Ouannas et al. *Advances in Difference Equations* (2018) 2018:309
https://doi.org/10.1186/s13662-018-1772-y

Advances in Difference Equations
a SpringerOpen Journal

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Function-based hybrid synchronization types and their coexistence in non-identical fractional-order chaotic systems

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Abstract

This paper presents new results related to the coexistence of *function-based hybrid synchronization* types between non-identical incommensurate fractional-order systems characterized by different dimensions and orders. Specifically, a new theorem is illustrated, which ensures the coexistence of the *full-state hybrid function projective synchronization* (*FSHFPS*) and the *inverse full-state hybrid function projective synchronization* (*IFSHFPS*) between wide classes of three-dimensional master systems and four-dimensional slave systems. In order to show the capability of the approach, a numerical example is reported, which illustrates the coexistence of *FSHFPS* and *IFSHFPS* between the incommensurate *chaotic* fractional-order *unified system* and the incommensurate *hyperchaotic* fractional-order *Lorenz system*.

MSC: 34H10; 26A33; 34A08

Keywords: Coexistence of synchronization types; Full-state hybrid function projective synchronization; Inverse full-state hybrid function projective synchronization; Incommensurate fractional-order systems; Non-identical systems

1 Introduction

Chaos synchronization refers to a process wherein two dynamical systems (master and slave systems, respectively) adjust their motion to achieve a common behavior, mainly due to a coupling or control input [1]. The issue was firstly studied in dynamical systems described by *integer-order* differential equations [2]. By considering the historical timeline of the topic, it can be observed that a large variety of synchronization types has been proposed [3, 4]. Among the different methods, *projective synchronization* provides the slave system variables consisting in scaled replicas of the master system variables [5]. Recently, the *full-state hybrid projective synchronization* (*FSHPS*) has been introduced, wherein each slave system variable synchronizes with a linear combination of master system variables. On the other hand, when the inverted scheme is implemented, i.e., each master system state synchronizes with a linear combination of slave system states, the *inverse full-state hybrid projective synchronization* (*IFSHPS*) is obtained [6]. Moreover, when the scaling factors are replaced by scaling functions, *function-based* hybrid synchronization schemes are obtained, i.e., the *full-state hybrid function projective synchronization*

(*FSHFPS*) and the *inverse full-state hybrid function projective synchronization* (*IFSHFPS*), respectively. Note that some synchronization types may coexist in chaotic systems.

In recent years, fractional calculus has been noticed for having superior characteristics over conventional calculus in the modeling dynamics of natural phenomena [7–13]. The recent development in fractional calculus has been focused on dynamical systems in fractional sense, for example the nonlinear fractional-order systems with disturbances [14]. These systems are characterized by the fact that the order of the derivative is a non-integer number. In particular, it has been shown that may also have complex dynamics such as chaos and bifurcation [15–17]. Some efforts have been recently made devoted to the synchronization of fractional-order chaotic system [18, 19]. It is worth noting that, differently from integer-order systems, most of the approaches for fractional-order systems are related to the synchronization of *identical* systems rather than *non-identical* systems [20]. On the other hand, referring to *function-based* hybrid synchronization schemes, only few schemes have been proposed to date; see [21–23]. Similar considerations hold for the coexistence of different synchronization types in fractional-order systems, given that very few attempts have been made. For example, in [24] the coexistence of some synchronization types has been illustrated, including the *inverse generalized synchronization* and the *Q-S synchronization*. However, no function-based hybrid synchronization schemes have been analyzed in [25].

Based on these considerations, this paper aims to give a further contribution to the topic by considering the coexistence of *function-based hybrid synchronization* in *non-identical* fractional-order (incommensurate) systems characterized by *different dimensions* and *different orders*. Specifically, the paper illustrates a new theorem, which proves the coexistence of the *full-state hybrid function projective synchronization* (*FSHFPS*) and the *inverse full-state hybrid function projective synchronization* (*IFSHFPS*) between a three-dimensional master system and a four-dimensional slave system. These fractional-order master-slave systems belong to general classes, which include several chaotic (hyperchaotic) incommensurate systems characterized by different dimensions and orders. Numerical examples of coexistence of synchronization types are illustrated, with the aim to show the effectiveness of the approach developed herein. The manuscript is organized as follows. In Sect. 2, the basic notions on fractional calculus are given, whereas in Sect. 3 the coexistence of *FSHFPS* and *IFSHFPS* for fractional-order systems is introduced. The scheme is general and the only restriction on the scaling functions is that they must be differentiable functions. In Sect. 4, by exploiting the stability of both integer-order and fractional-order linear error systems, a new theorem is illustrated, which proves the coexistence of *FSHFPS* and *IFSHFPS* between a three-dimensional incommensurate master system and a four-dimensional incommensurate slave system. Finally, in Sect. 5, numerical simulations show that the coexistence of *FSHFPS* and *IFSHFPS* is successfully achieved between the incommensurate *chaotic* fractional-order *unified system* and the incommensurate *hyperchaotic* fractional-order *Lorenz system*.

2 Basic concepts on fractional calculus

Both the Riemann-Liouville operator and the Caputo fractional derivative are considered through the paper. The Caputo fractional derivative is defined as follows [26]:

$$D_t^p x(t) = J^{m-p} x^{(m)}(t) \quad \text{with } 0 < p \leq 1, \tag{1}$$

where $m = [p]$ (i.e., m is the first integer which is not less than p), $x^{(m)}$ is the mth-order derivative in the usual sense, and J^q ($q > 0$) is the qth-order Riemann-Liouville integral operator given by

$$J^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} y(\tau)\, d\tau, \tag{2}$$

where Γ denotes Gamma function [27].

Lemma 1 ([28]) *Suppose $f(t)$ has a continuous kth derivative on $[0,t]$ ($k \in \mathbb{N}$, $t > 0$), and let $p,q > 0$ such that there exist some $l \in \mathbb{N}$ with $l \leq k$ and $(p, p+q) \in [l-1, l]$. Then it results that $D_t^p D_t^q f(t) = D_t^{p+q} f(t)$.*

Lemma 2 ([29]) *The n-dimensional fractional-order linear system*:

$$D_t^p X(t) = AX(t), \tag{3}$$

where $D_t^p = [D_t^{p_1}, D_t^{p_2}, \dots, D_t^{p_n}]$, $0 < p_i \leq 1$, $1 < i \leq n$ and $A \in \mathbb{R}^{n\times n}$ is asymptotically stable if all roots λ of the equation

$$\det\bigl(\operatorname{diag}\bigl(\lambda^{Mp_1}, \lambda^{Mp_2}, \dots, \lambda^{Mp_n}\bigr) - A\bigr) = 0 \tag{4}$$

satisfy the condition that $|\arg(\lambda)| > \frac{\pi}{2M}$, where M is the least common multiple of the denominators of p_i's.

3 Coexistence of function-based hybrid synchronization schemes

Consider the master system in the form

$$D_t^{p_i} x_i(t) = f_i\bigl(X(t)\bigr), \quad i = 1, 2, 3, \tag{5}$$

where $X(t) = (x_1(t), x_2(t), x_3(t))^T$ is the state vector, $f_i : \mathbb{R}^3 \to \mathbb{R}$, $0 < p_i \leq 1$ and $D_t^{p_i}$ is the Caputo fractional derivative of order p_i, for $i = 1, 2, 3$. The slave system is defined as

$$D_t^{q_i} y_i(t) = \sum_{j=1}^{4} b_{ij} y_j(t) + g_i\bigl(Y(t)\bigr) + u_i, \quad i = 1, 2, 3, 4, \tag{6}$$

where $Y(t) = (y_1(t), \dots, y_4(t))^T$ is the state vector, $g_i : \mathbb{R}^4 \to \mathbb{R}$, $0 < q_i \leq 1$, $D_t^{q_i}$ is the Caputo fractional derivative of order q_i and u_i are synchronization controllers ($i = 1, 2, 3, 4$).

In the following two different synchronization types are considered, where the *scaling factors* between the state variables of master and slave systems are replaced by *scaling functions*, indicating that *function-based projective synchronization* schemes are introduced.

Definition 1 The master system (5) and the slave system (6) are in *full-state hybrid function projective synchronization* (*FSHFPS*) when, for an initial condition, there exist controllers u_i, $0 \leq i \leq 4$ and differentiable functions $\lambda_{ij}(t)$: $\mathbb{R} \to \mathbb{R}$ ($1 \leq i \leq 4$; $1 \leq j \leq 3$) so that the synchronization errors:

$$e_i(t) = y_i(t) - \sum_{j=1}^{3} \lambda_{ij}(t) x_j(t), \quad i = 1, 2, 3, 4. \tag{7}$$

Satisfy the condition that $\lim_{t\to+\infty} \|e_i(t)\| = 0$.

Definition 2 The master system (5) and the slave system (6) are in *inverse full-state hybrid function projective synchronization* (*IFSHFPS*) when, for an initial condition, there exist controllers u_i, $0 < i \leq 4$ and differentiable functions $\mu_{ij}(t)$: $\mathbb{R} \to \mathbb{R}$ ($1 \leq i \leq 3$; $1 \leq j \leq 4$) so that the synchronization errors:

$$e_i(t) = \sum_{j=1}^{4} \mu_{ij}(t) y_j(t) - x_i(t), \quad i = 1, 2, 3. \tag{8}$$

Satisfy the condition that $\lim_{t\to+\infty} \|e_i(t)\| = 0$.

Now the coexistence of such two different synchronization types is considered.

Definition 3 *Full-state hybrid function projective synchronization* (*FSHFPS*) and *inverse full-state hybrid function projective synchronization* (*IFSHFPS*) coexist between the master system (5) and the slave system (6), if there exist controllers u_i ($1 \leq i \leq 4$) and differentiable functions $(\alpha_j(t))_{1\leq i\leq 4}$, $(\beta_j(t))_{1\leq i\leq 3}$, $(\gamma_j(t))_{1\leq i\leq 4}$ and $(\theta_j(t))_{1\leq i\leq 3}$, such that the following synchronization errors:

$$\begin{aligned}
e_1(t) &= \sum_{j=1}^{4} \alpha_j(t) y_j(t) - x_1(t), \\
e_2(t) &= y_2(t) - \sum_{j=1}^{3} \beta_j(t) x_j(t), \\
e_3(t) &= \sum_{j=1}^{4} \gamma_j(t) y_j(t) - x_3(t), \\
e_4(t) &= y_4(t) - \sum_{j=1}^{3} \theta_j(t) x_j(t).
\end{aligned} \tag{9}$$

Satisfy the condition that $\lim_{t\to+\infty} \|e_i(t)\| = 0$, $i = 1, 2, 3, 4$.

4 A theorem for the coexistence of FSHFPS and IFSHFPS

In this section the coexistence of two different function-based synchronization types is proved, i.e., *FSHFPS* and *IFSHFPS* are proved to coexist between a three-dimensional master system and a four-dimensional slave system.

The error system (9) between the master system (5) and the slave system (6) can be derived as follows:

$$\begin{aligned}
\dot{e}_1(t) &= \sum_{j=1}^{4} \dot{\alpha}_j(t) y_j(t) + \sum_{j=1}^{4} \alpha_j(t) \dot{y}_j(t) - \dot{x}_1(t), \\
D_t^{q_2} e_2(t) &= D_t^{q_2} y_2(t) - D_t^{q_2} \left[\sum_{j=1}^{3} \beta_j(t) x_j(t) \right], \\
\dot{e}_3(t) &= \sum_{j=1}^{4} \dot{\gamma}_j(t) y_j(t) + \sum_{j=1}^{4} \gamma_j(t) \dot{y}_j(t) - \dot{x}_3(t), \\
D_t^{q_4} e_4(t) &= D_t^{q_4} y_4(t) - D_t^{q_4} \left[\sum_{j=1}^{3} \theta_j(t) x_j(t) \right].
\end{aligned} \tag{10}$$

Let us suppose that the controllers u_i $(i = 1, 2, 3, 4)$ can be designed as follows:

$$\begin{aligned} u_1 &= -\sum_{j=1}^{4} b_{1j} y_j(t) - g_1\big(Y(t)\big) + J^{1-q_1}(v_1), \\ u_2 &= v_2, \\ u_3 &= -\sum_{j=1}^{4} b_{3j} y_j(t) - g_3\big(Y(t)\big) + J^{1-q_3}(v_3), \\ u_4 &= v_4, \end{aligned} \tag{11}$$

where v_i $(1 \leq i \leq 4)$ are new controllers to be determined. By substituting Eqs. (11) into Eqs. (6), the slave system can be written as follows:

$$D_t^{q_i} y_i(t) = J^{1-q_i}(v_i), \quad i = 1, 3, \tag{12}$$

and

$$D_t^{q_i} y_i(t) = \sum_{j=1}^{4} b_{ij} y_j(t) + g_i\big(Y(t)\big) + v_i, \quad i = 2, 4. \tag{13}$$

By applying the Caputo fractional derivative of order $1 - q_i$ $(i = 1, 3)$ to both the left and the right sides of Eq. (12), the following result is obtained:

$$\begin{aligned} \dot{y}_i(t) &= D_t^{1-q_i}\big(D_t^{q_i} y_i(t)\big) \\ &= D_t^{1-q_i} J^{1-q_i}(v_i) = v_i, \quad i = 1, 3. \end{aligned}$$

Note that $1 - q_i$ satisfies $1 - q_i \in [0, 1]$. According to *Lemma* 1 the above statement holds. Furthermore, the error system (10) can be written as

$$\begin{aligned} \dot{e}_1(t) &= -|b_{11}| e_1(t) + \alpha_1(t) v_1 + \alpha_3(t) v_3 + R_1, \\ D_t^{q_2} e_2(t) &= -|b_{22}| e_2(t) + v_2 + R_2, \\ \dot{e}_3(t) &= -|b_{33}| e_3(t) + \gamma_1(t) v_1 + \gamma_3(t) v_3 + R_3, \\ D_t^{q_4} e_4(t) &= -|b_{44}| e_4(t) + v_4 + R_4, \end{aligned} \tag{14}$$

where

$$\begin{aligned} R_1 &= |b_{11}| e_1(t) + \sum_{j=1}^{4} \dot{\alpha}_j(t) y_j(t) + \alpha_2(t) \dot{y}_2(t) + \alpha_4(t) \dot{y}_4(t) - \dot{x}_1(t), \\ R_2 &= |b_{22}| e_2(t) + \sum_{j=1}^{4} b_{2j} y_j(t) + g_2\big(Y(t)\big) - D_t^{q_2}\left[\sum_{j=1}^{3} \beta_j(t) x_j(t)\right], \\ R_3 &= |b_{33}| e_3(t) + \sum_{j=1}^{4} \dot{\gamma}_j(t) y_j(t) + \gamma_2(t) \dot{y}_2(t) + \gamma_4(t) \dot{y}_4(t) - \dot{x}_3(t), \\ R_4 &= |b_{44}| e_4(t) + \sum_{j=1}^{4} b_{4j} y_j(t) + g_4\big(Y(t)\big) - D_t^{q_4}\left[\sum_{j=1}^{3} \theta_j(t) x_j(t)\right]. \end{aligned}$$

The error system (14) can be written in the compact form as

$$\dot{e}_I(t) = B_I e(t) + M \times V_I + R_I \tag{15}$$

and

$$D_t^q e_{II}(t) = B_{II} e(t) + V_{II} + R_{II}, \tag{16}$$

where

$$\dot{e}_I(t) = \bigl(\dot{e}_1(t), \dot{e}_3(t)\bigr)^T, \qquad D_t^q e_{II}(t) = \bigl(D_t^{q_2} e_2(t), D_t^{q_4} e_4(t)\bigr)^T,$$
$$B_I = \begin{pmatrix} -|b_{11}| & 0 \\ 0 & -|b_{33}| \end{pmatrix}, \qquad B_{II} = \begin{pmatrix} -|b_{22}| & 0 \\ 0 & -|b_{44}| \end{pmatrix}, \qquad M = \begin{pmatrix} \alpha_1(t) & \alpha_3(t) \\ \gamma_1(t) & \gamma_3(t) \end{pmatrix},$$
$$R_I = (R_1, R_3)^T, \qquad R_{II} = (R_2, R_4)^T, \qquad V_I = (v_1, v_3)^T \quad \text{and} \quad V_{II} = (v_2, v_4)^T.$$

Now the following theorem can be proved.

Theorem Full-state hybrid function projective synchronization (FSHFPS) *and* inverse full-state hybrid function projective synchronization (IFSHFPS) *coexist between the master system* (5) *and the slave system* (6), *provided that the control signals V_I and V_{II} in the error system* (15)–(16) *are selected as*

$$V_I = -M^{-1} \times R_I, \tag{17}$$
$$V_{II} = -R_{II}, \tag{18}$$

where $M = \left(\begin{smallmatrix} \alpha_1(t) & \alpha_3(t) \\ \gamma_1(t) & \gamma_3(t) \end{smallmatrix}\right)$ is assumed to be an invertible matrix.

Proof By applying the control law (17) to Eq. (15), it follows that the resulting error dynamics is described by

$$\dot{e}_I(t) = B_I e(t). \tag{19}$$

Since all the eigenvalues of B_I have negative real part, it can be readily concluded that the integer-order linear continuous-time systems (19) is asymptotically stable, i.e., $\lim_{t\to+\infty} e_1(t) = \lim_{t\to+\infty} e_3(t) = 0$. Successively, by substituting the control law (18) into Eq. (16), it follows that

$$D_t^q e(t) = B_{II} e(t). \tag{20}$$

By computing the roots of the equation $\det(\operatorname{diag}(\lambda^{Mq_2}, \lambda^{Mq_4}) - B_{II}) = 0$, the obtained results are $\lambda_i = |b_{ii}|^{\frac{1}{Mq_i}}(\cos \frac{\pi}{Mq_i} + \sin \frac{\pi}{Mq_i})$, where M is the least common multiple of the denominators of q_2 and q_4 ($i = 2, 4$). By computing $\arg(\lambda_i)$, it can be readily shown that $|\arg(\lambda_i)| > \frac{\pi}{2M}$, $i = 2, 4$. According to *Lemma* 2, the fractional-order system (20) is asymptotically stable, indicating that $\lim_{t\to+\infty} e_2(t) = \lim_{t\to+\infty} e_4(t) = 0$. Since the two error systems (19) and (20) are asymptotically stable, it can be concluded that *FSHFPS* and *IFSHFPS* coexist between the master system (5) and the slave system (6). □

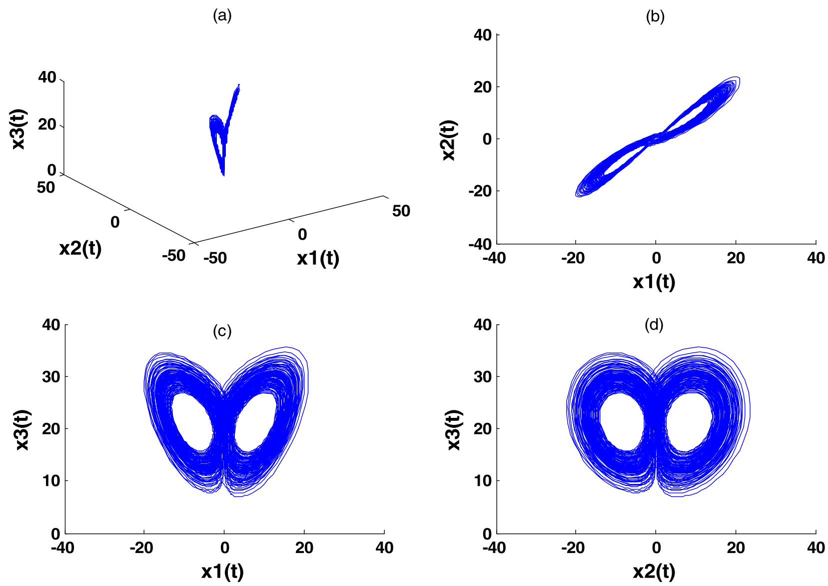

Figure 1 Chaotic attractors of the fractional-order unified system: (**a**) plot in the (x_1,x_2,x_3)-space; (**b**) plot in the (x_1,x_2)-plane; (**c**) plot in the (x_1,x_3)-plane; (**d**) plot in the (x_2,x_3)-plane

5 Numerical example

The aim of this section is to show the effectiveness of the conceived method. The selected master system is the incommensurate chaotic fractional-order *unified system*, whereas the slave system is the incommensurate hyperchaotic fractional-order *Lorenz system*. In particular, the considered master system is defined as

$$\begin{cases} D_t^{p_1} x_1 = (25r + 10)(x_2 - x_1), \\ D_t^{p_2} x_2 = (28 - 35r)x_1 + (29r - 1)x_2 + x_1x_3, \\ D_t^{p_3} x_3 = \frac{-(r+8)}{3} x_3 + x_1x_2. \end{cases} \tag{21}$$

System (21) exhibits chaotic behavior when $(p_1,p_2,p_3) = (0.85, 0.9, 0.95)$ and $r = 1$ [16]. The chaotic attractors of the incommensurate fractional-order unified system are shown in Fig. 1.

The slave system is the fractional-order Lorenz system described by

$$\begin{cases} D_t^{q_1} y_1 = a(y_2 - y_1) + y_4 + u_1, \\ D_t^{q_2} y_2 = cy_1 - y_2 - y_1y_3 + u_2, \\ D_t^{q_3} y_3 = -by_3 + y_1y_2 + u_3, \\ D_t^{q_4} y_4 = dy_4 + y_2y_3 + u_4, \end{cases} \tag{22}$$

where $U = (u_1,u_2,u_3,u_4)^T$ is the vector controller. This system exhibits hyperchaotic behavior when $(u_1,u_2,u_3,u_4) = (0,0,0,0)$, $(q_1,q_2,q_3,q_4) = (0.94, 0.96, 0.97, 0.99)$ and $(a,b,c,d) = (10, \frac{8}{3}, 28, -1)$ [17]. Some plots of the hyperchaotic Lorenz attractor are shown in Fig. 2.

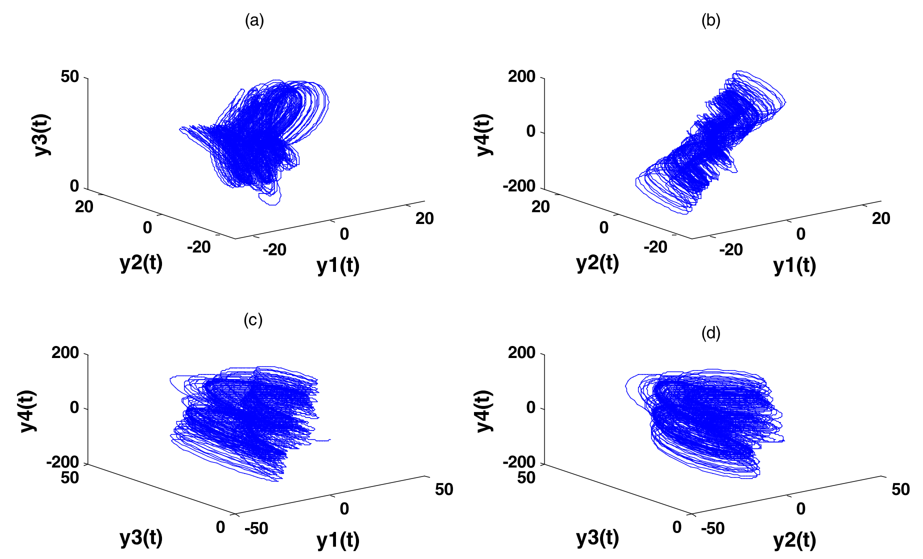

Figure 2 Hyperchaotic attractors of the fractional-order Lorenz system: (**a**) plot in the (y_1, y_2, y_3)-space; (**b**) plot in the (y_1, y_2, y_4)-space; (**c**) plot in the (y_1, y_3, y_4)-space; (**d**) plot in the (y_2, y_3, y_4)-space

By putting system (22) in the form (6), we have

$$B = (b_{ij}) = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 28 & -1 & 0 & 0 \\ 0 & 0 & -8/3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad (g_i)_{1\leq i\leq 4} = \begin{pmatrix} 0 \\ -y_1 y_3 \\ y_1 y_2 \\ y_2 y_3 \end{pmatrix}.$$

According to the conceived approach, *FSHFPS* and *IFSHFPS* coexist between the master system (21) and the slave system (22) if the following synchronization errors:

$$\begin{aligned} e_1 &= \sum_{j=1}^{4} \alpha_j(t) y_j - x_1, \\ e_2 &= y_2 - \sum_{j=1}^{3} \beta_j(t) x_j, \\ e_3 &= \sum_{j=1}^{4} \gamma_j(t) y_j - x_3, \\ e_4 &= y_4 - \sum_{j=1}^{3} \theta_j(t) x_j, \end{aligned} \tag{23}$$

asymptotically approach zero, where the functions $(\alpha_j(t))_{1\leq i\leq 4}$, $(\beta_j(t))_{1\leq i\leq 3}$, $(\gamma_j(t))_{1\leq i\leq 4}$ and $(\theta_j(t))_{1\leq i\leq 3}$ have been selected as

$$\alpha_1(t) = t^2 + 1, \qquad \alpha_2(t) = t, \qquad \alpha_3(t) = 0, \qquad \alpha_4(t) = -3,$$

$$\beta_1(t) = 0, \qquad \beta_2(t) = 2t, \qquad \beta_3(t) = 0,$$

$$\gamma_1(t)=0,\qquad \gamma_2(t)=t,\qquad \gamma_3(t)=\exp(t),\qquad \gamma_4(t)=0,$$
$$\theta_1(t)=t^2+1,\qquad \theta_2(t)=0,\qquad \theta_3(t)=t.$$

By applying the theorem illustrated in the previous section, we have

$$B_I=\begin{pmatrix}-10 & 0\\ 0 & -\frac{8}{3}\end{pmatrix},\qquad B_{II}=\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},$$
$$M=\begin{pmatrix}t^2+1 & 0\\ 0 & \exp(t)\end{pmatrix},\qquad M^{-1}=\begin{pmatrix}\frac{1}{t^2+1} & 0\\ 0 & \exp(-t)\end{pmatrix}.$$

As a consequence, the controllers u_i $(1\leq i\leq 4)$ in (22) can be written as

$$\begin{aligned}
u_1 &= -a(y_2-y_1)-y_4+J^{0.06}\left[\frac{-1}{t^2+1}\left(10e_1(t)+2ty_1+y_2+t\dot{y}_2-3\dot{y}_4-\dot{x}_1\right)\right],\\
u_2 &= -e_2-cy_1+y_2+y_1y_3+2D^{0.96}[tx_2],\\
u_3 &= by_3-y_1y_2+J^{0.03}\left[-\exp(-t)\left(\frac{8}{3}e_3+y_2+\exp(t)y_3+t\dot{y}_2-\dot{x}_3\right)\right],\\
u_4 &= -e_4-dy_4-y_2y_3+D^{0.99}\left[\left(t^2+1\right)x_1+tx_3\right],
\end{aligned}\tag{24}$$

indicating that the error sub-system (15) can be written in the form

$$\begin{aligned}
\dot{e}_1(t) &= -10e_1(t),\\
\dot{e}_3(t) &= -\frac{8}{3}e_3(t),
\end{aligned}\tag{25}$$

whereas the error sub-system (16) can be written as

$$\begin{aligned}
D_t^{0.96}e_2(t) &= -e_2(t),\\
D_t^{0.99}e_4(t) &= -e_4(t).
\end{aligned}\tag{26}$$

Numerical simulations have been carried out for solving systems (25) and (26). In particular, the fourth-order Runge-Kutta integration method has been applied to the integer-order system (25), whereas the fractional Euler integration method has been used for solving the incommensurate fractional-order system (26). The initial states of the master system and the slave system are $(x_1(0),x_2(0),x_3(0))=(-1,1,10)$ and $(y_1(0),y_2(0),y_3(0),y_4(0))=(2,-2,1,-1)$, respectively. The initial states of the error system (25)–(26) are $(e_1(0),e_2(0),e_3(0),e_4(0))=(1,-2,-10,-1)$. Figures 3 and 4 display the synchronization errors between the master system (21) and the slave system (22), indicating that the coexistence of *full-state hybrid function projective synchronization* (*FSHFPS*) and *inverse full-state hybrid function projective synchronization* (*IFSHFPS*) is effectively achieved.

6 Conclusion and future work

When analyzing the synchronization of fractional-order chaotic systems, an interesting phenomenon that may occur is the coexistence of some synchronization types.

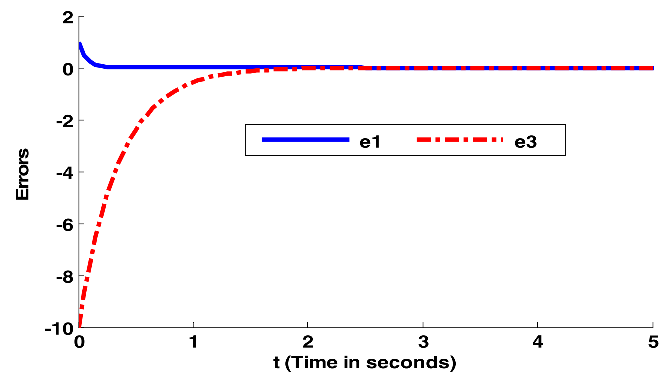

Figure 3 Time evolution of the errors $e_1(t)$ and $e_3(t)$

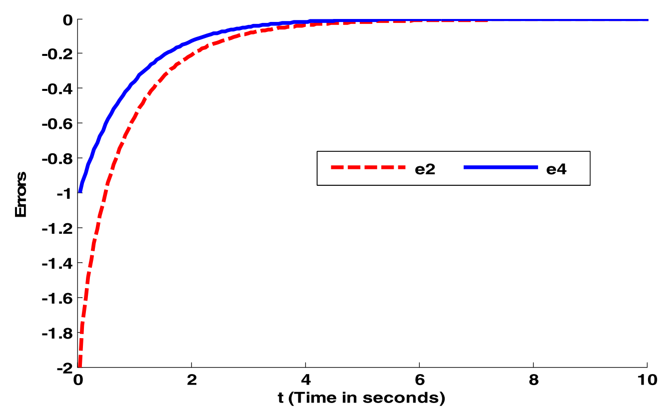

Figure 4 Time evolution of the errors $e_2(t)$ and $e_4(t)$

Based on these considerations, this paper has presented new results related to the coexistence of *function-based hybrid synchronization* types between non-identical incommensurate fractional-order systems characterized by different dimensions and orders. Specifically, the manuscript has proposed a new theorem, which ensures the coexistence of *FSHFPS* and *IFSHFPS* between a three-dimensional master system and a four-dimensional slave system. Note that the approach developed herein enables to prove the coexistence of *FSHFPS* and *IFSHFPS* in several cases. Specifically, the approach can be applied to: (i) wide classes of chaotic (hyperchaotic) fractional-order master-slave systems; (ii) non-identical incommensurate fractional-order systems with different dimensions; (iii) schemes wherein the scaling factor of the linear combination can be any arbitrary differentiable function. A numerical example, describing the coexistence of *FSHFPS* and *IFSHFPS* between the incommensurate *chaotic* fractional-order *unified system* and the incommensurate *hyperchaotic* fractional-order *Lorenz system*, has clearly highlighted the effectiveness of the approach proposed herein.

Finally, we would make some comments on future developments of the present work. In particular, we are conscious that circuit implementations of synchronization schemes are an important issue. For this reason, we are preparing a forthcoming paper where all the details related to the circuit implementation of the conceived synchronization scheme will be provided. Further developments and extended analysis related to the application of the new hybrid synchronization to secure communication systems and new complex fractional schemes of synchronization will be investigated in a future work.

Acknowledgements
The authors acknowledge Prof. GuanRong Chen, Department of Electronic Engineering, City University of Hong Kong for suggesting many helpful references.

Funding
The author Xiong Wang was supported by the National Natural Science Foundation of China (No. 61601306) and Shenzhen Overseas High Level Talent Peacock Project Fund (No. 20150215145C).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
AO and VTP made substantial contributions to conception and design. AO, XW, VTP, GG and TZ carried out results interpretation and analysis. XW, GG and TZ helped to evaluate, revise and edit the manuscript. AO and VTP drafted the article. All authors read and approved the final manuscript.

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Publisher's Note

Received: 7 March 2018 Accepted: 21 August 2018 Published online: 04 September 2018

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