Zhong et al. *Advances in Difference Equations* (2018) 2018:335
https://doi.org/10.1186/s13662-018-1795-4

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Energy decay rate of multidimensional inhomogeneous Landau–Lifshitz–Gilbert equation and Schrödinger map equation on the sphere

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Abstract

We consider the multidimensional dimensional inhomogeneous Landau–Lifshitz–Gilbert (ILLG) equation and its degenerate case, the Schrödinger map equation. We investigate the special solutions (under large initial values) and their energy property of the ILLG and Schrödinger map equations. Until now, we had not seen a paper presenting an explicit dynamic solution of the multidimensional ILLG. Using the stereographic method, an equivalent equation of ILLG is obtained. Based on this equivalent system, we obtain some exact solutions of the ILLG equation and present some implicit solutions of the Schrödinger map equation. Based on these solutions, by a careful estimation we give the decay rate of energy density.

Keywords: Landau–Lifshitz–Gilbert equation; Schrödinger map equation; Decay rate

1 Introduction

Long-wavelength spin motions in diverse ferromagnetic structures are commonly described by the Landau–Lifshitz–Gilbert equation (or the LLG equation), which was first derived by Landau and Lifshitz [14]. The LLG equation is a very fundamental equation in describing the evolution for the magnetization in ferromagnetic materials such as Navier–Stokes equations in fluid dynamics. Without the effects of anisotropy and external magnetic field, the LLG equation can be written as

$$\frac{\partial}{\partial t}S = \alpha S \times \Delta S - \beta S \times (S \times \Delta S), \tag{1}$$

where $S = (S_1, S_2, S_3)$, $S \in S^2 \hookrightarrow R^3$, $\alpha^2 + \beta^2 = 1$ ($\alpha \geq 0$ and $\beta \geq 0$), and $\times$ denotes the cross product.

The LLG equation is a mixture of two famous equations, the Schrödinger map equation ($\alpha = 1$) and the harmonic map heat flow ($\beta = 1$). The term multiplied with α represents the exchange interaction. From a physical point of view, the exchange interaction constitutes an indispensable part of the LLG system. However, the β-term denotes to the Gilbert damping considered as a dissipation factor in the LLG equation.

In the setting of the LLG equation, of particular importance is to consider the effect of inhomogeneity on the spin system. This extension is somewhat like the the inhomogeneous Schrödinger map equation proposed by Balakrishnan [1]. If the dissipation and inhomogeneity are considered in LLG, (1) can be generalized into the ILLG equation. Motivated by these considerations, in this work, we consider the n-dimensional ILLG equations

$$\begin{aligned}\frac{\partial}{\partial t}S &= \alpha\big[q(t,\overrightarrow{x})S \times \Delta S + S \times \big(\nabla q(t,\overrightarrow{x})\nabla S\big)\big] \\ &\quad - \beta S \times \big[q(t,\overrightarrow{x})S \times \Delta S + S \times \big(\nabla q(t,\overrightarrow{x})\nabla S\big)\big],\end{aligned} \tag{2}$$

where the scalar function $q(t,\overrightarrow{x})$ is the inhomogeneous term.

In the setting of the LLG equation (1), we list several known results. Liu [13] proved that the concentration set for fixed t is an H^{m-2}-rectifiable set for almost all t for LLG. For a smallness initial condition on the gradient, the global well-posedness results for (1) have been established in $n \geq 2$ by Melcher [17]. Some further works about the smallness condition for well-posedness of (1) were done by Lin, Lai, and Wang [15] in the Morrey space. The smallness condition does not always mean that the global solution exists. Ding and Wang [7] proved that the solution of (1) develops a finite time singularity in dimensions 3 or 4 under small special initial data. In the critical dimension $n = 2$, (1) contains a blowup solution, and the exact blowup rate is predicted by formal analysis [26] for some special large data. If $\alpha = 0$, then LLG degenerates into a harmonic map heat flow. Similarly, comparing a harmonic map heat flow with some general harmonic system [22, 32–34], it is clear that the mapping one is more difficult to deal with than a nonmapping system. The most difficulty of a mapping system is caused by the curvature flow of the Riemannian manifolds.

The results on the Schrödinger map equation are much more fruitful than those on the LLG equation. There are a lot fruitful results on the existence, uniqueness, and the blowup property and soliton solution of the nonlinear Schrödinger equation [11, 19, 24, 31]. Even for the fractional Schrödinger equations, there are some recent fruitful results [2, 8, 18, 20]. However, very few results can be seen for the Schrödinger map equation, and some further work still needs to be done. We refer the reader to [25] for some results about the local existence theorem. Some progress of small initial data existence results can be found in [4] and [3] for $n \geq 2$. Especially, the classical solution with small energy is global in time for the radial case [4]. For some special large initial data, the possibility of a finite time blowup and the blowup rate has been proved [21, 23]. In 2008, Huh [10] constructed infinite energy explicit blowup solutions for the modified Schrödinger map equation. Ding [5] constructed an infinite energy blowup solution for the Schrödinger map equation on a hyperbolic target.

In spite of these developments on the LLG equation and Schrödinger map equation, little progress has been made in the case of ILLG equation. Comparing to many existence results of the homogeneous case, in which $q(t,\overrightarrow{x})$ (shortly, q) is a constant, the inhomogeneous case is not so clear (even the local solution). However, these existence (or blowup) results of the LLG equation or Schrödinger map equation are expected to be extended to the inhomogeneous case.

For $\beta = 0$, Daniel et al. [6] analyzed the singularity structure of solutions of (2). More exactly, (2) identified the integrable model under the inhomogeneity as follows:

$$q = \frac{\varepsilon_1}{r^{2(n-1)}} + \frac{\varepsilon_2}{r^{n-2}},$$

where r is the radial spatial variable.

The Lax pair of this integrable case was found, and some soliton like solutions are also presented in [6]. However, Li and Wang [16] proved that blowup occurs in some specific format of the inhomogeneous term. In fact, their did not provide an exact form of the solution of the inhomogeneous Schrödinger map equation. If q takes some decay forms, then this equation contains some finite-time blowup solutions such as the solutions first presented in [28, 29]. Similarly, this equation ill also develops some finite-time blowup solution on the hyperbolic target [30]. More clearly, in [28, 29], the authors provided a solution of the form

$$S(t, \overrightarrow{x}) = \begin{pmatrix} \sin(F_1(\overrightarrow{x}))\cos(\frac{F_2(\overrightarrow{x})}{G_1(t)}) \\ \sin(F_1(\overrightarrow{x}))\sin(\frac{F_2(\overrightarrow{x})}{G_1(t)}) \\ \cos(F_1(\overrightarrow{x})) \end{pmatrix}, \tag{3}$$

where $F_1(\overrightarrow{x})$ and $F_2(\overrightarrow{x})$ are functions of $\overrightarrow{x}$, and $G_1(t)$ is a function of t such that $G_1(t) \to +\infty$ as $t \to T$ (T is a positive constant).

These finite-time blowup solutions indicate that some special solutions of the equation can develop a finite-time singularity from smooth initial data with finite energy in finite (or infinite) spatial domain. It is a natural question to ask whether the ILLG equation contains some blowup or nonblowup solution. However, we cannot obtain any explicit or implicit solution under the constraint of (3). Therefore, in this paper, we are devoted to solve the ILLG equation under another type of solution,

$$S(t, \overrightarrow{x}) = \begin{pmatrix} \sin(F_1(\overrightarrow{x}))\cos(F_2(\overrightarrow{x}) + G_1(t))) \\ \sin(F_1(\overrightarrow{x}))\sin(F_2(\overrightarrow{x}) + G_1(t))) \\ \cos(F_1(\overrightarrow{x})) \end{pmatrix}, \tag{4}$$

where $F_1(\overrightarrow{x})$ and $F_2(\overrightarrow{x})$ are functions of $\overrightarrow{x}$, and $G_1(t)$ is a function of t.

We use the variable separation ansatz (4) to construct the solution of ILLG equation in Sect. 3. At the same time, we employ this form to construct an implicit solution of the Schrödinger map equation in Sect. 4.

Up to now, we have never seen any paper discussing how to exactly solve the ILLG equation. As far as we know, even for the LLG equation and Schrödinger map equation, the exact treatment is very scarce. The ILLG and Schrödinger map equations are nonintegrable systems in most cases. Based on the property of the Hamiltonian systems [39–41], the bifurcation structure of the general integrable systems and autonomous differential systems [35–38] are clear. However, the property of the nonintegrability of a system needs to be further explored. In this paper, we present some special solutions of the ILLG equation and Schrödinger map equation and discuss their properties to enrich the solutions of these equations. It is well known that the main barrier to the ILLG equation and Schrödinger map equation is that there is no energy monotonicity inequality for them. Hence, it is difficult to determine how the solution will evolve after a long time or in the remote area. We present some results, which may be expected to result in a better understanding of these two systems. More exactly, we study the special solutions in the form of (4) and present some explicit or implicit structures of it, which will be useful in studying their properties.

Based on the smallness initial condition, we can prove the global existence of weak (or even smooth) solutions of the Schrödinger map equation (or even ILLG). However, the question of regularity and uniqueness of weak solutions is a delicate question depending on the initial data. In this paper, we study some special large initial data solutions. For the ILLG, some properties of the global type solution are clearer because of their straightforward and exact form. We also present some implicit solution for the Schrödinger map equation. At the same time, we analyze a type of special large initial data solution, which indicates a global solution.

The plan of the paper is as follows. In Sect. 2, we construct the equation of motion to represent the ILLG equation and then obtain an equivalent generalized nonlinear modification equation by the stereographic projection method. In Sect. 3, we construct the solution of this equivalent system and deduce the solution of the ILLG equation and analyze the energy density of these two equivalent equations. We study the variable separation solutions under the cylindrical coordinates and the normal coordinates in Sect. 4. In this section, we also carry out the decay analysis to identify the asymptotic behavior of the energy density.

2 Equivalent equation of ILLG equation

In this section, we deduce an equivalent system of (2), which is helpful in constructing its solution. As we know, the Hasimoto transformation is usually employed to convert some geometrical partial differential equation to another complex equation. However, it is always difficult to avoid the emergence of some integral term in the deduced new system. Always, it is difficult to solve this new deduced system due to the new integral term difficult to deal with. The stereographic projection method avoids some defects such as the appearing of integral section when we do the equivalent transformation between the two systems. Furthermore, if we obtain a solution of the new deriving equation, then we can usually obtain the solution of the original equation by the opposite transformation of the stereographic projection. However, it is difficult to do this opposite process for the equivalent deriving system under the Hasimoto transformation. Considering this situation, we use the stereographic projection to deduce the equivalent system that does not contain a nonlocal term. First, we calculate the expanded form of (2):

$$\alpha\Psi_1 - \beta(S_2\Psi_3 - S_3\Psi_2) - S_{1t} = 0, \tag{5}$$

$$\alpha\Psi_2 - \beta(-S_1\Psi_3 + S_3\Psi_1) - S_{2t} = 0, \tag{6}$$

$$\alpha\Psi_3 - \beta(S_1\Psi_2 - S_2\Psi_1) - S_{3t} = 0, \tag{7}$$

where

$$\Psi_1 = q(S_2\Delta S_3 - S_3\Delta S_2) + \nabla q(S_2\nabla S_3 - S_3\nabla S_2), \tag{8}$$

$$\Psi_2 = q(-S_1\Delta S_3 + S_3\Delta S_1) + \nabla q(-S_1\nabla S_3 + S_3\nabla S_1), \tag{9}$$

$$\Psi_3 = q(S_1\Delta S_2 - S_2\Delta S_1) + \nabla q(S_1\nabla S_2 - S_2\nabla S_1). \tag{10}$$

The couple system (5)–(10) is complicated. If we could simplify it, the resolving progress would become much simpler. As we will see, (5)–(10) will adopt another single complex form. We definite

$$W = \frac{S_1 + iS_2}{1 + S_3}.$$

Furthermore, $\overline{W}$ denotes the conjugate complex numbers of W; the real and imaginary parts of the complex number W are $\Re(W)$ and $\Im(W)$, respectively. The spin vector S of LLG equation (2) is laid on $\mathbb{S}^2$. Hence, we can use the conversion

$$(S_1, S_2, S_3) = \left(\frac{2\Re(W)}{1 + W\overline{W}}, \frac{2\Im(W)}{1 + W\overline{W}}, \frac{1 - W\overline{W}}{1 + W\overline{W}}\right). \tag{11}$$

The derivative of each component of S_1 is

$$S_{1t} = \Phi\left[-W^2\overline{W_t} - (\overline{W})^2 W_t + W_t + \overline{W_t}\right], \tag{12}$$

$$\nabla S_1 = \Phi\left[-W^2\overline{\nabla W} - (\overline{W})^2\nabla W + \nabla W + \overline{\nabla W}\right], \tag{13}$$

where $\Phi = (1 + W\overline{W})^{-2}$.

Similarly, the first-order derivative of S_2 is

$$S_{2t} = i\Phi\left[W^2\overline{W_t} - (\overline{W})^2 W_t - W_t + \overline{W_t}\right], \tag{14}$$

$$\nabla S_2 = i\Phi\left[W^2\overline{\nabla W} - (\overline{W})^2\nabla W - \nabla W + \overline{\nabla W}\right], \tag{15}$$

and the derivative of S_3 is

$$S_{3t} = -2\Phi(W_t\overline{W} + W\overline{W_t}), \tag{16}$$

$$\nabla S_3 = -2\Phi(\nabla W\overline{W} + W\overline{\nabla W}). \tag{17}$$

Setting $\langle A, B\rangle = A \cdot B$ and $\langle A\rangle = A \cdot A$, the derivatives of (13), (15), and (17) are

$$\begin{aligned} \Delta S_1 &= \Phi\left[-W^2\overline{\Delta W} - (\overline{W})^2\Delta W + \Delta W + \overline{\Delta W}\right] \\ &\quad + 2\Phi^{3/2}\left[\langle\nabla W\rangle(\overline{W})^3 + \langle\overline{\nabla W}\rangle W^3 - \langle\nabla W\rangle\overline{W}\right. \\ &\quad \left. - 2\langle\nabla W, \overline{\nabla W}\rangle W - 2\langle\nabla W, \overline{\nabla W}\rangle\overline{W} - \langle\overline{\nabla W}\rangle W\right], \end{aligned} \tag{18}$$

$$\begin{aligned} \Delta S_2 &= i\Phi\left[\overline{\Delta W}W^2 - (\overline{W})^2\Delta W + \overline{\Delta W} - \Delta W\right] \\ &\quad + 2i\Phi^{3/2}\left[-\langle\overline{\nabla W}\rangle W^3 + (\overline{W})^3\langle\nabla W\rangle - \langle\overline{\nabla W}\rangle W\right. \\ &\quad \left. - 2\langle\overline{\nabla W}, \nabla W\rangle\overline{W} + 2\langle\overline{\nabla W}, \nabla W\rangle W + \overline{W}\langle\nabla W\rangle\right], \end{aligned} \tag{19}$$

and

$$\Delta S_3 = -2\Phi[W\overline{\Delta W} + \Delta W\overline{W}] + 4\Phi^{3/2}\left[W^2\langle\overline{\nabla W}\rangle + W\overline{W}\langle\nabla W, \overline{\nabla W}\rangle\right.$$

$$+ (\overline{W})^2\langle\nabla W\rangle - \langle\nabla W, \overline{\nabla W}\rangle\big], \tag{20}$$

respectively.

Substituting (11)–(20) into (5), the complex equation of W is

$$\begin{aligned} &W_t\overline{W} + W\overline{W}_t \\ &\quad= -(\alpha - \beta i)iq\left[W\overline{\Delta W} - \Delta W\overline{W} + 2\frac{\langle\nabla W\rangle(\overline{W})^2}{W\overline{W}+1} - 2\frac{\langle\overline{\nabla W}\rangle W^2}{W\overline{W}+1}\right] \\ &\qquad - (\alpha - \beta i)i\nabla q\cdot(W\overline{\nabla W} - \nabla W\overline{W}), \end{aligned}$$

which can be simplified as

$$\begin{aligned} \Re(W_t\overline{W}) = -(\alpha - \beta i)i\bigg[&-iq\Im(\Delta W\overline{W}) + iq\Im\left(\frac{2\langle\nabla W\rangle(\overline{W})^2}{W\overline{W}+1}\right) \\ &- i\nabla q\cdot\Im(\nabla W\overline{W})\bigg] \end{aligned} \tag{21}$$

or

$$\begin{aligned} \Re(\overline{W}_t W) = -(\alpha - \beta i)i\bigg[&iq\Im(\Delta\overline{W}W) - iq\Im\left(\frac{2\langle\nabla\overline{W}\rangle(W)^2}{W\overline{W}+1}\right) \\ &+ i\nabla q\cdot\Im(\nabla\overline{W}W)\bigg]. \end{aligned} \tag{22}$$

Similarly to (21) or (22), (6) transforms into

$$\begin{aligned} &\overline{W_t}(W^2 - 1) + W_t(\overline{W}^2 - 1) \\ &\quad= -(\alpha - \beta i)iq\left[\overline{\Delta W}(W^2 - 1) - \Delta W(\overline{W}^2 - 1) - 2\frac{\langle\overline{\nabla W}\rangle W(W^2-1)}{1 + W\overline{W}}\right. \\ &\qquad\left. + 2\frac{\langle\nabla W\rangle\overline{W}(\overline{W}^2 - 1)}{1 + W\overline{W}}\right] - (\alpha - \beta i)i\nabla q\cdot\left[\overline{\nabla W}(W^2 - 1) - \nabla W(\overline{W}^2 - 1)\right], \end{aligned}$$

which can be rearranged as

$$\begin{aligned} &\Re\big(W_t(\overline{W}^2 - 1)\big) \\ &\quad= -(\alpha - \beta i)i\left[-iq\Im\big(\Delta W(\overline{W}^2 - 1)\big) + qi\Im\left(\frac{2\langle\nabla W\rangle\overline{W}(\overline{W}^2 - 1)}{W\overline{W}+1}\right)\right. \\ &\qquad\left. - i\nabla q\cdot\Im\big(\nabla W(\overline{W}^2 - 1)\big)\right] \end{aligned} \tag{23}$$

or

$$\begin{aligned} &\Re\big(\overline{W}_t(W^2 - 1)\big) \\ &\quad= -(\alpha - \beta i)i\left[iq\Im\big(\Delta\overline{W}(W^2 - 1)\big) - iq\Im\left(\frac{2\langle\nabla\overline{W}\rangle W(W^2 - 1)}{W\overline{W}+1}\right)\right. \end{aligned}$$

$$+ i\nabla q \cdot \Im\big(\nabla\overline{W}\big(W^2-1\big)\big)\bigg]. \tag{24}$$

Similarly, we obtain the equivalent equation of (7)

$$\begin{aligned}
&\overline{W_t}\big(W^2+1\big) - W_t\big(\overline{W}^2+1\big) \\
&\quad = -(\alpha-\beta i)iq\bigg[\overline{\Delta W}\big(W^2+1\big) + \Delta W\big(\overline{W}^2+1\big) - 2\frac{\langle\overline{\nabla W}\rangle W(W^2+1)}{1+W\overline{W}} \\
&\qquad - 2\frac{\langle\nabla W\rangle\overline{W}(\overline{W}^2+1)}{1+W\overline{W}}\bigg] - (\alpha-\beta i)i\nabla q\cdot\big[\overline{\nabla W}\big(W^2+1\big) + \nabla W\big(\overline{W}^2+1\big)\big],
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\Im\big(W_t\big(\overline{W}^2+1\big)\big) \\
&\quad = -(\alpha-\beta i)i\bigg[-iq\Re\big(\Delta W\big(\overline{W}^2+1\big)\big) + iq\Re\bigg(\frac{2\langle\nabla W\rangle\overline{W}(\overline{W}^2+1)}{W\overline{W}+1}\bigg) \\
&\qquad - i\nabla q\cdot\Re\big(\nabla W\big(\overline{W}^2+1\big)\big)\bigg]
\end{aligned} \tag{25}$$

or

$$\begin{aligned}
&-\Im\big(\overline{W}_t\big(W^2+1\big)\big) \\
&\quad = -(\alpha-\beta i)i\bigg[-iq\Re\big(\Delta\overline{W}\big(W^2+1\big)\big) + iq\Re\bigg(\frac{2\langle\nabla\overline{W}\rangle W(W^2+1)}{W\overline{W}+1}\bigg) \\
&\qquad - i\nabla q\cdot\Re\big(\nabla\overline{W}\big(W^2+1\big)\big)\bigg].
\end{aligned} \tag{26}$$

According to (21), (23), and (25) (or (22), (24), and (26)), we obtain the complex equation of W:

$$-(\alpha+\beta i)iW_t = q\Delta W + \nabla q\cdot\nabla W - \frac{2q\overline{W}}{1+|W|^2}\langle\nabla W\rangle. \tag{27}$$

Under the cylindrical coordinates, it is not difficult to verify that (27) satisfies

$$-(\alpha+\beta i)iW_t = q\bigg(W_{rr} + \frac{n-1}{r}W_r\bigg) - \frac{2q\overline{W}\langle W_r\rangle}{1+|W|^2} + q_rW_r, \tag{28}$$

where $r = |\overrightarrow{x}|$.

In the above deducing process, we obtain an equivalent equation of (2), which is of the form (27). This fact indicates that we can do a transformation between $\mathbb{S}^2$ and the extend complex plane C_∞ by (11), which is a relationship between W and S. If we get a solution of (27), then we can transform it into the solution of (2) by (11). In the next section, the complex equation (27) will be used to construct the solution of (2), which is useful to analyze its energy property.

Table 1 Variable separation solution, inhomogeneity terms, and decay rate of ILLG equation (2)

Solutions I–III	$q(r)$ and decay rate
Solution I: $\begin{pmatrix} \cos(\Theta_1)\sin(B) \\ \sin(\Theta_1)\sin(B) \\ \cos(B) \end{pmatrix}$,	$-\frac{C_1 r^{2-2n}(C_2 r^n + C_3 n)^2 \cos(B)\beta^2}{\alpha C_2^2 n^2}$
$\Theta_1 = -\frac{\alpha}{\cos(B)\beta}\ln(-\frac{\cos(B)\beta(C_2 r^n + C_3 n)}{\alpha n}) + C_1 t$	and $O(1)$
Solution II: $\begin{pmatrix} \cos(\Theta_2)\sin(B) \\ \sin(\Theta_2)\sin(B) \\ \cos(B) \end{pmatrix}$,	$-\frac{C_1 \cos(B)\beta^2(\overline{r} + K_{n+1})^2}{\alpha(\overrightarrow{K} \cdot \overrightarrow{K})}$
$\Theta_2 = C_1 t - \frac{\alpha \ln(\overline{r} + K_{n+1})}{\cos(B)\beta} + K_{n+2}$	and $O(1)$

Table 2 Variable separation solution, inhomogeneity terms, and decay rate of equivalent ILLG equation (27)

Solutions I–III	$q(r)$ and decay rate
Solution I: $A\exp[i\Theta_1']$,	$\frac{C_1 r^{-2n+2}(C_2 r^n + C_3 n)^2 \beta^2 (A^2-1)}{\alpha(A^2+1)C_2^2 n^2}$
$\Theta_1' = C_1 t + \frac{\alpha(A^2+1)}{\beta(A^2-1)}\ln(\frac{\beta(A^2-1)(C_2 r^n + C_3 n)}{n\alpha(A^2+1)})$	and $O(1)$
Solution II: $\frac{\sin(B)}{1+\cos(B)}\exp[i\Theta_2]$,	$-\frac{C_1 \cos(B)\beta^2(\overline{r} + K_{n+1})^2}{\alpha(\overrightarrow{K} \cdot \overrightarrow{K})}$
$\Theta_2 = C_1 t - \frac{\alpha \ln(\overline{r} + K_{n+1})}{\cos(B)\beta} + K_{n+2}$	and $O(1)$

3 Solution of ILLG equation

In this section, we present some solutions of ILLG equation. These solutions are all in explicit forms, which can be seen in theorems and corollaries of this section. For convenience, we list all these solutions in the Tables 1 and 2, which demonstrate the exact solutions and their decay rates.

3.1 Solution under the cylindrical coordinates

We construct and analyze the variable separation type solution of (4) under the cylindrical coordinates. According to (4) and (11), we use the special variable separation ansatz of the solution as follows:

$$W = F(r)\mathrm{e}^{iM(t,r)}, \tag{29}$$

where the functions $F(r)$ and $M(t,r)$ (dependent variables in parentheses) are to be determined.

In the real physical model of the spin system, the inhomogeneous term of (2) and (28) are usually independent of t. Hence we just consider the case $q(t\,\overrightarrow{x}) = q(r)$. This setting and (29) lead to the derivatives of time as follows:

$$-(\alpha + \beta i)iW_t = (\alpha + i\beta)FM_t\mathrm{e}^{iM}. \tag{30}$$

The structure of $q_r W_r$ is

$$q_r W_r = q_r \mathrm{e}^{iM}(iM_r F + F_r). \tag{31}$$

Similarly, the item associated with the Laplace and nonlinear derivatives is

$$q\left[\Delta W - \frac{2\bar{W}W_r^2}{1+|W|^2}\right]$$

$$= \frac{1}{r} q e^{iM} \bigl(2iF_r M_r r + iM_r F n + iFM_{rr} r - FM_r^2 r - iM_r F \\ + F_{rr} r + F_r n - F_r\bigr) - 2qF e^{iM} (iM_r F + F_r)^2 / \bigl(F^2 + 1\bigr). \tag{32}$$

Adding (30)–(32) together and multiplying with e^{-iM}, we separate their real and imaginary parts as

$$\begin{cases} -qM_r^2 F^3 r + F^3 M_t \alpha r - qF_r F^2 n - qF_{rr} F^2 r - F_r F^2 q_r r + 2qF_r^2 Fr \\ \quad + qM_r^2 Fr + qF_r F^2 + FM_t \alpha r - qF_r n - qF_{rr} r - F_r q_r r + qF_r = 0, \\ F^3 M_t \beta r - qM_r F^3 n - qF^3 M_{rr} r - M_r F^3 q_r r + 2qF_r M_r F^2 r + qM_r F^3 \\ \quad + FM_t \beta r - qM_r Fn - qFM_{rr} r - M_r F q_r r - 2qF_r M_r r + qM_r F = 0. \end{cases} \tag{33}$$

If we use $M = P(r)Q(t)$, then we obtain a blowup solution of the case $\beta = 0$ of (2). However, if $\alpha\beta \neq 0$, then this special ansatzs is difficult to find out the solution of (2). In fact, we use this assumption to construct the solution of the ILLG equation, but we could not find any nontrivial solution. Hence we use another special form to find the solution of (2). We set M in the variable separation form

$$M = P(r) + Q(t).$$

Then (33) transforms into

$$\begin{cases} -qP_r^2 F^3 r + F^3 Q_t \alpha r - qF_r F^2 n - qF_{rr} F^2 r - F_r F^2 q_r r + 2qF_r^2 Fr \\ \quad + qP_r^2 Fr + qF_r F^2 + FQ_t \alpha r - qF_r n - qF_{rr} r - F_r q_r r + qF_r = 0, \\ F^3 Q_t \beta r - qP_r F^3 n - qF^3 P_{rr} r - P_r F^3 q_r r + 2qF_r P_r F^2 r + qP_r F^3 \\ \quad + FQ_t \beta r - qP_r Fn - qFP_{rr} r - P_r F q_r r - 2qF_r P_r r + qP_r F = 0. \end{cases} \tag{34}$$

From the first and second equations of (34) we have

$$Q = C_1 t, \tag{35}$$

where C_1 is a constant.

Although Q is clear from (35), it is still difficult to solve (34) under the hypothesis that F is a function of r. However, if we set $F = A$, where A is a constant, we can simplify (34) as follows:

$$\begin{cases} -qP_r^2 A^3 r + A^3 C_1 \alpha r + qP_r^2 Ar + AC_1 \alpha r = 0, \\ -A^3 C_1 \beta r - qP_r A^3 n - qA^3 P_{rr} r - P_r A^3 q_r r + qP_r A^3 \\ \quad + AC_1 \beta r - qP_r An - qAP_{rr} r - P_r A q_r r + qP_r A = 0. \end{cases} \tag{36}$$

From the first equation of (36) we obtain

$$q = \frac{\alpha C_1 (A^2 + 1)}{P_r^2 (A^2 - 1)}. \tag{37}$$

By (37) the second equation of (36) transforms into

$$
\begin{aligned}
&-A^4P_r^2\beta r + A^4 n\alpha P_r - A^4 P_{rr} r\alpha - A^4\alpha P_r + 2A^2 P_r n\alpha - 2A^2 P_{rr} r\alpha \\
&\quad - 2A^2 P_r\alpha + P_r^2\beta r + P_r n\alpha - P_{rr} r\alpha - P_r\alpha = 0.
\end{aligned} \tag{38}
$$

Solving (38), we can find out the solution

$$
P = \frac{\alpha(A^2+1)}{\beta(A^2-1)} \ln\left(\frac{\beta(A^2-1)(C_2 r^n + C_3 n)}{n\alpha(A^2+1)}\right), \tag{39}
$$

where C_2 and C_3 are arbitrary constants.

Substituting (39) into (37), we obtain the exact expression of q.

According to the above analysis, we obtain the following:

Theorem 3.1 *If the inhomogeneity term is*

$$
q = \frac{C_1 r^{-2n+2}(C_2 r^n + C_3 n)^2 \beta^2 (A^2-1)}{\alpha(A^2+1) C_2^2 n^2}, \tag{40}
$$

then the solution of (28) *is*

$$
W = A\exp\left[i\frac{\alpha(A^2+1)}{\beta(A^2-1)} \ln\left(\frac{\beta(A^2-1)(C_2 r^n + C_3 n)}{n\alpha(A^2+1)}\right) + iC_1 t\right], \tag{41}
$$

where A, C_1, C_2, and C_3 are constants.

Based on (40) and (41), we analyze the energy of this solution in the following contents. If $q = 1$ and $n = 2$, then the total energy of the solution of (28) is a constant due to $n = 2$ is the critical case for the LLG equation. However, if the inhomogeneity term q is not a constant, then ILLG may not be in critical situation. We denote by $E = \int_\Omega w_E \, dx$ the total energy of the solution of a partial differential equation.

The mapping target manifold of (28) is $\mathbb{S}^2$. Under projection, the conformal factor is

$$
g = \frac{1}{(1+|W|^2)^2},
$$

whereas the energy is

$$
\int_0^\infty q|W_r|^2 g r^{n-1} \, dr.
$$

Hence, the energy density w_E of (41) is

$$
w_E(41) = \frac{\alpha C_1 A^2}{A^4 - 1}. \tag{42}
$$

According to (11) and Theorem 3.1, we obtain the solution of (2):

Corollary 3.2 *If the inhomogeneity term is*

$$q = -\frac{C_1 r^{2-2n}(C_2 r^n + C_3 n)^2 \cos(B)\beta^2}{\alpha C_2^2 n^2}, \tag{43}$$

then the solution of (2) *is*

$$\begin{cases} S_1 = \cos(-C_1 t + \frac{\alpha}{\cos(B)\beta} \ln(-\frac{\cos(B)\beta(C_2 r^n + C_3 n)}{\alpha n})) \sin(B), \\ S_2 = -\sin(-C_1 t + \frac{\alpha}{\cos(B)\beta} \ln(-\frac{\cos(B)\beta(C_2 r^n + C_3 n)}{\alpha n})) \sin(B), \\ S_3 = \cos(B), \end{cases} \tag{44}$$

where B is a constant, and C_1, C_2, and C_3 are the same constants as in Theorem 3.1.

It is clear that (44) does not tend to a trivial solution (for example, $S = (0, 0, 1)$) as $t \to +\infty$. Moreover, we cannot find out a constant value of $\lim_{t\to+\infty} S$. According to Corollary 3.2, (43) and (44) lead to

$$\begin{aligned} |S_r|^2 &= S_1^2 + S_2^2 + S_3^2 \\ &= -\frac{r^{2n-2}((\cos(B))^2 - 1)\alpha^2 n^2 C_2^2}{(C_2 r^n + C_3 n)^2 (\cos(B))^2 \beta^2} \end{aligned}$$

and

$$w_E(44) = q|S_r|^2 = -\frac{C_1 \alpha (\sin(B))^2}{\cos(B)}. \tag{45}$$

According to (42) and (45), the energy density of two systems (28) and (2) are constants that are independent of time. This fact indicates that the energy of two equivalent ILLG equations tends to $+\infty$ in the whole spacial area.

3.2 Solution under the normal coordinates

In the last subsection, we present the exact solution under the cylindrical coordinates. We study the solution under the noncylindrical cases. The solutions obtained in this section may regard as the plane wave type solution, which is different from the cylindrical case of last subsection.

First, we search the solution of (2) under $n = 3$. Denote by x, y, z the different directions of space. Inspired by the last subsection, we look for the solution of (2) in the form

$$\begin{cases} S_1(t, x, y, z) = \cos(M(t, x, y, z)) \sin(B), \\ S_2(t, x, y, z) = \sin(M(t, x, y, z)) \sin(B), \\ S_3 = \cos(B), \end{cases} \tag{46}$$

where $M(t, x, y, z)$ is the function to be determined.

From (46), (5)–(10) can be simplified to

$$\begin{cases} q_xM_x\alpha + q_yM_y\alpha + q_zM_z\alpha + qM_{xx}\alpha + qM_{yy}\alpha + qM_{zz}\alpha \\ \quad + M_y^2q\cos(B)\beta + M_z^2q\cos(B)\beta + M_x^2q\cos(B)\beta = 0, \\ -M_t - M_z^2q\cos(B)\alpha - {M_x}^2q\cos(B)\alpha - M_y^2q\cos(B)\alpha \\ \quad + q_xM_x\beta + q_yM_y\beta + q_zM_z\beta + qM_{xx}\beta + qM_{yy}\beta + qM_{zz}\beta = 0. \end{cases} \tag{47}$$

We use the ansatz of M as follows:

$$M = P(x,y,z) + Q(t) \quad \text{and} \quad Q(t) = C_1t, \tag{48}$$

where C_1 is a constant.

Substituting (48) into (47), we get

$$\begin{cases} q_xP_x\alpha + q_yP_y\alpha + q_zP_z\alpha + qP_{xx}\alpha + qP_{yy}\alpha + qP_{zz}\alpha \\ \quad + P_y^2q\cos(B)\beta + P_z^2q\cos(B)\beta + P_x^2q\cos(B)\beta = 0, \\ -C_1 - P_z^2q\cos(B)\alpha - {P_x}^2q\cos(B)\alpha - P_y^2q\cos(B)\alpha \\ \quad + q_xP_x\beta + q_yP_y\beta + q_zP_z\beta + qP_{xx}\beta + qP_{yy}\beta + qP_{zz}\beta = 0. \end{cases} \tag{49}$$

Solving (49), P and q are

$$\begin{cases} P = -\frac{\alpha\ln(xK_1+yK_2+zK_3+K_4)}{\cos(B)\beta} + K_5, \\ q = -\frac{C_1\cos(B)\beta^2(xK_1+yK_2+zK_3+K_4)^2}{\alpha(K_1^2+{K_2}^2+K_3^2)}, \end{cases} \tag{50}$$

where K_i $(i = 1,2,3,4,5)$ are arbitrary constants.

According to (46), (48), and (50), we obtain the solution of (2): If inhomogeneity term is

$$q = -\frac{C_1\cos(B)\beta^2(xK_1+yK_2+zK_3+K_4)^2}{\alpha(K_1^2+{K_2}^2+K_3^2)},$$

then the solution of (2) is

$$\begin{cases} S_1 = \cos(C_1t - \frac{\alpha\ln(xK_1+yK_2+zK_3+K_4)}{\cos(B)\beta} + K_5)\sin(B), \\ S_2 = \sin(C_1t - \frac{\alpha\ln(xK_1+yK_2+zK_3+K_4)}{\cos(B)\beta} + K_5)\sin(B), \\ S_3 = \cos(B), \end{cases} \tag{51}$$

where K_i $(i = 1,2,3,4,5)$, B, and C_1 are constants.

Expression (51) is a solution for $n = 3$. Under an arbitrary integer n, we can find the solution of (2) similarly. We search the solution of (2) in the form

$$\begin{cases} S_1(t,\overrightarrow{x}) = \cos(P(\overrightarrow{x}) + C_1t)\sin(B), \\ S_2(t,\overrightarrow{x}) = \sin(P(\overrightarrow{x}) + C_1t)\sin(B), \\ S_3 = \cos(B), \end{cases} \tag{52}$$

where $P(\overrightarrow{x})$ is the function to be determined.

Substituting (52) into (5)–(10), we obtain

$$\begin{cases} \alpha\nabla q\cdot\nabla P+\alpha q\Delta P+\beta\cos(B)q\nabla P\cdot\nabla P=0, \\ -C_1-\alpha q\cos(B)\nabla P\cdot\nabla P+\beta\nabla q\cdot\nabla P+\beta q\Delta P=0. \end{cases} \tag{53}$$

Solving (53), P and q are as follows:

$$\begin{cases} P=-\frac{\alpha\ln(\overline{r}+K_{n+1})}{\cos(B)\beta}+K_{n+2}, \\ q=-\frac{C_1\cos(B)\beta^2(\overline{r}+K_{n+1})^2}{\alpha(\overrightarrow{K}\cdot\overrightarrow{K})}, \end{cases} \tag{54}$$

where K_i ($i=1,2,3,\dots$) are arbitrary constants, $\overrightarrow{K}=(K_1,K_2,K_3,\dots,K_n)$, and $\overline{r}=\overrightarrow{K}\cdot\overrightarrow{x}$.

According to (52) and (54), we obtain the solution of (2) as follows.

Theorem 3.3 *If the inhomogeneity term is*

$$q=-\frac{C_1\cos(B)\beta^2(\overline{r}+K_{n+1})^2}{\alpha(\overrightarrow{K}\cdot\overrightarrow{K})}, \tag{55}$$

then the solution of (2) *is*

$$\begin{cases} S_1=\cos(C_1t-\frac{\alpha\ln(\overline{r}+K_{n+1})}{\cos(B)\beta}+K_{n+2})\sin(B), \\ S_2=\sin(C_1t-\frac{\alpha\ln(\overline{r}+K_{n+1})}{\cos(B)\beta}+K_{n+2})\sin(B), \\ S_3=\cos(B). \end{cases} \tag{56}$$

Similarly to (44), (56) does not tend to a trivial solution as $t\to+\infty$ too. Moreover, $\lim_{t\to+\infty}S$ does not exist. It is easy to find the solution of (27). Employing (56) and (11), we obtain the following:

Corollary 3.4 *If the inhomogeneity term satisfies* (55), *then the solution of* (27) *is*

$$W=\frac{\sin(B)}{1+\cos(B)}\exp\left[i\left(C_1t-\frac{\alpha\ln(\overline{r}+K_{n+1})}{\cos(B)\beta}+K_{n+2}\right)\right]. \tag{57}$$

The energy density of the solutions presented in Theorem 3.3 and Corollary 3.4 shows a similar property as the solutions of the previous subsection. According to Corollary 3.4, we obtain

$$|\nabla S|^2=\frac{(\sin(B))^2\alpha^2(|\overrightarrow{K}|)}{(\overline{r}+K_{n+1})^2(\cos(B))^2\beta^2}$$

and

$$w_E(56)=q|\nabla S|^2=-\frac{C_1\alpha(\sin(B))^2}{\cos(B)}. \tag{58}$$

Similarly to (45), the energy density (58) is constant. This situation will be the same for (41) and (57):

$$w_E(57) = q\frac{|\nabla W|^2}{(1+|W|^2)^2} = -\frac{C_1\alpha(\sin(B))^2(\cos(B)+1)^2}{4\cos(B)}.$$

In this section, we present two solutions of (2) (or (27)). For convenience, we summarize some information about these solutions in Tables 1 and 2, respectively. From these tables we can see the exact form of variable separation solution, inhomogeneity terms, and decay rates. According to the solutions of the ILLG equation in the tables, we can see that α and β cannot be 0 since the denominator cannot be 0. Although the solutions of the ILLG equation does not degenerate into the trivial solutions as $t \to +\infty$, their decay rates are all independent of the spatial variable $\overrightarrow{x}$. More exactly, the decay rate of the solutions for the ILLG equation are $O(1)$. Different from the solutions presented in our previous work [28, 29], the solutions in Tables 1 and 2 do not develop any blowup at any time.

4 Solution of Schrödinger map equation

In this section, we study the solution of the LLG equation without damping. More clearly, we set $q = 1$ and $\beta = 0$ in (2). In this setting, (2) degenerates into the simple form

$$\frac{\partial}{\partial t}S = S\wedge\Delta S. \tag{59}$$

Therefore W satisfies

$$-iW_t = \Delta W - \frac{2\overline{W}}{1+|W|^2}\langle\nabla W\rangle. \tag{60}$$

Although (59) is simpler than (2), the solution is very rare as far as we see. For the details of the exact solution of (59), we refer the reader to [9, 12, 27].

4.1 Solution under the cylindrical coordinates

In the last section, we obtain some variable separation solutions such as (44) (q as (43)) and (56) (q as (55)). According to these solutions, we can see that the extreme case $\alpha = 0$ and $\beta = 0$ are not included since the denominator is not zero. If we settle down on the variable separation form provided in the last section, what will be the solution like? Motivated by this, we consider the solution

$$\begin{cases} S_1(t,r) = \cos(N(t,r))\sin(G(r)), \\ S_2(t,r) = \sin(N(t,r))\sin(G(r)), \\ S_3(r) = \cos(G(r)), \end{cases} \tag{61}$$

where $N(t,r)$ and $G(r)$ are functions to be determined.

We consider the case where $q(t\overrightarrow{x})$ does not contain t. Then, by (61), (2) is greatly simplified and transforms into

$$-\sin(G)N_r + 2\cos(G)G_rN_rr + \sin(G)N_rn + \sin(G)N_{rr}r = 0 \tag{62}$$

and

$$\sin(G)\cos(G)N_r^2 r - G_r n - G_{rr} r + \sin(G)N_t r + G_r = 0, \tag{63}$$

which are nonlinear partial differential equations. To solve these equations, let us assume that N is a variable separation function,

$$N = M_1(r) + M_2(t), \tag{64}$$

where $M_1(r)$ and $M_2(t)$ depend on r and t, respectively. Substituting (64) into (62) and (63), we obtain

$$\begin{aligned} &M_2 = C_5 t, \\ &\quad -\sin(G)M_{1r} + 2\cos(G)G_r M_{1r} r + \sin(G)M_{1r} n + \sin(G)M_{1rr} r = 0, \end{aligned} \tag{65}$$

and

$$\sin(G)\cos(G)M_{1r}^2 r - G_r n - G_{rr} r + \sin(G)C_5 r + G_r = 0. \tag{66}$$

Equations (65)–(66) are ordinary differential equations (ODEs), which are hard to solve explicitly. However, we can find out some constraints of the solution that are useful for analysis of the decay rate of the energy density. If we use

$$G = 2\arctan\left(\mathrm{e}^F\right), \tag{67}$$

then (65)–(66) transform into

$$\begin{aligned} &2M_{1r}F_r\mathrm{e}^{2F}r - M_{1r}\mathrm{e}^{2F}n - M_{1rr}\mathrm{e}^{2F}r - 2M_{1r}rF_r \\ &\quad + M_{1r}\mathrm{e}^{2F} - M_{1r}n - M_{1rr}r + M_{1r} = 0 \end{aligned} \tag{68}$$

and

$$\begin{aligned} &M_{1r}^2\mathrm{e}^{2F}r - F_r^2\mathrm{e}^{2F}r + F_{rr}\mathrm{e}^{2F}r + F_r\mathrm{e}^{2F}n - C_5\mathrm{e}^{2F}r - M_{1r}^2 r \\ &\quad + rF_r^2 - F_r\mathrm{e}^{2F} + rF_{rr} + nF_r - C_5 r - F_r = 0. \end{aligned} \tag{69}$$

From (68) we obtain

$$M_1 = C_1 + C_2\int r^{-n+1}\left(\mathrm{e}^{4F} + 2\mathrm{e}^{2F} + 1\right)\mathrm{e}^{-2F}\,\mathrm{d}r. \tag{70}$$

Substituting (70) into (69), we have

$$\begin{aligned} &3r^{-2n+3}\mathrm{e}^{4F}C_2^2 + 2r^{-2n+3}\mathrm{e}^{2F}C_2^2 + r^{-2n+3}\mathrm{e}^{6F}C_2^2 - 2r^{-2n+3}C_2^2 \\ &\quad - 3r^{-2n+3}\mathrm{e}^{-2F}C_2^2 - r^{-2n+3}\mathrm{e}^{-4F}C_2^2 - F_r^2\mathrm{e}^{2F}r + \mathrm{e}^{2F}F_{rr}r + F_r\mathrm{e}^{2F}n \\ &\quad - \mathrm{e}^{2F}rC_5 + rF_r^2 + rF_{rr} - F_r\mathrm{e}^{2F} + nF_r - C_5 r - F_r = 0. \end{aligned} \tag{71}$$

We cannot find the solution of (71). However, there is some additional information about F. If we solve (34), then we can find

$$M_1 = \int \frac{C_3^2 r^{-n+1}}{(\sin(G))^2}\,\mathrm{d}r + C_4. \tag{72}$$

If we set $C_4 = C_1$ and $C_2 = 1/(2C_3)^2$, then (72) and (70) will be of the same form. Substituting (72) into (66), we obtain

$$G_{rr} + \frac{(n-1)G_r}{r} - \frac{\cos(G)C_3^4 r^{-2n+2}}{(\sin(G))^3} - \sin(G)C_5 = 0. \tag{73}$$

Equation (73) is just an ODE, which is like a sine-Gordon equation. Because (73) is not a standard sine-Gordon equation, its existence details are not so clear. In fact, we rewrite (73) as

$$G_{rr} = f(r, G, G_r), \tag{74}$$

where

$$f(r, G, G_r) = -\frac{(n-1)G_r}{r} + \frac{\cos(G)C_3^4 r^{-2n+2}}{(\sin(G))^3} + \sin(G)C_5.$$

We consider the Cauchy problem of (74): finding a solution of equation (74) satisfying the initial conditions

$$G(\varepsilon) = G_\varepsilon \quad \text{and} \quad G_r(\varepsilon) = G_{r\varepsilon} \quad (0 < \varepsilon < +\infty, 0 < G_\varepsilon < \pi). \tag{75}$$

Obviously, $f(r, G, G_r)$ is a continuous function of all its arguments in a neighborhood of a point $(\varepsilon, G_\varepsilon, G_{r\varepsilon})$. Furthermore, it is not difficult to verify that $f(r, G, G_r)$ have bounded partial derivatives $\frac{\partial f}{\partial G}$ and $\frac{\partial f}{\partial G_r}$ in this neighborhood. Hence, a solution of (74) (or (73)) satisfying the initial conditions (75) exists and is unique. How to construct a reasonable initial condition that satisfies (74)? We just need to employ the setting (67).

Due to (72), if we obtain the solution of (73), then we can find the solution of (71). According to the above results, we obtain the following:

Theorem 4.1 *Let F be a function satisfying* (71). *Tthen the solution of* (59) *is*

$$\begin{cases} S_1(t,r) = \frac{2\mathrm{e}^F}{1+\mathrm{e}^{2F}} \cos(C_1 + C_2 \int r^{-n+1}(\mathrm{e}^{2F}+1)^2 \mathrm{e}^{-2F}\,\mathrm{d}r + C_5 t), \\ S_2(t,r) = \frac{2\mathrm{e}^F}{1+\mathrm{e}^{2F}} \sin(C_1 + C_2 \int r^{-n+1}(\mathrm{e}^{2F}+1)^2 \mathrm{e}^{-2F}\,\mathrm{d}r + C_5 t), \\ S_3(r) = \frac{1-\mathrm{e}^{2F}}{1+\mathrm{e}^{2F}}, \end{cases} \tag{76}$$

whereas

$$W = \mathrm{e}^F \exp\left[i\left(C_1 + C_2 \int r^{-n+1}\left(\mathrm{e}^{2F}+1\right)^2 \mathrm{e}^{-2F}\,\mathrm{d}r + C_5 t\right)\right] \tag{77}$$

is a solution of (60).

Although we cannot solve (71), we can deduce some its scale as $r \to +\infty$. Substituting $F = K_1 \ln(Kr)$ ($K > 0$) into the left side of (71), we obtain

$$\begin{aligned}
&3r^{-2n+4+4K_1}K^{4K_1}C_2^2 + 2r^{-2n+4+2K_1}K^{2K_1}C_2^2 + r^{-2n+4+6K_1}K^{6K_1}C_2^2 \\
&\quad - 2r^{-2n+4}C_2^2 - 3r^{-2n+4-2K_1}K^{-2K_1}C_2^2 - r^{-2n+4-4K_1}K^{-4K_1}C_2^2 \\
&\quad - K_1^2K^{2K_1}r^{2K_1} - 2K^{2K_1}r^{2K_1}K_1 + K_1K^{2K_1}r^{2K_1}n - K^{2K_1}r^{2K_1+2}C_5 + K_1^2 \\
&\quad - 2K_1 + nK_1 - C_5r^2 \triangleq H(r).
\end{aligned} \tag{78}$$

If $C_5 > 0$ and $n \geq 2$, then the leading-order term of (78) is

$$r^{-2n+4+6K_1}K^{6K_1}C_2^2 - K^{2K_1}r^{2K_1+2}C_5 \triangleq L_{C_5>0},$$

and we obtain

$$K_1 = \frac{n}{2} - \frac{1}{2}.$$

So, as $r \to +\infty$, we have

$$\lim_{r\to+\infty} H(r) = \operatorname{sign}\left[K^{n-1}\left(K^{2n-2}C_2^2 - C_5\right)\right] \cdot \infty. \tag{79}$$

Relationship (79) means that K determines the limit to be a positive or negative value of $H(r)$ for sufficiently large r. In this situation, we have

$$\left(\frac{n}{2} - \frac{1}{2}\right)\ln(K_Ar) < F < \left(\frac{n}{2} - \frac{1}{2}\right)\ln(K_Br) \quad (0 < K_A < K_B). \tag{80}$$

Similarly, if $C_5 < 0$ and $n \geqq 2$, then the leading-order term of (78) is

$$-r^{-2n+4-4K_1}K^{-4K_1}C_2^2 - C_5r^2 \triangleq L_{C_5<0},$$

where

$$K_1 = -\frac{n}{2} + \frac{1}{2}.$$

In this case, we obtain

$$\lim_{r\to+\infty} H(r) = -\operatorname{sign}\left(K^{2n-2}C_2^2 + C_5\right) \cdot \infty. \tag{81}$$

If r is sufficiently large, then (81) indicates that

$$\left(-\frac{n}{2} + \frac{1}{2}\right)\ln(K_Cr) < F < \left(-\frac{n}{2} + \frac{1}{2}\right)\ln(K_Dr) \quad (0 < K_C < K_D). \tag{82}$$

If $n \geq 2$, then by (80) and (82) we have (for sufficiently large r)

$$\left(\frac{n}{2} - \frac{1}{2}\right)\ln(K_Er) < |F| < \left(\frac{n}{2} - \frac{1}{2}\right)\ln(K_Fr) \quad (0 < K_E < K_F).$$

From (61), (64), (67), and (70) we have

$$\begin{aligned} w_E(76) &= S_{1r}^2 + S_{2r}^2 + S_{3r}^2 \\ &= N_r^2\big(\sin(G)\big)^2 + G_r^2 \\ &= M_{1r}^2\big(\sin(G)\big)^2 + G_r^2 \\ &= \frac{C_3^4 r^{-2n+2}}{(\sin(G(r)))^2} + G_r^2 \\ &= \frac{4r^{2-2n}F^2C_2^2(\mathrm{e}^{2F}+4)(\mathrm{e}^{2F}+\mathrm{e}^{-2F})}{(F^2+1)^2} + \frac{24r^{2-2n}F^2C_2^2 + 4F_r^2}{(F^2+1)^2}. \end{aligned} \tag{83}$$

Substituting

$$F = \pm\left(\frac{n}{2} - \frac{1}{2}\right)\ln(Kr) \tag{84}$$

into (83), we obtain

$$\begin{aligned} w_E(76) = {} & 16K^{-2}r^{-2}(-1+n)^2\big(\ln(Kr)\big)^2\big(r^{-4n+6}K^{-2n+4}C_2^2 \\ & + 4r^{-3n+5}K^{-n+3}C_2^2 + 6r^{-2n+4}C_2^2K^2 + 4r^{-n+3}K^{n+1}C_2^2 \\ & + r^2K^{2n}C_2^2 + K^2\big)/\big(\ln(Kr)\big)^2n^2 - 2\big(\ln(Kr)\big)^2 n \\ & + \big(\ln(Kr)\big)^2 + 4)^2. \end{aligned} \tag{85}$$

According to (85), the scale of the energy density is

$$w_E(76) \sim O(1) \quad (n \geq 2)$$

as $r \to +\infty$.

The energy density of (77) is

$$w_E(77) = \frac{|W_r|^2}{(1+|W|^2)^2} = \frac{\mathrm{e}^{2F}((r^{-n+1})^2(\mathrm{e}^{4F}+2\mathrm{e}^{2F}+1)^2(\mathrm{e}^{-2F})^2C_2^2 + F_r^2)}{(1+\mathrm{e}^{2F})^2}. \tag{86}$$

Employing (84), the right side of (86) changes into

$$\begin{aligned} & 3r^{-2n+4+4K_1}K^{4K_1}C_2^2 + 2r^{-2n+4+2K_1}K^{2K_1}C_2^2 + r^{-2n+4+6K_1}K^{6K_1}C_2^2 \\ & - 2r^{-2n+4}C_2^2 - 3r^{-2n+4-2K_1}K^{-2K_1}C_2^2 - r^{-2n+4-4K_1}K^{-4K_1}C_2^2 \\ & - K_1^2K^{2K_1}r^{2K_1} - 2K^{2K_1}r^{2K_1}K_1 + K_1K^{2K_1}r^{2K_1}n - K^{2K_1}r^{2K_1+2}C_5 + K_1^2 \\ & - 2K_1 + nK_1 - C_5r^2 \triangleq H(r). \end{aligned} \tag{87}$$

If $C_5 > 0$ and $n \geq 2$, then the leading-order term of (87) is

$$r^{-2n+4+6K_1}K^{6K_1}C_2^2 - K^{2K_1}r^{2K_1+2}C_5 \triangleq L_{C_5>0},$$

and we obtain

$$K_1 = \frac{n}{2} - \frac{1}{2}.$$

So, as $r \to +\infty$, we have

$$\lim_{r\to+\infty} H(r) = \operatorname{sign}\bigl[K^{n-1}\bigl(K^{2n-2}C_2^2 - C_5\bigr)\bigr] \cdot \infty$$

and

$$\begin{aligned} w_E(77) = {} & (4Kr)^{-1}\bigl(4K^{-n+4}r^{-3n+6}C_2^2 + 16r^{5-2n}K^3C_2^2 \\ & + 24K^{n+2}r^{-n+4}C_2^2 + 16K^{2n+1}r^3{C_2}^2 + K^{n+2}r^n n^2 \\ & + 4K^{3n}r^{n+2}C_2^2 - 2K^{n+2}r^n n + K^{n+2}r^n\bigr)/\bigl(K^n r^n + Kr\bigr)^2. \end{aligned} \tag{88}$$

Relation (88) shows that the scale of the energy density is

$$w_E(77) \sim \frac{1}{r^{n-1}} \quad (n \geq 2)$$

as $r \to +\infty$.

If $n = 1$, then F presents a different behavior as $r \to +\infty$. Exactly, F is not a $K_1 \ln(Kr)$ scale, but

$$F \sim \ln\bigl(K \arctan(r)\bigr) \tag{89}$$

as $r \to +\infty$.

Substituting $n = 1$ and $F = K\ln(\arctan(r))$ into the left side of (71), we obtain

$$\begin{aligned} & r^2K^6\bigl(\arctan(r)\bigr)^6C_2^2 - \frac{r^2C_2^2}{K^4(\arctan(r))^4} + 3r^2K^4\bigl(\arctan(r)\bigr)^4C_2^2 - \frac{3r^2C_2^2}{K^2(\arctan(r))^2} \\ & - \frac{2K^2r^2}{(r^2+1)^2} + 2r^2K^2\bigl(\arctan(r)\bigr)^2C_2^2 - \frac{2r^3K^2\arctan(r)}{(r^2+1)^2} - r^2K^2\bigl(\arctan(r)\bigr)^2C_5 \\ & - 2r^2C_2^2 - \frac{2r^3}{(r^2+1)^2\arctan(r)} - C_5r^2 \triangleq H_1(r). \end{aligned} \tag{90}$$

According to (90), its leading-order term is

$$\begin{aligned} & r^2K^6\bigl(\arctan(r)\bigr)^6C_2^2 - \frac{r^2C_2^2}{K^4(\arctan(r))^4} + 3r^2K^4\bigl(\arctan(r)\bigr)^4C_2^2 - \frac{3r^2C_2^2}{K^2(\arctan(r))^2} \\ & + 2r^2K^2\bigl(\arctan(r)\bigr)^2C_2^2 - r^2K^2\bigl(\arctan(r)\bigr)^2C_5 - 2r^2C_2^2 - C_5r^2 \triangleq H_2(r). \end{aligned} \tag{91}$$

As $r \to +\infty$, $\arctan(r) \to \pi/2$. Substituting $\arctan(r) = \pi/2$ into (91), $H_2(r)$ becomes

$$\begin{aligned} & \frac{r^2K^6\pi^6C_2^2}{64} - \frac{16r^2C_2^2}{K^4\pi^4} + \frac{3}{16}r^2K^4\pi^4{C_2}^2 - \frac{12r^2C_2^2}{K^2\pi^2} \\ & + \frac{1}{2}r^2K^2\pi^2C_2^2 - \frac{1}{4}r^2K^2\pi^2C_5 - 2r^2C_2^2 - C_5r^2 \triangleq H_3(r). \end{aligned}$$

If $C_5 \neq 0$, then, as $r \to +\infty$, we have

$$\lim_{r\to+\infty} H_3(r) = \operatorname{sign}\bigl[K^8\pi^8C_2^2 + 8K^6\pi^6C_2^2 - 16K^4\pi^4C_5 - 128K^2\pi^2C_2^2 - 256C_2^2\bigr]\cdot\infty, \tag{92}$$

where $K > 0$.

For sufficiently large r, (92) means that $K > 0$ determines the limit to be a positive or negative value of $H_3(r)$ According to (92), we prove (89). The decay rate of energy density is clear due to (89) under $n = 1$. So, we obtain (here we set $A_r = \arctan(r)$)

$$\begin{aligned} w_E(76) \sim{}& 4\Bigl(\bigl(\ln(KA_r)\bigr)^2K^8{A_r}^8C_2^2 + 4\bigl(\ln(KA_r)\bigr)^2K^6{A_r}^6C_2^2 \\ &+ 6\bigl(\ln(KA_r)\bigr)^2K^4{A_r}^4C_2^2 + \frac{AR^2K^4}{(r^2+1)^2} + 4\bigl(\ln(KA_r)\bigr)^2K^2{A_r}^2C_2^2 \\ &+ \bigl(\ln(KA_r)\bigr)^2C_2^2\Bigr)\Big/\Bigl[\bigl(1+\bigl(\ln(KA_r)\bigr)^2\bigr)^2K^4{A_r}^4\Bigr] \\ \sim{}& O(1) \end{aligned}$$

as $r \to +\infty$.

Similarly, the energy density of (77) under $n = 1$ is

$$\begin{aligned} w_E(77) \sim{}& \Bigl(K^6A_r^6C_2^2 + 4K^4A_r^4C_2^2 + \frac{K^2}{(r^2+1)^2} + 6K^2A_r^2C_2^2 \\ &+ 4C_2^2 + \frac{C_2^2}{K^2A_r^2}\Bigr)\Big/\bigl(K^2A_r^2+1\bigr)^2 \sim O(1) \end{aligned}$$

as $r \to +\infty$.

From the above analysis, we can conclude that: if F satisfies (71), then we have

Theorem 4.2 *If $n \geq 2$, then the energy densities of (76) and (77) are*

$$w_E(76) \sim O(1) \quad \textit{and} \quad w_E(77) \sim \frac{1}{r^{n-1}}$$

as $r \to +\infty$, respectively, whereas

$$w_E\bigl((76) \text{ or } (77)\bigr) \sim O(1) \quad (n = 1).$$

4.2 Solution under the normal coordinates

Under the arbitrary integer n, we can similarly find the solution of (59). We search the solution of (59) of the form

$$\begin{cases} S_1(t,\vec{x}) = \cos(M_2(\vec{x}) + C_5t)\sin(G_2(\vec{x})), \\ S_2(t,\vec{x}) = \sin(M_2(\vec{x}) + C_5t)\sin(G_2(\vec{x})), \\ S_3(\vec{x}) = \cos(G_2(\vec{x})), \end{cases} \tag{93}$$

where $M_2(\vec{x})$ and $G_2(\vec{x})$ are functions to be determined.

Substituting (93) into (59), we obtain

$$\begin{cases} 2\cos(G_2)\nabla G_2 \cdot \nabla M_2 + \sin G_2 \Delta M_2 = 0, \\ \sin(G_2)\cos(G_2)\nabla M_2 \cdot \nabla M_2 + C_5 \sin(G_2) - \Delta G_2 = 0. \end{cases} \tag{94}$$

We study the plane wave solution of (94). Here we set L_i $(i = 1, 2, 3, \ldots)$ as arbitrary constants, $\overrightarrow{L} = (L_1, L_2, L_3, \ldots, L_n)$, and $\overline{r} = \overrightarrow{L} \cdot \overrightarrow{x}$. So, (94) transforms into

$$\begin{cases} 2\cos(G_2)G_{2\overline{r}}M_{2\overline{r}} + \sin G_2 M_{2\overline{r}\overline{r}} = 0, \\ |\overrightarrow{L}|^2 \sin(G)\cos(G)M_{2\overline{r}}^2 + C_5 \sin(G) - |\overrightarrow{L}|^2 G_{2\overline{r}\overline{r}} = 0. \end{cases} \tag{95}$$

Equation (95) is a nonlinear ODE. Solving the first equation of (95), we obtain

$$M_2 = C_1 + C_2 \int \bigl(\sin(G_2)\bigr)^{-2} \,\mathrm{d}\overline{r}. \tag{96}$$

Employing (96), the second equation of (95) transforms into

$$G_{2rr} - \sin(G_2)C_5 - \frac{1}{(\sin(G_2))^3}\left(\frac{C_5}{|\overrightarrow{L}|^2} - C_5 + \cos(G_2){C_2}^2\right) = 0, \tag{97}$$

where

$$L_C = \frac{C_5}{|\overrightarrow{L}|^2} - C_5.$$

If G_2 satisfies (97) (M_2 satisfies (96)), then we have

$$\begin{aligned} w_E(93) &= |\overrightarrow{L}|^2 \bigl(S_{1\overline{r}}^2 + S_{2\overline{r}}^2 + S_{3\overline{r}}^2\bigr) \\ &= |\overrightarrow{L}|^2 \bigl(M_{2\overline{r}}^2 \bigl(\sin(G_2)\bigr)^2 + G_{2\overline{r}}^2\bigr) \\ &= |\overrightarrow{L}|^2 \left(G_{2r}^2 + \frac{C_2^2}{(\sin(G_2))^2}\right). \end{aligned} \tag{98}$$

Equation (98) is similar to (73), whereas (96) is similar to (70) under $n = 1$. We also use the transformation like (67)

$$G_2 = 2\arctan\bigl(\mathrm{e}^{F_2}\bigr),$$

which transforms (97) into

$$\begin{aligned} &2\mathrm{e}^{F_2}C_2^2 + \mathrm{e}^{-3F_2}C_2^2 + 3\mathrm{e}^{-F_2}C_2^2 - 2\mathrm{e}^{3F_2}C_2^2 - 16\mathrm{e}^{F_2}F_{2r}^2 - 16\mathrm{e}^{F_2}F_{2rr} \\ &\quad + 16\mathrm{e}^{F_2}C_5 - \mathrm{e}^{7F_2}C_2^2 - 3\mathrm{e}^{5F_2}C_2^2 + 16\mathrm{e}^{3F_2}F_r^2 - 16\mathrm{e}^{3F_2}F_{2rr} + 16\mathrm{e}^{3F_2}C_5 \\ &\quad + L_C\bigl(\mathrm{e}^{7F_2} + 5\mathrm{e}^{5F_2} + 10\mathrm{e}^{3F_2} + \mathrm{e}^{-3F_2} + 10\mathrm{e}^{F_2} + 5\mathrm{e}^{-F_2}\bigr) = 0. \end{aligned} \tag{99}$$

For simplicity, we just consider the case of $C_2^2 > L_C$. Substituting $F_2 = \ln(K\arctan(\overline{r}))$ ($K > 0$) into (99) and then extracting the coefficient of the highest order of $\overline{r}$, we obtain the coefficient of $\overline{r}^4$ (setting $A_{\overline{r}} = \arctan(\overline{r})$),

$$\begin{aligned}&\left(A_{\overline{r}}{}^{10}C_2^2 - A_{\overline{r}}{}^{10}L_C\right)K^{10} + \left(2A_{\overline{r}}{}^{6}C_2^2 - 16A_{\overline{r}}{}^{6}C_5 - 10A_{\overline{r}}{}^{6}L_C\right)K^6\\ &\quad + \left(3A_{\overline{r}}{}^{8}C_2^2 - 5A_{\overline{r}}{}^{8}L_C\right)K^8 + \left(-2A_{r}^4C_2^2 - 16A_{\overline{r}}{}^{4}C_5 - 10A_{\overline{r}}{}^{4}L_C\right)K^4\\ &\quad + \left(-3A_{\overline{r}}{}^{2}C_2^2 - 5A_{\overline{r}}{}^{2}L_C\right)K^2 - C_2^2 - L_C \triangleq H_4(K).\end{aligned}$$

The term $H_4(K)\overline{r}^4$ is the leading-order term as $\overline{r} \to +\infty$. We can control the positive and negative values of $H_4(K)$ by K. Hence, if $C_2^2 > L_C$, then we obtain

$$F_2 \sim \ln\left(K\arctan(\overline{r})\right) \quad (K > 0) \tag{100}$$

as $\overline{r} \to +\infty$.

Employing (98) and (100), it is simple to check that

$$w_E(93) = \frac{4|\overrightarrow{L}|^2F_{2\overline{r}}^2}{(F_2{}^2+1)^2} + \frac{|\overrightarrow{L}|^2 C_2^2(F_2{}^2+1)^2}{4F_2{}^2} \sim O(1)$$

as $\overline{r} \to +\infty$.

Similarly to the proof of Theorem 4.2, there exists a function F satisfying (99). In this setting, we obtain the following:

Theorem 4.3 *Suppose there exists a function F_2 satisfying* (99). *Then the solution of* (59) *is*

$$\begin{cases} S_1(t,\overline{r}) = \frac{2e^{F_2}}{1+e^{2F_2}}\cos(C_1 + C_2\int(e^{2F_2}+1)^2e^{-2F_2}\,d\overline{r} + C_5t),\\ S_2(t,\overline{r}) = \frac{2e^{F_2}}{1+e^{2F_2}}\sin(C_1 + C_2\int(e^{2F_2}+1)^2e^{-2F_2}\,d\overline{r} + C_5t),\\ S_3(\overline{r}) = \frac{1-e^{2F_2}}{1+e^{2F_2}},\end{cases} \tag{101}$$

whereas

$$W = e^{F_2}\exp\left[i\left(C_1 + C_2\int\left(e^{2F_2}+1\right)^2e^{-2F_2}\,d\overline{r} + C_5t\right)\right] \tag{102}$$

is a solution of (60).

Furthermore, if $C_2^2 > L_C$, then the decay rate of energy density is

$$w_E\big((101)\text{ or }(102)\big) \sim O(1)$$

as $\overline{r} \to +\infty$.

In Tables 3 and 4, we present the variable separation solution (and energy density) of (59) and (60), respectively. Similarly to Tables 1 and 2, most of the decay rates of the solutions are in a scale of $O(1)$ in Tables 3 and 4. The only nonconstant decay rate of the solution is the solution I in Table 3. Its decay rate is related to the dimension n as $1/r^{n-1}$. If $n = 1$, then the decay rate degenerates to the constant scale $O(1)$. Similarly to the solutions in Tables 1 and 2, the solutions in Tables 3 and 4 develop no any singularity.

Table 3 Variable separation solution of the LLG equation without damping (see equation (59)). F is the solution of (71); F_2 satisfies (99)

Solutions I–II	Decay rate
Solution I: $\begin{pmatrix} \frac{2e^F}{1+e^{2F}}\cos(C_1+C_2\int r^{-n+1}(e^{2F}+1)^2e^{-2F}\,dr+C_5t) \\ \frac{2e^F}{1+e^{2F}}\sin(C_1+C_2\int r^{-n+1}(e^{2F}+1)^2e^{-2F}\,dr+C_5t) \\ \frac{1-e^{2F}}{1+e^{2F}} \end{pmatrix}$	$O(1)$
Solution II: $\begin{pmatrix} \frac{2e^{F_2}}{1+e^{2F_2}}\cos(C_1+C_2\int (e^{2F_2}+1)^2e^{-2F_2}\,d\bar{r}+C_5t) \\ \frac{2e^{F_2}}{1+e^{2F_2}}\sin(C_1+C_2\int (e^{2F_2}+1)^2e^{-2F_2}\,d\bar{r}+C_5t) \\ \frac{1-e^{2F_2}}{1+e^{2F_2}} \end{pmatrix}$	$O(1)$

Table 4 Variable separation solution of the equivalent LLG equation without damping (see equation (60)). F satisfies (71); F_2 is the solution of (99)

Solutions I–II	Decay rate
Solution I: $e^F \exp[i(C_1+C_2\int r^{-n+1}(e^{2F}+1)^2e^{-2F}\,dr+C_5t)]$	$\frac{1}{r^{n-1}}$, $(n \geq 2)$; $O(1)$, $(n=1)$
Solution II: $e^{F_2} \exp[i(C_1+C_2\int (e^{2F_2}+1)^2e^{-2F_2}\,d\bar{r}+C_5t)]$	$O(1)$

5 Conclusions

We investigate a variable separation solution of the multidimensional ILLG equation and Schrödinger map equation. Based on the stereographic method, we deduce an equivalent ILLG equation, which can be solved exactly. Under the different ansatzs of the solutions, we obtain some explicit solutions of the modification system. The solution of this system can be changed into the solution of the ILLG equation. Hence, for the ILLG equation, we obtain two different solutions, which are all in a constant decay scale.

If $q = 1$ and $\beta = 0$, then the ILLG equation degenerates into the Schrödinger map equation. The situation of this equation is somewhat different from the ILLG equation: the solutions obtained are in implicit forms, which are determined by nonlinear ODEs. This brings some difficulties to our analysis of the decay behavior of the solutions. Using the some heuristic method, we obtain the asymptotic scale of the functions contained in the nonlinear system. So, the decay rate of the energy density of the solutions can be characterized. Clearly, we obtain two different solutions of the Schrödinger map equation, which adopt the same decay rate $O(1)$. However, the situation of its equivalent system is somewhat different: the decay rate of radial type solution is in an r^{n-1} $(n \geq 2)$ and $O(1)$ $(n = 1)$ scale, whereas the plane wave type solution is $O(1)$.

All solutions obtained in this paper are periodic in time or spatial direction. The solution of ILLG equation (or Schrödinger map equation) of type (4) just shows a periodic behavior in the time direction. Hence, if the periodic condition is imposed in the initial condition, then the ILLG equation and Schrödinger map equation contain some smooth solutions for some large initial values. In this study, the properties of the variable separation type solutions are clearer due to their straightforward and exact form. These solutions will be useful in explaining some nonlinear dynamics of stimulation in the inhomogeneous or homogeneous system that comes from the ferromagnet.

Acknowledgements
The continuous support of the Guangdong University of Education is appreciated.

Funding
This work is supported by TianYuan Special Fund of the National Natural Science Foundation of China (No. 11426068), Project for Young Creative Talents of Ordinary University of Guangdong Province (No. 2014KQNCX228), the PhD Start-up Fund of Natural Science Foundation of Guangdong Province (No. 2014A030310330), and Fund for Science and Technology of Guangzhou (No. 201607010352). The author is also grateful to the sponsorship from China Scholarship Council.

Competing interests
The authors declare that there are no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 24 July 2018 Accepted: 7 September 2018 Published online: 20 September 2018

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