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Computing new solutions of algebro-geometric equation using the discrete inverse Sumudu transform

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Abstract

The discrete inverse Sumudu transform method is designed to solve ordinary differential equations and tested for an algebro-geometric equation. The two new sets of exact analytical and complex solutions are gotten through a discrete inverse Sumudu transform, and Maple complex graphs are drawn to show the new solution simulations in the complex plane which are compared to the existing solutions. The list of inverse Sumudu transforms is added in the sequel to strengthen the study.

MSC: 33E30; 44A10

Keywords: Inverse Sumudu transform; Complex solutions; Lommel S1 function; Struve function

1 Introduction

Algebro-geometric lattices were constructed using Rings and Fields to obtain the solutions of KdV equation and Toda equation in [1]. For solving sine-Gordon equation, Landau–Lifshitz equation, and reducing Abelian and hyperelliptic integrals, theta functions algebro-geometric principles were employed in [2]. Nonlinear integrable equations of mathematical physics, electrical systems were studied using the algebro-geometric method in [3]. KdV equation, Toda equation AKNS, and Hill's hierarchy were solved in [4, 5]. Soliton and quasi solutions of Dym type and water flow equations were solved for algebro-geometric solutions in [6]. Theta function notation of algebro-geometric solutions for Camassa–Holm equation and soliton solutions was given in [7–9]. Algebro-geometric Sturm–Liouville coefficients were calculated in [10]. Solutions without reflection for hierarchies of evolution equations were given in [11]. Endpoint classification of three forms of algebro-geometric equation (AGE) and eigenvalues of Sturm–Liouville differential equations were given in [12].

A Sumudu transform was applied for nonzero modulus Dixon elliptic functions to calculate their Hankel determinants in [13]. A discrete inverse Sumudu transform (DIST) method was proposed to solve ordinary differential equations to obtain their new exact solutions, and Whittaker and Zettl equations with a table of the inverse Sumudu transform of functions were solved in [14] as in [15], which gives special functions [16] as inverse Sumudu. Lane–Emden type differential equations were solved by the decomposition method using Sumudu in [17]. Fractional reaction-diffusion equation and delay differential equation were studied using the Sumudu transform in [18, 19]. Human relationships were studied numerically in [20]. Fuzzy differential equations were solved using a fuzzy Sumudu transform in [21]. Fractional differential equations, telegraph equations, and fuzzy differential equations were solved using the Sumudu transform respectively in [22–24]. Cattaneo–Vernotte with space fractional, time fractional, and space-time fractional equations were solved for integer values and rational values of φ in [25]. In [26] a reaction-diffusion equation with variable order fractionals was solved numerically by using a combined Adams and finite difference method where they took Liouville–Caputo and ABC (Atangana–Baleanu–Caputo) fractional derivatives. Liouville–Caputo, Caputo–Fabrizio–Caputo, and Mittag-Leffler kernel fractional derivatives were applied for the Bateman–Feshbach–Tikochinsky oscillator and Caldirola–Kanai oscillator and their individual behavior was studied in [27]. In [28] Atangana–Baleanu fractional derivatives were used for a nonlinear Bloch system, and the Adams–Moulton method was applied to solve it numerically. A homotopy perturbation transform method was applied to solve some nonlinear fractional differential equations in [29]. Apart from this, some of the very recent advancements in fractional calculus theories were given in [30, 31].

2 DIST method description

The Sumudu transform of the function $f(x)$ in the set $A = \{f(x) | \exists M, \tau_1, \tau_2 > 0, |f(x)| < Me^{\frac{|x|}{\tau_j}}, \text{if } x \in (-1)^j \times [0, \infty)\}$ is defined by the following integral equation:

$$\mathbb{S}\big[f(x)\big](u) = F(u) = \int_0^\infty e^{-x} f(ux)\,dx = \frac{1}{u}\int_0^\infty e^{-\frac{x}{u}} f(x)\,dx; \quad u \in (-\tau_1, \tau_2). \tag{1}$$

In the discrete definition, for the function $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)x^n}{n!}$,

$$\mathbb{S}\big[f(x)\big](u) = F(u) = \sum_{n=0}^{\infty} f^{(n)}(0)u^n. \tag{2}$$

The DIST definition is given by the following infinite series:

$$\mathbb{S}^{-1}\big[f(x)\big](w) = F_{-1}(w) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)w^n}{(n!)^2}, \tag{3}$$

where u is the (positive) Sumudu transform variable and w is the DIST (negative) Sumudu transform variable. For the functions from [15], the inverse Sumudu transform is calculated and provided in Table 1 and the corresponding special functions are defined in Table 2.

Theorem 1 *Let $F_{-1}(w)$ be the inverse Sumudu transform of $f(x)$.*

$$\mathbb{S}^{-1}\left[x^n \frac{d^n f(x)}{dx^n}\right] = w^n \frac{d^n F_{-1}(w)}{dw^n}. \tag{4}$$

$$\mathbb{S}^{-1}\left[x^n \frac{d^n x^n f(x)}{dx^n}\right] = w^n F_{-1}(w). \tag{5}$$

Table 1 Inverse Sumudu transform of elementary functions

S. No	$f(x)$	$\mathbb{S}^{-1}[f(x)] = F_{-1}(w)$
1	$\frac{1}{x+a}$	$\frac{1}{a}e^{-\frac{w}{a}}$
2	$\frac{1}{x-a}$	$-\frac{1}{a}e^{\frac{w}{a}}$
3	$\frac{1}{(x+a)^n}$	$\frac{e^{-\frac{w}{2a}}}{a^n w}[a(n+1)M_{n+1,\frac{1}{2}}(\frac{w}{a}) - (an-w)M_{n,\frac{1}{2}}(\frac{w}{a})]$
4	$\frac{x^n}{x+a}$	$\frac{(-1)^{-n}a^{n-1}e^{-\frac{w}{a}}}{n!}[\Gamma(n+1) - n\Gamma(n,-\frac{w}{a})]$
5	$\frac{Ax+Ba}{x^2-a^2}$	$\frac{1}{2a}[e^{-\frac{w}{a}}(A-B) - e^{\frac{w}{a}}(A+B)]$
6	$\frac{Ax+Ba}{x^2+a^2}$	$\frac{1}{2a}[e^{-\frac{iw}{a}}(iA+B) - e^{\frac{iw}{a}}(iA-B)]$
7	$\frac{1}{\sqrt{x+a}}$	$\frac{1}{\sqrt{a}}e^{-\frac{w}{2a}}\mathrm{I}_0(\frac{w}{2a})$
8	$\frac{1}{(x+a)^{\frac{3}{2}}}$	$\frac{e^{-\frac{w}{2a}}}{a^{\frac{5}{2}}}[(a-w)\mathrm{I}_0(\frac{w}{2a}) + w\mathrm{I}_1(\frac{w}{2a})]$
9	$\frac{\sqrt{x}}{x+a}$	$-\frac{e^{-\frac{w}{a}}}{\sqrt{a}}\mathrm{erf}(\frac{i\sqrt{w}}{\sqrt{a}})$
10	$\frac{\sqrt{x-b}}{x}$	$-\frac{ie^{\frac{w}{2b}}}{2\sqrt{b}}[\mathrm{I}_0(\frac{w}{2b}) - \mathrm{I}_1(\frac{w}{2b})]$
11	$\frac{1}{\sqrt{x}(x+a)}$	$\frac{e^{-\frac{w}{a}}}{a^2\sqrt{w\pi}}[ae^{\frac{w}{a}} + i\sqrt{aw\pi}\,\mathrm{erf}(\frac{i\sqrt{w}}{\sqrt{a}})]$
12	$\frac{1}{\sqrt{x(x+a)}}$	$\frac{e^{\frac{w}{a}}}{\sqrt{aw\pi}}$
13	$\frac{1}{x\sqrt{x-b}}$	$-\frac{ie^{\frac{w}{2b}}}{2b^{\frac{3}{2}}}[\mathrm{I}_0(\frac{w}{2b}) + \mathrm{I}_1(\frac{w}{2b})]$
14	$\frac{x}{\sqrt{x^2+a^2}}$	$-\frac{w\,\mathrm{csgn}(a)}{2a}[\mathrm{J}_0(\frac{w}{a})(\pi\mathbf{H}_1(\frac{w}{a}) - 2) - \pi\mathrm{J}_1(\frac{w}{a})\mathbf{H}_0(\frac{w}{a})]$
15	$\frac{x}{\sqrt{b^2-x^2}}$	$\frac{w\,\mathrm{csgn}(b)}{2b}[\mathrm{I}_0(\frac{w}{b})(\pi\mathbf{L}_1(\frac{w}{b}) + 2) - \pi\mathrm{I}_1(\frac{w}{b})\mathbf{L}_0(\frac{w}{b})]$
16	$\frac{x}{\sqrt{x^2-b^2}}$	$\frac{iw}{2b}[\mathrm{I}_0(\frac{w}{b})(\pi\mathbf{L}_1(\frac{w}{b}) + 2) - \pi\mathrm{I}_1(\frac{w}{b})\mathbf{L}_0(\frac{w}{b})]$
17	$\frac{1}{x+\sqrt{x^2+a^2}}$	$\frac{1}{2a^3}(2\,\mathrm{csgn}(a)\mathrm{J}_0(\frac{w}{a})(a^2 + w^2 - \frac{\pi w^2}{2}\mathbf{H}_1(\frac{w}{a})) - 2w(\mathrm{csgn}(a)\mathrm{J}_1(\frac{w}{a}) +$ $a(a - \frac{\pi w}{2}\mathbf{H}_0(\frac{w}{a}))))$
18	$(x+\sqrt{1+x^2})^n + (x-\sqrt{1+x^2})^n$	$(e^{in\pi}+1)_2F_3(\frac{n}{2},\frac{n}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4}) + nw(e^{in\pi} -$ $1)_2F_3(\frac{1+n}{2},\frac{1-n}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
19	$\frac{(x+\sqrt{1+x^2})^n}{\sqrt{1+x^2}}$	${}_2F_3(\frac{1-n}{2},\frac{1+n}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4}) + nw_2F_3(1-\frac{n}{2},1+\frac{n}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
20	$\frac{(x-\sqrt{1+x^2})^n}{\sqrt{1+x^2}}$	$(e^{in\pi})_2F_3(\frac{1-n}{2},\frac{1+n}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4}) - nw_2F_3(1-\frac{n}{2},1+\frac{n}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
21	$\frac{(x-b)^v}{x}$	$(-1)^v b^{v-1}[\mathrm{L}_v(\frac{w}{b}) - \mathrm{L}_v^1(\frac{w}{b})]$
22	$\frac{x^{v-1}}{1+x^2}$	$\frac{\sqrt{w}S1_{\frac{3}{2},\frac{1}{2}}(w)+v(v+1)w^{v-1}-w^{v+1}}{(v+1)!}$
23	$(1+x^2)^{v-\frac{1}{2}}$	${}_1F_2(\frac{1}{2}-v;\frac{1}{2},1;-\frac{w^2}{4})$
24	$(x^2-b^2)^{v-\frac{1}{2}}$	$(-b)^{v-\frac{1}{2}}{}_1F_2(\frac{1}{2}-v;\frac{1}{2},1;\frac{w^2}{4b^2})$
25	$(b^2-x^2)^{v-\frac{1}{2}}$	$\frac{2(b^2)^{v-\frac{1}{2}}}{b^2(2v+1)}[w^2\mathrm{L}^1_{v+\frac{1}{2}}(\frac{w^2}{b^2}) + ((v+\frac{1}{2})b^2 - w^2)\mathrm{L}_{v+\frac{1}{2}}(\frac{w^2}{b^2})]$
26	$x^{v-1}(x+a)^{\frac{1}{2}-v}$	$\frac{a^{\frac{1}{2}-v}w^{v-1}e^{-\frac{w}{2a}}(\frac{w}{a})^{-\frac{v}{2}}}{av(v-1)!(v+1)w}[a(av+w)(v+\frac{3}{2})\mathrm{M}_{\frac{v+3}{2},\frac{v+1}{2}}(\frac{w}{a}) +$ $(\frac{2w^2-2wa-va^2}{2})\mathrm{M}_{\frac{v+1}{2},\frac{v+1}{2}}(\frac{w}{a})]$
27	$x^{v-1}(x+a)^{-\frac{1}{2}-v}$	$\frac{e^{-\frac{w}{2a}}(\frac{w}{a})^{-\frac{v}{2}}a^{-(v+\frac{1}{2})}w^{v-1}}{vw(v+1)(v-1)!}[-(\frac{w+av}{2})\mathrm{M}_{\frac{v+1}{2},\frac{v+1}{2}}(\frac{w}{a}) + (v +$ $\frac{3}{2})av\mathrm{M}_{\frac{v+3}{2},\frac{v+1}{2}}(\frac{w}{a})]$
28	$(\sqrt{x^2+1}+x)^v$	${}_2F_3(\frac{v}{2},-\frac{v}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4}) + vw_2F_3(\frac{1+v}{2},\frac{1-v}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
29	$(\sqrt{x^2+1}-x)^v$	${}_2F_3(\frac{v}{2},-\frac{v}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4}) - vw_2F_3(\frac{1+v}{2},\frac{1-v}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
30	$\frac{(\sqrt{x^2+1}+x)^v}{\sqrt{x^2+1}}$	${}_2F_3(\frac{1-v}{2},\frac{1+v}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4}) + vw_2F_3(1-\frac{v}{2},1+\frac{v}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
31	$\frac{(\sqrt{x^2+1}-x)^v}{\sqrt{x^2+1}}$	${}_2F_3(\frac{1-v}{2},\frac{1+v}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4}) - vw_2F_3(1-\frac{v}{2},1+\frac{v}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
32	$\frac{(\sqrt{x^2-1}+x)^v+(\sqrt{x^2-1}+x)^{-v}}{\sqrt{x^2-1}}$	$-\frac{(e^{i\pi v}+1)}{i^{v-1}}{}_2F_3(\frac{1-v}{2},\frac{1+v}{2};\frac{1}{2},\frac{1}{2},1;\frac{w^2}{4}) - \frac{(e^{i\pi v}-1)vw}{i^v}{}_2F_3(1-\frac{v}{2},1+$ $\frac{v}{2};1,\frac{3}{2},\frac{3}{2};\frac{w^2}{4})$
33	$(\sqrt{x+2a}+\sqrt{x})^{2v} - (\sqrt{x+2a}-\sqrt{x})^{2v}$	$\frac{4v\sqrt{w}2^{v+\frac{1}{2}}a^{v-\frac{1}{2}}}{\sqrt{\pi}}{}_2F_2(\frac{1}{2}+v,\frac{1}{2}-v;\frac{3}{2},\frac{3}{2};-\frac{w}{2a})$
34	$(\sqrt{x+b}+\sqrt{x-b})^{2v} - (\sqrt{x+b}-\sqrt{x-b})^{2v}$	$-ivwb^{v-1}((1+i)^{2v} + (1-i)^{2v})_2F_3(\frac{1+v}{2},\frac{1-v}{2};1,\frac{3}{2},\frac{3}{2};\frac{w^2}{4b^2}) - b^v(-(1+$ $i)^{2v} + (1-i)^{2v})_2F_3(\frac{v}{2},-\frac{v}{2};\frac{1}{2},\frac{1}{2},1;,\frac{w^2}{4b^2})$

Table 1 ((*Continued*))

S. No	$f(x)$	$\mathbb{S}^{-1}[f(x)] = F_{-1}(w)$
35	$\frac{(2a)^{2v}(x+\sqrt{x^2+4a^2})^{2v}}{\sqrt{x^3+4a^2x}}$	$\frac{1}{24}(\mathrm{csgn}^{2v+1}(a)w^{\frac{3}{2}}a^{4v-3}16^v(v^2-\frac{1}{4})) \times {}_2F_5(\frac{3}{2}-v, \frac{3}{2}+v; \frac{5}{4}, \frac{3}{2}, \frac{3}{2}, \frac{7}{4}, 2; -\frac{w^2}{1024a^2}) + \frac{1}{2}(\mathrm{csgn}^{2v}(a)va^{4v+2}16^v\sqrt{w}) \times {}_2F_5(1+v, 1-v; \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}; -\frac{w^2}{1024a^2})$
36	$\frac{(x+\sqrt{x^2-1})^{2v}+(x-\sqrt{x^2-1})^{2v}}{\sqrt{x}\sqrt{x^2-1}}$	$\frac{8iv\sqrt{w}}{\sqrt{\pi}\sin(\pi v)}(\cos^2(\pi v)-1){}_2F_3(1+v, 1-v; \frac{3}{4}, \frac{5}{4}, \frac{3}{2}; \frac{w^2}{4}) - \frac{2i\cos(\pi v)}{\sqrt{\pi w}}{}_2F_3(\frac{1}{2}+v, \frac{1}{2}-v; \frac{1}{4}, \frac{1}{1}, \frac{3}{4}; \frac{w^2}{4})$
37	e^{-ax}	$\mathrm{J}_0(2\sqrt{aw})$
38	xe^{-ax}	$\frac{w\mathrm{J}_1(2\sqrt{aw})}{\sqrt{aw}}$
39	$x^{v-1}e^{-ax}$	$\frac{w^{v-1}\Gamma(v)\mathrm{J}_{v-1}(2\sqrt{aw})}{(v-1)!(aw)^{\frac{v-1}{2}}}$
40	$\frac{e^{-ax}-e^{-bx}}{x}$	$\frac{b\sqrt{a}\mathrm{J}_1(2\sqrt{bw})-a\sqrt{b}\mathrm{J}_1(2\sqrt{aw})}{\sqrt{abw}}$
41	$\frac{(1-e^{-ax})^2}{x^2}$	$\frac{-2a(\mathrm{J}_2(2\sqrt{aw})-\mathrm{J}_2(2\sqrt{2aw}))}{w}$
42	$\frac{1}{x}-\frac{(x+2)(1-e^{-x})}{2x^2}$	$\frac{w\mathrm{I}_3(2\sqrt{-w})}{2(-w)^{\frac{3}{2}}}$
43	$e^{-\frac{x^2}{4a}}$	${}_0F_2(;\frac{1}{2}, 1; -\frac{w^2}{16a})$
44	$xe^{-\frac{x^2}{4a}}$	$w{}_0F_2(;1, \frac{3}{2}; -\frac{w^2}{16a})$
45	$\frac{e^{-\frac{x^2}{4a}}}{\sqrt{x}}$	$\frac{1}{\sqrt{\pi w}}{}_0F_2(;\frac{1}{4}, \frac{3}{4}; -\frac{w^2}{16a})$
46	$x^{v-1}e^{-\frac{x^2}{8a}}$	$\frac{w^{v-1}}{(v-1)!}{}_0F_2(;\frac{v}{2}, \frac{v+1}{2}; -\frac{w^2}{32a})$
47	$e^{-\frac{x}{4a}}$	$\mathrm{I}_0(\sqrt{-\frac{w}{a}})$
48	$\sqrt{x}e^{-\frac{x}{4a}}$	$2\sqrt{\frac{a}{\pi}}\sin(\sqrt{\frac{w}{a}})$
49	$\frac{e^{-\frac{x}{4a}}}{\sqrt{x}}$	$\frac{\cos(\sqrt{\frac{w}{a}})}{\sqrt{\pi w}}$
50	$\frac{e^{-\frac{x}{4a}}}{x^{\frac{3}{2}}}$	$-\frac{(\sqrt{aw}\sin(\sqrt{\frac{w}{a}})+a\cos(\sqrt{\frac{w}{a}}))}{2a\sqrt{\pi}w^{\frac{3}{2}}}$
51	$x^{v-1}e^{-\frac{x}{4a}}$	$\frac{(2w)^{v-1}\Gamma(v)(-\frac{w}{a})^{\frac{1-v}{2}}\mathrm{I}_{v-1}(\sqrt{-\frac{w}{a}})}{(v-1)!}$
52	$\frac{(e^{-\frac{x}{4a}}-1)}{\sqrt{x}}$	$\frac{1}{\sqrt{w\pi}}[\cos(\sqrt{\frac{w}{a}})-1]$
53	$e^{-2\sqrt{a}\sqrt{x}}$	$-\frac{4\sqrt{aw}}{\sqrt{\pi}}{}_0F_2(;\frac{3}{2}, \frac{3}{2};, aw) + {}_0F_2(;\frac{1}{2}, 1;, aw)$
54	$\sqrt{x}e^{-2\sqrt{a}\sqrt{x}}$	$\frac{2\sqrt{w}}{\sqrt{\pi}}{}_0F_2(;\frac{1}{2}, \frac{3}{2};, aw) - 2w\sqrt{a}{}_0F_2(;\frac{3}{2}, 2;, aw)$
55	$\frac{e^{-2\sqrt{a}\sqrt{x}}}{\sqrt{x}}$	$\frac{1}{\sqrt{w\pi}}{}_0F_2(;\frac{1}{2}, \frac{1}{2};, aw) - 2\sqrt{a}{}_0F_2(;1, \frac{3}{2};, aw)$
56	$\frac{e^{-2\sqrt{a}\sqrt{x}}}{\sqrt{2x}}$	$\frac{1}{\sqrt{2w\pi}}{}_0F_2(;\frac{1}{2}, \frac{1}{2};, aw) - \sqrt{2a}{}_0F_2(;1, \frac{3}{2};, aw)$
57	$\log(1+ax)$	$\mathrm{Ei}_1(aw) + \ln(aw) + \gamma$
58	$\log(x+a)$	$\mathrm{Ei}_1(\frac{w}{a}) + \ln(w) + \gamma$
59	$\log(x^2-a^2)$	$2[\ln(w) + \gamma - \mathrm{Chi}(\frac{w}{a})] + i\pi$
60	$\log(x^2+a^2)$	$2[\ln(w) + \gamma - \mathrm{Ci}(\frac{w}{a})]$
61	$\frac{\log(x^2+a^2)-\log(a^2)}{x}$	$\frac{2}{w}[1-\cos(\frac{w}{a})]$
62	$\log(\frac{\sqrt{x+b}+\sqrt{x-b}}{\sqrt{2}\sqrt{b}})$	$\frac{1}{2b}[-iw{}_2F_3(\frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}, \frac{3}{2}; \frac{w^2}{4b^2}) - b(\ln(2) + 2\ln(i+1))]$
63	$\log(\frac{\sqrt{x}+\sqrt{x+2a}}{\sqrt{2}\sqrt{a}})$	$\sqrt{\frac{2w}{a\pi}}{}_2F_2(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{w}{2a})$
64	$\log(\frac{\sqrt{x+ib}+\sqrt{x-ib}}{\sqrt{2}\sqrt{b}})$	$\frac{1}{2b}[w{}_2F_3(\frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}, \frac{3}{2}; -\frac{w^2}{4b^2} + 2i\pi b)]$
65	$\sin(ax)$	$\mathrm{bei}_0(2\sqrt{aw})$
66	$x\sin(ax)$	$-\frac{\sqrt{aw}}{2}[\mathrm{ber}_1(2\sqrt{w}) + \mathrm{bei}_1(2\sqrt{w})]$
67	$x^n\sin(ax)$	$\frac{aw^{n+1}}{(n+1)!}{}_0F_3(;\frac{3}{2}, \frac{n}{2}+1, \frac{n+3}{2}; -\frac{(aw)^2}{16})$
68	$x^{v-1}\sin(ax)$	$\frac{aw^v}{v!}{}_0F_3(;\frac{3}{2}, \frac{v}{2}+1, \frac{v+1}{2}; -\frac{(aw)^2}{16})$
69	$\frac{\sin(ax)}{x}$	$-\sqrt{\frac{a}{2w}}[\mathrm{ber}_1(2\sqrt{aw}) - \mathrm{bei}_1(2\sqrt{aw})]$
70	$\frac{\sin^2(ax)}{x}$	$-\sqrt{\frac{a}{4w}}[\mathrm{ber}_1(2\sqrt{2aw}) + \mathrm{bei}_1(2\sqrt{2aw})]$
71	$\frac{\sin^3(ax)}{x}$	$-\frac{3\sqrt{a}}{8\sqrt{3iw}}[-\sqrt{3}(\mathrm{I}_1(2(-1)^{\frac{1}{4}}\sqrt{aw}) + \mathrm{J}_1(2(-1)^{\frac{1}{4}}\sqrt{aw})) + \mathrm{I}_1(2\sqrt{3iaw}) + \mathrm{J}_1(2\sqrt{3iaw})]$
72	$\frac{\sin^2(ax)}{x^2}$	$\frac{\sqrt{a}}{2w^{\frac{3}{2}}}[\mathrm{ber}_1(2\sqrt{2aw}) + \mathrm{bei}_1(2\sqrt{2aw}) + 2\sqrt{aw}\,\mathrm{bei}_0(2\sqrt{2aw})]$
73	$\sin(x^2)$	$\frac{w^2}{2}{}_0F_5(;\frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}, \frac{3}{2}; -\frac{w^2}{1024})$
74	$\frac{\sin(x^2)}{x}$	$w{}_0F_5(;\frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}; -\frac{w^2}{1024})$

Table 1 ((*Continued*))

S. No	$f(x)$	$\mathbb{S}^{-1}[f(x)] = F_{-1}(w)$
75	$\frac{\sin(x^2)}{x^2}$	$w_0F_5(;\frac{1}{4},\frac{1}{2},\frac{3}{2},1,\frac{3}{2};-\frac{w^2}{1024})$
76	$\sin(2\sqrt{a}\sqrt{x})$	$\frac{4\sqrt{aw}}{\sqrt{\pi}}\,{}_0F_2(;\frac{3}{2},\frac{3}{2};-aw)$
77	$x^n\sin(2\sqrt{a}\sqrt{x})$	$\frac{2\sqrt{a}w^{n+\frac{1}{2}}}{(n+\frac{1}{2})!}\,{}_0F_2(;\frac{3}{2},\frac{3}{2}+n;-aw)$
78	$\frac{\sin(2\sqrt{a}\sqrt{x})}{x}$	$\frac{2\sqrt{a}}{\sqrt{w\pi}}\,{}_0F_2(;\frac{1}{2},\frac{3}{2};-aw)$
79	$\sqrt{x}\sin(2\sqrt{a}\sqrt{x})$	$2w\sqrt{a}{}_0F_2(;\frac{3}{2},2;-aw)$
80	$\frac{\sin(2\sqrt{a}\sqrt{x})}{\sqrt{x}}$	$2\sqrt{a}{}_0F_2(;1,\frac{3}{2};-aw)$
81	$x^{v-1}\sin(2\sqrt{a}\sqrt{x})$	$\frac{2\sqrt{a}w^{v-\frac{1}{2}}}{(v-\frac{1}{2})!}\,{}_0F_2(;\frac{3}{2},v+\frac{1}{2};-aw)$
82	$\cos(ax)$	$\mathrm{ber}_0(2\sqrt{aw})$
83	$\cos^2(ax)$	$1-\frac{(aw)^2}{2}\,{}_1F_4(1;\frac{3}{2},\frac{3}{2},2,2;-\frac{(aw)^2}{4})$
84	$\cos^3(ax)$	$\frac{3}{8}[\mathrm{I}_0(2(-1)^{\frac{1}{4}}\sqrt{aw})+\mathrm{J}_0(2(-1)^{\frac{1}{4}}\sqrt{aw})]+\frac{1}{8}[\mathrm{I}_0(2\sqrt{3iaw})+\mathrm{J}_0(2\sqrt{3iaw})]$
85	$x\cos(x)$	$-\sqrt{\frac{w}{2}}[\mathrm{ber}_1(2\sqrt{w})-\mathrm{bei}_1(2\sqrt{w})]$
86	$x^n\cos(ax)$	$\frac{w^n}{n!}\,{}_0F_3(;\frac{1}{2},\frac{n}{2}+1,\frac{n+1}{2};-\frac{(aw)^2}{16})$
87	$x^{v-1}\cos(ax)$	$\frac{w^{v-1}}{(v-1)!}\,{}_0F_3(;\frac{1}{2},\frac{v}{2},\frac{v+1}{2};-\frac{(aw)^2}{16})$
88	$\frac{1-\cos(ax)}{x}$	$-\sqrt{\frac{a}{2w}}[\mathrm{ber}_1(2\sqrt{aw})+\mathrm{bei}_1(2\sqrt{aw})]$
89	$\cos(x^2)$	${}_0F_5(;\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},1;-\frac{w^2}{1024})$
90	$\cos(2\sqrt{a}\sqrt{x})$	${}_0F_2(;\frac{1}{2},1;-aw)$
91	$x\sqrt{x}\cos(2\sqrt{a}\sqrt{x})$	$\frac{2\sqrt{w}}{\sqrt{\pi}}\,{}_0F_2(;\frac{1}{2},\frac{3}{2};-aw)$
92	$\frac{\cos(2\sqrt{a}\sqrt{x})}{\sqrt{x}}$	$\frac{1}{\sqrt{w\pi}}\,{}_0F_2(;\frac{1}{2},\frac{1}{2};-aw)$
93	$x^{n-\frac{1}{2}}\cos(2\sqrt{a}\sqrt{x})$	$\frac{w^{n-\frac{1}{2}}}{(n-\frac{1}{2})!}\,{}_0F_2(;\frac{1}{2},n+\frac{1}{2};-aw)$
94	$x^{v-1}\cos(2\sqrt{a}\sqrt{x})$	$\frac{w^{v-1}}{(v-1)!}\,{}_0F_2(;\frac{1}{2},v;-aw)$
95	$\sin(ax)\sin(bx)$	$\frac{1}{4}[\mathrm{I}_0(2\sqrt{iw(a-b)})+\mathrm{I}_0(2\sqrt{-iw(a-b)})]-\frac{1}{4}[\mathrm{I}_0(2\sqrt{iw(a+b)})+\mathrm{I}_0(2\sqrt{-iw(a+b)})]$
96	$\cos(ax)\sin(bx)$	$\frac{1}{4}[i\mathrm{I}_0(2\sqrt{iw(a-b)})-i\mathrm{I}_0(2\sqrt{-iw(a-b)})]+\frac{1}{4}[i\mathrm{I}_0(2\sqrt{-iw(a+b)})-\mathrm{I}_0(2\sqrt{iw(a+b)})]$
97	$\cos(ax)\cos(bx)$	$\frac{1}{4}[\mathrm{I}_0(2\sqrt{iw(a+b)})+\mathrm{I}_0(2\sqrt{-iw(a+b)})]+\frac{1}{4}[\mathrm{I}_0(2\sqrt{iw(a-b)})+\mathrm{I}_0(2\sqrt{-iw(a-b)})]$
98	$\frac{2ax\sin(ax)\cos(ax)-\sin^2(ax)}{x^2}$	$\frac{1}{2\sqrt{a}w^{\frac{5}{2}}}[aw(2aw-1)\mathrm{bei}_1(2\sqrt{2aw})-aw(2aw+1)\mathrm{ber}_1(2\sqrt{2aw})-$ $2(aw)^{\frac{3}{2}}\mathrm{bei}_0(2\sqrt{2aw})]$
99	$\frac{ax\cos(ax)-\sin(ax)}{x^2}$	$\frac{1}{2a^{\frac{3}{2}}w^{\frac{7}{2}}}[\sqrt{2}(aw)^2(aw+1)\mathrm{bei}_1(2\sqrt{aw})+\sqrt{2}(aw)^2(aw-1)\mathrm{ber}_1(2\sqrt{aw})-$ $2(aw)^{\frac{5}{2}}\mathrm{ber}_0(2\sqrt{aw})]$
100	$\arcsin(x)$	$w_2F_3(\frac{1}{2},\frac{1}{2};1,\frac{3}{2},\frac{3}{2};\frac{w^2}{4})$
101	$x\arcsin(x)$	$\frac{w^2}{2}\,{}_2F_3(\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2},2;\frac{w^2}{4})$
102	$\arctan(\frac{x}{a})$	$\mathrm{Si}(\frac{W}{a})$
103	$\cot^{-1}(\frac{x}{a})$	$\frac{\pi}{2}-\mathrm{Si}(\frac{W}{a})$
104	$x\arctan(\frac{x}{a})$	$a[\cos(\frac{x}{a})-1]+w\mathrm{Si}(\frac{W}{a})$
105	$x\cot^{-1}(\frac{x}{a})$	$a[1-\cos(\frac{x}{a})]-w\mathrm{Si}(\frac{W}{a})+\frac{\pi w}{2}$
106	$\sinh(ax)$	$\frac{1}{2}[\mathrm{I}_0(2\sqrt{aw})-\mathrm{J}_0(2\sqrt{aw})]$
107	$\cosh(ax)$	$\frac{1}{2}[\mathrm{I}_0(2\sqrt{aw})+\mathrm{J}_0(2\sqrt{aw})]$
108	$\sinh^2(ax)$	$\frac{(aw)^2}{2}\,{}_1F_4(1;\frac{3}{2},\frac{3}{2},2,2;\frac{(aw)^2}{4})$
109	$\cosh^2(ax)$	$1+\frac{(aw)^2}{2}\,{}_1F_4(1;\frac{3}{2},\frac{3}{2},2,2;\frac{(aw)^2}{4})$
110	$\frac{2\sinh(ax)}{x}$	$\sqrt{\frac{a}{w}}[\mathrm{I}_1(2\sqrt{aw})+\mathrm{J}_1(2\sqrt{aw})]$
111	$\frac{2\cosh(ax)}{x}$	$\sqrt{\frac{a}{w}}[\mathrm{I}_1(2\sqrt{aw})-\mathrm{J}_1(2\sqrt{aw})]$
112	$x^{v-1}\sinh(ax)$	$\frac{w^{\frac{v}{2}-1}}{2a^{\frac{v}{2}}}[v\mathrm{I}_v(2\sqrt{aw})+\sqrt{aw}\mathrm{I}_{v+1}(2\sqrt{aw})-v\mathrm{J}_v(2\sqrt{aw})+\sqrt{aw}\mathrm{J}_{v+1}(2\sqrt{aw})]$
113	$x^{v-1}\cosh(ax)$	$\frac{w^{\frac{v-3}{2}}}{2a^{\frac{v+1}{2}}}[v\sqrt{aw}\mathrm{I}_v(2\sqrt{aw})+aw\mathrm{I}_{v+1}(2\sqrt{aw})-v\sqrt{aw}\mathrm{J}_v(2\sqrt{aw})-aw\mathrm{J}_{v+1}(2\sqrt{aw})]$
114	$\sin(ax)\sinh(ax)$	$\frac{(aw)^2}{2}\,{}_0F_7(;\frac{3}{4},\frac{3}{4},1,\frac{5}{4},\frac{5}{4},\frac{3}{2},\frac{3}{2};-\frac{(aw)^2}{16{,}384})$
115	$\cos(ax)\sinh(ax)aw$	${}_0F_7(;\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},1,\frac{5}{4},\frac{5}{4};-\frac{(aw)^2}{16{,}384})-\frac{(aw)^3}{18}\,{}_0F_7(;1,\frac{5}{4},\frac{5}{4},\frac{3}{2},\frac{3}{2},\frac{7}{4},\frac{7}{4};-\frac{(aw)^2}{16{,}384})$
116	$\sin(ax)\cosh(ax)$	$aw_0F_7(;\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},1,\frac{5}{4},\frac{5}{4};-\frac{(aw)^2}{16{,}384})+\frac{(aw)^3}{18}\,{}_0F_7(;1,\frac{5}{4},\frac{5}{4},\frac{3}{2},\frac{3}{2},\frac{7}{4},\frac{7}{4};-\frac{(aw)^2}{16{,}384})$
117	$\cos(ax)\cosh(ax)$	${}_0F_7(;\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4},1;-\frac{(aw)^2}{16{,}384})$

Table 1 (*(Continued)*)

S. No	$f(x)$	$\mathbb{S}^{-1}[f(x)] = F_{-1}(w)$
118	$\sinh(2\sqrt{a}\sqrt{x})$	$\frac{4\sqrt{aw}}{\sqrt{\pi}}{}_0F_2(;\frac{3}{2},\frac{3}{2};aw)$
119	$\cosh(2\sqrt{a}\sqrt{x})$	${}_0F_2(;\frac{1}{2},1;aw)$
120	$\sqrt{x}\sinh(2\sqrt{a}\sqrt{x})$	$2\sqrt{a}w_0F_2(;\frac{3}{2},2;aw)$
121	$\sqrt{x}\cosh(2\sqrt{a}\sqrt{x})$	$\frac{2\sqrt{w}}{\sqrt{\pi}}{}_0F_2(;\frac{1}{2},\frac{3}{2};aw)$
122	$\frac{\sinh(2\sqrt{a}\sqrt{x})}{\sqrt{x}}$	$2\sqrt{a}{}_0F_2(;1,\frac{3}{2};aw)$
123	$\frac{\cosh(2\sqrt{a}\sqrt{x})}{\sqrt{x}}$	$\frac{1}{\sqrt{w\pi}}{}_0F_2(;\frac{1}{2},\frac{1}{2};aw)$
124	$\frac{\sinh^2(2\sqrt{a}\sqrt{x})}{\sqrt{x}}$	$\frac{2a\sqrt{w}}{\sqrt{\pi}}{}_1F_3(1;\frac{3}{2},\frac{3}{2},2;aw)$
125	$\frac{\cosh^2(2\sqrt{a}\sqrt{x})}{\sqrt{x}}$	$\frac{1}{\sqrt{\pi w}}[1+2aw_1F_3(1;\frac{3}{2},\frac{3}{2},2;aw)]$
126	$\frac{\sinh(2\sqrt{a}\sqrt{x})}{x^{\frac{3}{4}}}$	$\frac{\sqrt{8a}}{w^{\frac{1}{4}}(-\frac{1}{4})!}{}_0F_2(;\frac{3}{4},\frac{3}{2};2aw)$
127	$\frac{\cosh(2\sqrt{a}\sqrt{x})}{x^{\frac{3}{4}}}$	$\frac{1}{w^{\frac{3}{4}}(-\frac{1}{4})!}{}_0F_2(;\frac{1}{2},\frac{3}{4};2aw)$
128	$x^{v-1}\sinh(2\sqrt{a}\sqrt{x})$	$\frac{\sqrt{2aw}^{v-\frac{1}{2}}}{(v-\frac{1}{2})!}{}_0F_2(;\frac{3}{2},v+\frac{1}{2};\frac{aw}{2})$
129	$x^{v-1}\cosh(2\sqrt{a}\sqrt{x})$	$\frac{w^{v-1}}{(v-1)!}{}_0F_2(;\frac{1}{2},v;\frac{aw}{2})$
130	$\sinh^{-1}(x)$	$w_2F_3(\frac{1}{2},\frac{1}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
131	$\cosh^{-1}(x)$	$-iw_2F_3(\frac{1}{2},\frac{1}{2};1,\frac{3}{2},\frac{3}{2};\frac{w^2}{4})+\frac{i\pi}{2}$
132	$\cosh^{-1}(1+\frac{x}{a})$	$\sqrt{\frac{8w}{a\pi}}{}_2F_2(\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2};-\frac{w}{2a})$
133	$x\sinh^{-1}(x)$	$\frac{w^2}{2}{}_2F_3(\frac{1}{2},\frac{1}{2};\frac{3}{2},\frac{3}{2},2;-\frac{w^2}{4})$
134	$\sinh((2n+1)\sinh^{-1}(x))$	$(2n+1)w_2F_3(-n,n+1;1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
135	$\cosh(2n\sinh^{-1}(x))$	${}_2F_3(n,-n;\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4})$
136	$\sinh(v\sinh^{-1}(x))$	$vw_2F_3(\frac{1+v}{2},\frac{1-v}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
137	$\cosh(v\sinh^{-1}(x))$	${}_2F_3(\frac{v}{2},-\frac{v}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4})$
138	$\sinh(v\cosh^{-1}(1+\frac{x}{a}))$	$v\sqrt{\frac{8w}{a\pi}}{}_2F_2(\frac{1}{2}+v,\frac{1}{2}-v;\frac{3}{2},\frac{3}{2};-\frac{w}{2a})$
139	$\frac{\exp(n\sinh^{-1}(x))}{\sqrt{1+x^2}}$	$nw_2F_3(1+\frac{n}{2},1-\frac{n}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})+{}_2F_3(\frac{1-n}{2},\frac{1+n}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4})$
140	$\frac{\exp(-n\sinh^{-1}(x))}{\sqrt{1+x^2}}$	$-nw_2F_3(1-\frac{n}{2},1+\frac{n}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})+{}_2F_3(\frac{1-n}{2},\frac{1+n}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4})$
141	$\frac{\sinh(v\sinh^{-1}(x))}{\sqrt{x^2+1}}$	$vw_2F_3(1-\frac{v}{2},1+\frac{v}{2};1,\frac{3}{2},\frac{3}{2};-\frac{w^2}{4})$
142	$\frac{\cosh(n\sinh^{-1}(x))}{\sqrt{x^2+1}}$	${}_2F_3(\frac{1-n}{2},\frac{1+n}{2};\frac{1}{2},\frac{1}{2},1;-\frac{w^2}{4})$
143	$\frac{\cosh(n\cosh^{-1}(x))}{\sqrt{x^2-1}}$	$-\frac{1}{2e^{\frac{in\pi}{2}}}[i(1+e^{in\pi})_2F_3(\frac{1-n}{2},\frac{1+n}{2};\frac{1}{2},\frac{1}{2},1;\frac{w^2}{4})+nw(1+e^{in\pi})_2F_3(1-\frac{n}{2},1+\frac{n}{2};1,\frac{3}{2},\frac{3}{2};\frac{w^2}{4})]$
144	$\frac{\exp(2v\sinh^{-1}(\frac{x}{2a}))}{\sqrt{x^3+4a^2x}}$	$\frac{\mathrm{csgn}(a)}{2a^2\sqrt{\pi w}}[a_2F_3(\frac{1}{2}+v,\frac{1}{2}-v;\frac{1}{4},\frac{1}{2},\frac{3}{4};-\frac{w^2}{16a^2})+2vw_2F_3(1-v,1+v;\frac{3}{4},\frac{5}{4},\frac{3}{2};-\frac{w^2}{16a^2})]$
145	$\frac{\exp(-2v\sinh^{-1}(\frac{x}{2a}))}{\sqrt{x^3+4a^2x}}$	$\frac{\mathrm{csgn}(a)}{2a^2\sqrt{\pi w}}[a_2F_3(\frac{1}{2}+v,\frac{1}{2}-v;\frac{1}{4},\frac{1}{2},\frac{3}{4};-\frac{w^2}{16a^2})-2vw_2F_3(1-v,1+v;\frac{3}{4},\frac{5}{4},\frac{3}{2};-\frac{w^2}{16a^2})]$
146	$\frac{1}{\sqrt{x^3+4a^2x}}(\cos((v+\frac{1}{4})\pi)\exp(-2v\sinh^{-1}(\frac{x}{2a}))+\sin((v+\frac{1}{4})\pi)\exp(2v\sinh^{-1}(\frac{x}{2a})))$	$\frac{\mathrm{csgn}(a)}{a^2\sqrt{2\pi w}}[a\cos(\pi v)_2F_3(\frac{1}{2}+v,\frac{1}{2}-v;\frac{1}{4},\frac{1}{2},\frac{3}{4};-\frac{w^2}{16a^2})+2vw\sin(\pi v)_2F_3(1-v,1+v;\frac{3}{4},\frac{5}{4},\frac{3}{2};-\frac{w^2}{16a^2})]$
147	$\frac{1}{\sqrt{x^3+4a^2x}}(\sin((v+\frac{1}{4})\pi)\exp(-2v\sinh^{-1}(\frac{x}{2a}))-\cos((v+\frac{1}{4})\pi)\exp(2v\sinh^{-1}(\frac{x}{2a})))$	$\frac{\mathrm{csgn}(a)}{a^2\sqrt{2\pi w}}[a\sin(\pi v)_2F_3(\frac{1}{2}+v,\frac{1}{2}-v;\frac{1}{4},\frac{1}{2},\frac{3}{4};-\frac{w^2}{16a^2})-2vw\cos(\pi v)_2F_3(1-v,1+v;\frac{3}{4},\frac{5}{4},\frac{3}{2};-\frac{w^2}{16a^2})]$
148	$\frac{\sinh(2v\sinh^{-1}(\frac{x}{2a}))}{\sqrt{x^3+4a^2x}}$	$\frac{v\sqrt{w}\,\mathrm{csgn}(a)}{a^2\sqrt{\pi}}{}_2F_3(1-v,1+v;\frac{3}{4},\frac{5}{4},\frac{3}{2};-\frac{w^2}{16a^2})$
149	$\frac{\cosh(2v\sinh^{-1}(\frac{x}{2a}))}{\sqrt{x^3+4a^2x}}$	$\frac{\mathrm{csgn}(a)}{2a\sqrt{\pi w}}{}_2F_3(\frac{1}{2}+v,\frac{1}{2}-v;\frac{1}{4},\frac{1}{2},\frac{3}{4};-\frac{w^2}{16a^2})$
150	$\frac{\sinh^{-1}(x)}{x}$	$\mathrm{J}_0(w)-\frac{\pi}{2}[\mathrm{J}_0(w)\mathbf{H}_1(w)-\mathrm{J}_1(w)\mathbf{H}_0(w)]$

Table 2 Special functions definition

S. No	Function	Definition
1	First kind Bessel function	$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{x}{2})^{2k+n}}{k!(n+k)!}$
2	Modified first kind Bessel function	$I_n(x) = \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k+n}}{k!(n+k)!}$
3	Kelvin real function	$\mathrm{ber}_n(x) = \mathrm{Re}\, J_n(i^{\frac{3}{2}}x)$
4	Kelvin imaginary function	$\mathrm{bei}_n(x) = \mathrm{Im}\, J_n(i^{\frac{3}{2}}x)$
5	Error function	$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2}\,dz$
6	Complementary error function	$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-z^2}\,dz$
7	Struve function	$\mathbf{H}_v(x) = (\frac{x}{2})^{v+1} \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{x}{2})^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+v+\frac{3}{2})}$
8	Modified Struve function	$\mathbf{L}_v(x) = (\frac{x}{2})^{v+1} \sum_{k=0}^{\infty} \frac{(\frac{x}{2})^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+v+\frac{3}{2})}$
9	Generalized hypergeometric function	${}_pF_q((a_p);(b_q);x) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdot(a_2)_k\cdots(a_p)_k x^k}{(b_1)_k\cdot(b_2)_k\cdots(b_q)_k k!}$
10	Lommel S1 function	$\mathbf{S}_{\mu,v}^{(1)}(x) = \frac{x^{\mu+1}\,{}_1F_2(1;\frac{\mu-v+3}{2},\frac{\mu+v+3}{2};-\frac{x^2}{4})}{(\mu+1)^2-v^2}$
11	Whittaker M function	$M_{\kappa,\mu}(x) = e^{-\frac{x}{2}} x^{\mu+\frac{1}{2}} M(\mu-\kappa+\frac{1}{2}, 1+2\mu; x)$
12	Kummer function	$M(a,b,c) = \sum_{n=0}^{\infty} \frac{a^{(n)}x^n}{b^{(n)}n!} = {}_1F_1(a;b;x)$
13	Sign function	$\mathrm{csgn}(x) = \begin{cases} 1; x < \mathcal{R}(x), \\ -1; x > \mathcal{R}(x) \end{cases}$
14	Sine integral	$\mathrm{Si}(x) = \int_0^x \frac{\sin(z)\,dz}{z}$
15	Cosine integral	$\mathrm{Ci}(x) = -\int_x^{\infty} \frac{\cos(z)\,dz}{z}$
16	Hyperbolic cosine integral	$\mathrm{Chi}(x) = \gamma + \ln(x) + \int_x^{\infty} \frac{(\cos(z)-1)\,dz}{z}$
17	Exponential integral	$\mathrm{Ei}(x) = \int_x^{\infty} \frac{e^{-z}\,dz}{z}$
18	Laguerre polynomials	$L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x})$
19	Euler's constant	$\gamma = 0.5772156$

$$\mathbb{S}^{-1}\left[x^n f(x)\right] = \underbrace{\int_0^w \cdots \int_0^w}_{n\text{-times}} F_{-1}(\eta)(d\eta)^n. \tag{6}$$

$$\mathbb{S}^{-1}\left[\frac{f(x)}{x^n}\right] = \frac{d^n F_{-1}(w)}{dw^n}. \tag{7}$$

Proof The proof is straightforward. □

The following steps of Algorithm 1 solve the ordinary differential equations [14].

Algorithm 1 DIST methodology

Step 1. Apply inverse Sumudu using Theorem 1 to a given differential equation.

Step 2. If the result of Step 1 is an integro-differential equation with n integrals, then convert to a differential equation by substituting:

$$F_{-1}(w) = \frac{d^n G_{-1}(w)}{dw^n}. \tag{8}$$

Step 3. Find the power series solution of Step 2.

Step 4. Convert $G_{-1}(w)$ back to $F_{-1}(w)$ using Eq. (8).

Step 5. Apply the Sumudu transform to Step 4 which leads to the solution $f(x)$ of the given differential equation.

3 New exact solutions of AGE-F1

In this work, the AGE-F1 (Sect. 40.1, pp. 308, [12]) ordinary differential equation is solved for integer coefficients to get their new exact solutions.

$$-\frac{d^2y(x)}{dx^2}+\frac{l(l+1)y(x)}{x^2}=\lambda y(x). \tag{9}$$

Applying the inverse Sumudu transform to Eq. (9) with $\mathbb{S}^{-1}[y(x)]=F_{-1}(w)$, using the properties of Theorem 1, and converting the resulting integro-differential equation and simplifying, we obtain

$$-w^2\frac{d^4G_{-1}(w)}{dw^4}+l(l+1)\frac{d^2G_{-1}(w)}{dw^2}=\lambda G_{-1}(w). \tag{10}$$

Example 1 For $l=1$ in Eq. (10), Step 3 of Algorithm 1 gives the following power series solution:

$$\begin{aligned} G_{-1}(w) := \frac{72}{\lambda^2}\Biggl[&\sum_{n=2}^{\infty}\frac{2^{4n}(n-1)(-\frac{\lambda}{16})^n w^{2n}}{n\Gamma(2n)^2} \\ &-\sum_{n=0}^{\infty}\frac{2^{4n}(2n-1)(2n+1)(-\frac{\lambda}{16})^n w^{2n+1}}{\Gamma(2n+2)^2}\Biggr]. \end{aligned} \tag{11}$$

Next converting Eq. (11) to $F_{-1}(w)$ and applying the Sumudu transform, we get

$$y(x) := \frac{1}{\lambda^{\frac{3}{2}}x}\bigl[\bigl(\lambda^2x+72\bigr)\sin(x\sqrt{\lambda})+\bigl(\lambda^{\frac{3}{2}}-72x\sqrt{\lambda}\bigr)\cos(x\sqrt{\lambda})\bigr]. \tag{12}$$

$y(x)$ in Eq. (12) is the new exact solution of Eq. (9) with $l=1$.

Example 2 For $l=2$ in Eq. (10), we have the following power series solution:

$$\begin{aligned} G_{-1}(w) := \frac{1}{3}&\sum_{n=0}^{\infty}\frac{2^{4n}(2n-3)(2n-1)(-\frac{\lambda}{16})^n w^{2n}}{\Gamma(2n+1)^2} \\ &+\frac{7200}{\lambda^2}\sum_{n=2}^{\infty}\frac{2^{4n}(n-1)n(-\frac{\lambda}{16})^n w^{2n+1}}{n\Gamma(2n+2)^2}. \end{aligned} \tag{13}$$

Differentiating twice w.r.t. w, (13) converts to $F_{-1}(w)$, then applying the Sumudu transform leads to

$$\begin{aligned} y(x) := -&\frac{1}{3x^2\lambda^{\frac{5}{2}}}\bigl[\bigl(5400\lambda x^2-3\lambda^3x-16{,}200\bigr)\sin(x\sqrt{\lambda}) \\ &+\bigl(\lambda^{\frac{7}{2}}x^2-16{,}200\sqrt{\lambda}x-3\lambda^{\frac{5}{2}}\bigr)\cos(x\sqrt{\lambda})\bigr]. \end{aligned} \tag{14}$$

Here (14) is the new exact solution of (9) with $l=2$.

Example 3 The power series solution of (10) for $l=3$ is given by

$$G_{-1}(w) := -\frac{4}{3\lambda}\sum_{n=1}^{\infty}\frac{2^{4n}(2n-5)(2n-3)(2n-1)n(-\frac{\lambda}{16})^n w^{2n-1}}{\Gamma(2n+1)^2}$$

$$-\frac{151{,}200}{\lambda^3}\sum_{n=3}^{\infty}\frac{2^{4n}(n-2)(n-1)(-\frac{\lambda}{16})^n w^{2n}}{n(2n+1)\Gamma(2n)^2}. \tag{15}$$

Converting (15) to $F_{-1}(w)$ and applying the Sumudu transform to resulting summation leads to

$$\begin{aligned} y(x) := {}& \frac{1}{3x^3\lambda^4}\Big[-2\big(\lambda^{\frac{9}{2}}x^3 + 680{,}400\lambda^{\frac{3}{2}}x^2 - 15\lambda^{\frac{7}{2}}x - 1{,}701{,}000\sqrt{\lambda}\big)\sin(x\sqrt{\lambda}) \\ & + \big(226{,}800\lambda^2x^3 - 12\lambda^4x^2 - 340{,}200\lambda x + 30\lambda^3\big)\cos(x\sqrt{\lambda})\Big]. \end{aligned} \tag{16}$$

$y(x)$ in (16) is the new exact solution of (9) with $l = 3$. The complex plot of (16) for $\lambda = 5, 10,$ and 15 is shown in Fig. 1.

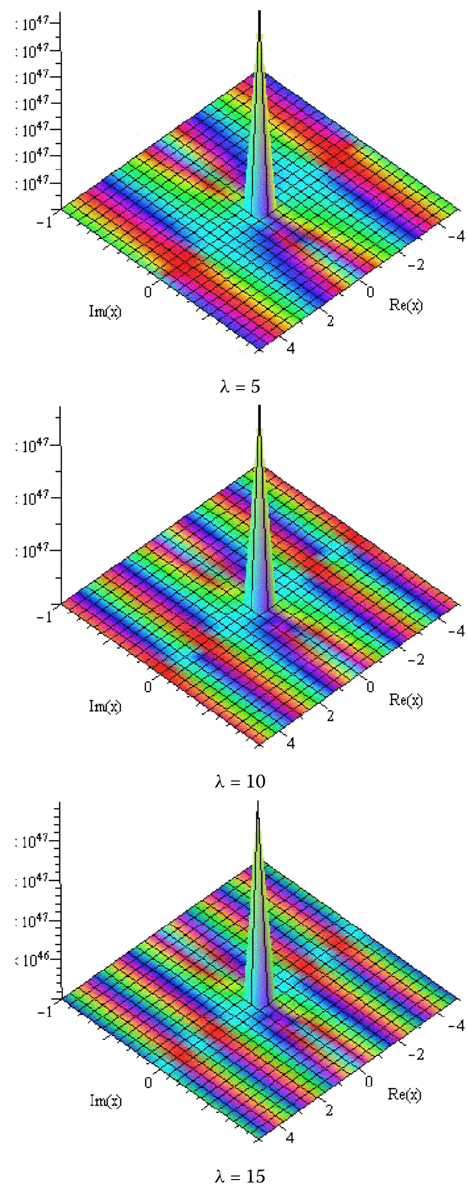


Figure 1 Solution $y(x)$ Eq. (16) of Example 3 in a complex plane $x \in [-5 - I, 5 + I]$ for different values of λ

Example 4 When $l = 4$ in (10), the power series solution is given by

$$G_{-1}(w) := \frac{1}{15\lambda}\sum_{n=1}^{\infty}\frac{2^{4n}(2n-7)(2n-5)(2n-3)(-\frac{\lambda}{16})^n w^{2n-2}}{n\Gamma(2n-1)^2} + \frac{38{,}102{,}400}{\lambda^4}\sum_{n=4}^{\infty}\frac{2^{4n}(n-3)(n-2)(n-1)(-\frac{\lambda}{16})^n w^{2n-1}}{(2n+1)\Gamma(2n)^2}. \tag{17}$$

Applying the Sumudu transform after converting (17) to $F_{-1}(w)$ leads to

$$\begin{aligned} y(x) := \frac{1}{15x^4\lambda^5}\Bigg[&71{,}442{,}000\left(\lambda^{\frac{5}{2}}x^4 - \frac{\lambda^{\frac{11}{2}}x^3}{3{,}572{,}100}\right.\\ &\left. - 45\lambda^{\frac{3}{2}}x^2 + \frac{\lambda^{\frac{9}{2}}x}{340{,}200} + 105\sqrt{\lambda}\right)\sin(x\sqrt{\lambda})\\ &+ 2\lambda\left(\lambda^5x^4 + 357{,}210{,}000\lambda x^3\right.\\ &\left. - 45\lambda^4x^2 - 3{,}750{,}705{,}000x + 105\lambda^3\right)\cos(x\sqrt{\lambda})\Bigg]. \end{aligned} \tag{18}$$

$y(x)$ in (18) is the new exact solution of (9) for $l = 4$. The complex plot of (18) for $\lambda = 5, 10,$ and 15 is shown in Fig. 2.

Example 5 In (10) for $l = 5$ the power series solution is

$$\begin{aligned} G_{-1}(w) := &-\frac{8}{15\lambda^2}\sum_{n=2}^{\infty}\frac{2^{4n}(2n-9)(2n-7)(2n-5)(2n-3)(n-1)(-\frac{\lambda}{16})^n w^{2n-3}}{n\Gamma(2n-1)^2}\\ &- \frac{6{,}706{,}022{,}400}{\lambda^5}\\ &\times\sum_{n=5}^{\infty}\frac{2^{4n}(2n-1)(n-4)(n-3)(n-2)(n-1)(-\frac{\lambda}{16})^n w^{2n-2}}{(2n+1)\Gamma(2n)^2}. \end{aligned} \tag{19}$$

Applying the Sumudu transform after converting (19) leads to

$$\begin{aligned} y(x) := \frac{1}{15x^5\lambda^6}\Bigg[&8\left(\lambda^{\frac{13}{2}}x^5 + 11{,}787{,}930{,}000\lambda^{\frac{5}{2}}x^4 - 105\lambda^{\frac{11}{2}}x^3\right.\\ &\left. - 330{,}062{,}040{,}000\lambda^{\frac{3}{2}}x^2 + 945\lambda^{\frac{9}{2}}x + 742{,}639{,}590{,}000\sqrt{\lambda}\right)\sin(x\sqrt{\lambda})\\ &- 6{,}286{,}896{,}000\left(\lambda^2x^5 - \frac{\lambda^5x^4}{52{,}390{,}800} - 105\lambda x^3\right.\\ &\left. + \frac{\lambda^4x^2}{1{,}871{,}100} + 945x - \frac{\lambda^3}{831{,}600}\right)\lambda\cos(x\sqrt{\lambda})\Bigg]. \end{aligned} \tag{20}$$

$y(x)$ in (20) is the new exact solution of (9) with $l = 5$.

Remark 1 Solutions in (12), (14), (16), (18), and (20) for respective $l = 1$ to 5 in (9) are verified in a Maple computer algebra system which satisfies the given differential equation

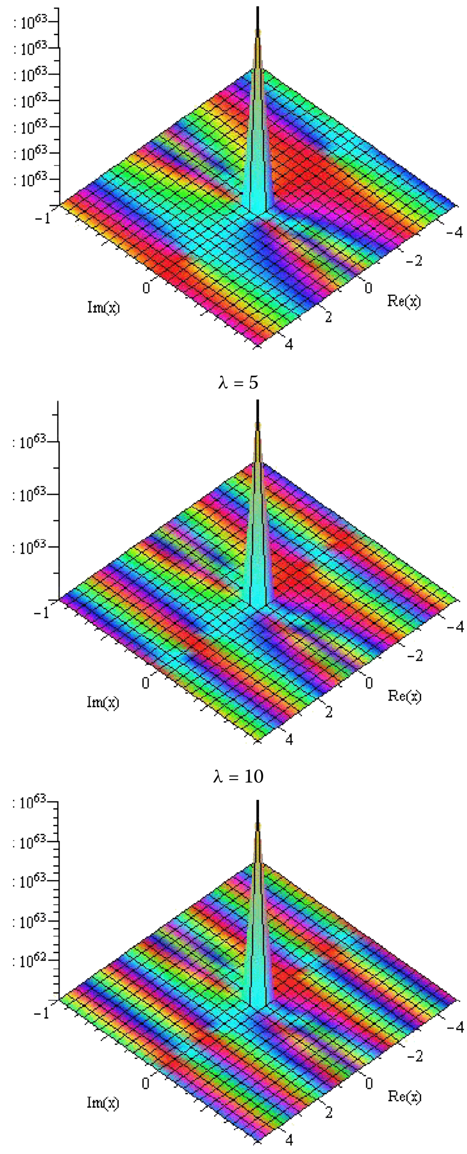


Figure 2 Solution $y(x)$ Eq. (18) of Example 4 in a complex plane $x \in [-5 - I, 5 + I]$ for different values of λ

and all the new solutions appear for the first time in this research work as per the literature surveyed. Comparing to solving directly, the DIST method gives new exact analytical solutions for ordinary differential equations. For instance $l = 5$ in (9), solving directly gives the following solution:

$$y(x) := \frac{1}{x^5}\Big[\big(\lambda^{\frac{9}{2}}x^5 - 15\lambda^4x^4 - 105\lambda^{\frac{7}{2}}x^3 + 420\lambda^3x^2 + 945\lambda^{\frac{5}{2}}x - 945\lambda^2\big)\sin(x\sqrt{\lambda}) + \big(\lambda^{\frac{9}{2}}x^5 + 15\lambda^4x^4 - 105\lambda^{\frac{7}{2}}x^3 - 420\lambda^3x^2 + 945\lambda^{\frac{5}{2}}x + 945\lambda^2\big)\cos(x\sqrt{\lambda})\Big]. \tag{21}$$

Comparative study of (20) and (21) shows that the DIST method solves the differential equations for new exact solutions, which is verified in a Maple computer algebra system both numerically and graphically, shown in a complex plot of Fig. 3.

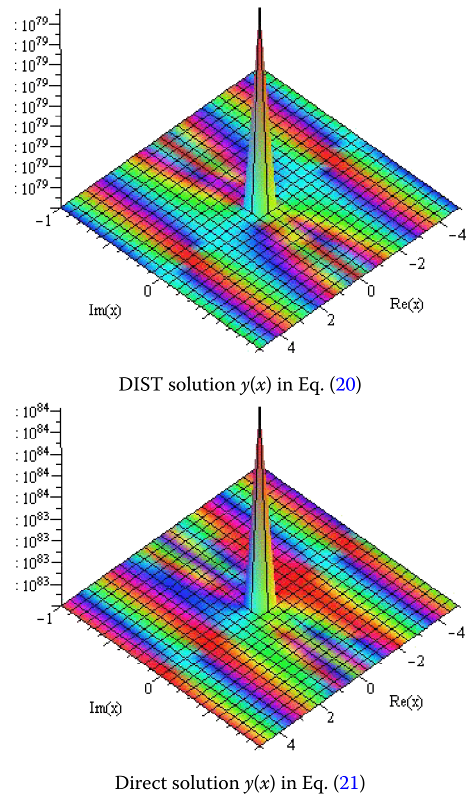


Figure 3 Comparison in a complex plane $x \in [-5 - I, 5 + I]$ of DIST solution and direct solution of Example 5 for $\lambda = 10$

4 New exact complex solutions of AGE-F1

By computing another power series solution of Step 3 of Algorithm 1, the second set of solutions of (9) is studied in this section which are new exact solutions. All the computations are worked through the Maple computer algebra system.

Example 6 When $l = 1$ in (10), the formal series solution is given by

$$G_{-1}(w) := -\frac{24i}{\lambda^{\frac{3}{2}}} \sum_{n=3}^{\infty} \frac{(n-2)e^{\frac{in\pi}{2}} \lambda^{\frac{n}{2}} w^n}{n\Gamma(n)^2}. \tag{22}$$

Converting $G_{-1}(w)$ in (22) to $F_{-1}(w)$ by differentiating twice w.r.t. w and then applying the Sumudu transform gives

$$y(x) := \frac{1}{\lambda x}\left[24(ix\sqrt{\lambda} - 1)e^{ix\sqrt{\lambda}}\right]. \tag{23}$$

Here (23) is the second new exact complex solution of (9) with $l = 1$.

Example 7 When $l = 2$ in (10), the formal series solution is

$$G_{-1}(w) := \frac{96(12 + \lambda w^2)}{\lambda^2} + \frac{384}{\lambda^2} \sum_{n=4}^{\infty} \frac{(n-3)(n-1)e^{\frac{in\pi}{2}} \lambda^{\frac{n}{2}} w^n}{n\Gamma(n+1)^2}. \tag{24}$$

Converting $G_{-1}(w)$ in (24) to $F_{-1}(w)$ by differentiating twice w.r.t. w and then applying the Sumudu transform leads to

$$y(x) := -\frac{192e^{-x\sqrt{-\lambda}}}{\lambda^2 x^2}\left[\lambda x^2 - 3x\sqrt{-\lambda} - 3 + \left(\lambda x^2 + 3x\sqrt{-\lambda} - 3\right)e^{2x\sqrt{-\lambda}}\right]. \tag{25}$$

$y(x)$ in (25) is the second new exact complex solution of (9) with $l = 2$.

Example 8 When $l = 3$ in (10), the formal series solution is given by

$$G_{-1}(w) := -\frac{11{,}520i}{\lambda^{\frac{5}{2}}}\sum_{n=5}^{\infty}\frac{(n-4)(n-2)e^{\frac{in\pi}{2}}\lambda^{\frac{n}{2}}w^n}{n(n+1)\Gamma(n)^2}. \tag{26}$$

Applying the Sumudu transform after converting (26) to $F_{-1}(w)$ gives

$$y(x) := \frac{11{,}520e^{ix\sqrt{\lambda}}}{\lambda^3 x^3}\left[i\lambda^{\frac{3}{2}}x^3 - 6\lambda x^2 - 15ix\sqrt{\lambda} + 15\right]. \tag{27}$$

$y(x)$ in (27) is the second new exact complex solution of (9) with $l = 3$.

Example 9 When $l = 4$ in (10), the formal series solution is

$$\begin{aligned} G_{-1}(w) := -\frac{480}{\lambda^3}\Biggl[&-\lambda^2 w^4 - 216\lambda w^2 \\ &- 8640 + 1152\sum_{n=6}^{\infty}\frac{(n-5)(n-3)(n-1)e^{\frac{in\pi}{2}}\lambda^{\frac{n}{2}}w^n}{(n+2)\Gamma(n+1)^2}\Biggr]. \end{aligned} \tag{28}$$

Converting (28) to $F_{-1}(w)$ and applying the Sumudu transform gives

$$y(x) := \frac{552{,}960e^{ix\sqrt{\lambda}}}{\lambda^{\frac{17}{2}}x^4}\left[\lambda^{\frac{13}{2}}x^4 + 10i\lambda^6 x^3 - 45\lambda^{\frac{11}{2}}x^2 - 105i\lambda^5 x + 105\lambda^{\frac{9}{2}}\right]. \tag{29}$$

Here (29) is the second new exact complex solution of (9) with $l = 4$. The complex plot of (29) for $\lambda = 5, 10$, and 15 is shown in Fig. 4.

Example 10 When $l = 5$ in (10), the formal series solution is given by

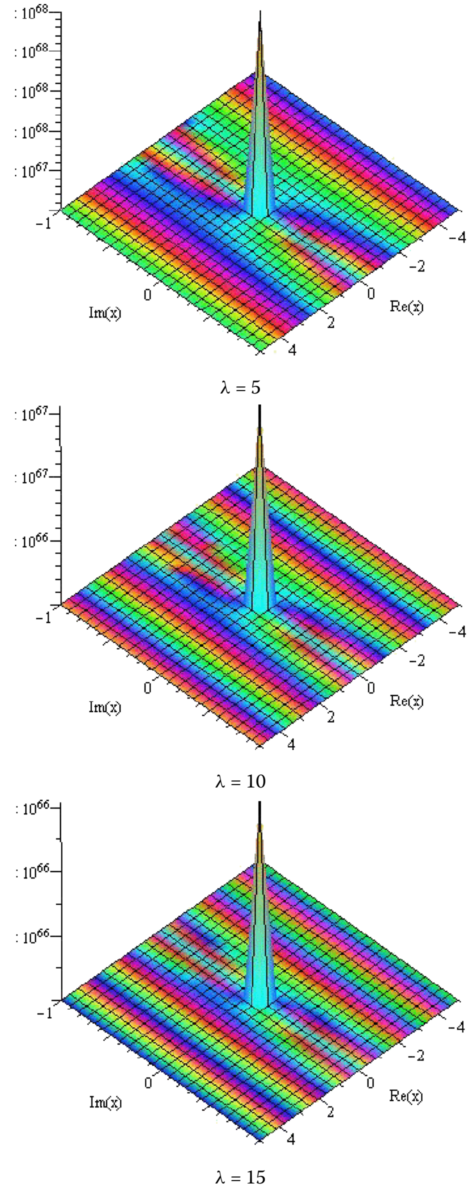
$$G_{-1}(w) := \frac{38{,}707{,}200i}{\lambda^{\frac{7}{2}}}\sum_{n=7}^{\infty}\frac{(n-6)(n-4)(n-2)e^{\frac{in\pi}{2}}\lambda^{\frac{n}{2}}w^n}{n(n+1)(n+3)\Gamma(n)^2}. \tag{30}$$

Converting (30) to $F_{-1}(w)$ by differentiating w.r.t. w and then applying the Sumudu transform leads to

$$\begin{aligned} y(x) := -\frac{38{,}707{,}200e^{ix\sqrt{\lambda}}}{\lambda^6 x^5}\Bigl[&i\lambda^{\frac{7}{2}}x^5 - 15\lambda^3 x^4 \\ &- 105i\lambda^{\frac{5}{2}}x^3 + 420\lambda^2 x^2 + 945i\lambda^{\frac{3}{2}}x - 945\lambda\Bigr]. \end{aligned} \tag{31}$$

Here $y(x)$ in (31) is the second new exact complex solution of (9) with $l = 5$. The complex plot of (31) for $\lambda = 5, 10$, and 15 is shown in Fig. 5.

Figure 4 Solution $y(x)$ Eq. (29) of Example 9 in a complex plane $x \in [-5 - I, 5 + I]$ for different values of λ

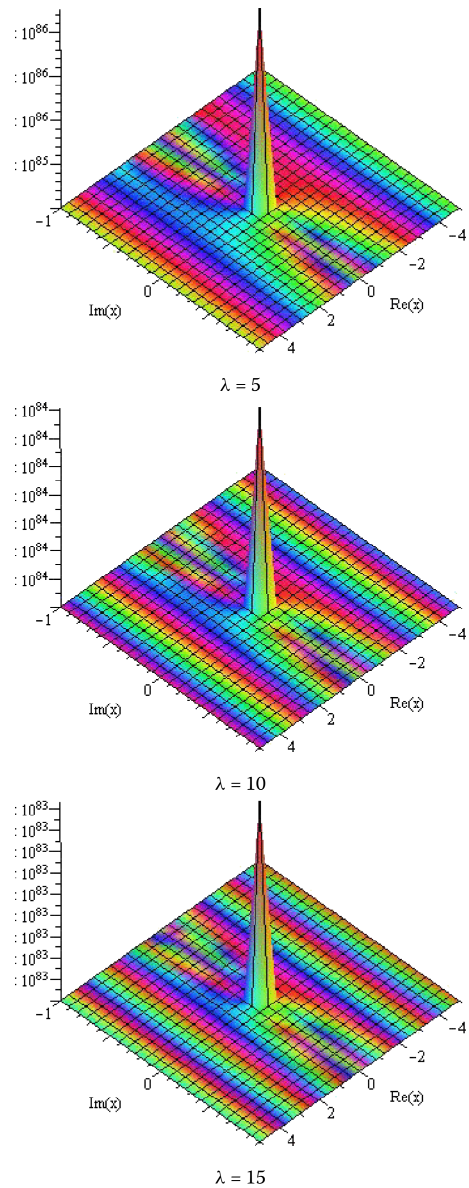


Remark 2 Solutions (23), (25), (27), (29), and (31) are verified in the Maple computer algebra system, and they are new exact complex solutions of (9) respectively for $l = 1$ to 5. These complex solutions appear for the first time in this research works as per the literature surveyed. For instance, solving directly $l = 1$ in (9) gives

$$y(x) := \frac{e^{x\sqrt{-\lambda}}(\sqrt{\lambda x^2 + 2i\sqrt{\lambda}x - 1}) + e^{-x\sqrt{-\lambda}}(\sqrt{\lambda x^2 - 2i\sqrt{\lambda}x - 1})}{x}. \tag{32}$$

Comparing solutions (23) and (32), DIST solves the differential equations to new exact complex solutions, which is verified in the Maple computer algebra system both numerically and graphically and shown in a complex plot of Fig. 6.

Figure 5 Solution $y(x)$ Eq. (31) of Example 10 in a complex plane $x \in [-5 - I, 5 + I]$ for different values of λ

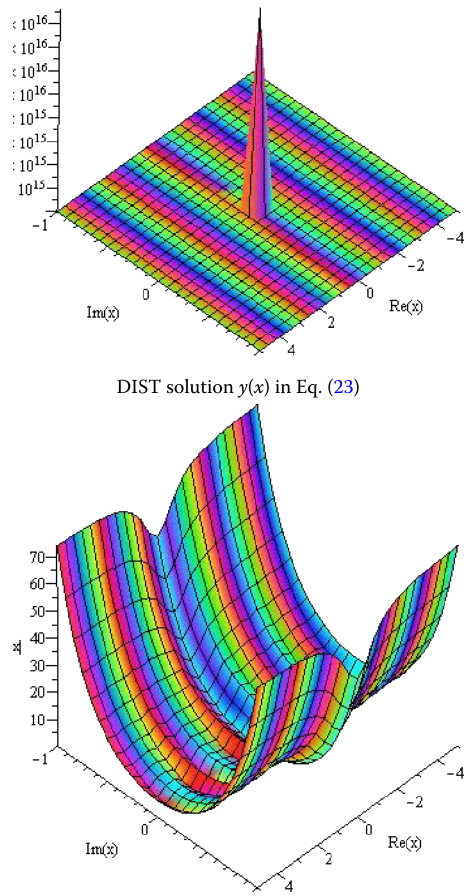


Next, when solving for general l, (9), DIST method in both the above methods gives the new approximate analytical solution in terms of Lommel S1 function which will be studied in a separate work numerically.

5 Conclusion

Through this research communication an algorithm based on the discrete inverse Sumudu Transform (DIST) was described to solve ordinary differential equations for their new exact analytical and complex solutions. An algebro-geometric equation for different integer value coefficients was studied with the algorithm and the method was proven by deriving their new solutions. Efficiency of the DIST method was shown via the comparative study in Remarks 1 and 2, Maple plots were shown graphically in Figs. 3 and 6. The enlarged list of functions and their inverse Sumudu transforms in Table 1 shows that inverting the

Figure 6 Comparison in a complex plane $x \in [-5 - I, 5 + I]$ of DIST solution and direct solution of Example 6 for $\lambda = 10$



DIST solution $y(x)$ in Eq. (23)

Direct solution $y(x)$ in Eq. (32)

elementary functions upon Sumudu discrete-wise gives the special functions in Table 2 and will help future research.

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Authors' contributions
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