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Two numerical methods for fractional partial differential equation with nonlocal boundary value problem

Mahmut Modanlı^{1*}

*Correspondence:

mmodanli@harran.edu.tr

¹Department of Mathematics,
Faculty of Science and Art, Harran
University, Şanlıurfa, Turkey

Abstract

The exact solution of fractional telegraph partial differential equation of nonlocal boundary value problem is obtained. The theorem of stability estimates is presented for this equation. Difference schemes are constructed for both the implicit finite difference scheme and the Dufort–Frankel finite difference scheme (DFFDS). The stability of these difference schemes for this problem are proved. With the help of these methods, numerical solutions of the fractional telegraph differential equation defined by Caputo fractional derivative for fractional orders $\alpha = 0.1, 0.5, 0.9$ are calculated. Numerical results are compared with the exact solution and the accuracy and effectiveness of the proposed methods are investigated.

Keywords: Fractional telegraph partial differential equations; Stability estimates; Nonlocal boundary value problem; Implicit finite difference scheme; Dufort–Frankel finite difference schemes

1 Introduction

Fractional differential equations lead to various significant applications in engineering, finance, physics, biology and seismology [1–4]. The present differential equation is solvable with respect to variables time and space. Some difference schemes are presented for the space-fractional heat equations in [5–7]. In [8, 9], the time-fractional equations are investigated. The solutions of fractional difference equations are obtained by different methods, which include the differential transform method [10], the exponential-function method [11], the variation iteration method [12], the homotopy perturbation method [13], the fractional sub-equation method [14] and analytical solutions in terms of the Mittag–Leffler function [15]. In developing those methods the usually used fractional derivative is Caputo derivative [16], Riemann–Liouville (R–L) [16], Jumarie’s left handed modification of R–L fractional derivative [17, 18]. In this paper, Caputo time fractional derivative is used. In [15], an algorithm was developed to solve the homogeneous fractional-order differential equations in terms of the Mittag–Leffler function and fractional sine and cosine functions. In [19], by finite difference and iteration methods, a fractional hyperbolic partial differential equation with the Neumann condition for $\alpha = \frac{1}{2}$ was studied. In [20], the theta method was used for time-fractional differential diffusion equation. In [21, 22], differential equations were investigated by the nonlocal boundary problem. Yuan and Chen [23]

worked on an expanded mixed finite element method for the two-sided time-dependent fractional diffusion problem with two-sided Riemann–Liouville fractional derivatives. In [24–28], alternative approaches were proposed to the corresponding fractional differential equations.

In the present paper, we shall investigate the following fractional telegraph partial differential equations for nonlocal boundary value problem:

$$\begin{cases} \frac{\partial^2 u(t, x)}{\partial t^2} + \frac{\partial^\alpha u(t, x)}{\partial t^\alpha} - \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x) = f(t, x), \\ 0 < x < L, \quad 0 < t < T, \quad 0 < \alpha < 1, \\ u(0, x) = \lambda u(T, x) + \varphi(x), \\ u_t(0, x) = \mu u_t(T, x) + \psi(x), \quad 0 \leq t \leq T, \\ u(t, X_L) = u(t, X_R) = 0, \quad X_L < x < X_R. \end{cases} \quad (1)$$

Implicit and Dufort–Frankel difference scheme methods (IFDS and DFFDS methods, respectively) are used to obtain numerical results for the problem (1) with the nonlocal conditions. Then we shall investigate the stability estimates of these methods.

In the literature, the notions of gamma function, Caputo fractional derivative and first-order approach method are defined as follows.

Definition 1 ([29]) For all complex numbers except the non-positive integers, the definition of the gamma function is given by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \quad (2)$$

Definition 2 The Caputo fractional derivative $D_t^\alpha u(t, x)$ of order α with respect to time is defined as

$$\begin{aligned} \frac{\partial^\alpha u(t, x)}{\partial t^\alpha} &= D_t^\alpha u(t, x) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{1}{(t-p)^{\alpha-n+1}} \frac{\partial^n u(p, x)}{\partial p^n} dp \quad (n-1 < \alpha < n) \end{aligned} \quad (3)$$

and for $\alpha = n \in \mathbb{N}$ is defined as

$$D_t^\alpha u(t, x) = \frac{\partial^\alpha u(t, x)}{\partial t^\alpha} = \frac{\partial^n u(t, x)}{\partial t^n}.$$

Definition 3 The first-order approach method for the calculation of the problem (3) is given by the formula

$$D_t^\alpha u_n^k \cong g_{\alpha, \tau} \sum_{j=1}^k w_j^{(\alpha)} (u_n^{k-j+1} - u_n^{k-j}), \quad (4)$$

where $g_{\alpha, \tau} = \frac{1}{\Gamma(2-\alpha)\tau^\alpha}$ and $w_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}$. Using these values, one has the following approximation [30]:

$$\frac{\partial^\alpha u(t_k, x_n)}{\partial t^\alpha} = g_{\alpha, \tau} \left[w_1 u(t_k, x_n) - w_k u(t_0, x_n) + \sum_{j=1}^{k-1} (w_{k-j+1} - w_{k-j}) u(t_j, x_n) \right]. \quad (5)$$

For more details of the above definitions, we refer to [16, 31]. This paper is organized as follows. In Sect. 2, the exact solution of the given problem is shown and the stability estimates are presented. In Sect. 3, IFDS and DFFDS for solving fractional partial differential equation are introduced. In Sect. 4, the error estimates of the IFDS and DFFDS schemes for a test example are shown and numerical results are investigated. In Sect. 5, the results of our paper are discussed.

2 Solution and stability estimates of the abstract form for fractional telegraph equation

Using the method [32, 33], we rewrite Eq. (1) in the following abstract form:

$$\begin{cases} \frac{d^2 u(t)}{dt^2} + Au(t) = F(t), & (0 < t < T), \\ u(0) = \lambda u(T) + \varphi, & u'(0) = \mu u'(T) + \psi \end{cases} \quad (6)$$

in a Hilbert space $H = L_2[0, L]$ with a self-adjoint positive definite operator A , $A \geq \delta I$ and $\delta > 0$. Here $f(t) = f(t, x)$ is given abstract function defined on $[0, T]$ with values in $H = L_2[0, L]$. $\varphi = \varphi(x)$ and $\psi = \psi(x)$ are the elements of $H = L_2[0, L]$. $u(t) = u(t, x)$ is the unknown abstract function defined on $[0, T]$ with values in $H = L_2[0, L]$.

$A : D(A) \rightarrow H$ is the differential operator defined by

$$Au(x) = -u''(x) + u(x)$$

with domain

$$D(A) = \{u : u, u_{xx} \in L_2[0, L]; u(0) = u(L) = 0\}.$$

Here, $F(t) = f(t) - D_t^\alpha u(t)$.

A function $u(t)$ is called a solution of the problem (6) if the following conditions are satisfied:

- (i) $u(t)$ is a twice continuously differentiable on the interval $[0, T]$. The derivatives at the endpoints of the interval are understood as the appropriate unilateral derivatives.
- (ii) The value of $u(t)$ relates to $D(A)$ for all $t \in [0, T]$ and the function $Au(t)$ is continuous on the interval $[0, T]$.
- (iii) $u(t)$ satisfies both the equation and the boundary conditions (6).

Now, we shall obtain the formula for the solution of the problem (6). For this purpose, we can rewrite the problem (6) in the following the first-order differential equations systems form:

$$\begin{cases} \frac{du(t)}{dt} + iBu(t) = z(t), \\ \frac{dz(t)}{dt} - iBz(t) = F(t), \\ z(0) = u'(0) + iBu(0), \end{cases} \quad (7)$$

where $B = A^{1/2}$. Here, $z(t)$ is a continuously differentiable on the interval $[0, T]$. Integrating Eq. (7), we get

$$\begin{cases} u(t) = e^{-iBt}u(0) + \int_0^t e^{-iB(t-s)}z(s)ds, \\ z(t) = e^{iBt}z(0) + \int_0^t e^{iB(t-s)}F(s)ds. \end{cases}$$

Applying the initial condition $z(0) = u'(0) + iBu(0)$, we can obtain

$$\begin{aligned} u(t) &= e^{-iBt}u(0) + \int_0^t e^{-iB(t-s)} \int_0^s e^{iB(s-p)}F(p)dpds \\ &\quad + \int_0^t e^{-iB(t-s)}e^{iBs}ds(u'(0) + iBu(0)) \\ &= e^{-iBt}u(0) + \left(iB \int_0^t e^{-iB(t-s)}e^{iBs}ds \right)u(0) + \left(\int_0^t e^{-iB(t-s)}e^{iBs}ds \right)u'(0) \\ &\quad + \int_0^t \int_0^s e^{-iB(t-s)}e^{iB(s-p)}F(p)dpds. \end{aligned}$$

By an interchange of the order of integration, we can write

$$\begin{aligned} u(t) &= \left(e^{-iBt} + \frac{e^{iBt} - e^{-iBt}}{2} \right)u(0) + \left(B^{-1} \frac{e^{iBt} - e^{-iBt}}{2i} \right)u'(0) \\ &\quad + \int_0^t \int_p^t e^{-iB(t-s)}e^{iB(s-p)}F(p)dsdp \\ &= \left(e^{-iBt} + \frac{e^{iBt} - e^{-iBt}}{2} \right)u(0) + \left(B^{-1} \frac{e^{iBt} - e^{-iBt}}{2i} \right)u'(0) \\ &\quad + \int_0^t B^{-1} \frac{e^{iB(t-s)} - e^{-iB(t-s)}}{2i} F(s)ds. \end{aligned}$$

Then we obtain

$$u(t) = c(t)u(0) + s(t)u'(0) + \int_0^t s(t-s)F(s)ds, \quad (8)$$

where

$$\begin{aligned} B &= A^{\frac{1}{2}}, \\ c(t) &= \frac{e^{iA^{\frac{1}{2}}t} + e^{-iA^{\frac{1}{2}}t}}{2}, \\ s(t) &= A^{-\frac{1}{2}} \frac{e^{iA^{\frac{1}{2}}t} - e^{-iA^{\frac{1}{2}}t}}{2i}. \end{aligned}$$

Using Eq. (8) and the formula $F(t) = f(t) - D_t^\alpha u(t)$, we obtain

$$u(t) = c(t)u(0) + s(t)u'(0) + \int_0^t s(t-z)f(z)dz - \int_0^t s(t-z)D_z^\alpha u(z)dz. \quad (9)$$

From Eq. (9), it follows that

$$\begin{aligned} u'(t) &= -As(t)u(0) + c(t)u'(0) + (s(0)f(t)) \times 1 - (s(t)f(0)) \times 0 + \int_0^t c(t-z)f(z) dz \\ &\quad - s(0)D_t^\alpha u(t) \times 1 + s(t)D_{0+}^\alpha u(0) \times 0 - \int_0^t c(t-z)D_z^\alpha u(z) dz \\ &= -As(t)u(0) + c(t)u'(0) + \int_0^t c(t-z)f(z) dz - \int_0^t c(t-z)D_z^\alpha u(z) dz \end{aligned} \quad (10)$$

and from that we obtain

$$\begin{aligned} u''(t) &= -Ac(t)u(0) - As(t)u'(0) + c(0)f(t) \times 1 - c(t)f(0) \times 0 + \int_0^t -As(t-z)f(z) dz \\ &\quad - \left[c(0)D_t^\alpha u(t) \times 1 - c(t)D_{0+}^\alpha u(0) \times 0 + \int_0^t -As(t-z)D_z^\alpha u(z) dz \right] \\ &= -Ac(t)u(0) - As(t)u'(0) + f(t) - D_t^\alpha u(t) \\ &\quad + \int_0^t -As(t-z)f(z) dz - \int_0^t -As(t-z)D_z^\alpha u(z) dz, \end{aligned} \quad (11)$$

where $D_{0+}^\alpha u(0) = [D_t^\alpha u(t)]_{t=0}$. Applying Eqs. (9), (10) and the conditions $u(0) = \lambda u(T) + \varphi$, $u'(0) = \mu u'(T) + \psi$, we get

$$\begin{aligned} u(0) &= \lambda \left[c(T)u(0) + s(T)u'(0) + \int_0^T s(T-z)f(z) dz \right. \\ &\quad \left. - \int_0^T s(T-z)D_z^\alpha u(z) dz \right] + \varphi, \end{aligned} \quad (12)$$

$$\begin{aligned} u'(0) &= \mu \left[-As(T)u(0) + c(T)u'(0) + \int_0^T c(T-z)f(z) dz \right. \\ &\quad \left. - \int_0^T c(T-z)D_z^\alpha u(z) dz \right] + \psi. \end{aligned} \quad (13)$$

We shall obtain $u(0)$ and $u'(0)$. We have

$$\begin{aligned} \Delta &= \begin{vmatrix} I - \lambda c(T) & -\lambda s(T) \\ \mu As(T) & I - \mu c(T) \end{vmatrix} \\ &= (I - \lambda c(T))(I - \mu c(T)) + \lambda \mu (c^2(T) + As^2(T)) \\ &= [I + \lambda \mu (c^2(T) + As^2(T)) - (\lambda + \mu)c(T)]. \end{aligned}$$

Using the fact that

$$c^2(T) + As^2(T) = \left\{ \frac{e^{iA^{1/2}T} + e^{-iA^{1/2}T}}{2} \right\}^2 + A \left\{ \frac{e^{iA^{1/2}T} + e^{-iA^{1/2}T}}{2i} \right\}^2 = I$$

we have

$$\Delta = (1 + \lambda \mu) - (\lambda + \mu)c(T).$$

The assumption that

$$|1 + \lambda\mu| > (|\lambda + \mu|), \quad (14)$$

implies that

$$P = \Delta^{-1}.$$

Using the operator P , solving Δ and (9), we obtain

$$\begin{aligned} u(0) = P \Big\{ & \lambda \int_0^T s(T-z)f(z) dz - \lambda \int_0^T s(T-z)D_z^\alpha u(z) dz \\ & + \lambda\mu \left[\int_0^T s(z)f(z) dz - \int_0^T s(z)D_z^\alpha u(z) dz \right] + \lambda s(T)\psi + (I - c(T)\mu)\varphi \Big\} \end{aligned} \quad (15)$$

and

$$\begin{aligned} u'(0) = P \Big\{ & \mu \int_0^T c(T-z)f(z) dz - \mu \int_0^T c(T-z)D_z^\alpha u(z) dz \\ & - \lambda\mu \left[\int_0^T c(z)f(z) dz - \int_0^T c(z)D_z^\alpha u(z) dz \right] + (I - \lambda c(T))\psi - \mu As(T)\varphi \Big\}. \end{aligned} \quad (16)$$

Consequently, using Eqs. (9), (10), (15) and (16), we obtain the solution of the problem (6):

$$\begin{aligned} u(t) = P \Big\{ & \lambda\mu \int_0^T [c(t)s(z) - s(t)c(z)]f(z) dz + \lambda \int_0^T c(t)s(T-z)f(z) dz \\ & - \lambda \int_0^T c(t)s(T-z)D_z^\alpha u(z) dz + \lambda\mu \int_0^T [c(t)s(z) - s(t)c(z)]D_z^\alpha u(z) dz \\ & + \mu \int_0^T s(t)c(T-z)f(z) dz - \mu \int_0^T s(t)c(T-z)D_z^\alpha u(z) dz + \lambda c(t)s(T)\psi \\ & + c(t)(I - \mu c(T))\varphi + s(t)(I - \lambda c(T))\psi - \mu As(T)s(t)\varphi \Big\} \\ & + \int_0^t s(t-z)f(z) dz - \int_0^t s(t-z)D_z^\alpha u(z) dz. \end{aligned}$$

Thus, we write

$$\begin{aligned} u(t) = P \Big\{ & -\lambda\mu \int_0^T s(t-z)f(z) dz + \lambda\mu \int_0^T s(t-z)D_z^\alpha u(z) dz \\ & + \lambda \int_0^T c(t)s(T-z)f(z) dz - \lambda \int_0^T c(t)s(T-z)D_z^\alpha u(z) dz \\ & + \mu \int_0^T s(t)c(T-z)f(z) dz - \mu \int_0^T s(t)c(T-z)D_z^\alpha u(z) dz \\ & + (c(t) - \mu c(t-T))\varphi + (s(t) - \lambda s(t-T))\psi \\ & + \int_0^t s(t-z)f(z) dz - \int_0^t s(t-z)D_z^\alpha u(z) dz \Big\}. \end{aligned} \quad (17)$$

Using the following formula for the fractional derivative of order $0 < \alpha < 1$:

$$D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(p) dp}{(t-p)^\alpha}, \quad \text{where } u(0) = 0, \quad (18)$$

we get

$$\begin{aligned} D_t^\alpha u(t) = & \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-p)^\alpha} \left\{ P \left[-\lambda \mu \int_0^T c(p-z)f(z) dz \right. \right. \\ & + \lambda \mu \int_0^T c(p-z)D_z^\alpha u(z) dz - \lambda \int_0^T As(p)s(T-z)f(z) dz \\ & + \lambda \int_0^T As(p)s(T-z)D_z^\alpha u(z) dz + \mu \int_0^T c(p)c(T-z)f(z) dz \\ & - \mu \int_0^T c(p)c(T-z)D_z^\alpha u(z) dz + (-As(p) + \mu As(p)c(T) - \mu As(T)c(p))\varphi \\ & \left. \left. + (-\lambda As(p)s(T) + c(p) - \lambda c(p)c(T))\psi \right] \right. \\ & \left. + \int_0^p c(p-z)f(z) dz - \int_0^p c(p-z)D_z^\alpha u(z) dz \right\} dp. \end{aligned} \quad (19)$$

Lemma 2.1 Suppose that $\varphi \in D(A)$, $\psi \in D(A^{1/2})$, $D_t^\alpha u(t)$ and $f(t)$ are continuously differentiable on $[0, T]$ and assumption (14) holds. Then the following stability inequalities for Eqs. (15), (16) and (19) are true simultaneously:

$$\|D_t^\alpha u(t)\|_H \leq M_1(\alpha, \lambda, \mu) \left\{ \|A^{1/2}\varphi\|_H + \|\psi\|_H + \max_{0 \leq t \leq T} \|f(t)\|_H \right\}, \quad (20)$$

$$\|u(0)\|_H \leq M_2(\alpha, \lambda, \mu) \left\{ \|\varphi\|_H + \|A^{-1/2}\psi\|_H + \max_{0 \leq t \leq T} \|A^{-1/2}f(t)\|_H \right\}, \quad (21)$$

$$\|A^{-1/2}u'(0)\|_H \leq M_3(\alpha, \lambda, \mu) \left\{ \|\varphi\|_H + \|A^{-1/2}\psi\|_H + \max_{0 \leq t \leq T} \|A^{-1/2}f(t)\|_H \right\}, \quad (22)$$

where M_1, M_2, M_3 do not depend on $f(t)$, $t \in [0, T]$, φ , and ψ .

Proof Firstly, we shall prove (20). Using Eq. (18), applying the triangle inequality to Eq. (19), and by the formula changing integral order and the inequalities

$$\|P\|_{H \rightarrow H} \leq 1, \quad \|c(t)\|_{H \rightarrow H} \leq 1, \quad \|A^{1/2}s(t)\|_{H \rightarrow H} \leq 1, \quad (23)$$

it follows that

$$\begin{aligned} \|D_t^\alpha u(t)\|_H \leq & \frac{1}{\Gamma(1-\alpha)} \left[|\lambda| |\mu| \int_0^t \frac{dp}{(t-p)^\alpha} \left[\max_{0 \leq p \leq T} \|f(p)\|_H + \|D_p^\alpha u(p)\|_H \right] \right. \\ & + (|\lambda| + |\mu|) \int_0^t \frac{dp}{(t-p)^\alpha} \left[\max_{0 \leq p \leq T} \|f(p)\|_H + \|D_p^\alpha u(p)\|_H \right] \\ & + \|A^{1/2}\varphi\|_H \int_0^t \frac{dp}{(t-p)^\alpha} (1 + 2|\mu|) + \|\psi\|_H \int_0^t \frac{dp}{(t-p)^\alpha} (1 + 2|\lambda|) \left. \right] \\ & + \frac{1}{\Gamma(1-\alpha)} \int_0^t \int_p^t \frac{dp}{(t-p)^\alpha} [\|f(z)\|_H + \|D_z^\alpha u(z)\|_H] dp dz. \end{aligned} \quad (24)$$

By a straightforward computation, we find

$$\begin{aligned} \left\| \int_p^t \frac{1}{(t-p)^\alpha} c(p-z) dp \right\|_{H \rightarrow H} &\leq \int_p^t \frac{\|c(p-z)\|_{H \rightarrow H}}{(p-z)^\alpha} dp \\ &\leq 2(t-z)^{1-\alpha} \end{aligned} \quad (25)$$

and

$$\int_0^t (t-z)^{1-\alpha} dz = \frac{t^{1-\alpha}}{\alpha-1}. \quad (26)$$

Putting (25) and (26) in (24), it follows that

$$\begin{aligned} \|D_t^\alpha u(t)\|_H &\leq \frac{2}{\Gamma(1-\alpha)} \int_0^t z^{1-\alpha} \|D_z^\alpha u(z)\|_H dz \\ &\quad + M_1(\alpha, \lambda, \mu) t^{1-\alpha} \left[\|A^{1/2} \varphi\|_H + \|\psi\|_H + \max_{0 \leq t \leq T} \|f(t)\|_H \right] \end{aligned}$$

for any $0 \leq t \leq T$. Consequently, it is obvious that

$$\begin{aligned} \max_{0 \leq t \leq T} t^{1-\alpha} \|D_t^\alpha u(t)\|_H \\ \leq M_1(\alpha, \lambda, \mu) \left[\|A^{1/2} \varphi\|_H + \|\psi\|_H + \max_{0 \leq t \leq T} \|f(t)\|_H \right]. \end{aligned} \quad (27)$$

Using the estimate (27), we obtain

$$\|D_t^\alpha u(t)\|_H \leq M_1(\alpha, \lambda, \mu) \left[\|A^{1/2} \varphi\|_H + \|\psi\|_H + \max_{0 \leq t \leq T} \|f(t)\|_H \right].$$

Secondly, we shall prove Eq. (21). Using Eqs. (15) and applying the triangle inequality, we obtain

$$\begin{aligned} \|u(0)\|_H &\leq \|P\|_{H \rightarrow H} \left\{ |\lambda| \int_0^T \|A^{1/2} s(T-z)\|_{H \rightarrow H} \|A^{-1/2} f(z)\|_H dz \right. \\ &\quad + |\lambda| \int_0^T \|A^{1/2} s(T-z)\|_{H \rightarrow H} \|A^{-1/2}\|_{H \rightarrow H} \|D_z^\alpha u(z)\|_H dz \\ &\quad + |\lambda| |\mu| \left[\int_0^T \|A^{1/2} s(z)\|_{H \rightarrow H} \|A^{-1/2} f(z)\|_H dz \right. \\ &\quad \left. \left. + \int_0^T \|A^{1/2} s(z)\|_{H \rightarrow H} \|A^{-1/2}\|_{H \rightarrow H} \|D_z^\alpha u(z)\|_H dz \right] \right. \\ &\quad \left. + |\lambda| \|A^{1/2} s(T)\|_{H \rightarrow H} \|A^{-1/2} \psi\|_H + \|(I - c(T))\|_{H \rightarrow H} |\mu| \|\varphi\|_H \right\}. \end{aligned} \quad (28)$$

Using Eqs. (20) and (23), we have

$$\|u(0)\|_H \leq M_2(\alpha, \lambda, \mu) \left\{ \|\varphi\|_H + \|A^{-1/2} \psi\|_H + \max_{0 \leq t \leq T} \|A^{-1/2} f(t)\|_H \right\}.$$

Finally, we shall show that Eq. (22) is true. Using Eqs. (16) and applying the triangle inequality, we get

$$\begin{aligned} \|A^{-1/2}u'(0)\|_H &\leq \|P\|_{H \rightarrow H} \{ |\mu| \int_0^T \|c(T-z)\|_{H \rightarrow H} \|A^{-1/2}f(z)\|_H dz \\ &\quad + |\mu| \int_0^T \|c(T-z)\|_{H \rightarrow H} \|A^{-1/2}D_z^\alpha u(z)\|_H dz \\ &\quad + |\lambda| |\mu| \int_0^T \|c(z)\|_{H \rightarrow H} \|A^{-1/2}f(z)\|_H dz \\ &\quad + |\lambda| |\mu| \int_0^T \|c(z)\|_{H \rightarrow H} \|A^{-1/2}D_z^\alpha u(z)\|_H dz \\ &\quad + \|(I - \lambda c(T))\|_{H \rightarrow H} \|A^{-1/2}\psi\|_H \\ &\quad + |\mu| \|\varphi\|_H \|A^{1/2}s(T)\|_{H \rightarrow H}. \end{aligned}$$

Using Eqs. (20) and (23), we obtain

$$\|A^{-1/2}u'(0)\|_H \leq M_3(\alpha, \lambda, \mu) \left\{ \|\varphi\|_H + \|A^{-1/2}\psi\|_H + \max_{0 \leq t \leq T} \|A^{-1/2}f(t)\|_H \right\}.$$

Thus, the proof of the lemma is completed. $\square$

Theorem 2.2 Let $\varphi \in D(A)$, $\psi \in D(A^{1/2})$ and $f(t)$ be continuously differentiable on $[0, T]$ and assumption (14) holds. Then there exists a unique solution of problem (6) and the following stability inequalities hold:

$$\begin{aligned} \max_{0 \leq t \leq T} \|u(t)\|_H &\leq M_4(\alpha, \lambda, \mu) \left\{ \|\varphi\|_H + \|A^{-1/2}\psi\|_H + \max_{0 \leq t \leq T} \|A^{-1/2}f(t)\|_H \right\}, \end{aligned} \quad (29)$$

$$\begin{aligned} \max_{0 \leq t \leq T} \|A^{1/2}u(t)\|_H &\leq M_5(\alpha, \lambda, \mu) \left\{ \|A^{1/2}\varphi\|_H + \|\psi\|_H + \max_{0 \leq t \leq T} \|f(t)\|_H \right\}, \end{aligned} \quad (30)$$

$$\begin{aligned} \max_{0 \leq t \leq T} \left\| \frac{d^2 u(t)}{dt^2} \right\|_H + \max_{0 \leq t \leq T} \|Au(t)\|_H &\leq M_6(\alpha, \lambda, \mu) \left\{ \|A\varphi\|_H + \|A^{1/2}\psi\|_H + \|f(0)\|_H + \int_0^T \|f'(t)\|_H dt \right\}, \end{aligned} \quad (31)$$

where M_4, M_5, M_6 do not depend on $f(t)$, $t \in [0, T]$, φ , and ψ .

Proof Using (17) and the estimates (20) and (23), we obtain the following estimates:

$$\begin{aligned} \|u(t)\|_H &\leq \|P\|_{H \rightarrow H} [|\lambda| |\mu| \int_0^T \|A^{1/2}s(t-z)\|_{H \rightarrow H} \|A^{-1/2}f(z)\|_H dz \\ &\quad + |\lambda| |\mu| \int_0^T \|A^{1/2}s(t-z)\|_{H \rightarrow H} \|A^{-1/2}D_z^\alpha u(z)\|_H dz \\ &\quad + |\lambda| \int_0^T \|A^{\frac{1}{2}}s(T-z)\|_{H \rightarrow H} \|c(t)\|_{H \rightarrow H} \|A^{-1/2}f(s)\|_H dz \end{aligned}$$

$$\begin{aligned}
& + |\lambda| \int_0^T \|A^{\frac{1}{2}} s(T-z)\|_{H \rightarrow H} \|c(t)\|_{H \rightarrow H} \|A^{-1/2} D_z^\alpha u(z)\|_H dz \\
& + |\mu| \int_0^T \|A^{\frac{1}{2}} s(t)\|_{H \rightarrow H} \|c(T-z)\|_{H \rightarrow H} \|A^{-1/2} f(z)\|_H dz \\
& + |\mu| \int_0^T \|A^{\frac{1}{2}} s(t)\|_{H \rightarrow H} \|c(T-z)\|_{H \rightarrow H} \|A^{-1/2} D_z^\alpha u(z)\|_H dz \\
& + \|(c(t) - \mu c(t-T))\|_{H \rightarrow H} \|\varphi\|_H + \|A^{\frac{1}{2}} (s(t) - \lambda s(t-T))\|_{H \rightarrow H} \|A^{-1/2} \psi\|_H, \\
& \int_0^t \|A^{\frac{1}{2}} s(t-z)\|_{H \rightarrow H} \|A^{-1/2} f(z)\|_H dz + \int_0^t \|A^{\frac{1}{2}} s(t-z)\|_{H \rightarrow H} \|D_z^\alpha u(z)\|_H dz \\
& \leq M_4(\alpha, \lambda, \mu) \left[\|\varphi\|_H + \|A^{-1/2} \psi\|_H + \max_{0 \leq t \leq T} \|A^{-1/2} f(t)\|_H \right]
\end{aligned}$$

for any $t \in [0, T]$.

Hence, we obtain

$$\begin{aligned}
& \max_{0 \leq t \leq T} \|u(t)\|_H \\
& \leq M_4(\alpha, \lambda, \mu) \left[\|\varphi\|_H + \|A^{-1/2} \psi\|_H + \max_{0 \leq t \leq T} \|A^{-1/2} f(t)\|_H \right].
\end{aligned} \tag{32}$$

Applying $A^{\frac{1}{2}}$ to the estimates (32) and using the estimates (23), we obtain

$$\|A^{\frac{1}{2}} u(t)\|_H \leq M_5(\alpha, \lambda, \mu) \left[\|A^{\frac{1}{2}} \varphi\|_H + \|\psi\|_H + \max_{0 \leq t \leq T} \|f(t)\|_H \right] \tag{33}$$

for any $t \in [0, T]$.

Now, we shall obtain an estimate for $\|Au(t)\|_H$. Applying A to Eq. (9) and using an integration by parts, we can write the formula as

$$\begin{aligned}
Au(t) &= c(t)Au(0) + A^{\frac{1}{2}} s(t)A^{\frac{1}{2}} u'(0) \\
& - \int_0^t -As(t-z)f(z) dz + \int_0^t -As(t-z)D_z^\alpha u(z) dz \\
& = c(t)Au(0) + A^{\frac{1}{2}} s(t)A^{\frac{1}{2}} u'(0) - c(0)f(t) \times 1 + c(t)f(0) \times 0 \\
& + \int_0^t c(t-z)f'(z) dz - \int_0^t A^{\frac{1}{2}} s(t-z)A^{\frac{1}{2}} D_z^\alpha u(z) dz.
\end{aligned}$$

Using the previous formula and the estimate (23), we obtain

$$\begin{aligned}
\|Au(t)\|_H &\leq \|c(t)\|_{H \rightarrow H} \|Au(0)\|_H \\
& + \|A^{\frac{1}{2}} s(t)\|_{H \rightarrow H} \|A^{\frac{1}{2}} u'(0)\|_H + \|c(t)\|_{H \rightarrow H} \|f(0)\|_H + \|f(t)\|_H \\
& + \int_0^t \|c(t-z)\|_{H \rightarrow H} \|f'(z)\|_H dz \\
& + \int_0^t \|A^{\frac{1}{2}} s(t-z)\|_{H \rightarrow H} \|A^{\frac{1}{2}} D_z^\alpha u(z)\|_H dz.
\end{aligned}$$

Applying $A^{\frac{1}{2}}$ to the formulas $\|A^{1/2}u(0)\|_H$, $\|A^{\frac{1}{2}}u'(0)\|_H$ and $\|D_t^\alpha u(t)\|_H$, we obtain

$$\|Au(t)\|_H \leq M_6(\alpha, \lambda, \mu) \left[\|A\varphi\|_H + \|A^{\frac{1}{2}}\psi\|_H + \|f(0)\|_H + \max_{0 \leq t \leq T} \|f'(t)\|_H \right]$$

for any $t \in [0, T]$.

Then we get

$$\begin{aligned} \max_{0 \leq t \leq T} \|Au(t)\|_H \\ \leq M_6(\alpha, \lambda, \mu) \left[\|Au(0)\|_H + \|A^{\frac{1}{2}}u'(0)\|_H + \|f(0)\|_H + \max_{0 \leq t \leq T} \|f'(t)\|_H \right]. \end{aligned} \quad (34)$$

The estimate (31) follows from the estimates (34). Finally, the estimate for $\max_{0 \leq t \leq T} \|\frac{d^2u}{dt^2}\|_H$ follows from the previous estimate and the triangle inequality, which completes the proof of the theorem. $\square$

3 Discretization of two numerical methods

The first method used in this section is the IFDS method. For this method, let us suppose that $h = \frac{L}{M}$ for the x -axis and $\tau = \frac{T}{N}$ for the t -axis as grid mess, then we get

$$x_n = x_L + nh; \quad n = 1, 2, \dots, M, \quad t_k = k\tau, \quad k = 1, 2, \dots, N.$$

Using the method of [34] and the discrete formula (4) for the fractional partial differential equation (1), we construct the following difference schemes:

$$\begin{cases} \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + g_{\alpha, \tau} \sum_{j=1}^k w_j^{(\alpha)} (u_n^{k-j+1} - u_n^{k-j}) + u_n^k - \frac{1}{h^2} (u_{n+1}^k - 2u_n^k + u_{n-1}^k) = f_n^k, \\ f_n^k = f(t_k, x_n), \\ u_n^0 = \lambda u_n^N + \varphi, \\ \frac{u_n^1 - u_n^0}{\tau} = \mu \frac{u_n^N - u_n^{N-1}}{\tau} + \psi. \end{cases} \quad (35)$$

The second method used in this section is DFFDS. The first-order difference scheme of this method is

$$u_x = \frac{u_n^{k+1} - u_n^{k-1}}{2h} \quad (36)$$

and the second-order difference scheme is

$$u_{xx} = \frac{u_{n+1}^k - (u_n^{k+1} + u_n^{k-1}) + u_{n-1}^k}{h^2}. \quad (37)$$

Writing (36) and (37) in Eq. (1), we get

$$\begin{cases} \frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\tau^2} + g_{\alpha, \tau} \sum_{j=1}^k w_j^{(\alpha)} (u_n^{k-j+1} - u_n^{k-j}) + \frac{1}{2} (u_n^{k+1} + u_n^{k-1}) \\ - \frac{1}{h^2} (u_{n+1}^k - (u_n^{k+1} + u_n^{k-1}) + u_{n-1}^k) = f_n^k, \\ f_n^k = f(t_k, x_n), \\ u_n^0 = \lambda u_n^N + \varphi, \\ \frac{u_n^1 - u_n^0}{\tau} = \mu \frac{u_n^N - u_n^{N-1}}{\tau} + \psi. \end{cases} \quad (38)$$

Now, we shall show convergence and stability for the two methods.

3.1 The convergence of the methods

The error vector at level $t = t_{k+1}$ is denoted by $e^{k+1} = \bar{u}^{k+1} - u^{k+1}$, $e^0 = 0$, where $\bar{u}^k = (\bar{u}_{n-1}^k, \bar{u}_{n-2}^k, \dots, \bar{u}_1^k)^T$ is the exact solution of the problem (6).

First, we shall show that the equation for IFDS shows convergence. Using Eq. (5), we can write Eq. (35) in the following form:

$$\begin{aligned} u_n^{k+1} = & \left(2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} w_1^{(\alpha)} - \tau^2 - 2\frac{\tau^2}{h^2}\right) u_n^k + \left(\frac{\tau^2}{h^2}\right) u_{n-1}^k + \left(\frac{\tau^2}{h^2}\right) u_{n+1}^k - u_n^{k-1} \\ & + \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} w_k^{(\alpha)} u_n^0 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k-1} (w_{k-j+1} - w_{k-j}) u_n^j \\ & + \tau^2 f_n^k, \quad k > 1, w_1^{(\alpha)} = 1, \end{aligned} \quad (39)$$

where $u_n^k = u(t_k, x_n)$, $u_n^0 = \varphi(x_n)$, $\frac{u_n^1 - u_n^0}{\tau} = \psi(x_n)$ for $n = 0, 1, \dots, N$. From Eq. (39), we obtain

$$u^{k+1} = A^k u^k - u^{k-1} + B^k + F^k, \quad (40a)$$

where $u^k = (u_{n-1}^k, u_{n-2}^k, \dots, u_1^k)^T$, $F^k = (\tau^2 f(u_{n-1}^{k-1}, x_{n-1}, t_k), \dots, \tau^2 f(u_1^{k-1}, x_1, t_k))^T$,

$$A^k = \begin{bmatrix} a & b & 0 & \dots & 0 & 0 & 0 \\ b & a & b & \dots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \dots & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & b & a & b \\ 0 & 0 & 0 & \dots & 0 & b & b \end{bmatrix}_{(N-1) \times (N-1)},$$

where $a = (2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} w_1^{(\alpha)} - \tau^2 - 2\frac{\tau^2}{h^2})$ and $b = \frac{\tau^2}{h^2}$,

$$\begin{aligned} B^k = & \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} \begin{bmatrix} w_2^{(\alpha)} u_n^0 \\ w_3^{(\alpha)} u_n^0 \\ \cdot \\ \cdot \\ w_k^{(\alpha)} u_n^0 \end{bmatrix}_{(N-1) \times 1} \\ & + \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} \begin{bmatrix} (w_2^{(\alpha)} - w_1^{(\alpha)}) u_n^1 \\ (w_3^{(\alpha)} - w_2^{(\alpha)}) u_n^1 + (w_2^{(\alpha)} - w_1^{(\alpha)}) u_n^2 \\ \cdot \\ \cdot \\ (w_k^{(\alpha)} - w_{k-1}^{(\alpha)}) u_n^1 + \dots + (w_2^{(\alpha)} - w_1^{(\alpha)}) u_n^{k-1} \end{bmatrix}_{(N-1) \times 1}. \end{aligned}$$

Theorem 3.1 *The system (40a) is convergent and $|u_n^k - \bar{u}_n^k| = O(O(\tau) + O(h))$ for any n, k where*

$$\frac{2\tau^2 \Gamma(2-\alpha)}{2\Gamma(2-\alpha) - \tau^{2-\alpha} - \tau^2 \Gamma(2-\alpha)} \leq h^2.$$

Proof From (40a), we have $e^{k+1} = A^k e^k - e^{k-1} + F_e^k + (O(\tau) + O(h))$;

$$\begin{aligned} F_e^k &= \left(\left| \tau^2 f(u_{n-1}^k, x_{n-1}, t_k) - \tau^2 f(\bar{u}_{n-1}^k, x_{n-1}, t_k) \right|, \dots, \left| \tau^2 f(u_1^k, x_1, t_k) - \tau^2 f(\bar{u}_1^k, x_1, t_k) \right| \right)^T \\ &\leq \left(\tau^2 L_{n-1}^k e_{n-1}^k, \dots, \tau^2 L_1^k e_1^k \right)^T = \Delta F^k e^k, \end{aligned}$$

when $\Delta F^k = \text{diag}((\tau^2 L_{n-1}^k, \dots, \tau^2 L_1^k)^T)$. Note that $|L_n^k| \leq L$, for any k, n .

For given h , we choose τ to satisfy the condition

$$2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} w_1^{(\alpha)} - \tau^2 - 2 \frac{\tau^2}{h^2} \geq 0. \quad (41)$$

From Eq. (41), we obtain $\frac{2\tau^{2-\alpha}\Gamma(2-\alpha)}{2\Gamma(2-\alpha)-\tau^{2-\alpha}-\tau^2\Gamma(2-\alpha)} \leq h^2$.

$$\begin{aligned} \|A^k\|_\infty &= \left| 2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} w_1^{(\alpha)} - \tau^2 - 2 \frac{\tau^2}{h^2} \right| + \left| \frac{\tau^2}{h^2} \right| + \left| \frac{\tau^2}{h^2} \right| \\ &= 2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} w_1^{(\alpha)} - \tau^2 - 2 \frac{\tau^2}{h^2} + \frac{\tau^2}{h^2} + \frac{\tau^2}{h^2} = 2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} w_1^{(\alpha)} - \tau^2 \leq 2. \end{aligned}$$

For any constant $C > 0$ independent of τ, h , we obtain

$$\begin{aligned} \|e^{k+1}\|_\infty &\leq \|A^k + \Delta F^k\|_\infty \|e^k\|_\infty + \|e^{k-1}\|_\infty + C(h^2 + \tau) \\ &\leq (2 + \tau^2 L) \|e^k\|_\infty + \|e^{k-1}\|_\infty + C(h^2 + \tau). \end{aligned}$$

Taking $r_1 = 2 + \tau^2 L$ and $r_2 = C(h^2 + \tau)$, then

$$\|e^{k+1}\|_\infty \leq r_1 \|e^k\|_\infty + \|e^{k-1}\|_\infty + r_2.$$

In a similar manner, using the method in [35], we obtain

$$\|e^{k+1}\|_\infty \leq C \|e^0\|_\infty + (O(\tau) + O(h)).$$

From the second initial condition we have $\|e^1\|_n \leq \|e^0\|_n$. Thus, the proof of the theorem is completed. $\square$

Second, we shall present the convergence of the DFFDS formula. From Eq. (38), we get

$$\begin{aligned} \left(1 + \frac{\tau^2}{h^2} \right) u_n^{k+1} &= \frac{\tau^2}{h^2} u_{n-1}^k + \left(2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} w_1 \right) u_n^k + \frac{\tau^2}{h^2} u_{n+1}^k - \left(1 + \frac{\tau^2}{h^2} \right) u_n^{k-1} \\ &\quad + \frac{\tau^{2(-\alpha)}}{\Gamma(2-\alpha)} w_k^{(\alpha)} u_n^0 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k-1} (w_{k-j+1} - w_{k-j}) u_n^j \\ &\quad + \tau^2 f_n^k, \quad k > 1, w_1^{(\alpha)} = 1. \end{aligned}$$

Using a similar procedure to above and the method of [35], the previous formula can be written

$$M^k u^{k+1} = N^k u_n^k - R^k u_n^{k-1} + K^k + F^k. \quad (42)$$

Theorem 3.2 *The system (42) is convergent and $|u_n^k - \bar{u}_n^k| = O(O(\tau) + O(h))$ for any n, k where*

$$\tau^{2-\alpha} \leq 2\Gamma(2-\alpha).$$

Proof We can write Eq. (42) in the following form:

$$u^{k+1} = (M^k)^{-1} N^k u_n^k - (M^k)^{-1} R^k u_n^{k-1} + (M^k)^{-1} K^k + (M^k)^{-1} F^k.$$

Using the method in [36], we can obtain

$$\begin{aligned} \|(M^k)^{-1} N^k\| &\leq \|(M^k)^{-1}\| \|N^k\| \\ &= \frac{|2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)}| + |\frac{\tau^2}{h^2}| + |\frac{\tau^2}{h^2}|}{|1 + \frac{\tau^2}{h^2}|} = \frac{2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} + 2\frac{\tau^2}{h^2}}{1 + \frac{\tau^2}{h^2}} \\ &= 2 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} \leq 2, \\ \|(M^k)^{-1} R^k\| &\leq \|(M^k)^{-1}\| \|R^k\| = \frac{1 + \frac{\tau^2}{h^2}}{1 + \frac{\tau^2}{h^2}} = 1. \end{aligned}$$

Taking into consideration the previous theorem and the previous inequalities, the proof of this theorem is straightforward. $\square$

As a consequence of the above facts, we have the following theorem.

3.2 The stability of the methods

Theorem 3.3

- (i) Equation (35) is unconditionally stable.
- (ii) If $\frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} < 1$, then Eq. (38) is unconditionally stable.

Proof To prove this theorem, we shall consider the Fourier method of analyzing stability and the von Neumann criterion in the method [37]; a small error is given as follows:

$$Q_n^k = u_n^k - \bar{u}_n^k, \quad (43)$$

with

$$Q_n^k = \lambda^k e^{i\beta nh}, \quad i = \sqrt{-1}, \quad (44)$$

where λ is a complex and β is a real number. Due to the von Neumann criterion for stability, the condition $|\lambda| \leq 1$ has to be satisfied.

First, we shall present stability analysis for Eq. (35). From (43) and (44), the stability of IFDS (35) is

$$A\lambda^2 - 2B\lambda + C = 0, \quad (45)$$

where

$$A = 1, \quad B = 1 - \frac{1}{2} \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} - \frac{\tau^2}{2} - 2 \frac{\tau^2}{h^2} \sin^2 \frac{\beta h}{2}, \quad C = 1 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)}. \quad (46)$$

The condition $|\lambda| \leq 1$ is guaranteed if

$$-C \leq B \leq C \quad (47)$$

is satisfied. Considering (46) in the left-hand side of (47), the following form is obtained:

$$\left(\frac{1}{2} + \frac{2}{h^2} \sin^2 \frac{\beta h}{2} \right) \tau^2 + \frac{3}{2} \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} - 2 \leq 0, \quad (0 < \alpha < 1), \quad (48)$$

which is true if

$$\left(\frac{1}{2} + \frac{2}{h^2} \right) \tau^2 + \frac{3}{2} \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} - 2 \leq 0. \quad (49)$$

Considering (46) in the right-hand of (47), the following form is also obtained:

$$\frac{1}{2} \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} - \left(\frac{1}{2} + \frac{2}{h^2} \sin^2 \frac{\beta h}{2} \right) \tau^2 \leq 0, \quad (50)$$

which is true if

$$\frac{1}{2} \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} - \left(\frac{1}{2} + \frac{2}{h^2} \right) \tau^2 \leq 0. \quad (51)$$

From Eqs. (49) and (51), $\tau \leq 1$ is obtained. Thus, the scheme (35) is stable.

Second, we shall present stability analysis for Eq. (38). Using Eqs. (43) and (44), we get the following stability equation:

$$\overline{A} \lambda^2 - 2 \overline{B} \lambda + \overline{C} = 0, \quad (52)$$

where

$$\overline{A} = 1 + \frac{\tau^2}{h^2}, \quad \overline{B} = 1 - \frac{\tau^{2-\alpha}}{2\Gamma(2-\alpha)} - \frac{\tau^2}{2} \cos \beta h + \frac{\tau^2}{h^2} \cos \beta h, \quad \overline{C} = 1 + \frac{\tau^2}{h^2} - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)}.$$

Assume that $\cos \beta h > 0$, $1 - \frac{\tau^{2-\alpha}}{\Gamma(2-\alpha)} > 0$, if $\overline{A} > 0$, $\overline{C} > 0$ and $\overline{A} > \overline{C}$, then $\overline{B}^2 - \overline{A}\overline{C} \leq 0$ always holds. Hence the solutions

$$\lambda_{1,2} = \frac{\overline{B} \pm \sqrt{\overline{B}^2 - \overline{A}\overline{C}}}{\overline{A}}$$

of (52) satisfy $|\lambda_{1,2}| \leq 1$, which implies that the scheme (38) is stable. $\square$

Applications of this theorem are shown in the following numerical examples.

4 Numerical implementation

In this section, we shall present two examples which are applications of Theorem 3.3.

Example 1 Consider the following fractional telegraph partial differential equation for nonlocal boundary value problems:

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} - \frac{\partial^2 u(t,x)}{\partial x^2} + u(t,x) \\ \quad = \sin x(4t + 6\frac{t^{3-\alpha}}{\Gamma(4-\alpha)} - \frac{t^{3-\alpha}}{\Gamma(4-\alpha)} + 2(t^3 + 1)), & 0 < x < \pi, 0 < t < 1, 0 < \alpha < 1, \\ u(0,x) = \frac{1}{2}u(1,x) + \frac{1}{2}\sin x, \\ u_t(0,x) = \frac{1}{3}u_t(1,x) - \frac{5}{3}\sin x, & 0 \leq t \leq 1, \\ u(t,0) = u(t,\pi) = 0, & 0 \leq x \leq \pi. \end{cases} \quad (53)$$

Example 2 Consider the following fractional telegraph partial differential equation for a nonlocal boundary value problem:

$$\begin{cases} \frac{\partial^2 u(t,x)}{\partial t^2} + \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} - \frac{\partial^2 u(t,x)}{\partial x^2} + 6u(t,x) \\ \quad = 6(x^2 - 3x + 2)(t^3 + t + 1 + \frac{t^{3-\alpha}}{\Gamma(4-\alpha)}) \\ \quad \quad - 2(t^3 + 1), & 1 < x < 2, 0 < t < 1, 0 < \alpha < 1, \\ u(0,x) = \frac{1}{3}u(1,x) + \frac{1}{3}(x^2 - 3x + 2), \\ u_t(0,x) = \frac{1}{3}u_t(0,x) - (x^2 - 3x + 2), & 0 \leq t \leq 1, \\ u(t,1) = u(t,2) = 0, & 1 \leq x \leq 2. \end{cases} \quad (54)$$

Using the Laplace transform method, we see that the exact solution of the problem (53) is $u(t,x) = (t^3 - t + 1)\sin x$ and Eq. (54) is obtained: $u(t,x) = (t^3 + 1)(x^2 - 3x + 2)$. To solve problem (53) we applied two difference schemes method given by Eqs. (35) and (38). We use a procedure of modified Gauss elimination method for the difference equations (53) and (54). Then we calculate the maximum norm of error of the numerical solution as

$$\varepsilon = \max_{\substack{0 \leq n \leq M, \\ 0 \leq k \leq N}} |u(t,x) - u(t_k, x_n)|,$$

where $u_n^k = u(t_k, x_n)$ is the approximate solution and $u(t,x)$ is the exact solution. Table 1 shows the error analysis for the IFDS method and Table 2 presents the error analysis for the DFFDS method.

Remark 4.1 We note that in example (53), when we use IFDS, we see that the maxerror values are 0.0028(CPU : 2.171760), 0.0025(CPU : 0.801173), 0.0018(CPU : 0.79339) for $\alpha = 0.5, 0.1, 0.9$, respectively, at $N = 200$ and $M = 100$. Furthermore, in example (54) when we use DFFDS, we see that the maxerror values are 2.4246×10^{-4} (CPU : 0.759911), 3.1245×10^{-4} (CPU : 0.714135), 3.2523×10^{-4} (CPU : 0.707563) for $\alpha = 0.5, 0.1, 0.9$, respectively, at $N = 200$ and $M = 100$. From these computations and Table 1 and Table 2, one finds more effective results for $\tau = h^2$.

Table 1 First error analysis

$\tau = \frac{1}{N}, h = \frac{\pi}{M} (h = \frac{1}{M})$	α	IFDS method (53)		IFDS method (54)	
		maxerror	CPU time	maxerror	CPU time
$N = 100, M = 10$	0.1	0.0028	0.098552	6.2342×10^{-4}	0.053605
	0.5	0.0031	0.084086	6.4051×10^{-4}	0.052642
	0.9	0.0015	0.085630	4.6868×10^{-4}	0.054644
$N = 225, M = 15$	0.1	0.0012	0.250553	2.7629×10^{-4}	0.192028
	0.5	0.0014	0.207235	2.8823×10^{-4}	0.189281
	0.9	7.6017×10^{-4}	0.209145	2.1545×10^{-4}	0.190512
$N = 400; M = 20$	0.1	6.9287×10^{-4}	0.865346	1.5651×10^{-4}	0.781243
	0.5	8.0721×10^{-4}	0.803135	1.6457×10^{-4}	0.786518
	0.9	4.6721×10^{-4}	0.801428	1.2512×10^{-4}	0.796842
$N = 625, M = 25$	0.1	4.1218×10^{-4}	2.523352	1.0002×10^{-4}	2.602625
	0.5	5.2127×10^{-4}	2.663734	1.0570×10^{-4}	2.620778
	0.9	3.1690×10^{-4}	2.688309	8.1457×10^{-5}	2.620732
$N = 900, M = 30$	0.1	3.0755×10^{-4}	7.401635	6.9628×10^{-5}	7.366031
	0.5	3.6546×10^{-4}	7.372820	7.3836×10^{-5}	7.383805
	0.9	2.3088×10^{-4}	7.361036	5.7533×10^{-5}	7.390021
$N = 1600; M = 40$	0.1	1.1793×10^{-4}	41.706692	3.9181×10^{-5}	41.866478
	0.5	2.0757×10^{-4}	41.839637	4.1731×10^{-5}	41.602334
	0.9	1.3883×10^{-4}	41.987633	2.6572×10^{-5}	41.437862

Table 2 Second error analysis

$\tau = \frac{1}{N}, h = \frac{\pi}{M} (h = \frac{1}{M})$	α	DFFDS method (53)		DFFDS method (54)	
		maxerror	CPU time	maxerror	CPU time
$N = 100, M = 10$	0.1	0.0037	0.058788	9.3583×10^{-4}	0.053881
	0.5	0.0039	0.058378	9.4699×10^{-4}	0.053244
	0.9	0.0023	0.060307	7.6247×10^{-4}	0.053522
$N = 225, M = 15$	0.1	0.0016	0.209456	4.1057×10^{-4}	0.188579
	0.5	0.0019	0.211921	4.2000×10^{-4}	0.193100
	0.9	0.0011	0.210560	3.4152×10^{-4}	0.197266
$N = 400; M = 20$	0.1	9.0086×10^{-4}	0.842888	2.3168×10^{-4}	0.782940
	0.5	0.0011	0.833370	2.3836×10^{-4}	0.783129
	0.9	6.5204×10^{-4}	0.828719	1.9566×10^{-4}	0.792377
$N = 625, M = 25$	0.1	5.7434×10^{-4}	2.649214	1.4782×10^{-4}	2.573540
	0.5	7.0500×10^{-4}	2.483463	1.5264×10^{-4}	2.563704
	0.9	4.3467×10^{-4}	2.456029	1.2630×10^{-4}	2.574958
$N = 900, M = 30$	0.1	3.9925×10^{-4}	7.453583	1.0280×10^{-4}	7.316265
	0.5	4.9746×10^{-4}	6.869959	1.0641×10^{-4}	7.299569
	0.9	3.1281×10^{-4}	6.832467	8.8656×10^{-5}	7.294004
$N = 1600; M = 40$	0.1	2.2437×10^{-4}	41.880124	5.7796×10^{-5}	41.476774
	0.5	2.8486×10^{-4}	41.768273	6.0134×10^{-5}	41.488002
	0.9	1.8595×10^{-4}	41.786979	5.0550×10^{-5}	41.670310

5 Conclusion

In this paper, the solution of the fractional differential equation (6) is obtained. The stability estimates are shown for the abstract form of this differential equation. The first-order implicit and Dufort–Frankel difference schemes for Eq. (1) are constructed. Stability inequalities are proved for given difference schemes methods. With the help of the difference scheme method with the Caputo fractional definition, we construct and obtain the accuracy algorithms for solving partial differential equations. Approximate solutions for

numerical experiments are investigated by these methods. These results are compared with the exact solutions. MATLAB is used for all numerical calculations. Two different error analysis tables are obtained and compared. It was seen that the IFDS method is more effective than the DFFDS method in these numerical result tables.

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