M., Al-Askar, F.: Positive solution for boundary value problem of nonlinear fractional q-difference equation. ISRN Math. Anal. **2011**, Article ID 385459 (2011)
16. Darzi, R., Agheli, B.: Existence results to positive solution of fractional BVP with q-derivatives. J. Appl. Math. Comput. **55**, 353–367 (2017)
17. Zhai, C.B., Hao, M.R.: Fixed point theorems for mixed monotone operators with perturbation and applications to fractional differential equation boundary value problems. Nonlinear Anal. **75**, 2542–2551 (2012)
18. Zhai, C., Yang, C., Zhang, X.: Positive solutions for nonlinear operator equations and several classes of applications. Math. Z. **266**, 43–63 (2010)
19. Ahmad, B., Ntouyas, S.K., Purnaras, I.K.: Existence results for nonlocal boundary value problems of nonlinear fractional q-difference equations. Adv. Differ. Equ. **2012**, 140 (2012)
20. Graef, J.R., Kong, L.: Positive solutions for a class of higher order boundary value problems with fractional q-derivatives. Appl. Math. Comput. **218**, 9682–9689 (2012)
21. Almeida, R., Martins, N.: Existence results for fractional q-difference equations of order $\alpha \in [2,3]$ with three-point boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **19**, 1675–1685 (2014)

22. Yang, W.: Positive solution for fractional q-difference boundary value problems with Φ-Laplacian operator. Bull. Malays. Math. Sci. Soc. **36**, 1195–1203 (2013)
23. Ahmad, B., Etemad, S., Ettefagh, M., Rezapour, S.: On the existence of solutions for fractional q-difference inclusions with q-antiperiodic boundary conditions. Bull. Math. Soc. Sci. Math. Roum. **59**, 119–134 (2016)
24. Agarwal, R.P., Ahmad, B., Alsaedi, A., Al-Hutami, H.: Existence theory for q-antiperiodic boundary value problems of sequential q-fractional integro-differential equations. Abstr. Appl. Anal. **2014**, Article ID 207547 (2014)
25. Wang, J.R., Zhang, Y.R.: On the concept and existence of solutions for fractional impulsive systems with Hadamard derivatives. Appl. Math. Lett. **39**, 85–90 (2015)
26. Zhai, C.B., Yan, W.P., Yang, C.: A sum operator method for the existence and uniqueness of positive solution to Riemann–Liouville fractional differential equation boundary value problems. Commun. Nonlinear Sci. Numer. Simul. **18**, 858–866 (2013)
27. Zhao, Y., Ye, G., Chen, H.: Multiple positive solutions of a singular semipositone integral boundary value problem for fractional q-derivatives equation. Abstr. Appl. Anal. **2013**, Article ID 643571 (2013). https://doi.org/10.1155/2013/643571
28. Ahmad, B., Ntouyas, S.K., Alsaedi, A., Al-Hutami, H.: Nonlinear q-fractional differential equations with nonlocal and sub-strip type boundary conditions. Electron. J. Qual. Theory Differ. Equ. **2014**, 26 (2014)
29. Almeida, R., Martins, N.: Existence results for fractional q-difference equations of order $\alpha \in [2,3]$ with three-point boundary conditions. Commun. Nonlinear Sci. Numer. Simul. **19**, 1675–1685 (2014)
30. Sitthiwirattham, T.: On nonlocal fractional q-integral boundary value problems of fractional q-difference equations and fractional q-integrodifference equations involving different numbers of order and q. Bound. Value Probl. **2016**, Article ID 12 (2016)
31. Patanarapeelert, N., Sriphanomwan, U., Sitthiwirattham, T.: On a class of sequential fractional q-integrodifference boundary value problems involving different numbers of q in derivatives and integrals. Adv. Differ. Equ. **2016**, Article ID 148 (2016)
32. Sriphanomwan, U., Tariboon, J., Patanarapeelert, N., Sitthiwirattham, T.: Existence results of nonlocal boundary value problems for nonlinear fractional q-integral difference equations. J. Nonlinear Funct. Anal. **2017**, Article ID 28 (2017)