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Positive solutions for periodic boundary value problem of fractional differential equation in Banach spaces

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Abstract

This paper discusses the existence and uniqueness of positive solutions for a periodic boundary value problem of a fractional differential equation in an ordered Banach space E. The existence and uniqueness results of solutions for the associated linear periodic boundary value problem of the fractional differential equation are established, and the norm estimation of resolvent operator is accurately obtained. With the aid of this estimation, the existence and uniqueness results of positive solutions are obtained by using a monotone iterative technique.

MSC: 26A33; 34B15; 34K30

Keywords: Fractional differential equation; Periodic boundary value problem; Existence and uniqueness

1 Introduction

Fractional derivatives and integrals are generalizations of traditional integer-order differential and integral calculus. The history of fractional calculus reaches back to the end of 17th century, this idea has been a subject of interest not only among mathematicians but also among physicists and engineers; see [1–17] and the references therein for more comments and citations. Since fractional-order models are more accurate than integer-order models, there is a higher degree of freedom in the fractional-order models. Furthermore, fractional derivatives provide an excellent instrument for the description of memory and hereditary properties of various materials and processes due to the existence of a memory term in the model. This memory term ensures the history and its impact to the present and future. Hence, fractional differential equations have been frequently used in economics, bioscience [18], system control theory [19], electrochemistry [20], diffusion process [21], signal and image processing, and so on. Recently, the monotone iterative technique in the presence of upper and lower solutions has appeared to be an important method for seeking solutions of nonlinear differential equations.

In [22], by means of the method of upper and lower solutions and the associated monotone iterative, the author proved the existence and uniqueness of the solution for the initial value problem

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t)), \quad t \in (0, T], \\ t^{1-\alpha}u(t)|_{t=0} = u_0, \end{cases}$$

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where $0 < T < +\infty$, $0 < \alpha \leq 1$ is a real number, D^α is the Riemann–Liouville fractional derivative. In 2010, Wei, Dong and Che [23], using the method of upper and lower solutions and its associated monotone iterative technique, proved the existence and uniqueness of the solution to the periodic boundary value problem for a class of fractional differential equations in real space $\mathbb{R}$.

Motivated by the aforementioned work, in this paper, we consider the existence and uniqueness of positive solutions for the following periodic boundary value problem (PBVP) of a nonlinear fractional differential equation in Banach space E:

$$\begin{cases} D^\alpha u(t) = f(t, u(t)), & t \in (0, \omega], \\ t^{1-\alpha} u(t)|_{t=0} = t^{1-\alpha} u(t)|_{t=\omega}, \end{cases} \tag{1.1}$$

where $0 < \alpha \leq 1$ is a real number, D^α is the Riemann–Liouville fractional derivative, and $f : [0, \omega] \times E \to E$ is a continuous function.

In the general case, the authors always established the upper and lower solution criteria under the assumption that for the studied problem there exist a couple of ordered lower and upper solutions, which is a strong assumption. The main purpose of this paper is to obtain the existence of positive solutions for the periodic boundary value problem of a nonlinear fractional differential equation directly from the characteristics of the nonlinear term $f(t, u)$, without assuming the existence of the upper and lower solutions. In this paper, we first of all derive the corresponding fractional Green's function. Then the corresponding linear periodic boundary value problem is reduced to an equivalent integral equation by using the Green's function. Finally, we derive the sufficient conditions for nonlinear function f under which for the periodic boundary value problem (1.1) there exists a unique positive solution by using a monotone iterative technique.

2 Preliminaries

For the convenience of the reader, first we present the necessary definitions and some basic results.

Definition 2.1 ([24]) The Riemann–Liouville fractional integral of order $\delta > 0$ of a function $u(t)$ is defined by

$$I_a^\delta u(t) = \frac{1}{\Gamma(\delta)} \int_a^t (t-s)^{\delta-1} u(s)\,ds, \quad t > a,$$

provided that the right-hand side is defined pointwise, where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2 ([24]) The Riemann–Liouville fractional derivative of order $\delta > 0$ of a function $u(t)$ is defined by

$$D_a^\delta u(t) = \frac{1}{\Gamma(n-\delta)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\delta-1} u(s)\,ds, \quad t > a,$$

where n is the smallest integer greater than or equal to δ, provided that the right-hand side is defined pointwise. In particular, if $\delta = n$, then $D_a^n u(t) = u^{(n)}(t)$.

The Mittag-Leffler function plays a similar role in fractional calculus to the exponential function in the theory of integer-order differential equation. Thus, the Mittag-Leffler function in two parameters is defined as [25]

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\quad z\in\mathbb{R},\alpha,\beta>0.$$

Note that the series converges uniformly in $\mathbb{R}$.

Lemma 2.3 ([26]) *Let* $0<\alpha\leq 1$, $\beta,\gamma>0$, $M\in\mathbb{R}$ *and* $a\in\mathbb{R}$. *Then*:

(i) $E_{\alpha,2\alpha}(Mt^\alpha)=M^{-1}t^{-\alpha}(E_{\alpha,\alpha}(Mt^\alpha)-1/\Gamma(\alpha))$;

(ii) $I_a^\gamma(t-a)^{\beta-1}E_{\alpha,\beta}(M(t-a)^\alpha)=(t-a)^{\beta+\gamma-1}E_{\alpha,\beta+\gamma}(M(t-a)^\alpha)$ *for* $t>a$;

(iii) $E_{\alpha,\alpha}(Mt^\alpha)$ *is decreasing in* t *for* $M<0$ *and increasing for* $M>0$ *for all* $t>0$.

Let $I=[0,\omega]$, we use $C(I,E)$ to denote the Banach space of all continuous function on interval I with the norm $\|u\|_C=\max_{t\in I}\|u(t)\|$. In our further consideration we utilize its generalization, namely, $C_{1-\alpha}(I,E)=\{u\in C(I,E)|t^{1-\alpha}u(t)\in C(I,E),t\in I\}$ equipped with the norm $\|u\|_{C_{1-\alpha}}=\|t^{1-\alpha}u(t)\|_C$. It is easy to verify that $C_{1-\alpha}(I,E)$ is a Banach space.

Obviously, the periodic boundary value problem (1.1) is equivalent to the following:

$$\begin{cases}D^\alpha u(t)+Mu(t)=f(t,u(t))+Mu(t), & t\in I,\\ t^{1-\alpha}u(t)|_{t=0}=t^{1-\alpha}u(t)|_{t=\omega},\end{cases}\tag{2.1}$$

where $M>0$ is real number.

To prove our main results, for any $h\in C_{1-\alpha}(I,E)$, we consider the periodic boundary value problem (PBVP) of the linear equation in E,

$$\begin{cases}D^\alpha u(t)+Mu(t)=h(t), & t\in I,\\ t^{1-\alpha}u(t)|_{t=0}=t^{1-\alpha}u(t)|_{t=\omega}.\end{cases}\tag{2.2}$$

Lemma 2.4 *For any* $h\in C_{1-\alpha}(I,E)$, *the linear periodic boundary value problem* (2.2) *has a unique solution* $u\in C_{1-\alpha}(I,E)$ *given by*

$$u(t)=\int_0^\omega G_{\alpha,M}(s,t)h(s)\,ds:=Ph(t),\tag{2.3}$$

where the Green's function is given by

$$G_{\alpha,M}(s,t)=\begin{cases}\frac{\omega^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)E_{\alpha,\alpha}(-M(\omega-s)^\alpha)}{(1-\Gamma(\alpha)E_{\alpha,\alpha}(-M\omega^\alpha))(\omega-s)^{1-\alpha}t^{1-\alpha}}+\frac{E_{\alpha,\alpha}(-M(t-s)^\alpha)}{(t-s)^{1-\alpha}}, & 0\leq s<t\leq\omega,\\ \frac{\omega^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)E_{\alpha,\alpha}(-M(\omega-s)^\alpha)}{(1-\Gamma(\alpha)E_{\alpha,\alpha}(-M\omega^\alpha))(\omega-s)^{1-\alpha}t^{1-\alpha}}, & 0<t\leq s<\omega.\end{cases}\tag{2.4}$$

Moreover, the operator $P:C_{1-\alpha}(I,E)\to C_{1-\alpha}(I,E)$ *is a linear bounded operator.*

Proof We can verify directly that the function $u\in C_{1-\alpha}(I,E)$ defined by Eq. (2.3) is a solution of the linear periodic boundary value problem (2.2). Next, we prove that u is unique as a solution. Assume that $u_1,u_2\in C_{1-\alpha}(I,E)$ are two solutions of the linear periodic boundary value problem (2.2). From (2.3) one can easily see that $u_1(t)=u_2(t)$ on I. Hence, the linear periodic boundary value problem (2.2) has a unique solution $u(t)$ given by (2.3). Obviously, $P: C_{1-\alpha}(I,E) \to C_{1-\alpha}(I,E)$ is a linear bounded operator. □

Remark 2.5 In Lemma 2.4, for all $t \in (0,\omega]$, $s \in [0,\omega)$, and for $M > 0$, we have $G_{\alpha,M}(s,t) > 0$. Hence, for any $h \in C^+_{1-\alpha}(I,E)$, periodic resolvent operator $P: C_{1-\alpha}(I,E) \to C_{1-\alpha}(I,E)$ is positive linear operator.

Lemma 2.6 *Let* $0 < \alpha \leq 1$ *and* $M > 0$, *then the Green's function* (2.4) *satisfies*

$$t^{1-\alpha}\int_0^\omega G_{\alpha,M}(s,t)s^{\alpha-1}\,ds = \frac{1}{M}, \quad t,s \in [0,\omega].$$

Proof Employing the results of Lemma 2.3, we have

$$\begin{aligned}
t^{1-\alpha}\int_0^\omega G_{\alpha,M}(s,t)s^{\alpha-1}\,ds &= \int_0^\omega \frac{\omega^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)E_{\alpha,\alpha}(-M(\omega-s)^\alpha)}{(1-\Gamma(\alpha)E_{\alpha,\alpha}(-M\omega^\alpha))(\omega-s)^{1-\alpha}}s^{\alpha-1}\,ds \\
&\quad + t^{1-\alpha}\int_0^t \frac{E_{\alpha,\alpha}(-M(t-s)^\alpha)}{(t-s)^{1-\alpha}}s^{\alpha-1}\,ds \\
&= \frac{\omega^{1-\alpha}\Gamma(\alpha)E_{\alpha,\alpha}(-Mt^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(-M\omega^\alpha)}\int_0^\omega \frac{E_{\alpha,\alpha}(-M(\omega-s)^\alpha)}{(\omega-s)^{1-\alpha}}s^{\alpha-1}\,ds \\
&\quad + t^{1-\alpha}\int_0^t \frac{E_{\alpha,\alpha}(-M(t-s)^\alpha)}{(t-s)^{1-\alpha}}s^{\alpha-1}\,ds \\
&= \frac{\omega^{1-\alpha}\Gamma(\alpha)^2E_{\alpha,\alpha}(-Mt^\alpha)}{1-\Gamma(\alpha)E_{\alpha,\alpha}(-M\omega^\alpha)}\bigl(I_0^\alpha t^{\alpha-1}E_{\alpha,\alpha}\bigl(-Mt^\alpha\bigr)\bigr)|_{t=\omega} \\
&\quad + \Gamma(\alpha)t^{1-\alpha}\bigl(I_0^\alpha t^{\alpha-1}E_{\alpha,\alpha}\bigl(-Mt^\alpha\bigr)\bigr) \\
&= \frac{-\Gamma(\alpha)^2E_{\alpha,\alpha}(-Mt^\alpha)}{M(1-\Gamma(\alpha)E_{\alpha,\alpha}(-M\omega^\alpha))}\left(E_{\alpha,\alpha}\bigl(-M\omega^\alpha\bigr)-\frac{1}{\Gamma(\alpha)}\right) \\
&\quad - \frac{\Gamma(\alpha)(E_{\alpha,\alpha}(-Mt^\alpha)-\frac{1}{\Gamma(\alpha)})}{M} \\
&= \frac{1}{M}.
\end{aligned}$$

This concludes the proof. □

Lemma 2.7 *For any* $h \in C_{1-\alpha}(I,E)$, *the norm of the solution operator* P *satisfies*

$$\|P\|_{C_{1-\alpha}} \leq \frac{1}{M}. \tag{2.5}$$

Proof For any $h \in C_{1-\alpha}(I,E)$, due to the definition of the operator P and Lemma 2.6, we get

$$\begin{aligned}
\bigl\|t^{1-\alpha}Ph(t)\bigr\| &= \left\|t^{1-\alpha}\int_0^\omega G_{\alpha,M}(s,t)h(s)\,ds\right\| \\
&= \left\|t^{1-\alpha}\int_0^\omega G_{\alpha,M}(s,t)s^{\alpha-1}s^{1-\alpha}h(s)\,ds\right\| \\
&\leq t^{1-\alpha}\int_0^\omega G_{\alpha,M}(s,t)s^{\alpha-1}\bigl\|s^{1-\alpha}h(s)\bigr\|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le t^{1-\alpha}\int_0^{\omega} G_{\alpha,M}(s,t)s^{\alpha-1}\,ds\|h\|_{C_{1-\alpha}}\\
&=\frac{1}{M}\|h\|_{C_{1-\alpha}},
\end{aligned}
$$

which means that $\|Ph\|_{C_{1-\alpha}} \le \frac{1}{M}\|h\|_{C_{1-\alpha}}$. Hence $\|P\|_{C_{1-\alpha}} \le \frac{1}{M}$, namely (2.5) holds. □

3 Main results

Theorem 3.1 *Let E be an ordered Banach space, whose positive cone K is normal, let $f : I \times E \to E$ be a continuous mapping which is ω-periodic in t, and for any $t \in I$, and $f(t,\theta) \ge \theta$. Suppose that the following conditions are satisfied*:

(H1) *There exists a constant $M > 0$, such that $\theta \le x_1 \le x_2$, we have*

$$f(t,x_2) - f(t,x_1) \ge -M(x_2 - x_1), \quad t \in I.$$

(H2) *There exists a constant $0 < L < M$, such that $\theta \le x_1 \le x_2$, we have*

$$f(t,x_2) - f(t,x_1) \le -L(x_2 - x_1), \quad t \in I.$$

Then the periodic boundary value problem (1.1) *has a unique positive solution.*

Proof Let the positive cone K be a normal with normal constant N in E; evidently, the closed convex of cone K_C in $C_{1-\alpha}(I,E)$ is deduced by cone K, namely

$$K_C = \{u \in C_{1-\alpha}(I,E) : u(t) \in K, t \in I\},$$

then K_C is also normal with the same normal constant N. Hence, $C_{1-\alpha}(I,E)$ is an order Banach space with the semi-order reduced by the normal cone K_C. In the following, E comes with partial order $\le$.

Denote $h_0(t) = f(t,\theta)$, then $h_0 \ge \theta$ and $h_0 \in C_{1-\alpha}(I,E)$, we consider the existence of solution for the linear periodic boundary value problem

$$\begin{cases} D^{\alpha}u(t) + Lu(t) = h_0(t), & t \in I, \\ t^{1-\alpha}u(t)|_{t=0} = t^{1-\alpha}u(t)|_{t=\omega}. \end{cases} \tag{3.1}$$

By Lemma 2.4, for $h_0 \in C_{1-\alpha}(I,E)$, we find that the linear periodic boundary value problem (3.1) has a unique solution $\omega_0 \in C_{1-\alpha}(I,E)$ with $\omega_0 \ge \theta$.

We reconsider the linear periodic boundary value problem (2.2). By Lemma 2.4, for $h \in C_{1-\alpha}(I,E)$, we see that the linear periodic boundary value problem (2.2) has a unique solution $u = Ph$, and $P : C_{1-\alpha}(I,E) \to C_{1-\alpha}(I,E)$ is a positive linear boundary operator with $\|P\| \le \frac{1}{M}$.

Set $F(u) = f(t,u) + Mu$, then $F : C_{1-\alpha}(I,E) \to C_{1-\alpha}(I,E)$ is a continuous mapping, and $F(\theta) = h_0 \ge \theta$. We defined an order interval $[\theta,\omega_0]$ in $C_{1-\alpha}(I,E)$, by condition (H1), we see that F is restricted: it is an increasing operator on $[\theta,\omega_0]$. Setting $\upsilon_0 = \theta$, we make the iterative scheme

$$\upsilon_n = P \circ F(\upsilon_{n-1}), \qquad \omega_n = P \circ F(\omega_{n-1}), \quad n = 1,2,\ldots. \tag{3.2}$$

Since ω_0 is a solution of problem (3.1), we add $M\omega_0 - L\omega_0$ on both sides of Eq. (3.1), thus, we see that ω_0 is also the corresponding solution of (2.2), when $h = h_0 + M\omega_0 - L\omega_0$, namely

$$\omega_0 = P(h_0 - L\omega_0 + M\omega_0). \tag{3.3}$$

In condition (H2), setting $x_1 = \theta$, $x_2 = \omega_0(t)$, we have

$$f\bigl(t, \omega_0(t)\bigr) \le h_0(t) - L\omega_0(t),$$

adding both sides of this inequality by $M\omega_0$, we can obtain

$$\theta \le F(\theta) \le F(\omega_0) \le h_0 - L\omega_0 + M\omega_0. \tag{3.4}$$

Acting on (3.4) by P, combining this with the positivity of P and (3.3), we have

$$\theta = \upsilon_0 \le \upsilon_1 \le \omega_1 \le \omega_0.$$

Since $P \circ F$ is an increasing operator on $[\theta, \omega_0]$, repeated acting to this inequality by $P \circ F$ means that

$$\theta \le \upsilon_1 \le \upsilon_2 \le \cdots \le \upsilon_n \le \cdots \le \omega_n \le \cdots \le \omega_2 \le \omega_1 \le \omega_0, \tag{3.5}$$

so we have

$$\begin{aligned} \theta \le \omega_n - \upsilon_n &= P\bigl(F(\omega_{n-1}) - F(\upsilon_{n-1})\bigr) \\ &= P\bigl(f(\cdot, \omega_{n-1}) - f(\cdot, \upsilon_{n-1}) + M(\omega_{n-1} - \upsilon_{n-1})\bigr) \\ &\le P\bigl((M-L)(\omega_{n-1} - \upsilon_{n-1})\bigr) \\ &= (M-L)P(\omega_{n-1} - \upsilon_{n-1}). \end{aligned}$$

Using the recursive approach, it follows that

$$\theta \le \omega_n - \upsilon_n \le (M-L)^n P^n(\omega_0 - \upsilon_0) = (M-L)^n P^n(\omega_0),$$

by this and the normality of the cone K_C, we conclude that

$$\begin{aligned} \|\omega_n - \upsilon_n\|_{C_{1-\alpha}} &\le N \bigl\| (M-L)^n P^n(\omega_0) \bigr\|_{C_{1-\alpha}} \\ &\le N(M-L)^n \bigl\| P^n \bigr\|_{C_{1-\alpha}} \|\omega_0\|_{C_{1-\alpha}}. \end{aligned} \tag{3.6}$$

On the other hand, since $0 < M - L < M$, choose $\epsilon > 0$, such that $M - L + \epsilon < M$. By (2.5), there exists $N_0 \in \mathbb{N}$, such that, for $n \ge N_0$,

$$\|P\|^n_{C_{1-\alpha}} \le \left(\frac{1}{M - L + \epsilon} \right)^n.$$

Hence, for $n \geq N_0$, from (3.6) it follows that

$$\begin{aligned}\|\omega_n - \upsilon_n\|_{C_{1-\alpha}} &\leq N(M-L)^n \|P\|^n_{C_{1-\alpha}} \|\omega_0\|_{C_{1-\alpha}} \\ &= N\|\omega_0\|_{C_{1-\alpha}} \big((M-L)\|P\|_{C_{1-\alpha}}\big)^n \\ &\leq N\|\omega_0\|_{C_{1-\alpha}} \left(\frac{M-L}{M-L+\epsilon}\right)^n \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.7}$$

By (3.5) and the above inequality, combining this with the principle of nested intervals, we can obtain the existence of a unique solution, $u^* \in \bigcap_{n=0}^{\infty}[\upsilon_n, \omega_n]$, such that

$$\lim_{n\to\infty} \upsilon_n = \lim_{n\to\infty} \omega_n = u^*,$$

consequently, letting $n \to \infty$ in (3.2), we see that $u^* = P \circ F(u^*)$. By the definition of P, it is easy to see that u^* is the corresponding solution of the linear periodic boundary value problem (2.2), when $h(t) = f(t, u^*(t)) + Mu^*(t)$, and therefore, it is a positive solution of the periodic boundary value problem (1.1).

Next, we prove the uniqueness. Let u_1, u_2 be two arbitrary positive solutions of the PBVPs (1.1). Let P and F is the operator of the M corresponding in the above existence argumentation, then the operator F is order increasing on $[\theta, u_i]$ $(i = 1, 2)$ of the order interval. In the iterative scheme of (3.2), the initial element ω_0 is replaced by u_i, and we repeat the above argumentation process. Since $P \circ F(u_i) = u_i$, we have $u_i = \omega_n$. By (3.7), letting $n \to \infty$, we obtain $\|u_i - \upsilon_n\|_{C_{1-\alpha}} \to 0$, which means that $u_1 = u_2 = \lim_{n\to\infty} \upsilon_n$. Therefore, PBVP (1.1) has a unique positive solution. □

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Competing interests
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Authors' contributions
Each of the authors contributed to each part of this study equally and approved the final version of this manuscript.

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