Wang Advances in Difference Equations (2018) 2018:405
https://doi.org/10.1186/s13662-018-1784-7

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Conservative Fourier spectral scheme for the coupled Schrödinger–Boussinesq equations

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Abstract
In the paper, the conservative Fourier spectral scheme is presented for the coupled Schrödinger–Boussinesq equations. We apply the Fourier collocation scheme to spatial derivatives and the Crank–Nicolson scheme to the system in time direction, respectively. We find that the scheme can preserve mass and energy conservation laws. Moreover, the existence, uniqueness, stability and convergence of the scheme are discussed, and it is shown that the scheme is of the accuracy $O(\tau^2 + J^{-r})$. The numerical experiments are given to show that verify the correctness of theoretical results and the efficiency of the scheme.

Keywords: Schrödinger–Boussinesq equations; Conservative Fourier spectral method; Conservation laws; Convergence

1 Introduction

The coupled Schrödinger–Boussinesq (CSB) equations were given in the form [1–3]

$$iu_t + u_{xx} - uv = 0, \tag{1}$$

$$v_{tt} = v_{xx} - \alpha v_{xxxx} + f(v)_{xx} + \omega|u|^2_{xx}, \tag{2}$$

where α, ω are constants, $f(x)$ is a sufficiently smooth real function, $u(x,t)$ represents the complex Schrödinger field, and $v(x,t)$ represents the real Boussinesq field. Many researchers have devoted much energy to the study of the CSB system for a long time. As a result, many important properties such that existence and uniqueness of the solutions [4], exact solitary wave solutions [5, 6] and global attractors [7, 8] of some CSB systems have been discovered.

The nonlinear CSB equations are composed of a Schrödinger equation and a Boussinesq equation. Many scholars have given much numerical schemes to solve the Schrödinger equation. Numerical contributions for the good Boussinesq and nonlinear CSB equations have been topics of concern during the past decades. In [9], a quadratic B-spline finite element scheme was presented, and the corresponding results of the error estimates were obtained. In [10], a conservative multisymplectic scheme was presented and one simulated solitary waves for long times. In [11], Zhang obtained an optimal error estimate for an implicit conservative difference scheme with order $O(\tau^2 + h^2)$. In [12], two conserved compact finite difference schemes were given, and the conservation property, and the existence, convergence, and stability of the difference schemes were theoretically analyzed.

As a class of high accuracy methods, spectral methods are often chosen to solve differential equations. In [13], the authors have given a Fourier pseudospectral method to solve the good Boussinesq equation with second-order temporal accuracy, and they have discussed the nonlinear stability and convergence of the scheme. In [14], the authors showed a second-order operator splitting numerical scheme and a Fourier pseudospectral scheme to solve the good Boussinesq equation, and they gave a stability and convergence analysis. In [15], the authors derived the fourth-order average vector field method and the Fourier pseudospectral method for the good Boussinesq equation. However, to the best of our knowledge, there was little attention paid to the spectral method for the CSB system. In [16], a time-splitting Fourier spectral method was presented, but there was no theoretical analysis available for the approximation error. In the paper, we construct a conservative Fourier spectral scheme to solve nonlinear CSB equations, and we give a rigorous numerical analysis of the scheme.

Let $v_t = \phi_{xx}$, we consider the following initial-boundary value problem:

$$iu_t + u_{xx} - uv = 0, \quad x \in \Omega, 0 < t \leq T, \tag{3}$$

$$v_t = \phi_{xx}, \quad x \in \Omega, 0 < t \leq T, \tag{4}$$

$$\phi_t = v - \alpha v_{xx} + f(v) + \omega|u|^2, \quad x \in \Omega, 0 < t \leq T, \tag{5}$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x), \qquad \phi(x,0) = \phi_0(x), \quad x \in \Omega, \tag{6}$$

$$u(a,t) = u(b,t), \qquad v(a,t) = v(b,t), \qquad \phi(a,t) = \phi(b,t), \tag{7}$$

where $\Omega = (a,b)$. The system (3)–(7) has the following mass and energy conservation laws:

$$\text{Mass}: \quad M(t) = \|u\|^2 = M(0),$$

$$\text{Energy}: \quad E(t) = \|v\|^2 + \|\phi_x\|^2 + 2\omega\|u_x\|^2 + \alpha\|v_x\|^2 + 2\langle F(v), 1\rangle + 2\omega\langle |u^2|, v\rangle = E(0),$$

where $F(v) > 0$ is a primitive function of $f(v)$.

It is well known that the conservative schemes perform better than the nonconservative ones. Thus, it is of interest to investigate conservative schemes for the CSB system. The Fourier pseudospectral method has attracted much attention in recent years due to its high accuracy and efficiency. In this paper, the Fourier pseudospectral method [17–20] is utilized to discretize the CSB equations in space direction. In [14, 15], the authors employed the fourth-order average vector field method and the second-order operator splitting scheme, respectively, to solve the Boussinesq equation. However, the two method cannot satisfy the energy conservation law. Here, we apply the Crank–Nicolson method to solve the CSB equations in time direction. We find that the scheme can preserve mass and energy conservation laws simultaneously. Moreover, the existence, uniqueness and boundedness of the scheme are proved, and the convergence order is of $O(\tau^2 + J^{-r})$. Finally, we give numerical experiments to show the efficiency of the conservative scheme.

The remainder of this article is structured as follows. In Sect. 2, we give some useful lemmas and a conservative scheme, and we prove that the scheme preserve mass and energy conservation laws. Moreover, the existence, uniqueness, boundedness and convergence of the scheme are proved. In Sect. 3, we give the iterative algorithm of the scheme. In Sect. 4, numerical experiments are given, and the results verify the efficiency of the conservative schemes. Finally, a conclusion and some discussions are given in Sect. 5.

2 Conservative Fourier spectral scheme for the CSB system

2.1 Some useful lemmas

Let $\tau = T/N, h = (b-a)/J$, and define

$$\Omega_h = \{x_j | 0 \le j \le J-1\}, \qquad \Omega_\tau = \{t_n | 0 \le n \le N-1\}.$$

Suppose $w = \{w_j^n; j = 0,1,2,\ldots,J, n = 0,1,2,\ldots,N\}$ be a discrete function, and define operators

$$\left(w_j^n\right)_x = \frac{w_{j+1}^n - w_j^n}{h}, \qquad w_j^{n+\frac{1}{2}} = \frac{w_j^{n+1} + w_j^n}{2}, \qquad \left(w_j^n\right)_t = \frac{w_j^{n+1} - w_j^n}{\tau}.$$

Let U_j^n, V_j^n, Φ_j^n denote the numerical approximations to $u(x_j,t_n), v(x_j,t_n), \phi(x_j,t_n)$, respectively. Denote

$$\langle U, V\rangle = h\sum_{j=0}^{J-1} U_j \overline{V}_j, \qquad \|U\|^2 = \langle U, U\rangle, \qquad |U|_{h,1}^2 = \langle U_x, U_x\rangle,$$

$$\|U\|_{l_h^p}^p = h\sum_{j=0}^{J-1} |U_j|^p, \quad 1 \le p < +\infty,$$

$$\|U\|_{l_h^\infty} = \sup_{j\in Z} |U_j|.$$

For $\forall r > 0$, let $H^r(R) = W^{r,2}(R)$ be Sobolev space. Define $H_p^r(\Omega)$ as a subspace composed by periodic functions with period $L = b - a$ on $H^r(R)$, and

$$H_p^r(\Omega) = \left\{u | u \in H^r(R), u(x+a) = u(x+b)\right\}.$$

Let equivalent norm and semi-norm of $H_p^r(\Omega)$ be

$$\|u\|_r = \left[\sum_{l=-\infty}^{\infty} \left(1 + |l|^2\right)^r |\hat{u}_l|^2\right]^{1/2},$$

$$|u|_r = \left[\sum_{l=-\infty}^{\infty} |l|^{2r} |\hat{u}_l|^2\right]^{1/2},$$

where

$$u(x) = \sum_{l=-\infty}^{\infty} \hat{u}_l e^{il\mu(x-a)}, \qquad \hat{u}_l = \frac{1}{b-a}\int_\Omega u(x)e^{-il\mu(x-a)}\,dx, \quad \mu = \frac{2\pi}{L}.$$

Denote the orthogonal projector $P_J : L^2(\Omega) \to V_J$, where

$$V_J = \left\{u(x) = \sum_{|k|\le J/2} \hat{u}_k e^{ik\mu(x-a)}\right\}.$$

We have following conclusions:

$$P_J\partial_x u = \partial_x P_J u, \quad P_J u = u, \forall u \in V_J.$$

Denote the interpolation operator $I_J : L^2(\Omega) \to V_J''$ by

$$I_J u(x,t) = \sum_{j=0}^{J-1} u_j g_j(x),$$

where

$$V_J'' = \left\{ u(x) = \sum_{|l|\le J/2} \tilde{u}_l e^{il\mu(x-a)}, \tilde{u}_{J/2} = \tilde{u}_{-J/2} \right\},$$

$$\tilde{u}_l = \frac{1}{Jc_l}\sum_{j=0}^{J-1} u(x_j)e^{-ik(x_j-a)},$$

$$g_j(x) = \frac{1}{J}\sum_{l=-\frac{J}{2}}^{\frac{J}{2}-1} \frac{1}{c_l} e^{il\mu(x-x_j)}, \quad c_l = 1 \left(|l| \neq \frac{J}{2}\right), c_{\frac{J}{2}} = c_{-\frac{J}{2}} = 2. \tag{8}$$

We have the following conclusions:

$$I_J\partial_x u \neq \partial_x I_J u, \quad I_J u = u, \forall u \in V_J''.$$

The values for the derivatives $I_J u(x,t)$ at the collocation points x_j are obtained by [21]

$$\frac{\partial^k I_J u(x_i,t)}{\partial x^k} = \sum_{j=0}^{J-1} u_j \frac{d^k g_j(x_i)}{dx^k} = (D_k u)_i,$$

$$(D_k)_{j,n} = \frac{d^k g_n(x_j)}{dx^k}, \tag{9}$$

where D_k represents Fourier spectral differential matrix.

Lemma 2.1 ([21]) *Let $r > 0, u \in H_p^r(\Omega)$,*

$$\|P_J u - u\|_l \le CJ^{l-r}|u|_r, \quad 0 \le l \le r,$$

$$\|P_J u\|_l \le C\|u\|_l.$$

Lemma 2.2 ([21]) *Let $r > \frac{1}{2}, u \in H_p^r(\Omega)$,*

$$\|I_J u - u\|_l \le CJ^{l-r}|u|_r, \quad 0 \le l \le r,$$

$$\|I_J u\|_l \le C\|u\|_l.$$

Lemma 2.3 ([21]) *Assume $u^* = P_{J-2}u, u \in H_p^r(\Omega), r > \frac{1}{2}$, then $\|u^* - u\| \le CJ^{-r}|u|_r$.*

Lemma 2.4 ([22]) *For any discrete function $U \in w$, we can obtain*

$$\|U\|_{l^p} \le C\big(|U|_{h,1}^{\alpha}\|U\|^{1-\alpha} + \|U\|\big), \tag{10}$$

where $\alpha = \frac{1}{2} - \frac{1}{p}, p \in [2,+\infty)$, and C is a constant independent of h.

Here, we define a new seim-norm $|U|_h = \sqrt{\langle -D_2U, U\rangle}$. Noting that $|U|_{h,1} = \sqrt{\langle -AU, U\rangle}$, where

$$A = \frac{1}{h^2}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & \cdots & 0 & -1 & 2 & -1 \\ -1 & \cdots & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Lemma 2.5 ([23]) *For a real circulant matrix $A_1 = C(a_0, a_1, \dots, a_{n-1})$, all eigenvalues of A_1 are given by*

$$f(\varepsilon_k), \quad k = 0, 1, 2, \dots, n-1,$$

where $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$, and $\varepsilon_k = e^{i\frac{2k\pi}{n}}$.

Lemma 2.6 ([23]) *For matrices A, D_2, there exist relations*

$$A = F_1^T\Lambda_1F_1, \qquad D_2 = F_2^T\Lambda_2F_2,$$

where

$$\Lambda_1 = \operatorname{diag}(\lambda_{A,0}, \lambda_{A,1}, \dots, \lambda_{A,J-1}),$$
$$\Lambda_2 = \operatorname{diag}(\lambda_{D_2,0}, \lambda_{D_2,1}, \dots, \lambda_{D_2,J-1})$$

and

$$0 \le -\frac{4}{\pi^2}\lambda_{D_2,j} \le -\lambda_{A,j} \le -\lambda_{D_2,j}.$$

Lemma 2.7 ([12]) *For any grid function $U \in V_J''$, the following inequalities hold*:

$$|U|_{h,1} \le |U|_h \le \frac{\pi}{2}|U|_{h,1}.$$

Lemma 2.8 *For any grid functions $U^n \in w$, we can obtain*

$$\operatorname{Im}\big\langle D_2U^{n+\frac{1}{2}}, U^{n+\frac{1}{2}}\big\rangle = 0, \tag{11}$$

$$\operatorname{Re}\big\langle D_2U^{n+\frac{1}{2}}, U_t^n\big\rangle = -\frac{1}{2\tau}\big(\big|U^{n+1}\big|_h^2 - \big|U^n\big|_h^2\big), \tag{12}$$

where $\operatorname{Im}(s), \operatorname{Re}(s)$ are for the imaginary part and the real part of a complex number s, respectively.

Lemma 2.9 ([12]) *Suppose that $g(x) \in C^2[d_1, d_2]$ and $a_1, a_2, b_1, b_2 \in [d_1, d_2]$, there exist constants $\theta \in (-1, 1)$ and $\eta \in [d_1, d_2]$ such that*

$$\begin{aligned}\frac{g(a_2)-g(a_1)}{a_2-a_1} - \frac{g(b_2)-g(b_1)}{b_2-b_1} &= g'\left(\frac{1-\theta}{2}a_1 + \frac{1+\theta}{2}a_2\right) - g'\left(\frac{1-\theta}{2}b_1 + \frac{1+\theta}{2}b_2\right) \\ &= g''(\eta)\left(\frac{1-\theta}{2}(a_1-b_1) + \frac{1+\theta}{2}(a_2-b_2)\right).\end{aligned}$$

2.2 Conservative Fourier spectral scheme

In the paper, we give following conservative scheme for the CSB system:

$$i(U_j^n)_t + (D_2 U^{n+\frac{1}{2}})_j - U_j^{n+\frac{1}{2}} V_j^{n+\frac{1}{2}} = 0, \quad 0 \le j \le J-1, 0 \le n \le N-1, \tag{13}$$

$$(V_j^n)_t = (D_2 \Phi^{n+\frac{1}{2}})_j, \quad 0 \le j \le J-1, 0 \le n \le N-1, \tag{14}$$

$$\begin{aligned}(\Phi_j^n)_t &= V_j^{n+\frac{1}{2}} + \frac{F(V_j^{n+1}) - F(V_j^n)}{V_j^{n+1} - V_j^n} - \alpha(D_2 V^{n+\frac{1}{2}})_j + \frac{\omega}{2}(|U_j^{n+1}|^2 + |U_j^n|^2), \\ &\quad 0 \le j \le J-1, 1 \le n \le N-1,\end{aligned} \tag{15}$$

$$U_j^0 = u_0(x_j), \qquad V_j^0 = v_0(x_j), \qquad \Phi_j^0 = \phi_0(x_j), \quad 0 \le j \le J, \tag{16}$$

$$U_0^n = U_J^n, \qquad V_0^n = V_J^n, \qquad \Phi_0^n = \Phi_J^n, \quad 0 \le n \le N. \tag{17}$$

Theorem 2.1 *The scheme (13)–(17) is conservative in the sense*

$$\begin{aligned}&\text{Mass}: \quad M^n = M^{n-1} = \cdots = M^0, \\ &\text{Energy}: \quad E^n = E^{n-1} = \cdots = E^0,\end{aligned}$$

where

$$\begin{aligned}M^n &= \|U^n\|^2, \\ E^n &= \|V^n\| - \langle D_2\Phi^n, \Phi^n\rangle - 2\omega\langle D_2 U^n, U^n\rangle - \alpha\langle D_2 V^n, V^n\rangle + 2\langle F(V^n), I\rangle + 2\omega\langle V^n, |U^n|^2\rangle.\end{aligned}$$

Proof Computing the inner product of Eq. (13) with $2U^{n+\frac{1}{2}}$, we can obtain

$$\langle iU_t^n + D_2 U^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} V^{n+\frac{1}{2}}, 2U^{n+\frac{1}{2}}\rangle = 0.$$

By Lemma 2.8 and taking the imaginary part, we have

$$\|U^{n+1}\|^2 = \|U^n\|^2.$$

This means that $M^n = M^{n-1} = \cdots = M^0$.

Computing the inner product of Eq. (13) with $2\tau U_t^n$, we can obtain

$$\langle iU_t^n + D_2 U^{n+\frac{1}{2}} + U^{n+\frac{1}{2}} V^{n+\frac{1}{2}}, 2\tau U_t^n\rangle = 0. \tag{18}$$

Taking the real part yields

$$\langle D_2U^{n+1}, U^{n+1}\rangle - \langle D_2U^n, U^n\rangle = \frac{1}{2}\langle V^{n+1} + V^n, |U^{n+1}|^2\rangle - \frac{1}{2}\langle V^{n+1} + V^n, |U^n|^2\rangle. \tag{19}$$

Making the inner product of (14) and (15) with $2\tau\Phi_t^n, 2\tau V_t^n$, respectively, then we have

$$\begin{aligned}
2\tau\langle V_t^n, \Phi_t^n\rangle &= \langle D_2\Phi^{n+1}, \Phi^{n+1}\rangle - \langle D_2\Phi^n, \Phi^n\rangle, \\
2\tau\langle \Phi_t^n, V_t^n\rangle &= \left\langle V^{n+\frac{1}{2}} + \frac{F(V^{n+1}) - F(V^n)}{V^{n+1} - V^n} - \alpha\left(D_2V^{n+\frac{1}{2}}\right) + \frac{\omega}{2}\left(|U^{n+1}|^2 + |U^n|^2\right), 2\tau V_t^n\right\rangle \\
&= \|V^{n+1}\|^2 + \omega\langle V^{n+1}, |U^{n+1}|^2\rangle - \alpha\langle D_2V^{n+1}, V^{n+1}\rangle \\
&\quad + 2\langle F(V^{n+1}), I\rangle - \omega\langle V^n, |U^{n+1}|^2\rangle \\
&\quad - \|V^n\|^2 - \omega\langle V^n, |U^n|^2\rangle + \alpha\langle D_2V^n, V^n\rangle + 2\langle F(V^n), I\rangle - \omega\langle V^{n+1}, |U^n|^2\rangle.
\end{aligned}$$

It follows from the above equations and (19) that

$$\begin{aligned}
&\|V^{n+1}\| - \langle D_2\Phi^{n+1}, \Phi^{n+1}\rangle - 2\omega\langle D_2U^{n+1}, U^{n+1}\rangle - \alpha\langle D_2V^{n+1}, V^{n+1}\rangle \\
&\quad + 2\langle F(V^{n+1}), I\rangle + 2\omega\langle V^{n+1}, |U^{n+1}|^2\rangle \\
&= \|V^n\| - \langle D_2\Phi^n, \Phi^n\rangle - 2\omega\langle D_2U^n, U^n\rangle - \alpha\langle D_2V^n, V^n\rangle + 2\langle F(V^n), I\rangle + 2\omega\langle V^n, |U^n|^2\rangle.
\end{aligned}$$

This yields $E^n = E^{n-1} = \cdots = E^0$. □

Theorem 2.2 *The scheme* (13)–(17) *is bounded in the discrete* l_h^∞.

Proof It follows from Theorem 2.1 that $\|U^n\| = C_1$, and

$$\begin{aligned}
E^n &= \|V^n\| - \langle D_2\Phi^n, \Phi^n\rangle - 2\omega\langle D_2U^n, U^n\rangle - \alpha\langle D_2V^n, V^n\rangle \\
&\quad + 2\langle F(V^n), I\rangle + 2\omega\langle V^n, |U^n|^2\rangle.
\end{aligned} \tag{20}$$

According to the Young inequality, we can obtain

$$\langle V^n, |U^n|^2\rangle = h\sum_{j=0}^{J-1}|U_j^n|^2V_j^n \le h\sum_{j=0}^{J-1}\left(|U_j^n|^4 + \frac{1}{4}|V_j^n|^2\right) = \|U^n\|_{l_h^4}^4 + \frac{1}{4}\|V^n\|^2. \tag{21}$$

It follows from Lemma 2.4 that

$$\begin{aligned}
\|U^n\|_{l_h^4}^4 &\le C_2\left(|U^n|_{h,1}^{\frac{1}{4}}\|U^n\|^{\frac{3}{4}} + \|U^n\|\right)^4 \le 8C_2\left(|U|_{h,1}\|U^n\|^3 + \|U^n\|^4\right) \\
&\le 4C_2\left(\epsilon|U^n|_{h,1}^2 + \frac{1}{\epsilon}\|U^n\|^6 + 2\|U^n\|^4\right).
\end{aligned} \tag{22}$$

By Lemma 2.7, there exist constants C_3, C_4, C_5 such that

$$\begin{aligned}
&\langle -D_2\Phi^n, \Phi^n\rangle = C_3|\Phi^n|_{h,1}^2, \\
&2\omega\langle -D_2U^n, U^n\rangle = C_4|U^n|_{h,1}^2, \qquad \alpha\langle -D_2V^n, V^n\rangle = C_5|V^n|_{h,1}^2.
\end{aligned} \tag{23}$$

Substituting Eqs. (21)–(23) into (20), we have

$$\begin{aligned}
&\|V^n\|^2 + C_3|\Phi^n|_{h,1}^2 + C_4|U^n|_{h,1}^2 + C_5|V^n|_{h,1}^2 \\
&\quad \le E^n + 2\omega\left|\langle V^n, |U^n|^2\rangle\right| \\
&\quad \le E^n + 2\omega\left(\|U^n\|_{l_h^4}^4 + \frac{1}{2}\|V^n\|^2\right) \\
&\quad \le E^n + 8\omega C_2\left(\epsilon|U^n|_{h,1}^2 + \frac{1}{\epsilon}\|U^n\|^6 + 2\|U^n\|^4\right) + \frac{1}{4}\|V^n\|^2.
\end{aligned}$$

When $\epsilon < \frac{C_4}{8\omega C_2}$, there exists a constant C_6 such that

$$\|V^n\| \le C_6, \qquad |\Phi^n|_{h,1}^2 \le C_6, \qquad |U^n|_{h,1}^2 \le C_6, \qquad |V^n|_{h,1}^2 \le C_6.$$

By the Sobolev inequality, we have $\|U^n\|_{l_h^\infty} \le C_7$, $\|V^n\|_{l_h^\infty} \le C_7$, $\|\Phi^n\|_{l_h^\infty} \le C_7$. □

2.3 Existence and uniqueness

Theorem 2.3 *The numerical solutions of the scheme* (13)–(17) *exist.*

Proof Let $U = (U_0, U_1, \ldots, U_{J-1})^T$, $V = (V_0, V_1, \ldots, V_{J-1})^T$, $\Phi = (\Phi_0, \Phi_1, \ldots, \Phi_{J-1})^T$ and $X = (U^T, V^T, \Phi^T)^T$. Define a mapping $T_\lambda: R^{3J-3} \to R^{3J-3}$ with parameter $\lambda \in (0,1)$,

$$i\frac{U_j - u_j^n}{\tau} + \frac{\lambda}{2}\left(D_2\left(U + u^n\right)\right)_j + \frac{\lambda}{4}\left(U_j + u_j^n\right)\left(V_j + v_j^n\right) = 0, \quad 0 \le j \le J-1, \tag{24}$$

$$\frac{V_j - v_j^n}{\tau} = \frac{\lambda}{2}\left(D_2\left(\Phi + \phi^n\right)\right)_j, \quad 0 \le j \le J-1, \tag{25}$$

$$\begin{aligned}
\frac{\Phi_j - \phi_j^n}{\tau} &= \frac{\lambda}{2}\left(V_j + v_j^n\right) + \lambda\frac{F(V_j) - F(v_j^n)}{V_j - v_j^n} - \lambda\alpha\left(D_2\left(V + v^n\right)\right)_j + \frac{\omega\lambda}{2}\left(|U_j|^2 + |u_j^n|^2\right), \\
&\quad 0 \le j \le J-1.
\end{aligned} \tag{26}$$

It ie easy to see that the mapping $T_\lambda(X)$ is continuous, and $T_0(X)$ is a fixed point for any $X \in R^{3J-3}$. Next, we prove that X is uniformly bounded. Similar to the proof of Theorem 2.1, we have

$$\|U\|^2 = C_8.$$

Thus, U is uniformly bounded. Then we prove the uniformly bounded V and Φ. Computing the inner product of Eq. (25) and (26) with $\alpha\tau(V - v^n)$, $\tau(\Phi - \phi^n)$, respectively, we obtain

$$\left\langle \frac{V - v^n}{\tau}, \alpha\tau\left(V - v^n\right)\right\rangle = \left\langle \frac{\lambda}{2}\left(D_2\left(\Phi + \phi^n\right)\right), \alpha\tau\left(V - v^n\right)\right\rangle, \tag{27}$$

$$\begin{aligned}
\left\langle \frac{\Phi - \phi^n}{\tau}, \tau\left(\Phi - \phi^n\right)\right\rangle &= \left\langle \frac{\lambda}{2}\left(V + v^n\right) + \lambda\frac{F(V) - F(v^n)}{V - v^n} - \alpha\lambda\left(D_2\left(V + v^n\right)\right)\right. \\
&\quad \left. + \frac{\omega\lambda}{2}\left(|U|^2 + |u^n|^2\right), \tau\left(\Phi - \phi^n\right)\right\rangle.
\end{aligned} \tag{28}$$

Addition (27) and (28) yields

$$
\begin{aligned}
&\left(\alpha\|V\|^2+\|\Phi\|^2\right)-\left(\alpha\left\|v^n\right\|^2+\left\|\phi^n\right\|^2\right)\\
&\quad=\frac{\lambda\tau}{2}h\sum_{j=0}^{J-1}\left(V_j+v_j^n\right)\cdot\left(\Phi_j+\phi_j^n\right)\\
&\qquad+\frac{\lambda\omega\tau}{2}h\sum_{j=1}^{J-1}\left(|U_j|^2+\left|u_j^n\right|^2\right)\cdot\left(\Phi_j+\phi_j^n\right)++\lambda\tau h\sum_{j=1}^{J-1}\left(\frac{F(V_j)-F(v_j^n)}{V_j-v_j^n}\right)\cdot\left(\Phi_j+\phi_j^n\right).
\end{aligned}
$$

Similar to the proof of [12], we get

$$
\alpha\|V\|^2+\|\Phi\|^2=\lambda\tau C_9\left(\varepsilon\|U\|_{h,1}+C(\varepsilon)\|U\|^2\right)+C_9\le\tau rC_9\|U\|^2+C_9\left(r=\frac{\tau}{h^2}\right).
$$

Thus, $\|V\|$ and $\|\Phi\|$ are uniformly bounded. According to Lemma 4.1 of [12], we see that the numerical solutions of the scheme exist. This completes the proof. □

Theorem 2.4 *The numerical solution of the scheme* (13)–(17) *is unique.*

Proof Assume that (U^n,V^n,Φ^n) and $(\widetilde{U}^n,\widetilde{V}^n,\widetilde{\Phi}^n)$ satisfy scheme (13)–(17). Let $P^n=U^n-\widetilde{U}^n,Q^n=V^n-\widetilde{V}^n,S^n=\Phi^n-\widetilde{\Phi}^n$, we obtain

$$
i\frac{P^{n+1}-P^n}{\tau}+D_2P^{n+\frac{1}{2}}-F_1=0, \tag{29}
$$

$$
\frac{Q^{n+1}-Q^n}{\tau}=D_2S^{n+\frac{1}{2}}, \tag{30}
$$

$$
\frac{S^{n+1}-S^n}{\tau}=Q^{n+\frac{1}{2}}-\alpha D_2Q^{n+\frac{1}{2}}+F_2, \tag{31}
$$

where

$$
\begin{aligned}
F_1&=U^{n+\frac{1}{2}}V^{n+\frac{1}{2}}-\widetilde{U}^{n+\frac{1}{2}}\widetilde{V}^{n+\frac{1}{2}},\\
F_2&=\frac{F(V^{n+1}-F(V^n))}{V^{n+1}-V^n}-\frac{F(\widetilde{V}^{n+1}-F(\widetilde{V}^n))}{\widetilde{V}^{n+1}-\widetilde{V}^n}\\
&\quad+\frac{\omega}{2}\left(\left|U^{n+1}\right|^2+\left|U^n\right|^2\right)-\frac{\omega}{2}\left(\left|\widetilde{U}^{n+1}\right|^2+\left|\widetilde{U}^n\right|^2\right).
\end{aligned}
$$

Computing the inner product of (29) with $2P^{n+\frac{1}{2}}$ and taking the imaginary part, we have

$$
\frac{\|P^{n+1}\|^2-\|P^n\|^2}{\tau}-\operatorname{Im}\left\langle F_1,2P^{n+\frac{1}{2}}\right\rangle=0. \tag{32}
$$

Noting that

$$
\begin{aligned}
(F_1)_j&=U_j^{n+\frac{1}{2}}V_j^{n+\frac{1}{2}}-\widetilde{U}_j^{n+\frac{1}{2}}\widetilde{V}_j^{n+\frac{1}{2}}\\
&=\left(U_j^{n+\frac{1}{2}}-\widetilde{U}_j^{n+\frac{1}{2}}\right)V_j^{n+\frac{1}{2}}+\widetilde{U}_j^{n+\frac{1}{2}}\left(V_j^{n+\frac{1}{2}}-\widetilde{V}_j^{n+\frac{1}{2}}\right),
\end{aligned}
$$

and according to Theorem 2.2, we get

$$\|F_1\|^2 \le C_{10}\left(\left\|P^{n+1}\right\|^2 + \left\|P^n\right\|^2 + \left\|Q^{n+1}\right\|^2 + \left\|Q^n\right\|^2\right).$$

It follows from Eq. (32) that there exists a constant C_{11} such that

$$\left\|P^{n+1}\right\|^2 - \left\|P^n\right\|^2 \le C_{11}\tau\left(\left\|P^{n+1}\right\|^2 + \left\|P^n\right\|^2 + \left\|Q^{n+1}\right\|^2 + \left\|Q^n\right\|^2\right). \tag{33}$$

Computing the inner product of (30) and (31) with $2\alpha Q^{n+\frac{1}{2}}, 2S^{n+\frac{1}{2}}$, we have

$$\left\langle \frac{Q^{n+1} - Q^n}{\tau}, 2\alpha Q^{n+\frac{1}{2}} \right\rangle = \left\langle D_2 S^{n+\frac{1}{2}}, 2\alpha Q^{n+\frac{1}{2}} \right\rangle, \tag{34}$$

$$\left\langle \frac{S^{n+1} - S^n}{\tau}, 2S^{n+\frac{1}{2}} \right\rangle = \left\langle Q^{n+\frac{1}{2}} - \alpha D_2 Q^{n+\frac{1}{2}} + F_2, 2S^{n+\frac{1}{2}} \right\rangle. \tag{35}$$

According to Lemma 2.9, and adding Eqs. (34) and (35), we get

$$\begin{aligned} &\left\|Q^{n+1}\right\|^2 - \left\|Q^n\right\|^2 + \left\|S^{n+1}\right\|^2 - \left\|S^n\right\|^2 \\ &\quad \le C_{12}\tau\left(\left\|P^{n+1}\right\|^2 + \left\|P^n\right\|^2 + \left\|Q^{n+1}\right\|^2 + \left\|Q^n\right\|^2 + \left\|S^{n+1}\right\|^2 + \left\|S^n\right\|^2\right). \end{aligned} \tag{36}$$

Noting that (28) holds, we have

$$\begin{aligned} &\left\|P^{n+1}\right\|^2 - \left\|P^n\right\|^2 + \left\|Q^{n+1}\right\|^2 - \left\|Q^n\right\|^2 + \left\|S^{n+1}\right\|^2 - \left\|S^n\right\|^2 \\ &\quad \le C_{13}\tau\left(\left\|P^{n+1}\right\|^2 + \left\|P^n\right\|^2 + \left\|Q^{n+1}\right\|^2 + \left\|Q^n\right\|^2 + \left\|S^{n+1}\right\|^2 + \left\|S^n\right\|^2\right). \end{aligned} \tag{37}$$

Let $B^n = \|P^n\|^2 + \|Q^n\|^2 + \|S^n\|^2$, then

$$B^{n+1} - B^n \le C_{13}\tau\left(B^{n+1} + B^n\right).$$

It follows from Gronwall's inequality [24] that

$$\max_{1\le n\le N} B^n \le B^0 e^{4C_{13}T}.$$

Noting that $P^0 = Q^0 = S^0 = 0$, then we get $P^n = Q^n = S^n = 0$. This completes the proof of the uniqueness for U^n, V^n, Φ^n. □

2.4 Convergence and error estimates

Theorem 2.5 *Suppose that $u_0(x), v_0(x), \phi_0(x) \in H^r(R), s \ge 1$. Then the solution U^n, V^n, Φ^n of the scheme (13)–(17) converges to the true solution u, v, ϕ with order $O(\tau^2 + J^{-r})$ by the $\|\cdot\|_1$ norm.*

Proof Define

$$P_{J-2}(iu_t + u_{xx} - uv) = 0, \tag{38}$$

$$P_{J-2}v_t = P_{J-2}\phi_{xx}, \tag{39}$$

$$P_{J-2}\big(\phi_t - v + \alpha v_{xx} - f(v) - \omega|u|^2\big) = 0. \tag{40}$$

Let $u^* = P_{J-2}u, v^* = P_{J-2}v, \phi^* = P_{J-2}\phi$, we have

$$iu^*_t + u^*_{xx} - P_{J-2}(uv) = 0, \tag{41}$$

$$v^*_t = \phi^*_{xx}, \tag{42}$$

$$\phi^*_t - v^* + \alpha v^*_{xx} - P_{J-2}\big(f(v) - \omega|u|^2\big) = 0. \tag{43}$$

Define

$$\xi^n_j = i\big(u^{*n}_j\big)_t + \big(D_2 u^{*n+\frac{1}{2}}\big)_j - P_{J-2}\big(u^{n+\frac{1}{2}}_j v^{n+\frac{1}{2}}_j\big),$$

$$\eta^n_j = \big(v^{*n}_j\big)_t - \big(D_2\phi^{*n+\frac{1}{2}}\big)_j,$$

$$\rho^n_j = \big(\phi^{*n}_j\big)_t - v^{*n+\frac{1}{2}} + \alpha\big(D_2 v^{*n+\frac{1}{2}}\big)_j - P_{J-2}\left(\frac{F(v^{n+1}_j) - F(v^n_j)}{v^{n+1}_j - v^n_j}\right) - \frac{\omega}{2}\big(\big|u^{n+1}\big|^2 + \big|u^n\big|^2\big).$$

Due to $u^*, v^*, \phi^* \in V''_N, u^*_{xx}(x_j, t_n) = (D_2u^{*n})_j, v^*_{xx}(x_j, t_n) = (D_2v^{*n})_j, \phi^*_{xx}(x_j, t_n) = (D_2\phi^{*n})_j$, we have

$$u^*_{xx}(x_j, t_n) = \big(D_2u^{*n+\frac{1}{2}}\big)_j + C_{14}\tau^2,$$

$$\phi^*_{xx}(x_j, t_n) = \big(D_2\phi^{*n+\frac{1}{2}}\big)_j + C_{14}\tau^2,$$

$$v^*_{xx}(x_j, t_n) = \big(D_2v^{*n+\frac{1}{2}}\big)_j + C_{14}\tau^2.$$

Using the Taylor expansion, we have

$$\big|\xi^n_j\big| \le C_{15}\tau^2, \qquad \big|\eta^n_j\big| \le C_{15}\tau^2, \qquad \big|\rho^n_j\big| \le C_{15}\tau^2. \tag{44}$$

Define $e^n_1 = (u^*)^n - U^n, e^n_2 = (v^*)^n - V^n, e^n_3 = (\phi^*)^n - \Phi^n$ we have

$$\xi^n = i\big(e^n_1\big)_t + D_2 e^{n+\frac{1}{2}}_1 - F^{n+\frac{1}{2}}, \quad n = 0, 1, 2, \ldots, \tag{45}$$

$$\eta^n = \big(e^n_2\big)_t - D_2 e^{n+\frac{1}{2}}_3, \quad n = 0, 1, 2, \ldots, \tag{46}$$

$$\rho^n = \big(e^n_3\big)_t - e^{n+\frac{1}{2}}_2 + \alpha D_2 e^{n+\frac{1}{2}}_2 - G^{n+\frac{1}{2}}, \quad n = 0, 1, 2, \ldots, \tag{47}$$

$$e^0_1 = u^{*0} - U^0, \qquad e^0_2 = v^{*0} - V^0, \qquad e^0_3 = \phi^{*^0} - \Phi^0, \tag{48}$$

where

$$F^{n+\frac{1}{2}} = P_{J-2}\big(u^{n+\frac{1}{2}}\phi^{n+\frac{1}{2}}\big) - U^{n+\frac{1}{2}}\Phi^{n+\frac{1}{2}},$$

$$G^{n+\frac{1}{2}} = P_{J-2}\left(\frac{F(v^{n+1}_j) - F(v^n_j)}{v^{n+1}_j - v^n_j} + \frac{\omega}{2}\big(\big|u^{n+1}\big|^2 + \big|u^n\big|^2\big)\right) - \left(\frac{F(V^{n+1}_j) - F(V^n_j)}{V^{n+1}_j - V^n_j} + \frac{\omega}{2}\big(\big|U^{n+1}_j\big|^2 + \big|U^n_j\big|^2\big)\right).$$

Let

$$F^{n+\frac{1}{2}} = (F_1)^{n+\frac{1}{2}} + (F_2)^{n+\frac{1}{2}} + (F_3)^{n+\frac{1}{2}},$$
$$G^{n+\frac{1}{2}} = (G_1)^{n+\frac{1}{2}} + (G_2)^{n+\frac{1}{2}} + (G_3)^{n+\frac{1}{2}},$$

where

$$\begin{aligned}
(F_1)^{n+\frac{1}{2}} &= P_{J-2}\big(u^{n+\frac{1}{2}}v^{n+\frac{1}{2}}\big) - u^{n+\frac{1}{2}}v^{n+\frac{1}{2}},\\
(F_2)^{n+\frac{1}{2}} &= u^{n+\frac{1}{2}}v^{n+\frac{1}{2}} - u^{*n+\frac{1}{2}}v^{*n+\frac{1}{2}},\\
(F_3)^{n+\frac{1}{2}} &= u^{*n+\frac{1}{2}}v^{*n+\frac{1}{2}} - U^{n+\frac{1}{2}}V^{n+\frac{1}{2}},\\
(G_1)^{n+\frac{1}{2}} &= P_{J-2}\bigg(\frac{F(v_j^{n+1}) - F(v_j^n)}{v_j^{n+1} - v_j^n}\\
&\quad + \frac{\omega}{2}\big(\big|u^{n+1}\big|^2 + \big|u^n\big|^2\big)\bigg) - \bigg(\frac{F(v^{n+1}) - F(v^n)}{v^{n+1} - v^n} + \frac{\omega}{2}\big(\big|u^{n+1}\big|^2 + \big|u^n\big|^2\big)\bigg),\\
(G_2)^{n+\frac{1}{2}} &= \frac{F(v_j^{n+1}) - F(v_j^n)}{v_j^{n+1} - v_j^n} + \frac{\omega}{2}\big(\big|u^{n+1}\big|^2 + \big|u^n\big|^2\big))\\
&\quad - \bigg(\frac{F(v^{*n+1}) - F(v^{*n})}{v^{*n+1} - v^{*n}} + \frac{\omega}{2}\big(\big|u^{*n+1}\big|^2 + \big|u^{*n}\big|^2\big)\bigg),\\
(G_3)^{n+\frac{1}{2}} &= \bigg(\frac{F(v^{*n+1}) - F(v^{*n})}{v^{*n+1} - v^{*n}} + \frac{\omega}{2}\big(\big|u^{*n+1}\big|^2 + \big|u^{*n}\big|^2\big)\bigg)\\
&\quad - \bigg(\frac{F(V_j^{n+1}) - F(V_j^n)}{V_j^{n+1} - V_j^n} + \frac{\omega}{2}\big(\big|U_j^{n+1}\big|^2 + \big|U_j^n\big|^2\big)\bigg).
\end{aligned}$$

It follows from Lemmas 2.1, 2.3 and 2.9 that

$$\begin{aligned}
&\big\|(F_1)^{n+\frac{1}{2}}\big\| \le C_{16}J^{-r}, \qquad \big\|(G_1)^n\big\| \le C_{16}J^{-r},\\
&\big\|(F_2)^{n+\frac{1}{2}}\big\| = \big\|u^{n+\frac{1}{2}}v^{n+\frac{1}{2}} - u^{*n+\frac{1}{2}}v^{*n+\frac{1}{2}}\big\|\\
&\qquad = \big\|u^{n+\frac{1}{2}}\big(v^{n+\frac{1}{2}} - v^{*n+\frac{1}{2}}\big) + \big(u^{n+\frac{1}{2}} - u^{*n+\frac{1}{2}}\big)\big(v^{*n+\frac{1}{2}}\big)\big\| \le C_{17}J^{-r},\\
&\big\|(G_2)^{n+\frac{1}{2}}\big\| = \bigg\|\frac{F(v_j^{n+1}) - F(v_j^n)}{v_j^{n+1} - v_j^n} + \frac{\omega}{2}\big(\big|u^{n+1}\big|^2 + \big|u^n\big|^2\big))\\
&\qquad - \bigg(\frac{F(v^{*n+1}) - F(v^{*n})}{v^{*n+1} - v^{*n}} + \frac{\omega}{2}\big(\big|u^{*n+1}\big|^2 + \big|u^{*n}\big|^2\big)\bigg)\bigg\| \le C_{18}J^{-r},\\
&\big\|(F_3)^{n+\frac{1}{2}}\big\| = C_{19}\big(\big\|e_1^{n+\frac{1}{2}}\big\| + \big\|e_2^{n+\frac{1}{2}}\big\|\big),\\
&\big\|(G_3)^{n+\frac{1}{2}}\big\| = C_{19}\big(\big\|e_1^{n+\frac{1}{2}}\big\| + \big\|e_2^{n+\frac{1}{2}}\big\|\big).
\end{aligned}$$

Computing the inner product of Eq. (45) with $2\tau e_1^{n+\frac{1}{2}}$, then taking the imaginary part, we obtain

$$\big(\big\|e_1^{n+1}\big\|^2 - \big\|e_1^n\big\|^2\big) - \operatorname{Im}\big\langle F_1^{n+\frac{1}{2}} + F_2^{n+\frac{1}{2}} + F_3^{n+\frac{1}{2}}, 2\tau e^{n+\frac{1}{2}}\big\rangle = \big\langle \xi^n, 2\tau e_1^{n+\frac{1}{2}}\big\rangle. \tag{49}$$

Using the Cauchy–Schwartz inequality, we obtain

$$\langle F_1^{n+\frac{1}{2}} + F_2^{n+\frac{1}{2}}, 2\tau e_1^{n+\frac{1}{2}}\rangle \le \tau\left(\|F_1^{n+\frac{1}{2}}\|^2 + \|F_2^{n+\frac{1}{2}}\|^2\right) + \frac{\tau}{2}\left(\|e_1^{n+1}\|^2 + \|e_1^n\|^2\right),$$
$$\left|\mathrm{Im}\langle F_3^{n+\frac{1}{2}}, 2\tau e_1^{n+\frac{1}{2}}\rangle\right| \le C_{20}\tau\left(\|e_1^{n+1}\|^2 + \|e_1^n\|^2 + \|e_2^{n+1}\|^2 + \|e_2^n\|^2\right),$$
$$\langle \xi^n, 2\tau e_1^{n+\frac{1}{2}}\rangle \le \tau\|\xi^n\|^2 + \frac{\tau}{2}\left(\|e_1^{n+1}\|^2 + \|e_1^n\|^2\right).$$

It is easy to see that

$$\begin{aligned}&\left(\|e_1^{n+1}\|^2 - \|e_1^n\|^2\right)\\ &\quad\le C_{21}\tau\left(\|e_1^{n+1}\|^2 + \|e_1^n\|^2 + \|e_2^{n+1}\|^2 + \|e_2^n\|^2 + \|F_1^{n+\frac{1}{2}}\|^2 + \|F_2^{n+\frac{1}{2}}\|^2 + \|\xi^n\|^2\right).\end{aligned} \tag{50}$$

Computing the inner product of Eqs. (46) and (47) with $2\alpha\tau e_2^{n+\frac{1}{2}}, 2\tau e_3^{n+\frac{1}{2}}$, respectively, we obtain

$$\langle \eta^n, 2\alpha\tau e_2^{n+\frac{1}{2}}\rangle = \langle (e_2^n)_t - D_2 e_3^{n+\frac{1}{2}}, 2\alpha\tau e_2^{n+\frac{1}{2}}\rangle, \tag{51}$$
$$\langle \rho^n, 2\tau e_3^{n+\frac{1}{2}}\rangle = \langle (e_3^n)_t - e_2^{n+\frac{1}{2}} + \alpha D_2 e_2^{n+\frac{1}{2}} - G^{n+\frac{1}{2}}, 2\tau e_3^{n+\frac{1}{2}}\rangle. \tag{52}$$

Noting that

$$\langle \eta^n, 2\alpha\tau e_2^{n+\frac{1}{2}}\rangle \le \alpha\tau\|\eta^n\|^2 + \frac{\alpha\tau}{2}\left(\|e_2^{n+1}\|^2 + \|e_2^n\|^2\right),$$
$$\langle \rho^n, 2\tau e_3^{n+\frac{1}{2}}\rangle \le \tau\|\rho^n\|^2 + \frac{\tau}{2}\left(\|e_3^{n+1}\|^2 + \|e_3^n\|^2\right),$$
$$\langle G_1^{n+\frac{1}{2}} + G_2^{n+\frac{1}{2}}, 2\tau e_3^{n+\frac{1}{2}}\rangle \le \tau\left(\|G_1^{n+\frac{1}{2}}\|^2 + \|G_2^{n+\frac{1}{2}}\|^2\right) + \frac{\tau}{2}\left(\|e_3^{n+1}\|^2 + \|e_3^n\|^2\right),$$
$$\left|\langle G_3^{n+\frac{1}{2}}, 2\tau e_3^{n+\frac{1}{2}}\rangle\right| \le C_{22}\tau\left(\|e_1^{n+1}\|^2 + \|e_1^n\|^2 + \|e_2^{n+1}\|^2 + \|e_2^n\|^2 + \|e_3^{n+1}\|^2 + \|e_3^n\|^2\right),$$

and adding (51)–(52), we get

$$\begin{aligned}&\left(\|e_2^{n+1}\|^2 - \|e_2^n\|^2 + \|e_3^{n+1}\|^2 - \|e_3^n\|^2\right)\\ &\quad\le C_{23}\tau\left(\|G_1^{n+\frac{1}{2}}\|^2 + \|G_2^{n+\frac{1}{2}}\|^2 + \|e_1^{n+1}\|^2 + \|e_1^n\|^2 + \|e_2^{n+1}\|^2 + \|e_2^n\|^2\right.\\ &\qquad\left. + \|e_3^{n+1}\|^2 + \|e_3^n\|^2 + \|\eta^n\|^2 + \|\rho^n\|^2\right).\end{aligned}$$

Add the equations and (50), we have

$$\begin{aligned}&\left(\|e_1^{n+1}\|^2 - \|e_1^n\|^2 + \|e_2^{n+1}\|^2 - \|e_2^n\|^2 + \|e_3^{n+1}\|^2 - \|e_3^n\|^2\right)\\ &\quad\le C_{22}\tau\left(\|e_1^{n+1}\|^2 + \|e_1^n\|^2 + \|e_2^{n+1}\|^2 + \|e_2^n\|^2 + \|e_3^{n+1}\|^2 + \|e_3^n\|^2\right.\\ &\qquad\left. + \|\eta^n\|^2 + \|\rho^n\|^2 + \|F_1^{n+\frac{1}{2}}\|^2 + \|F_2^{n+\frac{1}{2}}\|^2 + \|G_1^{n+\frac{1}{2}}\|^2 + \|G_2^{n+\frac{1}{2}}\|^2\right).\end{aligned}$$

Let $B^n = \|e_1^n\|^2 + \|e_2^n\|^2 + \|e_3^n\|^2$, we can obtain

$$B^{n+1} - B^n \le C_{24}\tau\left(\|\xi^n\|^2 + \|\eta^n\|^2 + \|F_1^n\|^2 + \|F_2^n\|^2 + \|G_1^n\|^2 + \|G_2^n\|^2\right)$$

$$+ C_{22}\tau\left(B^{n+1} + B^n\right).$$

It follows from Gronwall's inequality [24] that

$$\max_{1\le n\le N} B^n \le \left(B^0 + C_{25}T\left(J^{-2r} + \tau^4\right)\right)e^{4C_{24}T}.$$

Noting that $B^0 \le C_{26}J^{-r}$, we can obtain

$$\left\|e_1^n\right\| \le C_{27}\left(J^{-r} + \tau^2\right), \qquad \left\|e_2^n\right\| \le C_{27}\left(J^{-r} + \tau^2\right), \qquad \left\|e_3^n\right\| \le C_{27}\left(J^{-r} + \tau^2\right).$$

Finally, we prove $|e_1^n|_h = O(J^{-r} + \tau^2)$, $|e_2^n|_h = O(J^{-r} + \tau^2)$, $|e_3^n|_h = O(J^{-r} + \tau^2)$. Computing the inner product of Eqs. (46) and (47) with $e_3^{n+1} - e_3^n$, $e_2^{n+1} - e_2^n$, respectively, we obtain

$$\left\langle \eta^n, e_3^{n+1} - e_3^n\right\rangle = \left\langle \left(e_2^n\right)_t - D_2 e_3^{n+\frac{1}{2}}, e_3^{n+1} - e_3^n\right\rangle, \tag{53}$$

$$\left\langle \rho^n, e_2^{n+1} - e_2^n\right\rangle = \left\langle \left(e_3^n\right)_t - e_2^{n+\frac{1}{2}} + \alpha D_2 e_2^{n+\frac{1}{2}} - G^{n+\frac{1}{2}}, e_2^{n+1} - e_2^n\right\rangle. \tag{54}$$

It follows from (46) that

$$\left\langle -G^{n+\frac{1}{2}}, e_2^{n+1} - e_2^n\right\rangle = \left\langle -G^{n+\frac{1}{2}}, \tau\eta^n + \tau D_2 e_3^{n+\frac{1}{2}}\right\rangle = \tau\left[\left\langle -G^{n+\frac{1}{2}}, \eta^n\right\rangle + \left\langle -G^{n+\frac{1}{2}}, D_2 e_3^{n+\frac{1}{2}}\right\rangle\right].$$

Noting that

$$\begin{aligned}
&\left|\left\langle -G^{n+\frac{1}{2}}, D_2 e_3^{n+\frac{1}{2}}\right\rangle\right| \le C_{28}\left(\left|G^{n+\frac{1}{2}}\right|_h + \left|e_3^{n+1}\right|_h + \left|e_3^n\right|_h\right),\\
&\left|G^{n+\frac{1}{2}}\right|_h \le C_{28}\left(\left|e_1^{n+1}\right|_h + \left|e_1^n\right|_h + \left|e_2^{n+1}\right|_h + \left|e_2^n\right|_h + O\left(\tau^2 + J^{-r}\right)\right),
\end{aligned}$$

we get

$$\begin{aligned}
&\left|e_2^{n+1}\right|_h^2 - \left|e_2^n\right|_h^2 + \left|e_3^{n+1}\right|_h^2 - \left|e_3^n\right|_h^2\\
&\quad\le C_{29}\left(\left|e_1^{n+1}\right|_h^2 + \left|e_1^n\right|_h^2 + \left|e_2^{n+1}\right|_h^2 + \left|e_2^n\right|_h^2 + \left|e_3^{n+1}\right|_h^2 + \left|e_3^n\right|_h^2\right)\\
&\qquad + O\left(\tau^2 + J^{-r}\right)^2 + \left|\left\langle \eta^n, e_3^{n+1} - e_3^n\right\rangle\right| + \left|\left\langle \rho^n, e_2^{n+1} - e_2^n\right\rangle\right|.
\end{aligned} \tag{55}$$

Computing the inner product of Eqs. (45) with $e_1^{n+1} - e_1^n$, we obtain

$$\left\langle \xi^n, e_1^{n+1} - e_1^n\right\rangle = \left\langle i\left(e_1^n\right)_t + D_2 e_1^{n+\frac{1}{2}} - F^{n+\frac{1}{2}}, e_1^{n+1} - e_1^n\right\rangle. \tag{56}$$

Taking the real part of Eq. (56), we get

$$\operatorname{Re}\left\langle -D_2 e_1^{n+\frac{1}{2}}, e_1^{n+1} - e_1^n\right\rangle = -\operatorname{Re}\left\langle \xi^n, e_1^{n+1} - e_1^n\right\rangle - \operatorname{Re}\left\langle F^{n+\frac{1}{2}}, e_1^{n+1} - e_1^n\right\rangle.$$

It follows from Lemma 2.8 and Eq. (45) that

$$\operatorname{Re}\left\langle -D_2 e_1^{n+\frac{1}{2}}, e_1^{n+1} - e_1^n\right\rangle = \frac{1}{2}\left(\left|e_1^{n+1}\right|_h - \left|e_1^n\right|_h\right),$$

$$\begin{aligned}\mathrm{Re}\langle F^{n+\frac{1}{2}}, e_1^{n+1} - e_1^n\rangle &= \tau\,\mathrm{Re}\langle F^{n+\frac{1}{2}}, iD_2 e_1^{n+\frac{1}{2}} - iF^{n+\frac{1}{2}} - i\xi^n\rangle \\ &= \tau\,\mathrm{Im}\langle F^{n+\frac{1}{2}}, D_2 e_1^{n+\frac{1}{2}} - \xi^n\rangle.\end{aligned}$$

Then we get

$$\begin{aligned}&\left|e_1^{n+1}\right|_h^2 - \left|e_1^n\right|_h^2 \\ &\quad = C_{30}\left(\left|e_1^{n+1}\right|_h^2 + \left|e_1^n\right|_h^2 + \left|e_2^{n+1}\right|_h^2 + \left|e_2^n\right|_h^2\right) + O\left(\tau^2 + J^{-r}\right)^2 + \left|\langle \xi^n, e_1^{n+1} - e_1^n\rangle\right|.\end{aligned} \tag{57}$$

Adding Eqs. (55)–(57), we get

$$\begin{aligned}&\left|e_1^{n+1}\right|_h^2 - \left|e_1^n\right|_h^2 + \left|e_2^{n+1}\right|_h^2 - \left|e_2^n\right|_h^2 + \left|e_3^{n+1}\right|_h^2 - \left|e_3^n\right|_h^2 \\ &\quad \le C_{31}\left(\left|e_1^{n+1}\right|_h^2 + \left|e_1^n\right|_h^2 + \left|e_2^{n+1}\right|_h^2 + \left|e_2^n\right|_h^2 + \left|e_3^{n+1}\right|_h^2 + \left|e_3^n\right|_h^2\right) \\ &\qquad + O\left(\tau^2 + J^{-r}\right)^2 + \left|\langle \xi^n, e_1^{n+1} - e_1^n\rangle\right| + \left|\langle \eta^n, e_3^{n+1} - e_3^n\rangle\right| + \left|\langle \rho^n, e_2^{n+1} - e_2^n\rangle\right|.\end{aligned}$$

According to Gronwall's inequality [24], we get $|e_1^n|_h = O(J^{-r} + \tau^2)$, $|e_2^n|_h = O(J^{-r} + \tau^2)$, $|e_3^n|_h = O(J^{-r} + \tau^2)$. □

3 Iterative algorithm

In order to derive the algorithms conveniently, we also give some notations:

$$\begin{aligned}U_j^n &= \sum_{l=-\frac{J}{2}}^{\frac{J}{2}-1} \widehat{U}_l^n e^{i\mu_l(x_j-a)}, \quad j = 0, 1, \ldots, J-1, \\ \widehat{U}_j^n &= \frac{1}{J}\sum_{j=0}^{J-1} U_j^n e^{i\mu_l(x_j-a)}, \quad l = \frac{J}{2}, \ldots, \frac{J}{2} - 1,\end{aligned} \tag{58}$$

$$\begin{aligned}V_j^n &= \sum_{l=-\frac{J}{2}}^{\frac{J}{2}-1} \widehat{V}_l^n e^{i\mu_l(x_j-a)}, \quad j = 0, 1, \ldots, J-1, \\ \widehat{V}_j^n &= \frac{1}{J}\sum_{j=0}^{J-1} V_j^n e^{i\mu_l(x_j-a)}, \quad l = \frac{J}{2}, \ldots, \frac{J}{2} - 1,\end{aligned} \tag{59}$$

$$\begin{aligned}\Phi_j^n &= \sum_{l=-\frac{J}{2}}^{\frac{J}{2}-1} \widehat{\Phi}_l^n e^{i\mu_l(x_j-a)}, \quad j = 0, 1, \ldots, J-1, \\ \widehat{\Phi}_j^n &= \frac{1}{J}\sum_{j=0}^{J-1} \Phi_j^n e^{i\mu_l(x_j-a)}, \quad l = \frac{J}{2}, \ldots, \frac{J}{2} - 1.\end{aligned} \tag{60}$$

Substituting (58)–(60) into (13)–(15), we can obtain

$$\begin{aligned}&i\frac{\widehat{U}_l^{n+1} - \widehat{U}_l^n}{\tau} - \frac{\mu_l^2}{2}\left(\widehat{U}_l^{n+1} + \widehat{U}_l^n\right) - \frac{1}{4}\widehat{\left(U_j^{n+1} + U_j^n\right)\left(V_j^{n+1} + V_j^n\right)} = 0, \\ &\frac{\widehat{V}_l^{n+1} - \widehat{V}_l^n}{\tau} = -\frac{\mu_l^2}{2}\left(\widehat{\Phi}_l^{n+1} + \widehat{\Phi}_l^n\right),\end{aligned}$$

$$\frac{\widehat{\Phi}_l^{n+1}-\widehat{\Phi}_l^n}{\tau}=\frac{1}{2}\left(\widehat{V}_l^{n+1}+\widehat{V}_l^n\right)+\frac{\alpha\mu_l^2}{2}\left(\widehat{V}_l^{n+1}+\widehat{V}_l^n\right)-\mu_l^2\frac{\widehat{F(V_j^{n+1})-F(V_j^n)}}{V_j^{n+1}-V_j^n}$$
$$+\mu_l^2\frac{\omega}{2}\widehat{|U_j^{n+1}|^2+|U_j^n|^2}.$$

By direct calculation, we get

$$\widehat{U}_l^{n+1}=\frac{i+\frac{\mu_l^2\tau}{2}}{i-\frac{\mu_l^2\tau}{2}}\widehat{U}_l^n+\frac{1}{4}\widehat{\left(U_j^{n+1}+U_j^n\right)\left(V_j^{n+1}+V_j^n\right)},$$

$$\widehat{V}_l^{n+1}=\frac{1}{\frac{\tau}{2}+\frac{\alpha\mu_l^2\tau}{2}+\frac{2}{\mu_l^2\tau}}\left[\left(\frac{2}{\mu_l^2\tau}-\frac{\tau}{2}-\frac{\alpha\mu_l^2\tau}{2}\right)\widehat{V}_l^n-2\widehat{\Phi}_l^n\right.$$
$$\left.+\mu_l^2\tau\frac{\widehat{F(V_j^{n+1})-F(V_j^n)}}{V_j^{n+1}-V_j^n}-\frac{\mu_l^2\tau\omega}{2}\widehat{|U_j^{n+1}|^2+|U_j^n|^2}\right],$$

$$\widehat{\Phi}_l^{n+1}=\frac{2}{\mu_l^2\tau}\widehat{V}_l^n-\widehat{\Phi}_l^n-\frac{2}{\mu_l^2\tau}\widehat{V}_l^{n+1}.$$

Let $U_j^{n+1,0}=U_j^n$, $V_j^{n+1,0}=V_j^n$, $\Phi_j^{n+1,0}=\Phi_j^n$. We use the following iterative method to solve the algebraic systems:

$$\widehat{U}_l^{n+1,s+1}=\frac{l+\frac{\mu_l^2\tau}{2}}{i-\frac{\mu_l^2\tau}{2}}\widehat{U}_j^n+\frac{1}{4}\widehat{\left(U_j^{n+1,s}+U_j^n\right)\left(V_j^{n+1,s}+V_j^n\right)},$$

$$\widehat{V}_l^{n+1,s+1}=\frac{1}{\frac{\tau}{2}+\frac{\alpha\mu_l^2\tau}{2}+\frac{2}{\mu_l^2\tau}}\left[\left(\frac{2}{\mu_l^2\tau}-\frac{\tau}{2}-\frac{\alpha\mu_l^2\tau}{2}\right)\widehat{V}_l^n-2\widehat{\Phi}_l^n\right.$$
$$\left.+\mu_l^2\tau\frac{\widehat{F(V_j^{n+1,s})-F(V_j^n)}}{V_j^{n+1,s}-V_j^n}-\frac{\mu_l^2\tau\omega}{2}\widehat{|U_j^{n+1,s+1}|^2+|U_j^n|^2}\right],$$

$$\widehat{\Phi}_l^{n+1,s+1}=\frac{2}{\mu_l^2\tau}\widehat{V}_l^n-\widehat{\Phi}_l^n-\frac{2}{\mu_l^2\tau}\widehat{V}_l^{n+1,s+1}.$$

4 Numerical example

Taking $f(v)=\frac{1}{2}v^2, \alpha=1, \omega=\frac{1}{2}$, then we consider the following initial condition:

$$u(x,0)=u_0(x)=\frac{9}{10}\sec h^2\left(\frac{\sqrt{15}}{10}x\right)\exp\left(i\frac{\sqrt{10}}{10}x\right),$$

$$v(x,0)=v_0(x)=-\frac{9}{10}\sec h^2\left(\frac{\sqrt{15}}{10}x\right),$$

$$v_t(x,0)=v_1(x)=-\frac{9\sqrt{6}}{50}\sec h^2\left(\frac{\sqrt{15}}{10}x\right)\tanh\left(\frac{\sqrt{15}}{10}x\right).$$

The forms of the exact solutions are available as

$$u(x,t)=\frac{9}{10}\sec h^2\left(\frac{\sqrt{15}}{10}\left(x-\frac{2\sqrt{10}}{10}t\right)\right)\exp\left(i\left(\frac{\sqrt{10}}{10}x+\frac{1}{2}t\right)\right),$$

Table 1 Errors and orders in time for $J = 1024, t = 1$

		τ	0.05	0.025	0.0125	0.00625
u	l_h^2	error	8.750×10^{-5}	2.188×10^{-5}	5.470×10^{-6}	1.367×10^{-6}
		order	*	1.99	2.00	2.00
	l_h^∞	error	4.179×10^{-5}	1.060×10^{-5}	2.692×10^{-6}	6.765×10^{-7}
		order	*	1.97	1.97	2.00
v	l_h^2	error	1.150×10^{-4}	2.880×10^{-5}	7.211×10^{-6}	1.803×10^{-6}
		order	*	1.99	1.99	1.99
	l_h^∞	error	7.487×10^{-5}	1.92×10^{-5}	4.862×10^{-6}	1.223×10^{-6}
		order	*	1.96	1.98	1.99

Table 2 Errors and orders in space for $\tau = 0.001, t = 1$

		J	64	128	256	512
u	l_h^2	error	5.076×10^{-5}	3.501×10^{-8}	3.5015×10^{-8}	3.5015×10^{-8}
	l_h^∞	error	1.982×10^{-5}	1.425×10^{-8}	1.739×10^{-8}	1.739×10^{-8}
v	l_h^2	error	1.654×10^{-5}	4.615×10^{-8}	4.615×10^{-8}	4.615×10^{-8}
	l_h^∞	error	9.350×10^{-5}	3.146×10^{-8}	3.146×10^{-8}	3.146×10^{-8}

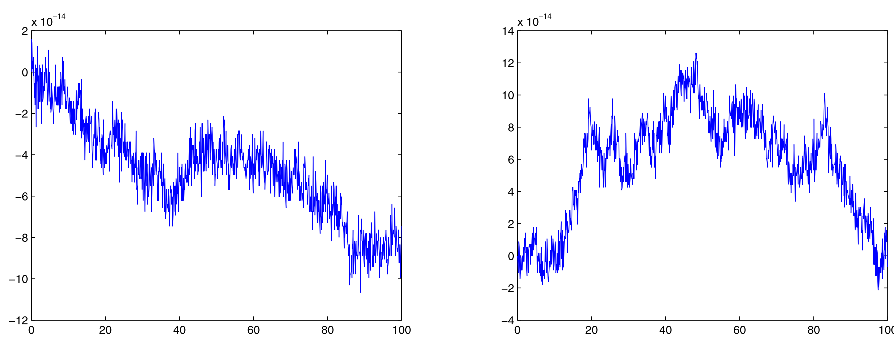

Figure 1 Errors of mass M^n (left column) and energy E^n (right column) of the numerical solutions

$$v(x,t) = -\frac{9}{10}\sec h^2\left(\frac{\sqrt{15}}{10}\left(x - \frac{2\sqrt{10}}{10}t\right)\right),$$

$$v_t(x,t) = -\frac{9\sqrt{6}}{50}\sec h^2\left(\frac{\sqrt{15}}{10}\left(x - \frac{2\sqrt{10}}{10}t\right)\right)\tanh\left(\frac{\sqrt{15}}{10}\left(x - \frac{2\sqrt{10}}{10}t\right)\right).$$

Firstly, the numerical accuracy of the scheme (13)–(17) is examined. Table 1 shows the time errors and the convergence orders in l_h^2 norms and l_h^∞ norms of the scheme (13)–(17), respectively. The data in Table 1 indicate that the scheme (13)–(17) are of second order in time, and confirm the theoretical accuracy in Theorem 3.5. Then we fix the time step $\tau = 0.001$ to test the space accuracy of the scheme (13)–(17). The results are listed in Table 2. From Table 2, it is found that the errors decrease as fast as the number of grid points J increases.

Secondly, we test the numerical performance for the long time computation with $x \in [-15, 15], t \in [0, 100], \tau = 0.01, J = 256$. Figure 1 show the errors of mass M^n and energy E^n at different time for the scheme (13)–(17). From Fig. 1, we find that the scheme (13)–(17) preserves the mass and energy conservation very well.

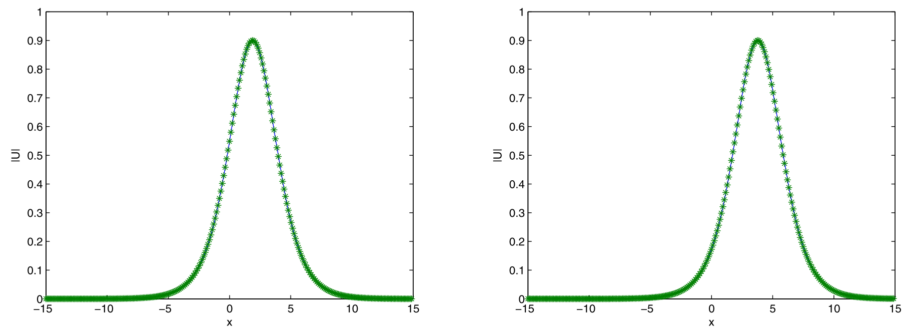

Figure 2 The wave forms of the numerical solutions for $t = 3$

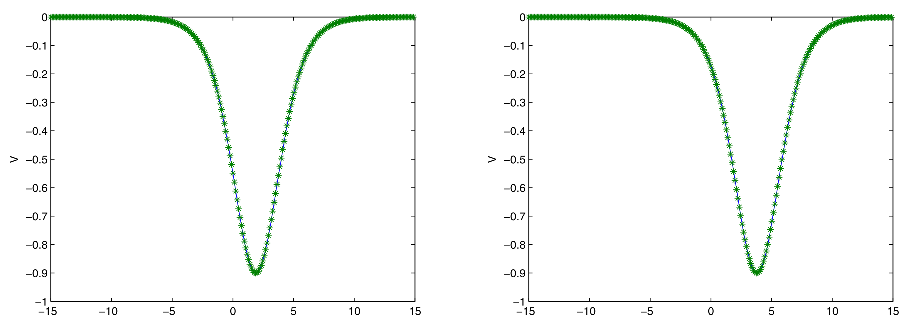

Figure 3 The wave forms of the numerical solutions for $t = 6$

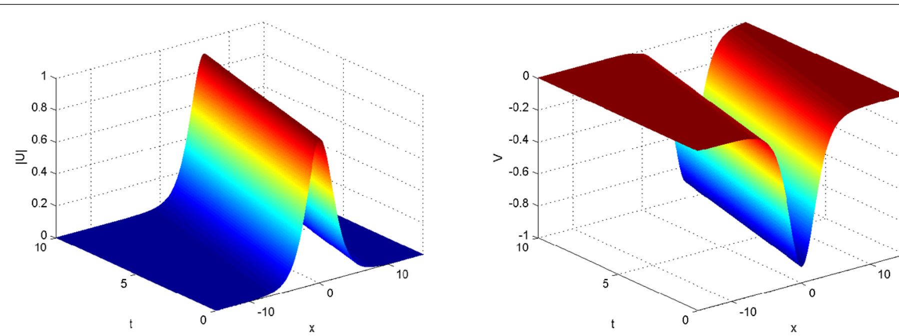

Figure 4 The wave forms of the numerical solutions

Finally, the numerical solutions for the system (13)–(17) are depicted. We simulate the solitary wave solutions with $x \in [-15, 15], t \in [0, 10], \tau = 0.01, J = 256$. Figures 2–4 show the wave forms of the numerical solutions.

5 Conclusion

In the paper, we introduce a conservative Fourier spectral scheme to solve the CSB system. We give the iterative algorithm of the scheme and prove that the scheme preserves the mass and energy conservation laws. The convergence of the scheme is discussed, and it is shown that the scheme is of the accuracy $O(\tau^2 + J^{-r})$. Numerical tests are presented to demonstrate the obtained theoretical results and the method availability.

Funding
This work is supported by Pu'er University innovation team (CXTD003).

Competing interests
The author declares to have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Publisher's Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 1 April 2018 Accepted: 31 August 2018 Published online: 02 November 2018

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