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Basic theory of initial value problems of conformable fractional differential equations

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Abstract

In this paper, we discuss the basic theory of the conformable fractional differential equation $T_{\alpha}^{a}x(t) = f(t, x(t)), t \in [a, \infty)$, subject to the local initial condition $x(a) = x_a$ or the nonlocal initial condition $x(a) + g(x) = x_a$, where $0 < \alpha < 1$, $T_{\alpha}^{a}x(t)$ denotes the conformable fractional derivative of a function $x(t)$ of order α, $f : [a, \infty) \times \mathbb{R} \mapsto \mathbb{R}$ is continuous and g is a given functional defined on an appropriate space of functions. The theory of global existence, extension, boundedness, and stability of solutions is considered; by virtue of the theory of the conformable fractional calculus and by the use of fixed point theorems, some criteria are established. Several concrete examples are given to illustrate the possible application of our analytical results.

Keywords: Conformable fractional derivatives; Fractional differential equations; Initial value problems; Fixed point theorems

1 Introduction

Fractional derivative is a generalization of the classical one to an arbitrary order, and it is as old as calculus. It has been applied to almost every field of science, engineering, and mathematics in the last three decades [1–10]. At present, there exist a number of definitions of fractional derivatives in the literature, each depending on a given set of assumptions, the most popular of which are the Riemann–Liouville and Caputo fractional derivatives. But it is worth noting that these two kinds of derivatives do not satisfy the classical chain rule.

Recently, Khalil et al. in [11] introduced a new well-behaved definition of a fractional derivative, called the conformable fractional derivative, which satisfies the chain rule. The new definition has attracted a great deal of attention from many researchers. And for the basic properties of the conformable fractional derivative, some results have been obtained [11–13]; its several applications and generalizations were also discussed [14–18]. But the investigation of the theory of conformable fractional differential equations has only been started quite recently.

The existence of solutions to the boundary value problems for some specific conformable fractional differential equations were discussed in [19–25]. For initial value problems, Abdourazek et al. in [26] studied the stability and asymptotic stability of conformable fractional-order non-linear systems by using Lyapunov functions; Abdeljawad et al. in [27] investigated abstract Cauchy problems of conformable fractional systems by introducing fractional semigroups of operators; and Bayour et al. in [28] studied the existence of solutions for the following problem by using the notion of tube solution:

$$\begin{cases} T_\alpha x(t) = f(t, x(t)), & t \in [a, b], a > 0, \\ x(a) = x_a, \end{cases}$$

where $T_\alpha x(t)$ denotes the conformable fractional derivative starting from zero of a function x of order α with α in $(0, 1)$. Some existence results of at least one solution for the above-mentioned problem were obtained by the use of Schauder's fixed point theorem.

In this paper, we consider the global existence, extension, boundedness and stabilities of solutions to the fractional differential equation

$$T_\alpha^a x(t) = f\big(t, x(t)\big), \quad t \in [a, \infty), 0 < \alpha < 1, \tag{1.1}$$

subject to the initial conditions

$$x(a) = x_a \tag{1.2}$$

or

$$x(a) + g(x) = x_a, \tag{1.3}$$

where $T_\alpha^a x(t)$ denotes the conformable fractional derivative starting from a of a function x of order α, $f : [a, \infty) \times \mathbb{R} \mapsto \mathbb{R}$ is continuous, and g is a given functional defined on an appropriate space of functions. The conditions (1.2) and (1.3) are often called local and nonlocal initial conditions, respectively.

There is a vast literature concerning the existence of solutions to an initial value problem for the differential equations with the Riemann–Liouville or Caputo fractional derivatives [29–32]. While in the setting of the conformable fractional derivatives, as far as we know, the global existence and extension and boundedness of solutions have not been discussed in the literature. It is worth pointing out that the global existence and boundedness of solutions play a prerequisite role in the discussion of the stabilities of solutions.

The rest of paper is organized as follows. Section 2 preliminarily provides some definitions and lemmas which are crucial to the following discussion. In Sect. 3, we first establish some criteria for the global existence, extension, and boundedness of solutions to the local initial value problem by means of some fixed point theorems and by the use of the conformable fractional calculus, and further discuss the stabilities of solutions; and then we investigate the existence of solutions to the nonlocal initial value problem. Finally, we give several concrete examples to illustrate the possible application of our analytical results.

2 Preliminaries

In this section, we preliminarily provide some definitions and lemmas which are useful in the following discussion. It is always assumed that $\alpha \in (0, 1]$ throughout this paper.

Definition 2.1 ([11, 13]) The conformable fractional derivative starting from a of a function $f : [a, \infty) \mapsto \mathbb{R}$ of order α is defined by

$$T_\alpha^a f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon(t-a)^{1-\alpha}) - f(t)}{\epsilon}.$$

If $T_\alpha^a f(t)$ exists on (a, b), then $T_\alpha^a f(a) = \lim_{t\to a} T_\alpha^a f(t)$.

Definition 2.2 ([11, 13]) The fractional integral starting from a of a function $f : [a, \infty) \mapsto \mathbb{R}$ of order α is defined by

$$I_\alpha^a f(t) = \int_a^t (s-a)^{\alpha-1} f(s)\,\mathrm{d}s.$$

Lemma 2.1 ([11, 13]) *If $f : [a, \infty) \mapsto \mathbb{R}$ is continuous. Then, for all $t > a$,*

$$T_\alpha^a I_\alpha^a f(t) = f(t).$$

Lemma 2.2 ([22]) *If $T_\alpha^a f(t)$ is continuous on $[a, b]$, then $I_\alpha^a T_\alpha^a f(t) = f(t) - f(a)$.*

Lemma 2.3 ([11, 13]) *If f is differentiable at t in $[a, b]$, then it is also α-differentiable at t and $T_\alpha^a f(t) = (t-a)^{1-\alpha} \frac{\mathrm{d}f(t)}{\mathrm{d}t}$.*

Lemma 2.4 ([11]) *If f is α-differentiable at t in $[a, b]$, then it is continuous at t.*

Lemma 2.5 (Chain rule [13]) *Let $f, g : (a, \infty) \mapsto \mathbb{R}$ be α-differential functions, where $\alpha \in (0, 1]$. Let $h(t) = f(g(t))$. Then $h(t)$ is an α-differential and for all $t \neq a$ and $g(t) \neq a$ we have*

$$\left(T_\alpha^a h\right)(t) = \left(T_\alpha^a f\right)\left(g(t)\right) \cdot \left(T_\alpha^a g\right)(t) \cdot \left(g(t) - a\right)^{\alpha-1}.$$

For $t = a$,

$$T_\alpha^a h(a) = \lim_{t\to a} \left(T_\alpha^a f\right)\left(g(t)\right) \cdot \left(T_\alpha^a g\right)(t) \cdot \left(g(t) - a\right)^{\alpha-1}.$$

Lemma 2.6 ([11]) *If f and g are α-differentiable at t in $[a, b]$, then fg is also α-differentiable at t and*

$$T_\alpha^a (fg)(t) = f(t) T_\alpha^a f(t) + g(t) T_\alpha^a f(t).$$

By an argument similar to the one used in [11], a general version of the mean value theorem for the conformable fractional derivative is yielded as follows [20]. It plays a crucial role in the study of the extension of solutions.

Lemma 2.7 *If $f : [a, b] \mapsto \mathbb{R}$ is continuous on the subinterval $[c, d]$ of $[a, b]$ and if $T_\alpha^a f(t)$ exists on (c, d). Then there exists a point ξ in (c, d) such that*

$$f(d) - f(c) = \frac{1}{\alpha} T_\alpha^a f(\xi)\left[(d-a)^\alpha - (c-a)^\alpha\right].$$

The following lemma is a direct consequence of the application of the mean value theorem [13].

Lemma 2.8 *If $T_\alpha^a f(t) \le 0$ on $[a,b]$, then f is decreasing on $[a,b]$.*

We next present an extended Gronwall's inequality, which generalizes the result in [13]; and it plays a key role in the discussion of the extension and stabilities of solutions.

Lemma 2.9 *Let f and g be continuous, nonnegative functions on $[a,b]$ and λ a nonnegative constant such that*

$$f(t) \le \lambda + I_\alpha^a(fg)(t) \quad \text{for } t \text{ in } [a,b],$$

then

$$f(t) \le \lambda \mathrm{e}^{I_\alpha^a g(t)} \quad \text{for } t \text{ in } [a,b].$$

Proof Let $F(t) = \lambda + I_\alpha^a(fg)(t)$ and $G(t) = \mathrm{e}^{-I_\alpha^a g(t)}$. Then the hypothesis of the inequality of f and g is equivalent to the inequality

$$f(t) \le F(t) \quad \text{for } t \text{ in } [a,b], \tag{2.1}$$

and the assumptions of continuities of f and g ensure that F, G and GF are α-differentiable; and thus from Lemma 2.1, the inequality (2.1) and the hypotheses of nonnegativity, it follows that

$$T_\alpha^a F(t) - F(t)g(t) = f(t)g(t) - F(t)g(t) \le f(t)g(t) - f(t)g(t) = 0.$$

Multiplying each side of the above inequality by $G(t)$, and using Lemmas 2.5 and 2.6, we get

$$T_\alpha^a (FG)(t) \le 0.$$

This, together with Lemma 2.8, implies that $F(t)G(t)$ is decreasing on $[a,b]$. Hence

$$F(t)G(t) \le F(a)G(a),$$

or, equivalently,

$$F(t) \le \lambda \mathrm{e}^{I_\alpha^a g(t)}.$$

Again, using the inequality (2.1), from the above inequality we obtain the desired conclusion. □

3 Main results

We first make the following hypotheses, which will be adopted in the following discussion. Let $\mathfrak{D} = [a, \infty) \times \mathbb{R}$.

(H1) The function $f : \mathfrak{D} \mapsto \mathbb{R}$ is continuous.

(H2) There exists a positive constant L such that, for any $(t, u), (t, v)$ in $\mathfrak{D}$,

$$\bigl|f(t,u) - f(t,v)\bigr| \le L|u - v|.$$

(H3) There exists a nonnegative function h such that, for any (t, u) in $\mathfrak{D}$,

$$\bigl|f(t,u)\bigr| \le h(t)|u|$$

for which $I_\alpha^a h(t)$ is bounded on $[a, \infty)$.

(H4) There exist a nonnegative function l and a positive constant L such that, for any $(t, u), (t, v)$ in $\mathfrak{D}$,

$$\bigl|f(t,u) - f(t,v)\bigr| \le l(t)|u - v| \le L|u - v|$$

for which $I_\alpha^a l(t)$ is bounded on $[a, \infty)$.

3.1 Local initial value problems

In this subsection, we establish some criteria for the global existence, extension, boundedness, and stabilities of solutions to the local initial value problem. By Lemmas 2.1 and 2.2, the initial value problem (1.1)–(1.2) is easily transformed into an equivalent integral equation.

Lemma 3.1 *If* (H1) *holds, then a function x in $C([a, b])$ is a solution of the initial value problem* (1.1)–(1.2) *if and only if it is a continuous solution of the following integral equation*:

$$x(t) = x_a + I_\alpha^a f\bigl(t, x(t)\bigr), \quad t \in [a, b].$$

Now, we are in a position to present a result of existence and uniqueness of the solution to the initial value problem (1.1)–(1.2).

Theorem 3.1 *If* (H1)–(H2) *hold, then the initial value problem* (1.1)–(1.2) *has exactly one solution defined on* $[a, b]$.

Proof Write $I = [a, b]$. The assertion will be proven by Banach's contraction principle on $C(I)$ equipped with an appropriate weighted maximum norm. To this end, given a positive number β in (L, ∞), define a function $e(t)$ by

$$e(t) = \mathrm{e}^{-\beta \frac{(t-a)^\alpha}{\alpha}},$$

and then, for x in $C(I)$, define

$$\|x\|_\beta = \bigl\|e(\cdot)x(\cdot)\bigr\|,$$

where $\|\cdot\|$ denotes the maximum norm on $C(I)$. It is easy to verify that $\|\cdot\|_\beta$ is also a norm on $C(I)$, which is equivalent to the maximum norm $\|\cdot\|$ since

$$\mathrm{e}(b)\|\cdot\| \leq \|\cdot\|_\beta \leq \|\cdot\|.$$

Hence $(C(I), \|\cdot\|_\beta)$ is a Banach space.

Define next an operator

$$\mathcal{T} : \bigl(C(I), \|\cdot\|_\beta\bigr) \mapsto \bigl(C(I), \|\cdot\|_\beta\bigr)$$

by

$$\mathcal{T}x(t) = x_0 + \int_a^t f\bigl(s, x(s)\bigr)(s-a)^{\alpha-1}\,\mathrm{d}s,$$

and then Lemma 3.1 ensures that the fixed points of the operator $\mathcal{T}$ are the solutions of the problem (1.1)–(1.2).

We now show that $\mathcal{T}$ is a contraction on $(C(I), \|\cdot\|_\beta)$. Indeed, let $x, y \in C(I)$ and observe

$$\mathcal{T}x(t) - \mathcal{T}y(t) = \int_a^t \bigl[f\bigl(s, x(s)\bigr) - f\bigl(s, y(s)\bigr)\bigr](s-a)^{\alpha-1}\,\mathrm{d}s.$$

Thus, by H2, a direct calculation gives, for every t in I,

$$\begin{aligned}
\bigl|\mathcal{T}x(t) - \mathcal{T}y(t)\bigr|e(t) &\leq Le(t)\int_a^t e^{-1}(s)e(s)\bigl|x(s) - y(s)\bigr|(s-a)^{\alpha-1}\,\mathrm{d}s \\
&\leq Le(t)\int_a^t e^{-1}(s)(s-a)^{\alpha-1}\,\mathrm{d}s\|x-y\| \\
&= Le(t)I_\alpha^a e^{-1}(t)\|x-y\| \\
&= \frac{L}{\beta}e(t)\bigl(e^{-1}(t) - 1\bigr)\|x-y\| \\
&\leq \frac{L}{\beta}\|x-y\|.
\end{aligned}$$

Hence

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \frac{L}{\beta}\|x-y\|.$$

Since $0 < \frac{L}{\beta} < 1$, the Banach contraction principle ensures that there is a unique x in $C(I)$ with $x = \mathcal{T}x$, and equivalently the problem (1.1)–(1.2) has a unique solution x in $C(I)$. The proof is complete. □

We next discuss the extension to the right of the solutions of Eq. (1.1) with initial condition (1.2).

Lemma 3.2 *If* (H1) *holds. Let $x(t)$ be a solution of the initial value problem* (1.1)–(1.2) *defined on $[a, t^+)$ with $t^+ \neq \infty$. If the limit of $x(t)$ exists as t tends to t^+, then the solution $x(t)$ can be extended to the closed interval $[a, t^+]$.*

Proof Let $\lim_{t\to t^+} x(t) = x^+$. Now let $J = [a,t^+)$ and define a function $\tilde{x}(t)$ by

$$\tilde{x} = \begin{cases} x(t), & t \in J, \\ x^+, & t = t^+. \end{cases}$$

By Lemma 2.4, the function $\tilde{x}(t)$ is obviously continuous on $[a,t^+]$.

We next show that the function $\tilde{x}(t)$ is also a solution of (1.1)–(1.2) defined on $[a,t^+]$, and clearly, it is sufficient to show

$$T_\alpha^a \tilde{x}(t^+) = f\big(t^+, \tilde{x}(t^+)\big).$$

Observe that the equation

$$T_\alpha^a \tilde{x}(t) = f\big(t, \tilde{x}(t)\big), \quad \text{for } t \in [a,t^+),$$

and the continuities of $\tilde{x}$ and f obviously imply

$$\lim_{t\to t^+} T_\alpha^a \tilde{x}(t) = f\big(t^+, \tilde{x}(t^+)\big). \tag{3.1}$$

Moreover, using Lemma 2.7, we see that, for every t in $[a,t^+)$, there exists a point η in (t,t^+) such that

$$T_\alpha^a \tilde{x}(\eta) = \alpha \cdot \frac{\tilde{x}(t) - \tilde{x}(t^+)}{t - t^+} \cdot \frac{t - t^+}{(t-a)^\alpha - (t^+ - a)^\alpha}. \tag{3.2}$$

Now letting $t \to t^+$, then it follows from (3.1) and (3.2) that the derivative of $\tilde{x}(t)$ at t^+ exists and

$$\tilde{x}'(t^+)(t^+ - a)^{1-\alpha} = f\big(t^+, \tilde{x}(t^+)\big).$$

Therefore, by Lemma 2.3,

$$T_\alpha^a \tilde{x}(t^+) = f\big(t^+, \tilde{x}(t^+)\big).$$

Consequently, we have shown that the function $\tilde{x}(t)$ is also a solution of (1.1)–(1.2) defined on $[a,t^+]$, and it is an extension of the solution $x(t)$ to $[a,t^+]$. Hence the desired assertion follows. □

Definition 3.1 Let J be the maximal existence interval of the solution $x(t)$ of the initial value problem (1.1)–(1.2), then the solutions $x(t)$ is called to come arbitrarily close to the boundary of $\mathfrak{D} = [a,\infty) \times \mathbb{R}$ to the right if for any closed and bounded domain $\mathfrak{D}_0$ in $\mathfrak{D}$, it is impossible that the point $(t,x(t))$ always remains in $\mathfrak{D}_0$ for every t in J.

Theorem 3.2 *If* (H1)–(H2) *hold, then the solution of the initial value problem* (1.1)–(1.2) *comes arbitrarily close to the boundary of* $\mathfrak{D} = [a,\infty) \times \mathbb{R}$ *to the right.*

Proof According to Theorem 3.1, the initial value problem (1.1)–(1.2) has a unique solution, and denote the solution by $x(t)$. Let J stand for the maximal existence interval of $x(t)$. Again, using Theorem 3.1, we infer that $J = [a, \infty)$ or $[a, t^+)$ with $t^+ \neq \infty$.

The desired result is obvious if $J = [a, \infty)$.

Next, consider the case $J = [a, t^+)$ with $t^+ \neq \infty$. Assume on the contrary that the desired assertion is not true, then there exists a closed and bounded domain $\mathfrak{D}_0 \subset \mathfrak{D}$ such that $(t, x(t)) \in \mathfrak{D}_0$ for every t in J. The continuity of f on $\mathfrak{D}_0$ implies that there exists a positive number M such that

$$\left|f\left(t, x(t)\right)\right| \le M \tag{3.3}$$

for every t in J.

Moreover, Lemma 2.7 ensures that, for any t_1, t_2 in J with $t_1 < t_2$, there exists a point ξ in (t_1, t_2) such that

$$x(t_2) - x(t_1) = \frac{T_\alpha^a x(\xi)}{\alpha}\left[(t_2 - a)^\alpha - (t_1 - a)^\alpha\right] = \frac{f(\xi, x(\xi))}{\alpha}\left[(t_2 - a)^\alpha - (t_1 - a)^\alpha\right].$$

This, together with (3.3), yields

$$\left|x(t_2) - x(t_1)\right| \le \frac{M}{\alpha}\left|(t_2 - a)^\alpha - (t_1 - a)^\alpha\right|,$$

which implies that $x(t)$ is uniformly continuous on J, and thus the limit of $x(t)$ exists as $t \to t^+$. And therefore, according to Lemma 3.2, the solution $x(t)$ can be extended to the closed interval $[a, t^+]$, which contradicts the fact that $[a, t^+)$ is the maximal existence interval of the solution $x(t)$. Hence the desired assertion follows. □

Using Theorems 3.1 and 3.2, we now give a result guaranteeing that the solution of Eq. (1.1) with the initial condition (1.2) is defined and bounded on $[a, \infty)$.

Theorem 3.3 *If* (H1)–(H3) *hold, then the solution of the initial value problem* (1.1)–(1.2) *is defined and bounded on* $[a, \infty)$.

Proof By Theorem 3.1, Eq. (1.1) with the initial condition (1.2) has a unique solution. Denote the solution by $x(t)$ for which its maximal existence interval is $[a, t^+)$. It remains to show that $t^+ = \infty$ and that $x(t)$ is bounded on $[a, \infty)$.

Observe that

$$x(t) = x_a + I_\alpha^a f\left(t, x(t)\right).$$

By assumption (H3),

$$\begin{aligned}\left|x(t)\right| &\le |x_a| + I_\alpha^a \left|f\left(t, x(t)\right)\right| \\ &\le |x_a| + I_\alpha^a\left[h(t)\left|x(t)\right|\right].\end{aligned}$$

Thus, Gronwall's inequality implies that

$$\left|x(t)\right| \le |x_a| \mathrm{e}^{I_\alpha^a h(t)} \quad \text{for every } t \text{ in } \left[a, t^+\right).$$

Noting that the assumption of boundedness of $I_\alpha^a h(t)$ implies that there exists a positive number M such that $I_\alpha^a h(t) \leq M$ for every t in $[a, t^+)$, and thus from the above inequality we get

$$\left|x(t)\right| \leq |x_a|e^M \quad \text{for every } t \text{ in } \left[a, t^+\right).$$

Hence $x(t)$ is bounded on $[a, t^+)$.

If $t^+ \neq \infty$, then Theorem 3.2 immediately implies $\lim_{t\to t^+} x(t) = \infty$, which contradicts the boundedness of $x(t)$ on $[a, t^+)$. Thus $t^+ = \infty$, and therefore the desired result follows. □

By Gronwall's inequality, we next investigate the stabilities of the solutions to the problem (1.1)–(1.2).

Definition 3.2 Let $x(t)$ be a solution to Eq. (1.1) defined on $[a, \infty)$ with $x(a) = x_a$. The solution $x(t)$ is said to be stable if, for any positive number ϵ, there exists a positive number δ such that every solution $y(t)$ with $|y(a) - x(a)| < \delta$ exists for all $t \geq a$ and satisfies the inequality

$$\left|y(t) - x(t)\right| < \epsilon$$

for $t \geq a$.

Theorem 3.4 *If* (H1), (H3) *and* (H4) *hold, then every solution to Eq.* (1.1) *with the local initial condition is always stable.*

Proof In the light of Theorem 3.3, the solution to Eq. (1.1) satisfying the local initial condition always exists and is defined on $[a, \infty)$. Let $x(t)$ be a solution with $x(a) = x_a$ and $y(t)$ a solution with $y(a) = y_a$. Then

$$x(t) = x_a + I_\alpha^a f\left(t, x(t)\right)$$

and

$$y(t) = y_a + I_\alpha^a f\left(t, y(t)\right).$$

By (H4),

$$\left|y(t) - x(t)\right| \leq |y_a - x_a| + I_\alpha^a\left[l(t)\left|y(t) - x(t)\right|\right]$$

and thus, using Gronwall's inequality, we obtain

$$\left|y(t) - x(t)\right| \leq |y_a - x_a|e^{I_\alpha^a l(t)} \quad \text{for every } t \text{ in } [a, \infty).$$

Furthermore, the boundedness of $I_\alpha^a l(t)$ ensures that the exists a positive number M such that $I_\alpha^a l(t) \leq M$ for t in $[a, \infty)$, and therefore

$$\left|y(t) - x(t)\right| \leq |y_a - x_a|e^M \quad \text{for every } t \text{ in } [a, \infty).$$

Hence $x(t)$ is stable on $[a, \infty)$. □

3.2 Nonlocal initial value problems

In this subsection, the existence of solutions to the nonlocal initial value problem is discussed. We next introduce a fixed theorem to be adopted to prove the main result in this subsection.

Lemma 3.3 ([33]) *Denoted by $\mathcal{U}$ an open set in a closed, convex set $\mathcal{C}$ of a Banach space $\mathcal{B}$. Assume $0 \in \mathcal{U}$. Also assume that $\mathcal{A}(\bar{\mathcal{U}})$ is bounded and that $\mathcal{A} : \bar{\mathcal{U}} \mapsto \mathcal{C}$ is given by $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$, in which $\mathcal{A}_1 : \bar{\mathcal{U}} \mapsto \mathcal{B}$ is completely continuous and $\mathcal{A}_2 : \bar{\mathcal{U}} \mapsto \mathcal{B}$ is a nonlinear contraction (i.e., there exists a nonnegative nondecreasing function $\phi : [0,\infty) \mapsto [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$, such that $\|\mathcal{A}_2(x) - \mathcal{A}_2(y)\| \le \phi(\|x - y\|)$ for all $x, y \in \bar{\mathcal{U}}$). Then either*

(C1) *$\mathcal{A}$ has a fixed point $u \in \bar{\mathcal{U}}$; or*

(C2) *there exist a point $u \in \partial\mathcal{U}$ and $\lambda \in (0,1)$ with $u = \lambda\mathcal{A}(u)$, where $\bar{\mathcal{U}}$ and $\partial\mathcal{U}$, respectively, represent the closure and boundary of $\mathcal{U}$.*

We further make the following assumptions.

(H5) f is a continuous function defined on $[a,b] \times \mathbb{R}$.

(H6) There exist a positive constant γ in $(0,1)$ and a nonnegative and nondecreasing function ϕ in $C([0,\infty))$ such that $\phi(z) < \gamma z$ for $z > 0$ and $|g(u) - g(v)| \le \phi(\|u - v\|)$ for any u, v in $C([a,b])$.

(H7) There exist a nonnegative function φ in $C([a,b])$ for which $\varphi > 0$ on a subinterval of $[a,b]$ and a nonnegative and nondecreasing function ψ in $C([0,\infty))$ such that

$$\left|f(t,u)\right| \le \varphi(t)\psi\left(|u|\right)$$

for any (t,u) in $[a,b] \times \mathbb{R}$ and

$$\sup_{r\in(0,\infty)} \frac{r}{|x_a| + \psi(r) I_\alpha^a \varphi(b)} > \frac{1}{1-\gamma}.$$

By Lemmas 2.1 and 2.2, it is easy to verify the following lemma.

Lemma 3.4 *If* (H5) *holds, then a function x in $C([a,b])$ is a solution of the nonlocal initial value problem* (1.1), (1.3) *if and only if it is a continuous solution of the following integral equation*:

$$x(t) = x_a - g(x) + I_\alpha^a f\left(t, x(t)\right), \quad t \in [a,b].$$

In order to utilize the fixed point theorem to discuss the existence of solutions to the nonlocal initial value problem, we first define some sets of functions in $C([a,b])$ and operators.

Given a positive number r, define the subset $\mathcal{U}_r$ of $C([a,b])$ by

$$\mathcal{U}_r = \left\{u \in C\left([a,b]\right) : \|u\| < r\right\}.$$

Also, define three operators from the space $C([a,b])$ to itself, respectively, by

$$\mathcal{A}_1 x(t) = I_\alpha^a f\left(t, x(t)\right), \tag{3.1}$$

$$\mathcal{A}_2x(t) = x_a - g(x), \tag{3.2}$$

$$\mathcal{A}x(t) = \mathcal{A}_1x(t) + \mathcal{A}_2x(t). \tag{3.3}$$

Using the standard arguments, the complete continuity of the operator $\mathcal{A}_1 : \bar{\mathcal{U}}_r \mapsto C([a,b])$ can be verified, and it is also easy to check that the operator $\mathcal{A}_2 : \bar{\mathcal{U}}_r \mapsto C([a,b])$ is a nonlinear contraction under the condition (H6). Here we omit their proofs.

Lemma 3.5 *If* (H5) *holds, then the operator* $\mathcal{A}_1 : \bar{\mathcal{U}}_r \mapsto C([a,b])$ *is completely continuous.*

Lemma 3.6 *If* (H6) *holds, then the operator* $\mathcal{A}_2 : \bar{\mathcal{U}}_r \mapsto C([a,b])$ *is a nonlinear contraction.*

We now present the main result in this subsection.

Theorem 3.5 *If* (H5)–(H7) *hold, then the nonlocal initial value problem* (1.1), (1.3) *exists at least one solution defined on* $[a,b]$.

Proof In the light of the assumption of the supremum in (H7), there exists a positive number r such that

$$\frac{r}{|x_a| + \psi(r)I_\alpha^a\varphi(b)} > \frac{1}{1-\gamma}. \tag{3.4}$$

And then we define the set $\mathcal{U}_r$ by $\mathcal{U}_r = \{u \in C([a,b]) : \|u\| < r\}$.

We first show that the operators $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ satisfy the corresponding conditions of Lemma 3.3; and owing to Lemmas 3.5 and 3.6, we only need to show the boundedness of $\mathcal{A}(\bar{\mathcal{U}}_r)$. Indeed, for every x in $\bar{\mathcal{U}}_r$, it follows from the assumptions (H5) and (H6) that

$$\left|\mathcal{A}_1x(t)\right| \le I_\alpha^a\left|f\left(t,x(t)\right)\right| \le \frac{(b-a)^\alpha}{\alpha} \cdot \sup\left\{\left|f(t,u)\right| : t \in [a,b], |u| \le r\right\}$$

and that

$$\left|\mathcal{A}_2x(t)\right| \le |x_a| + \left|g(x)\right| \le |x_a| + \gamma r.$$

Hence, according the definition of the operator $\mathcal{A}$,

$$\|\mathcal{A}x\| \le |x_a| + \gamma r + \frac{(b-a)^\alpha}{\alpha} \cdot \sup\left\{\left|f(t,u)\right| : t \in [a,b], |u| \le r\right\}.$$

This validates the unform boundedness of the set $\mathcal{A}(\bar{\mathcal{U}}_r)$.

Finally, it remains to show that the case (C2) in Lemma 3.3 does not occur. We argue by contradiction. Assume that the case (C2) holds. Then there exist λ in $(0,1)$ and x in $\partial\mathcal{U}_r$ such that $x = \lambda\mathcal{A}x$, i.e.,

$$x(t) = \lambda\left[x_a - g(x) + I_\alpha^a f\left(t,x(t)\right)\right].$$

This, combined with the hypotheses (H6)–(H7), further implies that

$$r \le |x_a| + \gamma r + \psi(r)I_\alpha^a\varphi(b)$$

or, equivalently,

$$\frac{r}{|x_a| + \psi(r)I_\alpha^a\varphi(b)} \leq \frac{1}{1-\gamma},$$

which contradicts the inequality (3.4). Consequently, we have shown that the operators $\mathcal{A}, \mathcal{A}_1$ and $\mathcal{A}_2$ satisfy all the conditions in Lemma 3.3, and therefore we conclude that the operator $\mathcal{A}$ has at least one fixed point x in $\bar{\mathcal{U}}_r$, which is a solution of the nonlocal initial value problem. □

3.3 Illustrative examples

Let $\mathfrak{D} = [a,\infty) \times \mathbb{R}, f(t,x) = \mathrm{e}^{-\frac{(t-a)^\alpha}{\alpha}}(x + \sin x), h(t) = l(t) = 2\mathrm{e}^{-\frac{(t-a)^\alpha}{\alpha}}$, and $L = 2$.

(I) Local initial value problems

It is easy to verify that, for any $(t,u),(t,v) \in \mathfrak{D}$,

$$\bigl|f(t,u) - f(t,v)\bigr| \leq L|u-v|$$

and

$$\bigl|f(t,u)\bigr| \leq h(t)|u|$$

for which $I_\alpha^a h(t) = 2 - h(t) \leq 2$. Hence, conditions (H1)–(H3) in Theorem 3.3 are satisfied for the above specified functions and parameters, which implies the solution to the local initial value problem (1.1)–(1.2) is defined and bounded on $[a,\infty)$.

(II) Stabilities

Similar to the case (I), we see that, for any $(t,u),(t,v) \in \mathfrak{D}$,

$$\bigl|f(t,u) - f(t,v)\bigr| \leq l(t)|u-v| \leq L|u-v|$$

for which $I_\alpha^a l(t) = 2 - l(t) \leq 2$. Thus, all the conditions in Theorem 3.4 are satisfied. By Theorem 3.4, we infer that every solution to the local initial value problem (1.1)–(1.2) is always stable.

(III) Nonlocal initial value problems

Choose the interval $[a,b]$ with $2 > \mathrm{e}^{\frac{(b-a)^\alpha}{\alpha}}$. Define the functions

$$\varphi(t) = 2\mathrm{e}^{-\frac{(t-a)^\alpha}{\alpha}}, \qquad \psi(z) = z \quad \text{and} \quad \phi(z) = \frac{\gamma}{2}z$$

with $0 < \gamma < \varphi(b) - 1$. For x in $C([a,b])$, define the functional $g(x) = \frac{\gamma}{2(b-a)}\int_a^b x(t)\,\mathrm{d}t$, and then it is easy to check that g is a contraction. Moreover, observe that

$$\bigl|f(t,u)\bigr| \leq \varphi(t)\psi\bigl(|u|\bigr)$$

for any (t,u) in $[a,b] \times \mathbb{R}$ and that a direct computation gives

$$\sup_{r\in(0,\infty)} \frac{r}{|x_a| + \psi(r)I_\alpha^a\varphi(b)} = \frac{1}{I_\alpha^a\varphi(b)} = \frac{1}{2-\varphi(b)} > \frac{1}{1-\gamma}.$$

Therefore, conditions (H5)–(H7) in Theorem 3.5 are satisfied for the above specified functions, functional and parameters, we conclude that the corresponding nonlocal initial value problem (1.1), (1.3) exists at least one solution defined on $[a,b]$.

4 Conclusion

By the use of the conformable fractional calculus and by means of fixed point theorems, some criteria are established for the global existence, extension, boundedness, and stabilities of solutions to the local initial value problem; and the existence result of solutions to the nonlocal initial value problem is also obtained. The obtained conditions are easy to satisfy and check. For $\alpha = 1$, the classical results corresponding to ordinary differential equations will be yielded.

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References

1. Oldham, K.B., Spanier, J.: Fractional Calculus: Theory and Applications, Differentiation and Integration to Arbitrary Order. Academic Press, New York (1974)
2. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integral and Derivatives: Theory and Applications. Gordon and Breach, Switzerland (1993)
3. Kilbas, A., Srivastava, H.M., Trujillo, J.J.: Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam (2006)
4. Diethelm, K.: The Analysis of Fractional Differential Equations. Lecture Notes in Mathematics, vol. 204. Springer, Berlin (2010)
5. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
6. Lundstrom, B.N., Higgs, M.H., et al.: Fractional differentiation by neocortical pyramidal neurons. Nat. Neurosci. **11**, 1335–1342 (2008)
7. West, B.J.: Colloquium: fractional calculus view of complexity: a tutorial. Rev. Mod. Phys. **86**, 1169–1184 (2014)
8. Zhong, W.Y., Lin, W.: Nonlocal and multiple-point boundary value problem for fractional differential equations. Comput. Math. Appl. **59**, 1345–1351 (2010)
9. Zhong, W.Y., Wang, L.F.: Monotone and concave positive solutions to three-point boundary value problems of higher-order fractional differential equations. Abstr. Appl. Anal. **2015**, Article ID 728491 (2015)
10. Zhong, W.Y.: Positive solutions for multipoint boundary value problem of fractional differential equations. Abstr. Appl. Anal. **2010**, Article ID 601492 (2010)
11. Khalil, R., Horani, M.A., et al.: A new definition of fractional derivative. J. Comput. Appl. Math. **264**, 65–70 (2014)
12. Jarad, F., Uğurlu, E., et al.: On a new class of fractional operators. Adv. Differ. Equ. (2017). https://doi.org/10.1186/s13662-017-1306-z
13. Abdeljawad, T.: On conformable fractional calculus. J. Comput. Appl. Math. **279**, 57–66 (2015)
14. Eslami, M.: Exact traveling wave solutions to the fractional coupled nonlinear Schrodinger equations. Appl. Math. Comput. **285**, 141–148 (2016)
15. Ekici, M., Mirzazadeh, M.: Optical soliton perturbation with fractional-temporal evolution by first integral method with conformable fractional derivatives. Optik **127**, 10659–10669 (2016)
16. Weberszpil, J., Helaël-Neto, J.A.: Variational approach and deformed derivatives. Physica A **450**, 217–227 (2016)
17. Anderson, D.R., Ulness, D.J.: Properties of the Katugampola fractional derivative with potential application in quantum mechanics. J. Math. Phys. **56**, 063502 (2015)
18. Katugampola, U.N.: A new fractional derivative with classical properties. arXiv:1410.6535
19. Al-Rifae, M., Abdeljawad, T.: Fundamental results of conformable Sturm–Liouville eigenvalue problems. Complexity **2017**, Article ID 3720471 (2017)
20. Asawasamrit, S., Ntouyas, S.K., Thiramanus, P., Tariboon, J.: Periodic boundary value problems for impulsive conformable fractional integrodifferential equations. Bound. Value Probl. **2016**, Article ID 122 (2016)

21. Abdeljawad, T., Alzabut, J.: A generalized Lyapunov-type inequality in the frame of conformable derivatives. Adv. Differ. Equ. (2017). https://doi.org/10.1186/s13662-017-1383-z
22. Dong, X., Bai, Z., Zhang, W.: Positive solutions for nonlinear eigenvalue problems with conformable fractional differential derivatives. J. Shandong Univ. Sci. Technol. Nat. Sci. **35**, 85–90 (2016)
23. Song, Q.L., Dong, X.Y., et al.: Existence for fractional Dirichlet boundary value problem under barrier strip conditions. J. Nonlinear Sci. Appl. **10**, 3592–3598 (2017)
24. He, L.M., Dong, X.Y., et al.: Solvability of some two-point fractional boundary value problems under barrier strip conditions. J. Funct. Spaces **2017**, Article ID 1465623 (2017)
25. Batarfi, H., Losada, J., et al.: Three-point boundary value problems for conformable fractional differential equations. J. Funct. Spaces **2015**, Article ID 706383 (2015)
26. Souahi, A., Makhlouf, A.B., et al.: Stability analysis of conformable fractional-order nonlinear systems. Indag. Math. **28**, 1265–1274 (2017)
27. Abdeljawad, T., AL Horani, M., Khalil, R.: Conformable fractional semigroups of operators. J. Semigroup Theory Appl. **2015**, Article ID 7 (2015)
28. Bayour, B., Torres, D.F.M.: Existence of solution to a local fractional nonlinear differential equation. J. Comput. Appl. Math. **312**, 127–133 (2017)
29. Lin, W.: Global existence theory and chaos control of fractional differential. J. Math. Anal. Appl. **332**, 709–726 (2007)
30. Lakshmikantham, V., Vatsala, A.S.: Basic theory of fractional differential equations. Nonlinear Anal. TMA **69**, 2677–2682 (2008)
31. N'Guérékata, G.M.: A Cauchy problem for some fractional abstract differential equation with non local conditions. Nonlinear Anal. TMA **70**, 1873–1876 (2009)
32. Balachandran, K., Park, J.Y.: Nonlocal Cauchy problem for abstract fractional semilinear evolution equations. Nonlinear Anal. TMA **71**, 4471–4475 (2009)
33. O'Regan, D.: Fixed-point theory for the sum of two operators. Appl. Math. Lett. **9**, 1–8 (1996)