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Existence and concentration of a nonlinear biharmonic equation with sign-changing potentials and indefinite nonlinearity

Qizhen Xiao[1], Hongliang Liu[1*] and Zigen Ouyang[1]

*Correspondence:
math_lhliang@163.com
[1]School of Mathematics and Physics, University of South China, Hengyang, P.R. China

Abstract

We consider the following nonlinear biharmonic equations:

$$\Delta^2 u - \Delta u + V_\lambda(x)u = f(x,u), \quad \text{in } \mathbb{R}^N,$$

where $V_\lambda(x)$ is allowed to be sign-changing and f is an indefinite function. Under some suitable assumptions, the existence of nontrivial solutions and the high energy solutions are obtained by using variational methods. Moreover, the phenomenon of concentration of solutions is explored. The results extend the main conclusions in recent literature.

MSC: 35J20; 35J60

Keywords: Biharmonic equation; Variational methods; High energy solutions; Concentration of solutions

1 Introduction and main results

This paper concerns the existence results and the phenomenon of concentration of solutions for the following biharmonic equation:

$$\Delta^2 u - \Delta u + V_\lambda(x)u = f(x,u), \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\Delta^2 = \Delta(\Delta)$ is the biharmonic operator, f is an indefinite function and the potential $V_\lambda(x) = \lambda V^+(x) - V^-(x)$ with V^+ having a bounded potential well Ω whose depth is controlled by λ and $V^-(x) \geq 0$ for all $x \in \mathbb{R}^N$. Such an equation may arise in many fields of physics, such as describing the traveling waves in suspension bridge [17] and describing the static deflection of an elastic plate in fluid [7]. For more physical background of problem (1.1), we refer the readers to [11] and the references therein.

In the last two decades, the existence of bound states, ground states, semi-classical states (where Δ^2 is replaced by $\varepsilon^4\Delta^2$ for $\varepsilon > 0$ small), and infinitely many nontrivial solutions of biharmonic equations have been widely discussed under various conditions no matter on a bounded domain or on the whole space. Here we just give some references which are close to the problem we consider in this paper. For instance, Yin and Wu [25] studied problem (1.1) with various sets of assumptions on the nonlinearity $f(x,u)$ (superquadraticity, subcriticality, etc.) and under the following conditions imposed on the potential $V(x)$:

- (V_1') $V \in C^1(\mathbb{R}^N,\mathbb{R})$, $\inf_{x\in\mathbb{R}^N} V(x) \geq a > 0$, where a is a constant;
- (V_2') For each $b > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \leq b\} < \infty$, where meas denotes the Lebesgue measure in $\mathbb{R}^N$.

They obtained the existence and infinitely many nontrivial solutions via variational methods. Soon after, Ye and Tang [23] improved these results. Here, we emphasize that conditions (V_1') and (V_2') are usually assumed to guarantee the compact embedding of the working space [33]. However, if (V_2') is replaced by the following more general condition (V_2''), the compactness of the embedding fails and this situation becomes more delicate.

- (V_2'') There exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) \leq b\}$ is nonempty and has finite measure.

As far as we observe, there are few papers concerning this case. We mention that the authors in [16] investigated the existence and multiplicity results of problem (1.1) when conditions (V_1') and (V_2'') hold and the nonlinearity $f(x,u)$ is superlinear at infinity and subcritical. Motivated by [16], Ye and Tang [24] studied problem (1.1) under the following more general case imposed on the potential $V(x)$:

- (V_1'') $V(x) \geq 0$ for all $x \in \mathbb{R}^N$;
- (V_2^*) There exists $b > 0$ such that the set $\{x \in \mathbb{R}^N : V(x) \leq b\}$ has finite measure.

Under conditions (V_1'') and (V_2^*) and more generic superlinear condition upon $f(x,u)$, the authors obtained some results which unify and significantly improve the results in [16]. Moreover, by applying a new version of the symmetric mountain pass lemma, they also investigated infinitely many small-energy solutions of problem (1.1) when the nonlinearity $f(x,u)$ is sublinear with mild assumptions different from [16]. For other interesting results on biharmonic equations, we refer readers to [4, 6, 8, 13–15, 19, 21, 22, 26–30, 32] and the references therein.

However, for most of these papers, the potential $V(x)$ is always assumed to be positive. To the authors' knowledge, there seems to be no result on the existence of solutions to problem (1.1) with sign-changing potential V_λ. Indeed, this is an interesting question, and we mainly consider the following two problems in the present paper:

- (i) The existence results of problem (1.1) when f is indefinite and satisfies the superquadratic linear conditions;
- (ii) The phenomenon of concentration of nontrivial solutions.

In order to give positive answers to the above problems, we shall assume that the potential function $V_\lambda(x) = \lambda V^+(x) - V^-(x)$, where $V^\pm = \max\{\pm V, 0\}$ satisfies the following conditions, which is quite different from the above cited papers.

- $(V1)$ $V_\lambda(x) \in C(\mathbb{R}^N,\mathbb{R})$ and $V_\lambda(x)$ is bounded from below;
- $(V2)$ There exists $b > 0$ such that $\{x \in \mathbb{R}^N \mid V^+(x) < b\}$ is nonempty and has finite measure;
- $(V3)$ $\Omega = \operatorname{int}\{x \in \mathbb{R}^N \mid V^+(x) = 0\}$ is nonempty and has smooth boundary with $\overline{\Omega} = \{x \in \mathbb{R}^N \mid V^+(x) = 0\}$;
- $(V4)$ There exists a constant $\mu_0 > 1$ such that

$$\mu_1(\lambda) := \inf_{u\in H^2(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} [|\Delta u|^2 + |\nabla u|^2 + \lambda V^+(x)u^2]\,dx}{\int_{\mathbb{R}^N} V^-(x)u^2\,dx} \geq \mu_0 \quad \text{for all } \lambda > 0.$$

Here, we point out that conditions $(V1)$–$(V3)$, which imply that $\lambda V^+(x)$ represents a potential well whose depth is controlled by λ, were firstly introduced by Bartsch and Wang [3], in which the authors studied the nonlinear Schrödinger equations. For $\lambda > 0$ large, one expects to find solutions which localize near its bottom Ω. Since the work [3], there have been many papers dealing with problems with potential well in different equations, see e.g. [1, 2, 9, 10, 31]. However, to the best of our knowledge, there seems to be no result on this case of problem (1.1) with sign-changing potential. This is the reason why we explore the phenomenon of concentration of solution in this paper as well.

Remark 1.1 Inspired by [20], we impose condition $(V4)$ in this paper. Obviously, there are cases when condition $(V4)$ is easily verifiable. For example, if one takes a function $V^- \in L^{\frac{2^*}{2^*-2}}(\mathbb{R}^N)$ with $\|V^-\|_{L^{\frac{2^*}{2^*-2}}} < \bar{\bar{S}}^2$, then a direct calculation from $(V1)$–$(V3)$, the fact $H^2(\mathbb{R}^N) \hookrightarrow H^1(\mathbb{R}^N)$, and the Hölder and Sobolev inequalities show that

$$\begin{aligned}\frac{\int_{\mathbb{R}^N}[|\Delta u|^2+|\nabla u|^2+\lambda V^+u^2]\,dx}{\int_{\mathbb{R}^N}V^-u^2\,dx} &\geq \frac{\int_{\mathbb{R}^N}[|\Delta u|^2+|\nabla u|^2+\lambda V^+u^2]\,dx}{\|V^-\|_{L^{\frac{2^*}{2^*-2}}}(\int_{\mathbb{R}^N}u^{2^*}\,dx)^{\frac{2}{2^*}}} \\ &\geq \frac{\int_{\mathbb{R}^N}|\nabla u|^2\,dx}{\|V^-\|_{L^{\frac{2^*}{2^*-2}}}(\int_{\mathbb{R}^N}u^{2^*}\,dx)^{\frac{2}{2^*}}} \\ &\geq \frac{\bar{\bar{S}}^2}{\|V^-\|_{L^{\frac{2^*}{2^*-2}}}} \quad \text{for all } \lambda \geq 0,\end{aligned}$$

which implies that

$$\mu_1(\lambda) \geq \frac{\bar{\bar{S}}^2}{\|V^-\|_{L^{\frac{2^*}{2^*-2}}}} > 1 \quad \text{for all } \lambda \geq 0,$$

where $2^* = \frac{2N}{N-2}$ for $N \geq 3$, $2^* = +\infty$ for $N = 1, 2$ and $\bar{\bar{S}}$ denotes the best Sobolev constant for the embedding of $D^{1,2}(\mathbb{R}^N)$ in $L^{2^*}(\mathbb{R}^N)$.

Before stating our main results, we also need to make some assumptions for the nonlinearity f and its primitive $F(x,u) = \int_0^u f(x,s)\,ds$.

$(F1)$ $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, and there exist $c_1 > 0$, $p \in (2, 2_*)$ such that

$$|f(x,u)| \leq c_1(1+|u|^{p-1}), \quad \text{for all } (x,u) \in (\mathbb{R}^N \times \mathbb{R}),$$

here and hereafter $2_* = \frac{2N}{N-4}$ for $N \geq 5$, $2_* = +\infty$ for $N < 5$;

$(F2)$ $\lim_{|u|\to 0}\frac{f(x,u)}{u} = 0$ uniformly for $x \in \mathbb{R}^N$;

$(F3)$ $\lim_{|u|\to\infty}\frac{F(x,u)}{|u|^q} = +\infty$ uniformly for $q \in (2,p)$ and $x \in \mathbb{R}^N$;

$(F4)$ There exist $\tau > 2$ and $C_2 > 0$ such that

$$\widetilde{F}(x,u) := \frac{1}{\tau}f(x,u) - F(x,u) \to +\infty, \quad \text{as } |u| \to +\infty \text{ uniformly in } x \in \mathbb{R}^N,$$

and

$$\widetilde{F}(x,u_n) \geq -C_2|u_n|^2, \quad \forall x \in \mathbb{R}^N.$$

$(F5)$ $f(x,-u) = -f(x,u)$ for all $(x,u) \in (\mathbb{R}^N \times \mathbb{R})$.

Remark 1.2 There are functions $f(x,u)$ satisfying conditions $(F1)$–$(F5)$ in this paper. For example, let

$$f(x,u)=\begin{cases} g(x)2\ln 2|u|^{p-2}u+\frac{u}{2} & \text{for } |u|>1,\\ g(x)2u\ln(1+|u|)+\frac{|u|u}{1+|u|} & \text{for } |u|\le 1,\end{cases}$$

where $2<p<2_*$ and $g(x)\in C(\mathbb{R}^N\times\mathbb{R})$ is a bounded function with $\inf_{x\in\mathbb{R}^N} g(x)>0$. Then

$$F(x,u)=\begin{cases} g(x)\frac{2\ln 2}{p}|u|^p+\frac{u^2}{4}-\frac{1}{4}+\frac{p-2}{p}\ln 2 & \text{for } |u|>1,\\ g(x)u^2\ln(1+|u|) & \text{for } |u|\le 1.\end{cases}$$

Hence, it is easy to check that conditions $(F1)$–$(F5)$ are satisfied.

Now, we give the main results as follows.

Theorem 1.1 *Assume that* $(V1)$–$(V4)$ *and* $(F1)$–$(F4)$ *are satisfied. There exists a constant* $\Lambda>0$ *such that problem* (1.1) *possesses a nontrivial solution for* $\lambda>\Lambda$.

Theorem 1.2 *Assume that* $(V1)$–$(V4)$ *and* $(F1)$–$(F5)$ *are satisfied. There exists a constant* $\Lambda>0$ *such that, for* $\lambda>\Lambda$, *problem* (1.1) *possesses infinitely many solutions* $\{u_k\}$ *satisfying*

$$\frac{1}{2}\int_{\mathbb{R}^N}\bigl[|\Delta u|^2+|\nabla u|^2+V_\lambda(x)u^2\bigr]\,dx-\int_{\mathbb{R}^N}F(x,u)\,dx\to+\infty \quad \text{as } k\to+\infty.$$

For $\lambda\to\infty$, we have further information on the solution u_λ which is obtained in Theorem 1.1.

Theorem 1.3 *Let* u_λ *be the solution obtained by Theorem* 1.1. *Then* $u_\lambda\to u_0$ *in* $H^2(\mathbb{R}^N)$ *as* $\lambda\to\infty$, *where* $u_0\in H_0^2(\Omega)$ *is the nontrivial solution of*

$$\begin{cases} \Delta^2u-\Delta u-V^-(x)u=f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega.\end{cases} \tag{1.2}$$

It is worth emphasizing that under conditions (V_1') and (V_2'), motivated by Lemma 3.4 in [33], we can prove that the working space $E\hookrightarrow L^s(\mathbb{R}^N)$ is compact for any $s\in[2,2_*)$, where $E:=\{u\in H^2(\mathbb{R}^N):\int_{\mathbb{R}^N}V^+(x)u^2\,dx<\infty\}$. Hence, the corresponding results in the present paper have been obtained by using variational techniques in a standard way [23, 25]. However, under conditions $(V1)$ and $(V2)$, the embedding lacks the compactness. This leads to a difficulty in using variational methods to get solutions of problem (1.1) since some techniques in compact cases do not work. To overcome this obstacle we have to search for other methods. Motivated by Brezis–Lieb lemma [5], we prove that the functional I_λ and its derivative I_λ' possess BL-splitting property (see Lemma 3.2). This important proposition paves the way for us to verify the boundedness of a Cerami sequence. Also, the term $\int_{\mathbb{R}^N}V^-(x)u^2\,dx$ is an issue for employing the variational methods. To get over this difficulty, some new inequalities are established. In addition, we consider the problem with more general potential, which includes the positive case in the aforementioned references. Moreover, from conditions $(F1)$–$(F4)$, one can see that the nonlinearity

$f(x,u)$ and its primitive $F(x,u)$ may change signs. That are the reasons why we say the sign-changing potential and indefinite nonlinearity on the title. Therefore, the corresponding results in the related papers are extended.

The remainder of this paper is organized as follows: In Sect. 2, some preliminaries and variational setting are presented; in Sect. 3, some important lemmas are given while the proofs of the main results are presented in Sect. 4.

2 Preliminaries and variational setting

In the present paper, we use the following notations:

- For any $\rho > 0$ and for any $z \in \mathbb{R}^N$, $B_\rho(z)$ denotes the ball of radius ρ centered at z.
- C and C_i denote various positive constants, which may vary from line to line.
- $\rightarrow$ denotes the strong convergence and $\rightharpoonup$ denotes the weak convergence.
- $o(1)$ denotes any quantity which tends to zero when $n \to \infty$.
- If we take a subsequence of a sequence $\{u_n\}$, we shall denote it again by $\{u_n\}$.
- $L^q(\mathbb{R}^N)$ denotes the weighted space of measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ satisfying

$$\|u\|_q = \left(\int_{\mathbb{R}^N} |u|^q \, dx \right)^{\frac{1}{q}} < \infty.$$

- $H^2(\mathbb{R}^N) := W^{2,2}(\mathbb{R}^N)$ denotes the space with the inner product and norm

$$\langle u, v \rangle_{H^2} = \int_{\mathbb{R}^N} [\Delta u \Delta v + \nabla u \nabla v + uv] \, dx, \qquad \|u\|_{H^2} = \langle u, u \rangle_{H^2}^{\frac{1}{2}}.$$

Followed by [31], set

$$E := \left\{ u \in H^2\bigl(\mathbb{R}^N\bigr) : \int_{\mathbb{R}^N} V^+(x) u^2 \, dx < \infty \right\}$$

be equipped with the inner product and norm

$$\langle u, v \rangle_E = \int_{\mathbb{R}^N} \bigl[\Delta u \Delta v + \nabla u \nabla v + V^+(x) uv\bigr] \, dx, \qquad \|u\|_E = \langle u, u \rangle_E^{\frac{1}{2}}.$$

Under conditions $(V1)$ and $(V2)$, Lemma 2.1 in [24] shows that $E \hookrightarrow H^2(\mathbb{R}^N)$ is continuous, i.e., there exists a positive constant C_E such that

$$\|u\|_{H^2} \le C_E \|u\|_E. \tag{2.1}$$

For $\lambda > 0$, we also need the following inner product and norm:

$$\langle u, v \rangle_\lambda = \int_{\mathbb{R}^N} \bigl[\Delta u \Delta v + \nabla u \nabla v + \lambda V^+(x) uv\bigr] \, dx, \qquad \|u\|_\lambda = \langle u, u \rangle_\lambda^{\frac{1}{2}}.$$

Obviously, $\|u\|_\lambda \ge \|u\|_E$ for all $\lambda \ge 1$. Furthermore, it follows from condition $(V4)$ that

$$\int_{\mathbb{R}^N} \bigl[|\Delta u|^2 + |\nabla u|^2 + V_\lambda(x) u^2\bigr] \, dx \ge \frac{\mu_0 - 1}{\mu_0} \|u\|_\lambda^2 \quad \text{for all } \lambda \ge 0. \tag{2.2}$$

Set $E_\lambda = (E, \|u\|_\lambda)$. For any $r \in [2, 2_*]$ and $\lambda \geq 1$, applying (2.1), $(V1)$, $(V2)$, the Hölder and Sobolev inequalities yields that

$$\begin{aligned}
\int_{\mathbb{R}^N} |u|^r\,dx &\leq \left(\int_{\{V^+(x)\geq b\}} |u|^2\,dx + \int_{\{V^+(x)<b\}} |u|^2\,dx\right)^{\frac{2_*-r}{2_*-2}} \left(\int_{\mathbb{R}^N} |u|^{2_*}\,dx\right)^{\frac{r-2}{2_*-2}} \\
&\leq \left[\frac{1}{\lambda b}\int_{\{V^+(x)\geq b\}} \lambda V^+(x)|u|^2\,dx + \left|\{V^+ < b\}\right|^{\frac{2_*-2}{2_*}} \left(\int_{\mathbb{R}^N} |u|^{2_*}\,dx\right)^{\frac{2}{2_*}}\right]^{\frac{2_*-r}{2_*-2}} \\
&\quad \cdot \left[\bar{S}^{-2_*}\left(\int_{\mathbb{R}^N} \left[|\triangle u|^2 + |\nabla u|^2 + u^2\right] dx\right)^{\frac{2_*}{2}}\right]^{\frac{r-2}{2_*-2}} \\
&\leq \left(\frac{1}{\lambda b}\|u\|_\lambda^2 + \left|\{V^+(x) < b\}\right|^{\frac{2_*-2}{2_*}} \bar{S}^{-2} C_E^2 \|u\|_\lambda^2\right)^{\frac{2_*-r}{2_*-2}} \left(\bar{S}^{-2_*} C_E^{2_*} \|u\|_\lambda^{2_*}\right)^{\frac{r-2}{2_*-2}} \\
&= \left(\frac{1}{\lambda b} + \left|\{V^+(x) < b\}\right|^{\frac{2_*-2}{2_*}} \bar{S}^{-2} C_E^2\right)^{\frac{2_*-r}{2_*-2}} \left(\bar{S}^{-2_*} C_E^{2_*}\right)^{\frac{r-2}{2_*-2}} \|u\|_\lambda^r,
\end{aligned} \tag{2.3}$$

which means that $E_\lambda \hookrightarrow L^r(\mathbb{R}^N)$ is continuous for $r \in [2, 2_*]$, where $\bar{S}$ is the best Sobolev constant for the imbedding of $H^2(\mathbb{R}^N) \hookrightarrow L^{2_*}(\mathbb{R}^N)$.

Define a functional I_λ on E_λ by

$$\begin{aligned}
I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N} \left[|\Delta u|^2 + |\nabla u|^2 + V_\lambda(x)u^2\right] dx - \int_{\mathbb{R}^N} F(x,u)\,dx \\
&= \frac{1}{2}\|u\|_\lambda^2 - \frac{1}{2}\int_{\mathbb{R}^N} V^-(x)u^2\,dx - \int_{\mathbb{R}^N} F(x,u)\,dx, \quad \forall u \in E_\lambda.
\end{aligned} \tag{2.4}$$

Followed by [24], the functional I_λ is of class $C^1(E_\lambda, \mathbb{R})$ and

$$\langle I_\lambda'(u), \varphi\rangle = \int_{\mathbb{R}^N} \left[\triangle u \triangle \varphi + \nabla u \nabla \varphi + V_\lambda(x) u\varphi\right] dx - \int_{\mathbb{R}^N} f(x,u)\varphi\,dx, \quad \forall u, \varphi \in E_\lambda. \tag{2.5}$$

Hence, if $u \in E_\lambda$ is a critical point of I_λ, then u is a solution of problem (1.1).

We shall end this section by giving the following definition and propositions which are applied to prove the main results.

Definition 2.1 Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. For some $c \in \mathbb{R}$, we say I satisfies the $(C)_c$ condition if any sequence $\{u_n\} \subset X$ such that $I(u_n) \to c$ and $\|I'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to \infty$ has a convergent subsequence.

Proposition 2.1 (Mountain Pass Theorem [18]) *Let X be a real Banach space, $I \in C^1(X, \mathbb{R})$ satisfies the $(C)_c$ condition for any $c > 0$, $I(0) = 0$ and*

(i) *there exist $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$;*

(ii) *there exists $e \in E \setminus B_\rho$ such that $I(e) \leq 0$.*

Then I has a critical value $c \geq \alpha$.

Proposition 2.2 (Symmetric Mountain Pass Theorem [18]) *Let X be an infinite dimensional Banach space, and let $I \in C^1(X, \mathbb{R})$ be even, satisfy $(C)_c$ condition and $I(0) = 0$. If $X = V \oplus W$, where V is finite dimensional, and I satisfies*

(A_1) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap W} \geq \alpha$, and*

(A_2) *for each finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X})$ such that $I \leq 0$ on $\widetilde{X} \setminus B_R(\widetilde{X})$.*

Then I possesses an unbounded sequence of critical values.

3 Some lemmas

To verify the main results, we need the following lemmas first.

Lemma 3.1 *Assume that $(V1)$–$(V4)$ and $(F1)$–$(F3)$ hold. Let $e \in E_\lambda$ with $e \neq 0$. Then*

(i) *there exist $\rho, \alpha > 0$ such that $I_\lambda|_{\partial B_\rho} \geq \alpha$;*

(ii) *$I_\lambda(te) \to -\infty$ as $t \to \infty$.*

Proof From $(F1)$ and $(F2)$, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\left|f(x,u)\right| \leq \varepsilon|u| + C(\varepsilon)|u|^{p-1}, \tag{3.1}$$

then

$$\left|F(x,u)\right| \leq \varepsilon|u|^2 + C(\varepsilon)|u|^p. \tag{3.2}$$

It deduces from (2.2), (2.3), (2.4), and (3.2) that

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\left[|\Delta u|^2 + |\nabla u|^2 + V_\lambda(x)u^2\right]dx - \int_{\mathbb{R}^N} F(x,u)\,dx \\ &\geq \frac{\mu_0 - 1}{2\mu_0}\|u\|_\lambda^2 - \int_{\mathbb{R}^N} F(x,u)\,dx \\ &\geq \frac{\mu_0 - 1}{2\mu_0}\|u\|_\lambda^2 - \int_{\mathbb{R}^N}\left[\varepsilon|u|^2 + C(\varepsilon)|u|^p\right]dx \\ &\geq \left[\frac{\mu_0 - 1}{2\mu_0} - \varepsilon\left(\frac{1}{\lambda b} + \left|\left\{V^+(x) < b\right\}\right|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2\right)\right]\|u\|_\lambda^2 - CC(\varepsilon)\|u\|_\lambda^p. \end{aligned} \tag{3.3}$$

Therefore, the conclusion (i) follows from taking $0 < \varepsilon < \frac{\mu_0-1}{2\mu_0}(\frac{1}{\lambda b} + |\{V^+(x) < b\}|^{\frac{2_*-2}{2_*}}\bar{S}^{-2} \times C_E^2)^{-1}$ and choosing $\|u\|_\lambda = \rho$ sufficiently small since $p > 2$.

Next, we shall show the conclusion (ii). From $(F3)$, for any $M > 0$, there exists $\delta = \delta(M) > 0$ such that

$$F(x,u) \geq M|u|^q \quad \text{for all } x \in \mathbb{R}^N \text{and } |u| > \delta. \tag{3.4}$$

By $(F1)$ and $(F2)$, there exists $M_1 = M_1(M) > 0$ such that

$$\frac{|f(x,u)u|}{|u|^2} \leq M_1 \quad \text{for all } x \in \mathbb{R}^N \text{and } 0 < |u| \leq \delta,$$

which combining with the mean value theorem gives that

$$\left|F(x,u)\right| \leq \frac{M_1}{2}|u|^2 \quad \text{for all } x \in \mathbb{R}^N \text{and } 0 < |u| \leq \delta. \tag{3.5}$$

Denote $\bar{M} = M|\delta|^{q-2} + \frac{M_1}{2}$. Then (3.4) and (3.5) imply that

$$F(x,u) \geq M|u|^q - \bar{M}|u|^2 \quad \text{for all } (x,u) \in (\mathbb{R}^N \times \mathbb{R}). \tag{3.6}$$

Therefore, for any given $e \in E_\lambda$, it follows from (2.4) and (3.6) that

$$\begin{aligned} I_\lambda(te) &= \frac{t^2}{2}\int_{\mathbb{R}^N}\left[|\Delta e|^2 + |\nabla e|^2 + V_\lambda(x)e^2\right]dx - \int_{\mathbb{R}^N} F(x,te)\,dx \\ &\leq \frac{t^2}{2}\|e\|_\lambda^2 - \int_{\mathbb{R}^N}\left[t^q M|e|^q - t^2\bar{M}|e|^2\right]dx \\ &\leq \frac{t^2}{2}\|e\|_\lambda^2 - t^q M\|e\|_q^q + t^2\bar{M}\|e\|_2^2 \\ &\to -\infty, \quad \text{as } t \to \infty, \end{aligned}$$

which means that the conclusion (ii) holds. This completes the proof. □

Lemma 3.2 *Suppose that* $(V1)$–$(V4)$, $(F1)$ *and* $(F2)$ *are satisfied. Moreover, if* $u_n \rightharpoonup u$ *in* E_λ, *then passing to a subsequence, the following conclusions*

$$I_\lambda(u_n - u) = I_\lambda(u_n) - I_\lambda(u) + o(1) \tag{3.7}$$

and

$$I'_\lambda(u_n - u) = I'_\lambda(u_n) - I'_\lambda(u) + o(1) \tag{3.8}$$

as $n \to \infty$ *are satisfied.*

Proof It follows from the assumption $u_n \rightharpoonup u$ in E_λ that $\langle u_n, u\rangle_\lambda \to \langle u,u\rangle_\lambda$ as $n \to \infty$, which yields that

$$\begin{aligned} \|u_n\|_\lambda^2 &= \langle u_n, u_n\rangle_\lambda \\ &= \langle u_n - u, u_n - u\rangle_\lambda + \langle u, u_n\rangle_\lambda + \langle u_n - u, u\rangle_\lambda \\ &= \|u_n - u\|_\lambda^2 + \|u\|_\lambda^2 + o(1). \end{aligned}$$

For all $\varphi \in E_\lambda$, it is clear that

$$\langle u_n, \varphi\rangle_\lambda = \langle u_n - u, \varphi\rangle_\lambda + \langle u, \varphi\rangle_\lambda.$$

Note that conditions $(V1)$ and $(V2)$ imply that $V^-(x) \geq 0$ for all $x \in \mathbb{R}^N$ and $V^-(x) \in L^\infty(\mathbb{R}^N)$. Moreover, from condition $(V2)$ it follows that $\{V^+(x) = 0\}$ has finite measure, which implies that $\{V^-(x) > 0\}$ has finite measure. Hence, applying the facts $u_n \rightharpoonup u$ in E_λ and $u_n \to u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ gives that

$$\begin{aligned} \left|\int_{\mathbb{R}^N} V^-(x)(u_n - u)^2\,dx\right| &= \left|\int_{\operatorname{supp} V^-} V^-(x)(u_n - u)^2\,dx\right| \\ &\leq \left\|V^-\right\|_\infty \int_{\operatorname{supp} V^-}(u_n - u)^2\,dx \to 0 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\left|\int_{\mathbb{R}^N} V^-(x)\left(u_n^2-u^2\right)dx\right| &= \left|\int_{\operatorname{supp} V^-} V^-(x)(u_n-u)(u_n+u)\,dx\right| \\ &\le \|V^-\|_\infty \int_{\operatorname{supp} V^-} |u_n-u||u_n+u|\,dx \\ &\le \|V^-\|_\infty \left(\int_{\operatorname{supp} V^-} |u_n-u|^2\,dx\right)^{\frac{1}{2}}\left(\int_{\operatorname{supp} V^-} |u_n+u|^2\,dx\right)^{\frac{1}{2}} \\ &\to 0. \end{aligned} \tag{3.10}$$

An easy calculation from (3.9) and (3.10) shows that

$$\int_{\mathbb{R}^N} V^-(x)(u_n-u)^2\,dx = \int_{\mathbb{R}^N} V^-(x)u_n^2\,dx - \int_{\mathbb{R}^N} V^-(x)u^2\,dx + o(1).$$

Similarly, for any $\varphi \in E_\lambda$, one can also obtain that

$$\int_{\mathbb{R}^N} V^-(x)(u_n-u)\varphi\,dx = \int_{\mathbb{R}^N} V^-(x)u_n\varphi\,dx - \int_{\mathbb{R}^N} V^-(x)u\varphi\,dx + o(1).$$

Therefore, to prove (3.7) and (3.8), it suffices to check that

$$\int_{\mathbb{R}^N}\left[F(x,u_n)-F(x,u_n-u)-F(x,u)\right]dx = o(1) \tag{3.11}$$

and

$$\sup_{\varphi\in E_\lambda,\|\varphi\|_\lambda=1}\int_{\mathbb{R}^N}\left[f(x,u_n)-f(x,u_n-u)-f(x,u)\right]\varphi\,dx = o(1). \tag{3.12}$$

Here, we only show (3.11) since the verification of (3.12) is similar. Inspired by [5, 24], let $w_n := u_n - u$. Then $w_n \rightharpoonup 0$ in E_λ and $w_n(x) \to 0$ a.e. $x \in \mathbb{R}^N$. It follows from (3.1) that

$$\begin{aligned}\left|F(x,w_n+u)-F(x,w_n)\right| &\le \int_0^1 \left|f(x,w_n+su)u\right|ds \\ &\le \int_0^1 \left(\varepsilon|w_n+su||u| + C(\varepsilon)|w_n+su|^{p-1}|u|\right)ds \\ &\le C\left(\varepsilon|w_n||u| + \varepsilon u^2 + C(\varepsilon)|w_n|^{p-1}|u| + C(\varepsilon)|u|^p\right).\end{aligned}$$

Then Young's inequality implies that

$$\left|F(x,w_n+u)-F(x,w_n)\right| \le C\left(\varepsilon|w_n|^2 + \varepsilon|u|^2 + C(\varepsilon)|w_n|^p + C(\varepsilon)|u|^p\right),$$

which combining with (3.2) yields that

$$\left|F(x,w_n+u)-F(x,w_n)-F(x,u)\right| \le C\left(\varepsilon|w_n|^2 + \varepsilon|u|^2 + C(\varepsilon)|w_n|^p + C(\varepsilon)|u|^p\right), \quad n\in\mathbb{N}.$$

Let

$$H_n(x) := \max\left\{\left|F(x,w_n+u)-F(x,w_n)-F(x,u)\right| - C\varepsilon\left(|w_n|^2+|w_n|^p\right), 0\right\}.$$

Then

$$0 \le H_n(x) \le C\varepsilon\bigl(|w_n|^2 + |u|^p\bigr) \in L^1\bigl(\mathbb{R}^N\bigr).$$

Thus, the Lebesgue dominated convergence theorem implies that

$$\int_{\mathbb{R}^N} H_n(x)\,dx \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

Furthermore, by the definition of $H_n(x)$, we have

$$\bigl|F(x, w_n + u) - F(x, w_n) - F(x, u)\bigr| \le C\varepsilon\bigl(|w_n|^2 + |w_n|^p\bigr) + H_n(x), \quad \forall n \in \mathbb{N},$$

which together with (2.3), (3.1), and (3.13) shows that

$$\begin{aligned}\int_{\mathbb{R}^N} \bigl|F(x, w_n + u) - F(x, w_n) - F(x, u)\bigr|\,dx &\le C\varepsilon\bigl(\|w_n\|_2^2 + \|w_n\|_p^p\bigr) + \varepsilon \\ &\le C\varepsilon\bigl(\|w_n\|_\lambda^2 + \|w_n\|_\lambda^p\bigr) + \varepsilon \\ &\le C\varepsilon\end{aligned}$$

for n sufficiently large. Hence, (3.11) holds. This completes the proof. □

Lemma 3.3 *Assume that* $(V1)$–$(V4)$ *and* $(F1)$–$(F4)$ *hold. Then any* $(C)_c$ *sequence of* I_λ *is bounded in* E_λ.

Proof Let $\{u_n\} \subset E_\lambda$ be a $(C)_c$ sequence of I_λ, that is,

$$I_\lambda(u_n) \to c, \qquad \bigl\|I'_\lambda(u_n)\bigr\|\bigl(1 + \|u_n\|_\lambda\bigr) \to 0, \quad \text{as } n \to \infty. \tag{3.14}$$

To prove the boundedness of $\{u_n\}$ in E_λ, arguing by contradiction, we suppose that $\|u_n\|_\lambda \to \infty$ as $n \to \infty$. Hence, for n sufficiently large, there exists a positive constant C_1 such that

$$\begin{aligned}C_1 + \|u_n\|_\lambda &\ge I_\lambda(u_n) - \frac{1}{\tau}\bigl\langle I'_\lambda(u_n), u_n\bigr\rangle \\ &\ge \frac{\mu_0 - 1}{\mu_0}\left(\frac{1}{2} - \frac{1}{\tau}\right)\|u_n\|_\lambda^2 + \int_{\mathbb{R}^N} \widetilde{F}(x, u)\,dx,\end{aligned}$$

which implies that

$$C_1 \ge \frac{\mu_0 - 1}{\mu_0}\left(\frac{1}{2} - \frac{1}{\tau}\right)\|u_n\|_\lambda^2 - \|u_n\|_\lambda + \int_{\mathbb{R}^N} \widetilde{F}(x, u)\,dx \ge \int_{\mathbb{R}^N} \widetilde{F}(x, u)\,dx \tag{3.15}$$

for n sufficiently large. Set $w_n = \frac{u_n}{\|u_n\|_\lambda}$. Then $\|w_n\|_\lambda = 1$. For $\lambda \ge 1$, noting that

$$\frac{\int_{\mathbb{R}^N} [|\Delta u_n|^2 + |\nabla u_n|^2 + V_\lambda(x) u_n^2]\,dx}{\|u_n\|_\lambda^q} \le \frac{\int_{\mathbb{R}^N} [|\Delta u_n|^2 + |\nabla u_n|^2 + V^+(x) u_n^2]\,dx}{\|u_n\|_\lambda^q} \le \frac{1}{\|u_n\|_\lambda^{q-2}}$$

and

$$\frac{\langle I'_\lambda(u_n), u_n\rangle}{\|u_n\|_\lambda^q} = \frac{\int_{\mathbb{R}^N}[|\Delta u_n|^2 + |\nabla u_n|^2 + V_\lambda(x)u_n^2]\,dx}{\|u_n\|_\lambda^q} - \int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|_\lambda^q}\,dx,$$

one has

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|_\lambda^q}\,dx = 0.$$

Note that $\|w_n\|_\lambda = 1$. Then up to a subsequence, we may assume $w_n \rightharpoonup w$ in E_λ and $w_n \to w$ a.e. $\mathbb{R}^N$. Set

$$\mathcal{B} = \{x \in \mathbb{R}^N \mid w(x) \neq 0\}.$$

Now, we show that meas$(\mathcal{B}) = 0$. Otherwise, $|u_n(x)| \to +\infty$ for a.e. $x \in \mathcal{B}$. For any constant $M_1 > 0$, $(F1)$, $(F2)$, and $(F3)$ imply that

$$f(x,u_n)u_n \geq M_1|u_n|^q - C(M_1)|u_n|^2 \quad \text{for all } x \in \mathbb{R}^N.$$

Then

$$\int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|_\lambda^q}\,dx \geq M_1\|w_n\|_q^q - C(M_1)\frac{\|w_n\|_2^2}{\|u_n\|_\lambda^{q-2}}.$$

Consequently,

$$0 = \lim_{n\to+\infty}\int_{\mathbb{R}^N} \frac{f(x,u_n)u_n}{\|u_n\|_\lambda^q}\,dx \geq M_1\|w\|_q^q = M_1\int_{\mathcal{B}} |w|^q\,dx > 0,$$

which is absurd. Hence, meas$(\mathcal{B}) = 0$. Therefore, $w(x) = 0$ for a.e. $x \in \mathbb{R}^N$. Then it follows from (2.2), (2.4), (3.14), and $(F4)$ that

$$\begin{aligned}
\frac{c + o(1)}{\|u_n\|_\lambda^2} &= \frac{1}{\|u_n\|_\lambda^2}\left[I_\lambda(u_n) - \frac{1}{\tau}\langle I'_\lambda(u_n), u_n\rangle\right] \\
&\geq \frac{\mu_0 - 1}{\mu_0}\left(\frac{1}{2} - \frac{1}{\tau}\right) + \int_{\mathbb{R}^N} \frac{\widetilde{F}(x,u_n)}{\|u_n\|_\lambda^2}\,dx \\
&\geq \frac{\mu_0 - 1}{\mu_0}\left(\frac{1}{2} - \frac{1}{\tau}\right) - C_2\|w_n\|_2^2,
\end{aligned}$$

which implies $0 \geq \frac{\mu_0-1}{\mu_0}(\frac{1}{2} - \frac{1}{\tau})$ as $n \to \infty$. This is a contradiction with $\frac{\mu-1}{\mu_0}(\frac{1}{2} - \frac{1}{\tau}) > 0$. Therefore, $\{u_n\}$ is bounded in E_λ. We complete the proof. □

Lemma 3.4 *Assume that* $(V1)$–$(V4)$, $(F1)$, $(F2)$, *and* $(F4)$ *are satisfied. Then there exists a constant* Λ *such that any* $(C)_c$ *sequence of* I_λ *possesses a convergent subsequence in* E_λ *for* $\lambda > \Lambda$.

Proof Let $\{u_n\}$ be a $(C)_c$ sequence. By the boundedness of $\{u_n\}$, there exist a subsequence $\{u_n\}$ and u_0 such that

$$\begin{aligned}
&u_n \rightharpoonup u_0, \quad \text{in } E_\lambda;\\
&u_n \to u_0, \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^N), \text{for } 2 \le p < 2_*;\\
&u_n \to u_0, \quad \text{a.e. } \mathbb{R}^N.
\end{aligned}$$

In what follows, we shall prove that $u_n \to u_0$ in E_λ. Let $v_n := u_n - u_0$. Then $v_n \rightharpoonup 0$ in E_λ. It deduces from $(V2)$ that

$$\begin{aligned}
\int_{\mathbb{R}^N} v_n^2\,dx &= \int_{\{x\in\mathbb{R}^N:V(x)+\ge b\}} v_n^2\,dx + \int_{\{x\in\mathbb{R}^N:V^+(x)<b\}} v_n^2\,dx\\
&\le \frac{1}{\lambda b}\int_{\mathbb{R}^N}\lambda V^+(x)v_n^2\,dx + \int_{\{x\in\mathbb{R}^N:V^+<b\}} v_n^2\,dx\\
&\le \frac{1}{\lambda b}\|v_n\|_\lambda^2 + o(1).
\end{aligned} \tag{3.16}$$

Set

$$\Lambda_0 := \max\left\{\frac{C_2}{b}\left[\left(\frac{1}{2}-\frac{1}{\tau}\right)\frac{\mu_0-1}{\mu_0}\right]^{-1}, 1\right\},$$

where C_2 is defined in $(F4)$. Then, for $\lambda > \Lambda_0$, a direct calculation from (2.2), (3.16), and the Hölder and Sobolev inequalities gives that

$$\begin{aligned}
\int_{\mathbb{R}^N}|v_n|^r\,dx &\le \left(\int_{\mathbb{R}^N}|v_n|^2\,dx\right)^{\frac{2_*-r}{2_*-2}}\left(\int_{\mathbb{R}^N}|v_n|^{2_*}\,dx\right)^{\frac{r-2}{2_*-2}}\\
&\le \left(\frac{1}{\lambda b}\|v_n\|_\lambda^2\right)^{\frac{2_*-r}{2_*-2}}\left[\bar{S}^{-2_*}C_E^2\left(\int_{\mathbb{R}^N}\left[|\Delta u|^2+|\nabla u|^2+|v_n|^2\right]dx\right)^{\frac{2_*}{2}}\right]^{\frac{r-2}{2_*-2}}\\
&\quad + o(1)\\
&\le \left(\frac{1}{\lambda b}\right)^{\frac{2_*-r}{2_*-2}}\left(\bar{S}^{-2_*}C_E^{2_*}\right)^{\frac{r-2}{2_*-2}}\|v_n\|_\lambda^r + o(1).
\end{aligned} \tag{3.17}$$

Moreover, for any given $c > 0$, let $M_2 := c - I_\lambda(u_0)$, then there exists $C_3 > 0$ such that $C_3 > M_2$. So, combining $(F4)$, (3.17), and Lemma 3.2 gives that

$$\begin{aligned}
C_3 &\ge c - I_\lambda(u_0) = I_\lambda(v_n) + o(1) = I_\lambda(v_n) - \frac{1}{\tau}\langle I_\lambda'(v_n), v_n\rangle + o(1)\\
&\ge \frac{\mu_0-1}{\mu_0}\left(\frac{1}{2}-\frac{1}{\tau}\right)\|v_n\|_\lambda^2 + \int_{\mathbb{R}^N}\widetilde{F}(x,v_n)\,dx + o(1)\\
&\ge \frac{\mu_0-1}{\mu_0}\left(\frac{1}{2}-\frac{1}{\tau}\right)\|v_n\|_\lambda^2 - C_2\int_{\mathbb{R}^N}|v_n|^2\,dx + o(1)\\
&\ge \left[\frac{\mu_0-1}{\mu_0}\left(\frac{1}{2}-\frac{1}{\tau}\right) - \frac{C_2}{\lambda b}\right]\|v_n\|_\lambda^2 + o(1).
\end{aligned}$$

This means that

$$\|v_n\|_\lambda^2 \le \left[\frac{\mu_0 - 1}{\mu_0}\left(\frac{1}{2} - \frac{1}{\tau}\right) - \frac{C_2}{\lambda b}\right]^{-1} C_3 + o(1) \quad \text{for } \lambda > \Lambda_0,$$

which together with (2.3) yields that

$$\begin{aligned}
\int_{\mathbb{R}^N} |v_n|^r \, dx &\le \left(\frac{1}{\lambda b} + \left|\{V^+(x) < b\}\right|^{\frac{2_*-2}{2_*}} \bar{S}^{-2} C_E^2\right)^{\frac{2_*-r}{2_*-2}} \left(\bar{S}^{-2_*} C_E^{2_*}\right)^{\frac{r-2}{2_*-2}} \|v_n\|_\lambda^r \\
&\le \left(\frac{1}{\lambda b} + \left|\{V^+(x) < b\}\right|^{\frac{2_*-2}{2_*}} \bar{S}^{-2} C_E^2\right)^{\frac{2_*-r}{2_*-2}} \left(\bar{S}^{-2_*} C_E^{2_*}\right)^{\frac{r-2}{2_*-2}} \\
&\quad \cdot \left[\left(\frac{\mu_0 - 1}{\mu_0}\left(\frac{1}{2} - \frac{1}{\tau}\right) - \frac{C_2}{\lambda b}\right)^{-1} C_3\right]^{\frac{r}{2}} + o(1).
\end{aligned} \tag{3.18}$$

Therefore, from (2.2), (3.1), (3.17), and (3.18), we have

$$\begin{aligned}
o(1) &= \int_{\mathbb{R}^N} \left[|\Delta v_n|^2 + |\nabla v_n|^2 + V_\lambda(x) v_n^2\right] dx - \int_{\mathbb{R}^N} f(x, v_n) v_n \, dx \\
&\ge \frac{\mu_0 - 1}{\mu_0} \|v_n\|_\lambda^2 - \varepsilon \int_{\mathbb{R}^N} |v_n|^2 \, dx - C(\varepsilon) \int_{\mathbb{R}^N} |v_n|^p \, dx \\
&\ge \left(\frac{\mu_0 - 1}{\mu_0} - \frac{\varepsilon}{\lambda b}\right) \|v_n\|_\lambda^2 - C(\varepsilon) \left(\int_{\mathbb{R}^N} |v_n|^p \, dx\right)^{\frac{p-2}{p}} \left(\int_{\mathbb{R}^N} |v_n|^p \, dx\right)^{\frac{2}{p}} \\
&\ge \left(\frac{\mu_0 - 1}{\mu_0} - \frac{\varepsilon}{\lambda b}\right) \|v_n\|_\lambda^2 \\
&\quad - C(\varepsilon) \left[\left(\frac{1}{\lambda b} + \left|\{V^+(x) < b\}\right|^{\frac{2_*-2}{2_*}} \bar{S}^{-2} C_E^2\right)^{\frac{2_*-r}{2_*-2}} \left(\bar{S}^{-2_*} C_E^{2_*}\right)^{\frac{r-2}{2_*-2}}\right]^{\frac{p-2}{p}} \\
&\ge \|v_n\|_\lambda^2 \left\{\left(\frac{\mu_0 - 1}{\mu_0} - \frac{\varepsilon}{\lambda b}\right)\right. \\
&\quad - C(\varepsilon) \left[\left(\frac{1}{\lambda b} + \left|\{V^+(x) < b\}\right|^{\frac{2_*-2}{2_*}} \bar{S}^{-2} C_E^2\right)^{\frac{2_*-r}{2_*-2}} \left(\bar{S}^{-2_*} C_E^{2_*}\right)^{\frac{r-2}{2_*-2}}\right]^{\frac{p-2}{p}} \\
&\quad \left. \cdot \left[\left(\frac{\mu_0 - 1}{\mu_0}\left(\frac{1}{2} - \frac{1}{\tau}\right) - \frac{C_2}{\lambda b}\right)^{-1} C_3\right]^{\frac{p-2}{2}} \left[\left(\frac{1}{\lambda b}\right)^{\frac{2_*-r}{2_*-2}} \left(\bar{S}^{-2_*} C_E^{2_*}\right)^{\frac{p-2}{2_*-2}}\right]^{\frac{2}{p}}\right\} + o(1).
\end{aligned}$$

Therefore, there exists $\Lambda = \Lambda(C_3) \ge \Lambda_0$ such that $v_n \to 0$ in E_λ for $\lambda > \Lambda$. This completes the proof. □

4 Proofs of the main results

In this section, we devote ourselves to giving the proofs of Theorems 1.1–1.3.

Proof of Theorem 1.1 Lemma 3.1 shows that the functional I_λ satisfies the geometric property of the mountain pass theorem. Moreover, Lemmas 3.3 and 3.4 imply that I_λ satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$. Then Theorem 1.1 follows from Proposition 2.1. This completes the proof. □

Proof of Theorem 1.2 Noting that $p > 2$ for $\lambda > \Lambda$ (where Λ is defined in Lemma 3.4), ε and $\|u\|_\lambda$ sufficiently small, one has

$$\begin{aligned}&\left[\frac{\mu_0-1}{2\mu_0}-\varepsilon\left(\frac{1}{\lambda b}+\left|\left\{V^+(x)<b\right\}\right|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2\right)\right]\|u\|_\lambda^2\\&\quad\geq 2C(\varepsilon)\left(\frac{1}{\lambda b}+\left|\left\{V^+(x)<b\right\}\right|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2\right)^{\frac{2_*-p}{2_*-2}}\left(\bar{S}^{-2_*}C_E^{2_*}\right)^{\frac{p-2}{2_*-2}}\|u\|_\lambda^p.\end{aligned}\tag{4.1}$$

Set

$$B_\rho:=\left\{u\in E_\lambda:\|u\|_\lambda<\rho\right\}.$$

Then, for any $u\in\overline{B}_\rho$, ε and ρ sufficiently small, it follows from (2.1), (2.3), (2.4), (3.2), and (4.1) that

$$\begin{aligned}I_\lambda(u)&\geq\frac{\mu_0-1}{2\mu_0}\|u\|_\lambda^2-\int_{\mathbb{R}^N}F(x,u)\,dx\\&\geq\frac{\mu_0-1}{2\mu_0}\|u\|_\lambda^2-\int_{\mathbb{R}^N}\left[\varepsilon|u|^2+C(\varepsilon)|u|^p\right]dx\\&\geq\left[\frac{\mu_0-1}{2\mu_0}-\varepsilon\left(\frac{1}{\lambda b}+\left|\left\{V^+(x)<b\right\}\right|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2\right)\right]\|u\|_\lambda^2\\&\quad-C(\varepsilon)\left(\frac{1}{\lambda b}+\left|\left\{V^+(x)<b\right\}\right|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2\right)^{\frac{2_*-p}{2_*-2}}\left(\bar{S}^{-2_*}C_E^{2_*}\right)^{\frac{p-2}{2_*-2}}\|u\|_\lambda^p\\&\geq\frac{1}{2}\left[\frac{\mu_0-1}{2\mu_0}-\varepsilon\left(\frac{1}{\lambda b}+\left|\left\{V^+(x)<b\right\}\right|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2\right)\right]\|u\|_\lambda^2.\end{aligned}$$

Hence,

$$I_\lambda|_{\partial B_\rho}\geq\frac{1}{2}\left[\frac{\mu_0-1}{2\mu_0}-\varepsilon\left(\frac{1}{\lambda b}+\left|\left\{V^+(x)<b\right\}\right|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2\right)\right]\rho^2:=\alpha>0.$$

Since E_λ is a separable Hilbert space, E_λ has a countable orthogonal basis $\{e_j\}$. Let $E_\lambda^k:=\operatorname{span}\{e_1,\dots,e_k\}$ and $Z_\lambda^k=(E_\lambda^k)^\perp$. Then $E_\lambda=E_\lambda^k\oplus Z_\lambda^k$. Therefore, for ε and ρ sufficiently small, we obtain

$$I_\lambda|_{\partial B_\rho\cap Z_\lambda^k}\geq\alpha>0.$$

Moreover, for any finite dimensional subspace $\overline{E}\subset E_\lambda$, there is a positive integral number m such that $\overline{E}\subset E_\lambda^m$. Note that all norms are equivalent in a finite dimensional space, then a direct calculation from (2.4) and (3.6) gives that

$$\begin{aligned}I_\lambda(u)&=\frac{1}{2}\|u\|_\lambda^2-\frac{1}{2}\int_{\mathbb{R}^N}V^-(x)u^2\,dx-\int_{\mathbb{R}^N}F(x,u)\,dx\\&\leq\frac{1}{2}\|u\|_\lambda^2-\int_{\mathbb{R}^N}\left|F(x,u)\right|dx\\&\leq\frac{1}{2}\|u\|_\lambda^2-M\|u\|_\lambda^q+\bar{M}\|u\|_\lambda^2.\\&\leq C_4\|u\|_\lambda^2-C_5\|u\|_\lambda^q.\end{aligned}$$

Consequently, there is large $\gamma > 0$ such that $I_\lambda(u) \le 0$ on $\overline{E} \setminus B_\gamma$. Therefore, there is a point $e \in E_\lambda$ with $\|e\|_\lambda > \rho$ such that $I_\lambda(e) < 0$.

Obviously, $I_\lambda(0) = 0$ and condition $(F5)$ implies that the functional I_λ is even. Therefore, combining the arguments above with Lemmas 3.3 and 3.4, Proposition 2.2 implies that the functional I_λ possesses an unbounded sequence of critical values, that is, problem (1.1) has infinitely many high energy solutions. □

Proof of Theorem 1.3 Following the argument in [1] (or see [31]), for any sequence $\lambda_n \to \infty$, we let $u_n := u_{\lambda_n}$ be the critical points of I_λ obtained in Theorem 1.1. By similar arguments of Lemma 3.3, we get that $\|u_n\|_{\lambda_n}$ is bounded in E_λ, that is,

$$\|u_n\|_{\lambda_n} \le C_6, \tag{4.2}$$

where C_6 is independent of λ_n. Therefore, we may assume that $u_n \rightharpoonup u_0$ in E and $u_n \to u_0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ for $2 \le s < 2_*$. Then Fatou's lemma implies that

$$\int_{\mathbb{R}^N} V^+(x)u_0^2\,dx \le \liminf_{n\to\infty} \int_{\mathbb{R}^N} V^+(x)u_n^2\,dx \le \liminf_{n\to\infty} \frac{\|u_n\|^2_{\lambda_n}}{\lambda_n} = 0,$$

which implies that $u_0 = 0$ a.e. in $\mathbb{R}^N \setminus V^{-1}(0)$ and $u_0 \in H^2_0(\Omega)$ by $(V3)$. For any $\varphi \in C^\infty_0(\Omega)$, it follows from $\langle I'_\lambda(u_n), \varphi\rangle = 0$ that

$$\int_{\mathbb{R}^N} \bigl[\triangle u_0 \triangle\varphi + \nabla u_0 \nabla\varphi - V^-(x)u_0\varphi\bigr]\,dx = \int_{\mathbb{R}^N} f(x,u_0)\varphi\,dx,$$

which means that u_0 is a weak solution of problem (1.2) by the density of $C^\infty_0(\Omega)$ in $H^2_0(\Omega)$.

Now, we show that $u_n \to u_0$ in $L^s(\mathbb{R}^N)$ for $2 \le s < 2_*$. If not, by Lions' vanishing lemma [12], there exist $\delta_0 > 0$, $R_0 > 0$, and $x_n \in \mathbb{R}^N$ such that

$$\int_{B_{R_0}(x_n)} (u_n - u_0)^2\,dx \ge \delta_0.$$

Moreover, $x_n \to \infty$, hence $|B_{R_0}(x_n) \cap \{x \in \mathbb{R}^N \mid V^+(x) < b\}| \to 0$. By the Hölder inequality, we have

$$\int_{B_{R_0}(x_n)\cap\{x\in\mathbb{R}^N \mid V^+(x)<b\}} (u_n - u_0)^2\,dx \to 0.$$

Consequently,

$$\begin{aligned}
\|u_n\|^2_\lambda &\ge \lambda_n b \int_{B_{R_0}(x_n)\cap\{x\in\mathbb{R}^N \mid V^+(x)\ge b\}} u_n^2\,dx \\
&\ge \lambda_n b \int_{B_{R_0}(x_n)\cap\{x\in\mathbb{R}^N \mid V^+(x)\ge b\}} (u_n - u_0)^2\,dx \\
&= \lambda_n b \biggl(\int_{B_{R_0}(x_n)} (u_n - u_0)^2\,dx - \int_{B_{R_0}(x_n)\cap\{x\in\mathbb{R}^N \mid V^+(x)<b\}} (u_n - u_0)^2\,dx + o(1) \biggr) \\
&\to \infty,
\end{aligned}$$

which contradicts (4.2). Therefore, $u_n \to u_0$ in $L^s(\mathbb{R}^N)$ for $2 \le s < 2_*$. Furthermore, applying $(F1)$, $(F2)$ and $u_n \to u_0$ in $L^s(\mathbb{R}^N)$ gives that

$$\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \to \int_{\mathbb{R}^N} f(x,u_0)u_0\,dx. \tag{4.3}$$

For $\varepsilon \in (0, \frac{\mu_0-1}{\mu_0}(\frac{1}{\lambda b} + |\{V^+(x)<b\}|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2)^{-1})$, (3.1) implies that

$$\left|\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx\right| \le \int_{\mathbb{R}^N}\left[\varepsilon|u_n|^2 + C(\varepsilon)|u_n|^p\right]dx, \tag{4.4}$$

which combining with the facts $u_n \ne 0$, (2.1), (2.3), and (4.3) yields that

$$\begin{aligned}\frac{\mu_0-1}{\mu_0}\|u_n\|_{\lambda_n}^2 &\le \int_{\mathbb{R}^N}\left[|\Delta u_n|^2 + |\nabla u_n|^2 + V_{\lambda_n}(x)u_n^2\right]dx = \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \\ &\le \int_{\mathbb{R}^N}\left[\varepsilon|u_n|^2 + C(\varepsilon)|u_n|^p\right]dx \\ &\le \varepsilon\left(\frac{1}{\lambda b} + \left|\left\{V^+(x)<b\right\}\right|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2\right)\|u_n\|_{\lambda_n}^2 \\ &\quad + C(\varepsilon)\left(\frac{1}{\lambda b} + \left|\left\{V^+(x)<b\right\}\right|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2\right)^{\frac{2_*-p}{2_*-2}}\left(\bar{S}^{-2_*}C_E^{2_*}\right)^{\frac{p-2}{2_*-2}}\|u\|_{\lambda_n}^p,\end{aligned}$$

which implies that

$$\|u_n\|_{\lambda_n} \ge \left[\frac{\frac{\mu_0-1}{\mu_0} - \varepsilon(\frac{1}{\lambda b} + |\{V^+(x)<b\}|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2)}{C(\varepsilon)(\frac{1}{\lambda b} + |\{V^+(x)<b\}|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2)^{\frac{2_*-p}{2_*-2}}(\bar{S}^{-2_*}C_E^{2_*})^{\frac{p-2}{2_*-2}}}\right]^{\frac{1}{p-2}} > 0. \tag{4.5}$$

Moreover, it follows from $\langle I'_{\lambda_n}(u_n), u_n\rangle = 0$ that

$$\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx = \int_{\mathbb{R}^N}\left[|\Delta u_n|^2 + |\nabla u_n|^2 + V_{\lambda_n}(x)u_n^2\right]dx \ge \frac{\mu_0-1}{\mu_0}\|u_n\|_{\lambda_n}^2. \tag{4.6}$$

Then, a direct calculation from (4.4), (4.5), and (4.6) shows that

$$\begin{aligned}\int_{\mathbb{R}^N} f(x,u_0)u_0\,dx &\ge \frac{\mu_0-1}{\mu_0}\left[\frac{\frac{\mu_0-1}{\mu_0} - \varepsilon(\frac{1}{\lambda b} + |\{V^+(x)<b\}|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2)}{C(\varepsilon)(\frac{1}{\lambda b} + |\{V^+(x)<b\}|^{\frac{2_*-2}{2_*}}\bar{S}^{-2}C_E^2)^{\frac{2_*-p}{2_*-2}}(\bar{S}^{-2_*}C_E^{2_*})^{\frac{p-2}{2_*-2}}}\right]^{\frac{2}{p-2}} \\ &> 0,\end{aligned}$$

which means that $u_0 \ne 0$.

In what follows, we shall show that $u_n \to u_0$ in E. Since $\langle I'_{\lambda_n}(u_n), u_n\rangle = \langle I'_{\lambda_n}(u_n), u_0\rangle = 0$, we have

$$\|u_n\|_{\lambda_n}^2 - \int_{\mathbb{R}^N} V^-(x)u_n^2\,dx = \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx \tag{4.7}$$

and

$$\langle u_n, u_0\rangle_{\lambda_n} - \int_{\mathbb{R}^N} V^-(x)u_nu_0\,dx = \int_{\mathbb{R}^N} f(x,u_n)u_0\,dx. \tag{4.8}$$

Similar to the proof of (3.10), we have

$$\begin{aligned}\int_{\mathbb{R}^N} V^-(x)\big(u_n^2 - u_n u_0\big)\,dx &\le \left|\int_{\mathbb{R}^N} V^-(x)\big(u_n^2 - u_n u_0\big)\,dx\right| \\ &= \left|\int_{\operatorname{supp} V^-} V^-(x)u_n(u_n - u_0)\,dx\right| \\ &\le \|V^-\|_\infty \left(\int_{\operatorname{supp} V^-} |u_n - u_0|^2\,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |u_n|^2\right)^{\frac{1}{2}} \\ &\to 0, \end{aligned} \tag{4.9}$$

since $u_n \to u_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N)$. Moreover, combining (4.4) with (4.7)–(4.9) shows that

$$\lim_{n\to\infty} \|u_n\|^2_{\lambda_n} = \lim_{n\to\infty} \langle u_n, u_0\rangle_{\lambda_n} = \lim_{n\to\infty} \langle u_n, u_0\rangle = \|u_0\|^2.$$

On the other hand, weak lower semi-continuity of norm implies that

$$\lim_{n\to\infty} \|u_n\|^2_{\lambda_n} \ge \liminf_{n\to\infty} \|u_n\|^2 \ge \|u_0\|^2.$$

Thus, $u_n \to u_0$ in E as $n \to \infty$. Therefore, u_0 is a nontrivial solution of problem (1.2). We complete the proof. □

5 Conclusions

A class of biharmonic equations with sign-changing potentials and an indefinite nonlinearity is studied in the present paper. Under some suitable conditions, the existence of nontrivial solutions and the high energy solutions are obtained by using variational methods. Moreover, the phenomenon of concentration of solutions is explored. The results extend the main conclusions in recent literature.

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Competing interests
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Authors' contributions
The authors contributed to each part of this study equally and read and approved the final version of the manuscript.

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