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Homoclinic solutions for Hamiltonian system with impulsive effects

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Abstract

In this article, we investigate a class of impulsive Hamiltonian systems with a *p*-Laplacian operator. By establishing a series of new sufficient conditions, the existence of homoclinic solutions to such type of systems is revealed. We show the existence of homoclinic orbit induced by impulses by introducing some conditions. To illustrate the applications of the main results in this article, we create an example.

Keywords: Impulsive Hamiltonian system; *p*-Laplacian operator; Homoclinic solution

1 Introduction

The aim of the article is to investigate the impulsive Hamiltonian systems with p-Laplacian operator of the form

$$\bigl(\Phi_p\bigl(u'(t)\bigr)\bigr)' - \nabla F\bigl(t,u(t)\bigr) = 0, \quad t \neq t_i, t \in R, \tag{1}$$

$$-\Delta\bigl(\Phi_p\bigl(u'(t_i)\bigr)\bigr) = g_i\bigl(u(t_i)\bigr), \quad i \in Z. \tag{2}$$

Here, we are particularly interested in the existence of homoclinic orbits for such systems. In the system, $u \in R$, $\Delta(\Phi_p(u'(t_i))) = |u'(t_i^+)|^{p-2}u'(t_i^+) - |u'(t_i^-)|^{p-2}u'(t_i^-)$ with $u'(t_i^{\pm}) = \lim_{t\to t_i^{\pm}} u'(t)$, $\nabla F(t,u) = \operatorname{grad}_u F(t,u)$, $g_i(u) = \operatorname{grad}_u G_i(u)$, $G_i \in C^1(R^n, R^n)$ for each $i \in Z$. There exist an $m \in N$ and a $T > 0$ such that $0 = t_0 < t_1 < \cdots < t_m = T$, $t_{i+m} = t_i + T$ and $g_{i+m} = g_i$ for all $i \in Z$.

The existence and multiplicity of homoclinic orbits attracted the attention of researchers from all over the world and as such have been extensively investigated in the literature [1–8]. Mathematical techniques such as the dual variational method [9], concentration compactness method and Ekeland variational principle [10, 11], and the approximation method [12] have been used in evaluating the existence of homoclinic orbits for Hamiltonian systems.

Real-world systems display a variety of abrupt changes, and such changes can be modeled using impulsive differential equations. Impulsive effects have been integrated into different types of differential equations to describe the consequences of abrupt changes. Such systems have been investigated in the literature [13–24].

A special case of system (1)–(2) where $p = 2$ has been considered by Zhang and Li [20]. The authors get the following result.

Theorem 1 ([20]) *Assume that, for $j = 1, 2, \ldots, m$, g_j is continuous and m-periodic in j, and F and g_j satisfy the following conditions*:

(H_1) *$F : R \times R^N \to R$ is continuously differentiable and T-periodic, and there exist positive constants $r_1, r_2 > 0$ such that*

$$r_1|u|^2 \le F(t,u) \le r_2|u|^2, \quad \forall (t,u) \in [0,T] \times R^n;$$

(H_2) *$F(t,u) \le (\nabla F(t,u), u) \le 2F(t,u)$, $\forall (t,u) \in [0,T] \times R^n$;*

(H_3) *there exists $\mu > 2$ such that*

$$0 < -\mu G_j(u) \le -u g_j(u), \quad \text{for } u \in R^n \setminus \{0\}, j = 1, 2, \ldots, m;$$

(H_4) $\lim_{|u|\to 0} \frac{g_j(u)}{|u|} = 0$ *for $j = 1, 2, \ldots, m$.*

Then the second order impulsive Hamiltonian system

$$\begin{aligned} &u''(t) - \nabla F\bigl(t, u(t)\bigr) = 0, \quad t \ne t_i, t \in R, \\ &-\Delta u'(t_i) = g_i\bigl(u(t_i)\bigr), \quad i \in Z, \end{aligned}$$

has at least one nonzero homoclinic solution generated by impulses.

For the general case of $p \ne 2$, due to the complex structure of problem (1) and (2), it is challenging to construct an appropriate functional such that the existence of its critical point implies a homoclinic orbit of the system. Since the domain under consideration is unbounded, the Sobolev embedding might not be compact. In order to complete the proof, we show that the homoclinic orbit u is obtained as the limit of $2kT$-periodic solutions u_k of (1)–(2) as $k \to \infty$. Due to the impulsive perturbation, the velocity is no longer continuous. Besides, if $p \ne 2$, the Sobolev space $W^{1,p}_{2kT}$ is not a Hilbert space. When we use the mountain pass theorem to prove the main results of the paper, it is necessary to guarantee that constants ρ and α are required to be independent of k. We also need to show that the approximating solution sequence $\{u_k\}$ has a bound, which is independent of k.

Before introducing the main results, we make the following assumptions.

Theorem 2 *Assume that F, g_i satisfies the following conditions*:

(A1) *$F \in C^1(R \times R^N, R)$, $F(t, 0) \equiv 0$ and F is T-periodic in its first variable;*

(A2) *$F(t,u) = \frac{1}{p}|u|^p + H(t,u)$, where $H \in C^1(R \times R^N, R)$;*

(A3) *for every $t \in R$ and $u \in R^n \setminus \{0\}$, there exists $\mu > p$ such that*

$$pH(t,u) \ge u \nabla H(t,u) > 0;$$

(A4) *there exists $\mu > p$, such that $g_i(u)u \le \mu G_i(u) \le 0$, $u \in R^n \setminus \{0\}$, $i = 1, 2, \ldots, m$.*

Then the system (1)–(2) possesses at least one nonzero homoclinic solution generated by impulses.

Here, a solution of problem (1)–(2) is said to be generated from the impulse if this solution emerges when the impulse is not zero, and on the other hand, the solution disappears when the impulse is zero.

The rest of the paper is organized as follows. In Sect. 2, we present some preliminaries and statements, which will be used in proving our main results. The proof of the main results of this article is given in Sect. 3. We then present an example to illustrate the applicability of our results in Sect. 4.

2 Preliminaries and statements

For each $k \in N$, set

$$E_k = \{u : R \to R^n \mid u, u' \in L^p([-kT, kT], R^n), u(t) = u(t + 2kT), t \in R\}.$$

Thus, E_k is a Hilbert space with the norm defined by

$$\|u\|_{E_k} = \left(\int_{-kT}^{kT} \left(\left|u'(t)\right|^p + \left|u(t)\right|^p\right) dt\right)^{\frac{1}{p}}.$$

Let $L^p_{2kT}(R, R^n)$ denote the Hilbert space of $2kT$-periodic functions on R with values in R^n under the norm

$$\|u\|_{L^p_{2kT}(R,R^n)} = \left(\int_{-kT}^{kT} \left|u(t)\right|^p dt\right)^{\frac{1}{p}},$$

and $L^\infty_{2kT}(R, R^n)$ be a space of $2kT$-periodic essentially bounded measurable functions from R to R^n under the norm

$$\|u\|_{L^\infty_{2kT}(R,R^n)} = \operatorname{esssup}\left\{\left|u(t)\right| : t \in [-kT, kT]\right\}.$$

Set $\Omega_k = \{-km + 1, -km + 2, \dots, 0, 1, 2, \dots, km - 1, km\}$ and define

$$\begin{aligned} I_k(u) &= \int_{-kT}^{kT} \left[\frac{1}{p}\left|u'(t)\right|^p + F\left(t, u(t)\right)\right] dt + \sum_{i\in\Omega_k} G_i\left(u(t_i)\right) \\ &= \frac{1}{p}\|u\|^p_{E_k} + \int_{-kT}^{kT} H\left(t, u(t)\right) dt + \sum_{i\in\Omega_k} G_i\left(u(t_i)\right). \end{aligned} \tag{3}$$

It follows that I_k is Frechet differentiable at any $u \in E_k$. For any $u \in E_k$, we thus have

$$\begin{aligned} I'_k(u)v = {} & \int_{-kT}^{kT} \left[\left|u'(t)\right|^{p-2} u'(t)v'(t) + \nabla F\left(t, u(t)\right)v(t)\right] dt \\ & + \sum_{i\in\Omega_k} g_i\left(u(t_i)\right)v(t_i). \end{aligned} \tag{4}$$

The above analysis shows that the critical points of the functional I_k are classical $2kT$-periodic solutions of system (1)–(2).

Lemma 1 ([25]) *There exists a positive constant C independent of k such that, for each $k \in N$ and $u \in E_k$, one has*

$$\|u\|_{L^\infty_{2kT}(R,R^n)} \le C\|u\|_{E_k}. \tag{5}$$

Lemma 2 ([26]) *Let $u : R \to R^n$ be a continuous mapping such that $u' \in L^p_{\mathrm{loc}}(R, R^n)$. For every $t \in R$, the following inequality holds*:

$$\left|u(t)\right| \le 2^{\frac{p-1}{p}} \left(\int_{t-\frac{1}{2}}^{t+\frac{1}{2}} \left(\left|u(s)\right|^p + \left|u'(s)\right|^p \right) ds \right)^{\frac{1}{p}}. \tag{6}$$

Lemma 3 ([27]) *Let X be a real Banach space and $I \in C'(X, R)$ satisfying the Palais–Smale (PS)-condition. Suppose that I satisfies the following conditions*:

(i) $I(0) = 0$;

(ii) *there exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \ge \alpha$*;

(iii) *there exists $e \in X \backslash \bar{B}_\rho(0)$ such that $I(e) \le 0$.*

Then I possesses a critical value $c \ge \alpha$ given by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} I\bigl(g(s)\bigr),$$

where $B_\rho(0)$ is an open ball in X of radius ρ centered at 0, and

$$\Gamma = \bigl\{ g \in C\bigl([0,1], X\bigr) : g(0) = 0, g(1) = e \bigr\}.$$

3 Proof of Theorem 2

Before starting the proof of Theorem 2, we recall some properties of the function $H(t, u)$.

Remark 1 If (A3) holds, then, for every $t \in [0, T]$, there exist $a_1, a_2 > 0$ such that:

$$H(t, u) \ge a_1 |u|^p, \quad \text{if } 0 < |u| < 1, \tag{7}$$

$$H(t, u) \le a_2 |u|^p, \quad \text{if } |u| \ge 1, \tag{8}$$

where $a_1 = \min_{t \in [0,T], |u|=1} H(t, u)$, $a_2 = \max_{t \in [0,T], |u|=1} H(t, u)$.

Proof To prove this fact, it suffices to show that, for every $u \neq 0$ and $t \in [0, T]$ the function $\zeta \to H(t, \zeta^{-1} u) \zeta^p$ (where $\zeta \in (0, +\infty)$) is nonincreasing, which is an immediate consequence of (A3). □

Remark 2 Assume that (A4) hold, then there exist $b_1, b_2 > 0$, such that

$$G_i(u) \ge -b_1 |u|^\mu, \quad \text{if } 0 < |u| < 1, \tag{9}$$

$$G_i(u) \le -b_2 |u|^\mu, \quad \text{if } |u| \ge 1. \tag{10}$$

Proof Since the proof is similar to that of Fact 2.2 of [11], here it is omitted. □

We divide the proof of Theorem 2 into several lemmas.

Lemma 4 *If F, g_i satisfies* (A1)–(A4), *then, for all $k \in N$, system* (1)–(2) *has a $2kT$-periodic solution.*

Proof By (A1) and (A4), we get $I_k(0) = 0$.

We first show that I_k, g_i satisfies the (PS)-condition. Assume that $\{u_j\}_{j\in N}$ in E_k is a sequence such that $\{I_k(u_j)\}_{j\in N}$ is bounded and $I_k'(u_j) \to 0$ as $j \to +\infty$. Then, for every $j \in N$, there is a constant $C_k > 0$ such that

$$\left|I_k(u_j)\right| \le C_k, \tag{11}$$

$$\left\|I_k'(u_j)\right\|_{E_k^*} \le C_k. \tag{12}$$

It follows from (4) that

$$\begin{aligned} I_k'(u_j)u_j &= \int_{-kT}^{kT} \left[\left|u_j'(t)\right|^p + \nabla F\big(t,u_j(t)\big)u_j(t)\right] dt + \sum_{i\in\Omega_k} g_i\big(u_j(t_i)\big)u_j(t_i) \\ &= \int_{-kT}^{kT} \left[\left|u_j'(t)\right|^p + \left|u_j(t)\right|^p\right] dt + \int_{-kT}^{kT} \nabla H\big(t,u_j(t)\big)u_j(t)\, dt \\ &\quad + \sum_{i\in\Omega_k} g_i\big(u_j(t_i)\big)u_j(t_i). \end{aligned} \tag{13}$$

By (A3), (A4), (3), (11), (12) and (13) we have

$$\begin{aligned} 2C_k &\ge I_k(u_j) - \frac{1}{\mu} I_k'(u_j)u_j \\ &= \int_{-kT}^{kT} \frac{1}{p}\left|u_j'(t)\right|^p + F\big(t,u_j(t)\big)\, dt + \sum_{i\in\Omega_k} G_i\big(u(t_i)\big) \\ &\quad - \int_{-kT}^{kT} \frac{1}{\mu}\left|u_j'(t)\right|^\mu - \frac{1}{\mu}\nabla F\big(t,u_j(t)\big)u_j(t)\, dt - \sum_{i\in\Omega_k} g_i\big(u_j(t_i)\big)u_j(t_i) \\ &\ge \int_{-kT}^{kT} \frac{1}{p}\left|u_j'(t)\right|^p + \frac{1}{p}\left|u_j(t)\right|^p - \frac{1}{\mu}\left|u_j'(t)\right|^\mu - \frac{1}{\mu}\left|u_j(t)\right|^\mu dt \\ &\quad + \int_{-kT}^{kT} H\big(t,u_j(t)\big) - \frac{1}{\mu}\nabla H\big(t,u_j(t)\big)u_j(t)\, dt \\ &\ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_j\|_{E_k}^p + \int_{-kT}^{kT} H\big(t,u_j(t)\big) - \frac{1}{p}\nabla H\big(t,u_j(t)\big)u_j(t)\, dt \\ &\ge \left(\frac{1}{p} - \frac{1}{\mu}\right)\|u_j\|_{E_k}^p. \end{aligned} \tag{14}$$

Since $\mu > p$, we know that $\{u_j\}_{j\in N}$ is bounded in E_k. Using the method proposed in [27], we can show that $\{u_n\}$ has a convergent subsequence. It thus follows that I_k satisfies the (PS)-condition.

We now show that the functional I_k satisfies Assumption (ii) in Lemma 3. Choose $0 < \delta < 1$ such that

$$\min\left\{\frac{1}{p}, \frac{1}{p} + a_1\right\} \frac{\delta^p}{C^p} - \frac{b_1\delta^\mu}{C^\mu} > 0.$$

If $\|u\|_{E_k} = \frac{\delta}{C} := \rho$, then it follows from (5) that $|u(t)| \le \delta < 1$ for $t \in [-kT, kT]$. From (3), (7) and (9) we have

$$I_k(u) \ge \int_{-kT}^{kT} \frac{1}{p}\left|u_j'(t)\right|^p + \frac{1}{p}\left|u_j(t)\right|^p dt + \int_{-kT}^{kT} H\big(t,u(t)\big)\, dt + \sum_{i\in\Omega_k} G_i\big(u(t_i)\big)$$

$$
\begin{aligned}
&\geq \int_{-kT}^{kT} \frac{1}{p}\left|u_j'(t)\right|^p + \frac{1}{p}\left|u_j(t)\right|^p + a_1\left|u(t)\right|^p\,dt - b_1\sum_{i\in\Omega_k}|u|^\mu \\
&\geq \min\left\{\frac{1}{p},\frac{1}{p}+a_1\right\}\|u\|_{E_k}^p - b_1\|u\|_{E_k}^\mu \\
&= \min\left\{\frac{1}{p},\frac{1}{p}+a_1\right\}\frac{\delta^p}{C^p} - \frac{b_1\delta^\mu}{C^\mu} \\
&:= \alpha > 0.
\end{aligned}
\tag{15}
$$

It follows from (15) that $\|u\|_{E_k} = \frac{\delta}{C} = \rho$, which implies that $I_k(u) \geq \alpha$.

It remains to prove that the functional I_k satisfies Assumption (iii) of Lemma 3. From (3), (8) and (10) we have

$$
\begin{aligned}
I_k(\zeta u) &= \frac{|\zeta|^p}{p}\|u\|_{E_k}^p + \int_{-kT}^{kT} H\bigl(t,\zeta u(t)\bigr)\,dt + \sum_{i\in\Omega_k} G_i\bigl(\zeta u(t_i)\bigr) \\
&\leq \frac{|\zeta|^p}{p}\|u\|_{E_k}^p + a_2|\zeta|^p\|u\|_{E_k}^p - b_2|\zeta|^\mu\sum_{i\in\Omega_k}\bigl|u(t_i)\bigr|^\mu.
\end{aligned}
\tag{16}
$$

Take $U \in E_1$ such that $U(\pm T) = 0$. Since $\mu > p$ and $b_1 > 0$, then by (16) there exists $\xi \in R\backslash\{0\}$ such that $\|\xi U\|_{E_1} > \rho$ and $I_1(\xi U) < 0$. For $k > 1$, set $e_1(t) = \xi U(t)$ and

$$
e_k(t) = \begin{cases} e_1(t), & |t| \leq T, \\ 0, & T < |t| \leq kT. \end{cases}
\tag{17}
$$

Then $e_k \in E_k$, $\|e_k\|_{E_k} = \|e_1\|_{E_1} > \rho$ and $I_k(e_k) = I_1(e_1) < 0$ for every $k \in N$.

By Lemma 3, I_k possesses a critical value $c_k \geq \alpha > 0$. Then, for every $k \in N$, there exists $u_k \in E_k$ satisfying

$$
I_k(u_k) = c_k, \qquad I_k'(u_k) = 0.
\tag{18}
$$

Hence, system (1)–(2) has a nontrivial $2kT$-periodic solution u_k. □

Lemma 5 *Let $\{u_k\}$ be the sequence given by (18). There exist a subsequence $\{u_{j,j}\}$ of $\{u_k\}$ and a function $u_0 \in W_{\mathrm{loc}}^{1,p} \cap L_{\mathrm{loc}}^\infty(R,R^n)$ such that $\{u_{j,j}\}$ converges to u_0 weakly in $W_{\mathrm{loc}}^{1,p}$ and strongly in $L_{\mathrm{loc}}^\infty(R,R^n)$.*

Proof Our first step is to show that the sequence $\{c_k\}_{k\in N}$ is bounded. For each $k \in N$, let $g_k : [0,1] \to E_k$ be a curve given by $g_k(s) = se_k$, where e_k is defined by (17). Then $g_k \in \Gamma_k$ and $I_k(g_k(s)) = I_1(g_1(s))$ for all $k \in N$ and $s \in [0,1]$. Therefore, it follows from the mountain pass theorem that

$$
c_k \leq \max_{s\in[0,1]} I_1\bigl(g_1(s)\bigr) \equiv C_0,
\tag{19}
$$

which is independent of $k \in N$. Since $I_k'(u_k) = 0$, from (A3), (A4) and (3) we obtain

$$
c_k = I_k(u_j) - \frac{1}{\mu}I_k'(u_j)u_j
$$

$$\begin{aligned}
&= \int_{-kT}^{kT} \frac{1}{p}\left|u_j'(t)\right|^p + F\big(t,u_j(t)\big)\,dt + \sum_{i\in\Omega_k} G_i\big(u(t_i)\big) \\
&\quad - \int_{-kT}^{kT} \frac{1}{\mu}\left|u_j'(t)\right|^\mu - \frac{1}{\mu}\nabla F\big(t,u_j(t)\big)u_j(t)\,dt - \sum_{i\in\Omega_k} g_i\big(u_j(t_i)\big)u_j(t_i) \\
&\geq \int_{-kT}^{kT} \frac{1}{p}\left|u_j'(t)\right|^p + \frac{1}{p}\left|u_j(t)\right|^p - \frac{1}{\mu}\left|u_j'(t)\right|^\mu - \frac{1}{\mu}\left|u_j(t)\right|^\mu\,dt \\
&\quad + \int_{-kT}^{kT} H\big(t,u_j(t)\big) - \frac{1}{\mu}\nabla H\big(t,u_j(t)\big)u_j(t)\,dt \\
&\geq \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_j\|_{E_k}^p + \int_{-kT}^{kT} H\big(t,u_j(t)\big) - \frac{1}{p}\nabla H\big(t,u_j(t)\big)u_j(t)\,dt \\
&\geq \left(\frac{1}{p}-\frac{1}{\mu}\right)\|u_j\|_{E_k}^p.
\end{aligned} \tag{20}$$

Since $\mu > p$ and all the constants in (20) are independent of k, then there exists a constant $L_1 > 0$ independent of k such that

$$\|u_k\|_{E_k} \leq L_1. \tag{21}$$

The boundedness of $\|u_k\|_{E_k}$ implies the boundedness of the set

$$\big\{\|u_k\|_{W^{1,p}((-kT,kT),R^n)}\big\}$$

for each positive integer k. In particular, when $k = 1$, since $\{u_k\}$ is a bounded sequence in

$$W^{1,p}\big((-T,T),R^n\big),$$

we can pick a subsequence $\{u_{1,k}\}$ such that $\{u_{1,k}\}$ converges weakly in

$$W^{1,p}\big((-T,T),R^n\big)$$

and strongly in $L^\infty((-T,T),R^n)$. For $k = 2$, since $\{u_{1,k}\}$ is bounded in $W^{1,p}((-2T,2T),R^n)$, we can pick a subsequence $\{u_{2,k}\}$ such that $\{u_{2,k}\}$ converges weakly in $W^{1,p}((-2T,2T),R^n)$ and strongly in $L^\infty((-2T,2T),R^n)$. We can repeat this process for $k = 3,4,\ldots$ and see that, for any positive integer m, there is a sequence $\{u_{m,k}\}$ which converges weakly in $W^{1,p}((-mT,mT),R^n)$ and strongly in $L^\infty((-mT,mT),R^n)$, and

$$\{u_k\} \supset \{u_{1,k}\} \supset \{u_{2,k}\} \supset \cdots \supset \{u_{m,k}\} \supset \cdots.$$

It follows that the sequence $\{u_{k,k}\}$ converges weakly in $W^{1,p}((-kT,kT),R^n)$ and strongly in $L^\infty((-kT,kT),R^n)$. Hence, there exists a function

$$u \in W^{1,p}_{\mathrm{loc}}\big(R,R^n\big) \cap L^\infty_{\mathrm{loc}}\big(R,R^n\big)$$

such that the sequence $\{u_{k,k}\}$ converges weakly in $W^{1,p}_{\mathrm{loc}}(R,R^n)$ and strongly in $L^\infty_{\mathrm{loc}}(R,R^n)$. □

Lemma 6 *The function u_0 determined by Lemma 5 is a nonzero homoclinic solution of the system (1)–(2).*

Proof The proof is divided into four steps.

First we show that u_0 is a solution of system (1)–(2). For convenience, we denote $\{u_{k,k}\}$ by $\{u_k\}$. For any given interval $(a,b) \subset (-kT,kT)$ and any $v \in W_0^{1,p}((a,b),R^n)$, define

$$v_1(t) = \begin{cases} v(t), & t \in (a,b), \\ 0, & t \in (-kT,kT) \setminus (a,b). \end{cases} \tag{22}$$

Then, for any $v \in W_0^{1,p}((a,b),R^n)$, one has

$$\begin{aligned} &\int_a^b \left(u_0'v' - \nabla F(t,u_0)v\right) dt + \sum_{t_i \in (a,b)} g_i\left(u_0(t_i)\right)v(t_i) \\ &\quad= \lim_{k\to+\infty}\left[\int_a^b \left(u_k'v' - \nabla F(t,u_k)v\right) dt + \sum_{t_i\in(a,b)} g_i\left(u_k(t_i)\right)v(t_i)\right] = 0. \end{aligned} \tag{23}$$

Let k_1 be a positive integer and set $a = -k_1T$ and $b = k_1T$. By a similar method to the one proposed in [20], using (23), we can show that $u_0|_{(t_i,t_{i+1})}$ satisfies (1) in classical sense for $i = -k_1m + 1, \dots, k_1m$, and u_0 satisfies

$$\left|u_0'(t_i^+)\right|^{p-2}u_0'(t_i^+) = \left|u_0'(t_i^-)\right|^{p-2}u_0'(t_i^-) + g_i\left(u_0(t_i)\right), \quad i = -k_1m+1, \dots, k_1m-1.$$

Since $v \in W_0^{1,p}((-k_1T,k_1T),R^n)$, we cannot show that

$$\left|u_0'(t_{k_1m}^+)\right|^{p-2}u_0'(t_{k_1m}^+) = \left|u_0'(t_{k_1m}^-)\right|^{p-2}u_0'(t_{k_1m}^-) + g_{k_1m}\left(u_0(t_{k_1m})\right)$$

from (23). Therefore, u_0 is a solution of (1)–(2) in $(-k_1T,k_1T)$. Since k_1 is arbitrary, u_0 is a solution of system (1)–(2) in R.

Next, we prove that $u_0(t) \to 0$, as $t \to \pm\infty$. Since $\{u_k\}$ is weakly continuous, it is weakly lower semicontinuous, and thus we have

$$\begin{aligned} \int_{-\infty}^{+\infty} \left(\left|u_0(t)\right|^p + \left|u_0'(t)\right|^p\right) dt &= \lim_{k\to+\infty}\int_{-kT}^{kT}\left(\left|u_0(t)\right|^p + \left|u_0'(t)\right|^p\right) dt \\ &\le \lim_{k\to+\infty}\lim_{j\to+\infty}\inf\int_{-kT}^{kT}\left(\left|u_j(t)\right|^p + \left|u_j'(t)\right|^p\right) dt \\ &\le L_1^p. \end{aligned}$$

Therefore,

$$\int_{|t|\ge r}\left(\left|u_0(t)\right|^p + \left|u_0'(t)\right|^p\right) dt \to 0, \quad \text{as } r \to +\infty. \tag{24}$$

By (6) and (24), we get $u_0(t) \to 0$ as $t \to \pm\infty$.

In the third step we show that $u_0'(t^{\pm}) \to 0$, as $t \to \pm\infty$. Since $u_0(t)$ is a solution of system (1)–(2), we obtain

$$\int_{t_{i-1}}^{t_i} \left|(\Phi_p(u_0'(t))'\right|^p \, dt = \int_{t_{i-1}}^{t_i} \left|\nabla F(t, u_0(t))\right|^p \, dt.$$

Since $\nabla F(t,0) = 0$ for all $t \in R$, $u_0(t) \to 0$ as $t \to \pm\infty$, we have

$$\int_{t_{i-1}}^{t_i} \left|(\Phi_p(u_0'(t))'\right|^p \, dt \to 0$$

as $i \to \pm\infty$. Then, from the definition of $\Phi_p(u)$, there exists $L_2 > 0$ such that

$$\int_{t_{i-1}}^{t_i} \left|u_0''(t)\right|^p \, dt < L_2.$$

By (6), we obtain

$$\begin{aligned} \left|u_0'(t)\right|^p &\le 2^{p-1} \int_{t_{i-1}}^{t_i} \left(\left|u_0'(t)\right|^p + \left|u_0''(t)\right|^p\right) dt \\ &\le 2^{p-1} \int_{t_{i-1}}^{t_i} \left(\left|u_0(t)\right|^p + \left|u_0'(t)\right|^p\right) dt + 2^{p-1} \int_{t_{i-1}}^{t_i} \left|u_0''(t)\right|^p \, dt, \quad t \in (t_{i-1}, t_i). \end{aligned}$$

Therefore we have $u_0'(t^{\pm}) \to 0$ as $t \to \pm\infty$.

Next, we show that $u_0 \neq 0$. $I_k'(u_k)u_k = 0$ implies

$$\int_{-kT}^{kT} \left[\left|u_k'(t)\right|^p + \nabla F(t, u_k(t))u_k(t)\right] dt = -\sum_{i \in \Omega_k} g_i(u_k(t_i))u_k(t_i). \tag{25}$$

Let $\underline{\theta} = \min_{i \in Z}\{t_i - t_{i-1}\}$ and $\overline{\theta} = \max_{i \in Z}\{t_i - t_{i-1}\}$. Then, by the Hölder inequality and $u_k(t_i) = u_k(\tau) + \int_\tau^{t_i} u_k'(t)\, dt$, $\tau \in [t_{i-1}, t_i]$, $i \in \Omega_k$, one has

$$\begin{aligned} &\sum_{i \in \Omega_k} \left|u_k(t_i)\right|^p \\ &\quad\le \frac{1}{\underline{\theta}} \sum_{i \in \Omega_k} \int_{t_{i-1}}^{t_i} \left|u_k(t_i)\right|^p \, d\tau \\ &\quad\le \frac{2^{p-1}}{\underline{\theta}} \sum_{i \in \Omega_k} \int_{t_{i-1}}^{t_i} \left(\left|u_k(\tau)\right|^p + \left|\int_\tau^{t_i} u_k'(s)\, ds\right|^p\right) d\tau \\ &\quad\le \frac{2^{p-1}}{\underline{\theta}} \int_{-kT}^{kT} \left|u_k(\tau)\right|^p \, d\tau + \frac{2^{p-1}}{\underline{\theta}} \sum_{i \in \Omega_k} \int_{t_{i-1}}^{t_i} \left[(t_i - \tau) \int_\tau^{t_i} \left|u_k'(s)\right|^p \, ds\right] d\tau \\ &\quad\le \frac{2^{p-1}}{\underline{\theta}} \|u_k\|_{L_{2kT}^2}^p + \frac{2^{p-1}\overline{\theta}^p}{\underline{\theta}} \|u_k'\|_{L_{2kT}^2}^p \\ &\quad\le \frac{2^{p-1}}{\underline{\theta}} \max\{1, \overline{\theta}^p\} \|u_k\|_{E_k}^p. \end{aligned} \tag{26}$$

Suppose that $\|u_k\|_{L^\infty} \to 0$, then, when k sufficiently large, by (A4), (9), (25) and (26), we have

$$\begin{aligned}
b_1\mu \sum_{i\in\Omega_k} \bigl|u_k(t_i)\bigr|^{\mu} &\geq -\sum_{i\in\Omega_k} g_i\bigl(u_k(t_i)\bigr)u_k(t_i) \\
&= \int_{-kT}^{kT} \bigl[\bigl|u_k'(t)\bigr|^p + \nabla F\bigl(t,u_k(t)\bigr)u_k(t)\bigr]\,dt \\
&= \int_{-kT}^{kT} \bigl[\bigl|u_k'(t)\bigr|^p + \bigl|u_k(t)\bigr|^p\bigr]\,dt + \int_{-kT}^{kT} \nabla H\bigl(t,u_k(t)\bigr)u_k(t)\,dt \\
&\geq \|u_k\|_{E_k}^p \\
&\geq \frac{\theta}{2^{p-1}\max\{1,\overline{\theta}^p\}} \sum_{i\in\Omega_k} \bigl|u_k(t_i)\bigr|^p.
\end{aligned}$$

This is impossible when $\|u_k\|_{L^\infty} \to 0$. Hence, the system (1)–(2) has a nontrivial homoclinic solution. □

Lemma 7 *The u_0 given in Lemma 6 is generated by impulses.*

Proof In order to complete the proof, we only need to show that under the conditions of Theorem 2, the system $(\Phi_p(u'(t)))' - \nabla F(t,u(t)) = 0$ has no nontrivial homoclinic solution.

If u is a homoclinic solution of $(\Phi_p(u'(t)))' - \nabla F(t,u(t)) = 0$, then

$$\lim_{t\to\to\pm\infty} u(t) = 0,$$
$$\lim_{t\to\to\pm\infty} u'(t) = 0.$$

By

$$\begin{aligned}
0 &= -\int_a^b \bigl(\bigl|u'(t)\bigr|^{p-2}u'(t)\bigr)'u(t) - \nabla F\bigl(t,u(t)\bigr)u(t)\,dt \\
&= \int_a^b \bigl|u'(t)\bigr|^p + \nabla F\bigl(t,u(t)\bigr)u(t)\,dt - \bigl|u'(t)\bigr|^{p-2}u'(t)u(t)\Big|_a^b \\
&= \int_a^b \bigl(\bigl|u'(t)\bigr|^p + \bigl|u(t)\bigr|^p\bigr)\,dt + \int_a^b H\bigl(t,u(t)\bigr)u(t)\,dt - \bigl|u'(t)\bigr|^{p-2}u'(t)u(t)\Big|_a^b,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_a^b \bigl(\bigl|u'(t)\bigr|^p + \bigl|u(t)\bigr|^p\bigr)\,dt + \int_a^b H\bigl(t,u(t)\bigr)u(t)\,dt \\
&\quad\leq \lim_{a\to-\infty}\lim_{b\to\infty} \bigl|u'(t)\bigr|^{p-2}u'(t)u(t)\Big|_a^b = 0.
\end{aligned}$$

Moreover, by (A3), we know that

$$\int_a^b H\bigl(t,u(t)\bigr)u(t)\,dt \geq 0.$$

Then

$$\int_a^b \left(\left|u'(t)\right|^p + \left|u(t)\right|^p\right) dt + \int_a^b H\bigl(t, u(t)\bigr) u(t)\, dt \geq 0.$$

Therefore, $u = 0$. It thus follows that the system $(\Phi_p(u'(t)))' - \nabla F(t, u(t)) = 0$ has only a trivial homoclinic solution. □

4 Example

In this section, we present an example to illustrate the application of the main results obtained in precious sections.

Example 1 Let

$$\begin{aligned} &p = 6, \qquad H(t,u) = (2 + \sin t)u^2, \\ &G_i\bigl(u(t_i)\bigr) = -\left|\cos \frac{t_i}{2}\right| u^{10}(t_i), \qquad g_i\bigl(u(t_i)\bigr) = -10\left|\cos \frac{t_i}{2}\right| u^9(t_i), \end{aligned} \tag{27}$$

where $t_i = \frac{2\pi i}{m}$, $i \in Z$.

For $\mu = 7$ and $p = 6$, we have $F(t,u) = \frac{1}{6}|u|^6 + (2 + \sin t)u^2$. It is easy to see that $F(t,0) = 0$. We notice that (A1), (A2) are satisfied.

By $H(t,u) = (2 + \sin t)u^2$, one has $\nabla H(t,u) = 2(2 + \sin t)u$. Therefore,

$$0 < u\nabla H(t,u) = 2(2 + \sin t)u^2 < 6(2 + \sin t)u^2 = pH(t,u),$$

which implies that condition (A3) is satisfied.

By $G_i(u(t_i)) = -|\cos \frac{t_i}{2}| u^{10}(t_i)$, $g_i(u(t_i)) = -10|\cos \frac{t_i}{2}| u^9(t_i)$, we have

$$g_i\bigl(u(t_i)\bigr)u(t_i) = -10\left|\cos \frac{t_i}{2}\right| u^{10}(t_i) < -7\left|\cos \frac{t_i}{2}\right| u^{10}(t_i) = \mu G_i\bigl(u(t_i)\bigr) < 0.$$

Thus, condition (A4) is satisfied.

By Theorem 2, system (1)–(2) with F, H, G_i and g_i defined in (27) has a nontrivial homoclinic solution.

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