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Error estimates for an augmented method for one-dimensional elliptic interface problems

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Abstract

Elliptic interface problems have many important scientific and engineering applications. Interface problems are encountered when the computational domain involves multi-materials with different conductivities, densities, or permeability. The solution or its gradient often has a jump across the interface due to discontinuous coefficients or singular sources. In this paper, optimal convergence of an augmented method is derived for one-dimensional interface problems. The dependence of the discontinuous coefficient in the error analysis is also considered. Numerical examples are presented to confirm the theoretical analysis and show that the estimate is sharp.

MSC: 65N15; 65N30; 35J60

Keywords: Interface problem; Immersed finite element; Augmented variable

1 Introduction

In scientific computation, we often encounter interface problems when multi-materials with different conductivities, densities, or permeability are involved. The solution or its gradient of the governing partial differential equation is often discontinuous due to discontinuous coefficients or singular sources across the interface. Traditional numerical methods can not achieve optimal convergence unless the used mesh fits the interface. The methods using fitted meshes are often called fitted mesh methods. There are many fitted mesh methods in the literature (see, for example, [2, 3, 10]). However, the fitted mesh is dependent on the shape and the location of the interface. It may be difficult and time consuming to generate a fitted mesh for a complicated interface. The difficulty becomes even severer for three-dimensional problems. Another disadvantage of the fitted mesh is encountered when solving moving interface problems. Since the interface is moving, a new fitted mesh has to be generated at each time step and an interpolation is required to transfer the numerical solutions solved on different meshes. From this point of view, it would be preferable to use an unfitted mesh in which the interface can be arbitrarily located with respect to the fixed background mesh. Note that many other methods [4, 18, 20, 21] can be used with fitted meshes when the problem is viewed as a problem with discontinuous coefficients. For unfitted mesh methods, the difficulty is that the interface can pass through the interior of elements of the mesh. Thus, special treatment needs to be done on these elements.

There are many unfitted mesh methods in the literature, for instance, the extended finite element method [5], the unfitted Nitsche's finite element method [6], the immersed interface method [9, 13], the immersed finite element/volume method [1, 7, 12, 16, 17, 19] and the augmented finite difference/element method [8, 11, 14, 15]. The extended finite element method (XFEM) enriches standard finite element space by adding extra functions near the interface to treat the jumps of the exact solution. The degrees of freedom of the XFEM often change with moving interfaces. In the unfitted Nitsche's finite element method, the function in finite element spaces is discontinuous across the interface and the interface conditions are absorbed in the bilinear form. A penalty is also added into the bilinear form to deal with the discontinuous of the finite element functions. The immersed finite element methods (IFEMs) are a class of unfitted mesh methods that modify the basis function on interface elements according to the interface conditions to capture the jumps of the exact solution. The bilinear form and the degrees of freedom are the same as if there was no interface. If the coefficient is a constant without jumps, then the stiffness matrix is the same as that obtained by traditional finite element for the problem without interfaces. And only the right-hand side needs to be modified according to the interface conditions. The augmented method is developed based on the above observations. In the augmented method, an augmented variable is introduced along the interface so that the original interface problem can be transferred to a new interface problem without discontinuous coefficients. Thus, the efficient method can be used by only modifying the right-hand side. The augmented variable should be chosen such that the original interface conditions are satisfied. Extensive numerical examples in [8] show that the augmented method achieves optimal convergence in the L^2, H^1 and L^∞ norms. In this work, we derive the optimal error estimates for the augmented method for one-dimensional interface problems. The dependence of the discontinuous coefficient is included. Numerical results show that the estimate is sharp.

The rest of the paper is organized as follows. In Sect. 2, we describe the model problem and some preliminaries. We choose the augmented variable and rewrite the interface problem. In Sect. 3, we analyze the method for interface problems only with singular sources where the augmented variable is assumed to be given. The augmented method and the error estimates are provided in Sect. 4. Finally, some numerical examples are presented in Sect. 5 to confirm the theoretical analysis.

2 Preliminaries

Let $\Omega = (a,b)$ be a finite interval. Assume that the domain Ω is separated into two subdomains $\Omega_1 = (a,\alpha)$ and $\Omega_2 = (\alpha,b)$ by an interface point $\alpha \in \Omega$. Consider the following one-dimensional second-order interface problem:

$$\begin{cases} -(\beta(x)\widetilde{u}'(x))' = f(x), & x \in \Omega_1 \cup \Omega_2, \\ \widetilde{u}(a) = \widetilde{u}(b) = 0, \end{cases} \tag{2.1}$$

where the diffusion coefficient $\beta(x)$ is assumed to have a finite jump across the interface α. We also assume that the coefficient $\beta(x) > 0$ is a piecewise constant, i.e.,

$$\beta(x) = \begin{cases} \beta_1, & x \in \Omega_1, \\ \beta_2, & x \in \Omega_2. \end{cases} \tag{2.2}$$

At the interface α, the solution is assumed to satisfy the interface conditions

$$[\![\widetilde{u}]\!]_\alpha = \widetilde{u}(\alpha^+) - \widetilde{u}(\alpha^-) = w, \qquad [\![\beta\widetilde{u}']\!]_\alpha = \beta_2\widetilde{u}'(\alpha^+) - \beta_1(\alpha)\widetilde{u}'(\alpha^-) = q, \tag{2.3}$$

where the notation $[\![v]\!]_\alpha = v(\alpha^+) - v(\alpha^-)$ denotes the jump of the quantity v across α. We assume that $f(x) \in C^0(a,\alpha) \cap C^0(\alpha,b)$. Thus, we have the exact solution $\widetilde{u}(x) \in C^2(a,\alpha)$ and $\widetilde{u}(x) \in C^2(\alpha,b)$.

Define $u(x) = \beta(x)\widetilde{u}(x)$. Then the original interface problem (2.1)–(2.3) can be written as

$$\begin{aligned} &-u'' = f \quad \text{in } \Omega_1 \cup \Omega_2, \\ &\left[\!\!\left[\frac{u}{\beta}\right]\!\!\right]_\alpha = w, \qquad [\![u']\!]_\alpha = q, \\ &u(a) = u(b) = 0. \end{aligned} \tag{2.4}$$

If we choose $g = [\![u]\!]_\alpha$ as an augmented variable, then we need to seek the solution u of the following problem:

$$\begin{aligned} &-u'' = f \quad \text{in } \Omega_1 \cup \Omega_2, \\ &[\![u]\!]_\alpha = g, \qquad [\![u']\!]_\alpha = q, \\ &u(a) = u(b) = 0. \end{aligned} \tag{2.5}$$

The augmented variable g should be chosen such that the augmented equation

$$\left[\!\!\left[\frac{u}{\beta}\right]\!\!\right]_\alpha = w \tag{2.6}$$

is satisfied.

The augmented method is to discretize (2.5) and (2.6), respectively. In the next section, we present the method to discretize (2.5) and give corresponding error analysis. The discretization of (2.6) is discussed in Sect. 4.

3 The method for interface problems only with singular sources

In this section, we discuss the method for the interface problem without discontinuous coefficients (2.5). Note that the jump conditions in the problem can be written using singular sources (see [13]). In this section, we assume that the augmented variable g is known. The weak formulation of the interface problem (2.5) is: find $u \in H^1(a,\alpha) \cup (\alpha,b)$ such that

$$\begin{aligned} &a(u,v) = (f,v) - qv(\alpha) \quad \forall v \in H_0^1(a,b), \\ &[\![u]\!]_\alpha = g, \\ &u(a) = u(b) = 0, \end{aligned} \tag{3.1}$$

where

$$a(u,v) = \int_a^\alpha u'(x)v'(x)\,dx + \int_\alpha^b u'(x)v'(x)\,dx, \qquad (f,v) = \int_a^b f(x)v(x)\,dx.$$

Let $a = x_0 < x_1 < \cdots < x_k < x_{k+1} < \cdots < x_N = b$ be a partition of $[a,b]$ independent of the interface point α. Assume that there exists a number k such that $\alpha \in [x_k, x_{k+1})$. We call the element $[x_k, x_{k+1}]$ the interface element. The rest of the elements $[x_i, x_{i+1}]$, $i \neq k$ are called non-interface elements. Let $h := \max_{0\leq i\leq N-1}(x_{i+1} - x_i)$. We assume that the partition is quasi-uniform, i.e., there exists a generic constant C independent of h such that $h \leq C \min_{0\leq i\leq N-1}(x_{i+1} - x_i)$.

Let V_h^L be the usual conforming linear finite element space which is defined as

$$\begin{aligned} V_h^L = \bigl\{ & v_h \in C^0(a,b) \mid v_h(x) \text{ is linear on } [x_k, x_{k+1}] \\ & \text{for } 0 \leq k \leq N-1, v_h(a) = v_h(b) = 0 \bigr\}. \end{aligned} \tag{3.2}$$

We also define the standard nodal interpolation operator I_h^L by

$$I_h^L u \in V_h^L, \qquad I_h^L u(x_i) = u(x_i), \quad i = 0, 1, \ldots, N. \tag{3.3}$$

Define

$$\chi(x) = \begin{cases} 1, & x > \alpha, \\ 0, & x \leq \alpha, \end{cases} \quad \text{and} \quad u_e(x) = (x-\alpha)q + g. \tag{3.4}$$

Thus, we can define

$$u_h^J(x) = \widehat{u} - I_h^L \widehat{u}, \quad \text{where } \widehat{u}(x) = \chi(x) u_e(x). \tag{3.5}$$

By the definition, we conclude that

$$\widehat{u}(x) = \begin{cases} (x-\alpha)q + g, & x > \alpha, \\ 0, & x \leq \alpha, \end{cases} \quad \text{and} \quad [\![\widehat{u}]\!]_\alpha = g, \ [\![(\widehat{u})']\!]_\alpha = q.$$

It is easy to verify that

$$u_h^J(x) = 0, \quad x \in [a, x_k] \cup [x_{k+1}, b], \tag{3.6}$$

and

$$u_h^J(x) \neq 0, \quad x \in (x_k, x_{k+1}), \qquad [\![u_h^J]\!]_\alpha = g, \qquad [\![(u_h^J)']\!]_\alpha = q. \tag{3.7}$$

The immersed finite element (IFE) space V_h^J is defined by

$$V_h^J = \bigl\{ v_h \in L^2(a,b) \mid v_h = v_h^L + u_h^J, v_h^L \in V_h^L \bigr\}. \tag{3.8}$$

The method for the interface problem (2.5) is: find $u_h \in V_h^J$ such that

$$\begin{aligned} & u_h = u_h^L + u_h^J, \\ & a\bigl(u_h^L, v_h\bigr) = (f, v_h) - q v_h(\alpha) - a\bigl(u_h^J, v_h\bigr), \quad \forall v_h \in v_h^L. \end{aligned} \tag{3.9}$$

The following lemma gives the bounds of interpolation error in the L^∞ norm.

Lemma 3.1 *There exists a generic constant $C > 0$ independent of h, α and β such that*

$$\left\| I_h^L u + u_h^J - u \right\|_{L^\infty(a,b)} \le Ch^2 \left\| u'' \right\|_{L^\infty(\Omega_1 \cup \Omega_2)}. \tag{3.10}$$

Proof On non-interface elements $[x_i, x_{i+1}]$, $i \neq k$, using (3.6), we have

$$\left\| I_h^L u + u_h^J - u \right\|_{L^\infty(x_i,x_{i+1})} = \left\| I_h^L u - u \right\|_{L^\infty(x_i,x_{i+1})} \le C \left\| u'' \right\|_{L^\infty(x_i,x_{i+1})}. \tag{3.11}$$

On the interface element $[x_k, x_{k+1}]$, we have

$$\begin{aligned} I_h^L u + u_h^J - u &= I_h^L u + \chi(x)u_e(x) - I_h^L\bigl(\chi(x)u_e(x)\bigr) - u \\ &= I_h^L\bigl(u - \chi(x)u_e(x)\bigr) - \bigl(u - \chi(x)u_e(x)\bigr). \end{aligned} \tag{3.12}$$

Define $\theta := u - \chi(x)u_e(x)$, then we have $[\![\theta]\!]_\alpha = 0$ and $[\![\theta']\!]_\alpha = 0$. Thus, we conclude that $\theta \in W^{2,\infty}(a,b)$. Using the standard interpolation error estimate, we obtain

$$\left\| I_h^L \theta - \theta \right\|_{L^\infty(a,b)} \le Ch^2 |\theta|_{W^{2,\infty}(a,b)} \le Ch^2 \left\| u'' \right\|_{L^\infty(\Omega_1 \cup \Omega_2)}, \tag{3.13}$$

where we have used $(u_e(x))'' = 0$ in the last inequality. Combining (3.12)–(3.13), we get the desired result. □

Theorem 3.2 *Assume that the augmented variable g is given exactly. We have the error estimate*

$$\|u_h - u\|_{L^\infty(a,b)} \le Ch^2 \left\| u'' \right\|_{L^\infty(\Omega_1 \cup \Omega_2)}, \tag{3.14}$$

where the constant $C > 0$ is independent of h, α and β.

Proof For any $v_h \in V_h^L$, from (3.1) and (3.9), we have

$$a\bigl(u - u_h^L - u_h^J, v_h\bigr) = 0. \tag{3.15}$$

Thus, we get

$$\begin{aligned} a\bigl(u_h^L - I_h^L u, v_h\bigr) &= a\bigl(u - I_h^L u - u_h^J, v_h\bigr) \\ &= \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} \bigl(u - I_h^L u - u_h^J\bigr)'(v_h)'\, dx. \end{aligned} \tag{3.16}$$

Since v_h is piecewise linear, using (3.6), we have, for $i \neq k$,

$$\begin{aligned} \int_{x_i}^{x_{i+1}} \bigl(u - I_h^L u - u_h^J\bigr)'(v_h)'\, dx &= (v_h)' \int_{x_i}^{x_{i+1}} \bigl(u - I_h^L u\bigr)'\, dx \\ &= (v_h)'\bigl(u - I_h^L u\bigr)\Big|_{x_i}^{x_{i+1}} = 0. \end{aligned} \tag{3.17}$$

On the interface element (x_k, x_{k+1}), we have

$$\begin{aligned}
&\int_{x_k}^{x_{k+1}} \bigl(u - I_h^L u - u_h^J\bigr)'(v_h)'\,dx \\
&\quad = (v_h)'\biggl(\int_{x_k}^{\alpha} \bigl(u - I_h^L u - u_h^J\bigr)'\,dx + \int_{\alpha}^{x_{k+1}} \bigl(u - I_h^L u - u_h^J\bigr)'\,dx\biggr) \\
&\quad = (v_h)'\bigl(\bigl(u - I_h^L u - u_h^J\bigr)\big|_{x_k}^{\alpha} + \bigl(u - I_h^L u - u_h^J\bigr)\big|_{\alpha}^{x_{k+1}}\bigr) \\
&\quad = (v_h)'\bigl(-[\![u]\!]_\alpha + [\![I_h^L u]\!]_\alpha + [\![u_h^J]\!]_\alpha\bigr) = 0.
\end{aligned} \tag{3.18}$$

Then we have

$$a\bigl(u_h^L - I_h^L u, v_h\bigr) = 0. \tag{3.19}$$

Taking $v_h = u_h^L - I_h^L u$, we conclude that

$$a\bigl(u_h^L - I_h^L u, u_h^L - I_h^L u\bigr) = 0, \tag{3.20}$$

which implies that

$$u_h^L = I_h^L u. \tag{3.21}$$

Thus,

$$u_h - u = u_h^L + u_h^J - u = I_h^L u + u_h^J - u, \tag{3.22}$$

which, together with Lemma 3.1, completes the proof. □

Next, we consider that the augmented variable is given with errors, i.e., $G = g + \mathcal{E}(G)$. Then, by the method (3.9), we need to replace g by G in (3.4). Now the function u_h^J becomes U_h^J and the numerical solution of the method is

$$U_h = U_h^L + U_h^J. \tag{3.23}$$

Theorem 3.3 *Assuming the augmented variable is given with the error $\mathcal{E}(G) = G - g$, then there exists a generic constant $C > 0$ independent of h, α and β such that*

$$\|U_h - u\|_{L^\infty(a,b)} \le Ch^2 \bigl\|u''\bigr\|_{L^\infty(\Omega_1 \cup \Omega_2)} + 2\bigl|\mathcal{E}(G)\bigr|. \tag{3.24}$$

Proof Obviously, we have

$$u_h^J - U_h^J = -\chi(x)\bigl(\mathcal{E}(G)\bigr) + I_h^L\bigl(\chi(x)\mathcal{E}(G)\bigr) \tag{3.25}$$

and

$$a\bigl(u_h^L - U_h^L, v_h\bigr) = -a\bigl(u_h^J - U_h^J, v_h\bigr). \tag{3.26}$$

From (3.25), we get

$$\left(u_h^J - U_h^J\right)' = \mathcal{E}(G)/(x_{k+1} - x_k), \quad x \in [x_k, \alpha) \cup (\alpha, x_{k+1}], \tag{3.27}$$

and

$$\begin{aligned}\left\|u_h^J - U_h^J\right\|_{L^\infty(a,b)} &= \max\left\{\left|\left(u_h^J - U_h^J\right)\left(\alpha^-\right)\right|, \left|\left(u_h^J - U_h^J\right)\left(\alpha^+\right)\right|\right\} \\ &\le \max\left\{\left|\frac{(x_{k+1} - \alpha)\mathcal{E}(G)}{x_{k+1} - x_k}\right|, \left|\frac{(\alpha - x_k)\mathcal{E}(G)}{x_{k+1} - x_k}\right|\right\} \\ &\le \left|\mathcal{E}(G)\right|. \end{aligned} \tag{3.28}$$

Thus, (3.26) becomes

$$\begin{aligned} a\left(u_h^L - U_h^L, v_h\right) &= \frac{-\mathcal{E}(G)}{x_{k+1} - x_k} \int_{x_k}^{x_{k+1}} (v_h)'\, dx = \frac{-\mathcal{E}(G)(v_h(x_{x+1}) - v_h(x_k))}{x_{k+1} - x_k} \\ &= -\int_a^b \frac{\mathcal{E}(G)}{x_{k+1} - x_k} \delta(x - x_{k+1}) v_h(x)\, dx \\ &\quad + \int_a^b \frac{\mathcal{E}(G)}{x_{k+1} - x_k} \delta(x - x_k) v_h(x)\, dx, \end{aligned} \tag{3.29}$$

which leads to

$$\begin{aligned} &-\left(u_h^L - U_h^L\right)'' = -\frac{\mathcal{E}(G)}{x_{k+1} - x_k} \delta(x - x_{k+1}) + \frac{\mathcal{E}(G)}{x_{k+1} - x_k} \delta(x - x_k), \\ &\left(u_h^L - U_h^L\right)(a) = \left(u_h^L - U_h^L\right)(b) = 0. \end{aligned} \tag{3.30}$$

For the problem

$$\begin{aligned} &-u'' = \delta(x - x_i), \quad x \in (a, b) \\ &u(a) = u(b) = 0, \end{aligned} \tag{3.31}$$

the solution is the well-known Green's function $\mathcal{G}(x; x_i)$, which is defined as

$$\mathcal{G}(x; x_i) = \begin{cases} -(x_i - b)(x - a)/(b - a), & a \le x \le x_i, \\ -(x - b)(x_i - a)/(b - a), & x_i \le x \le b. \end{cases} \tag{3.32}$$

Hence, we have

$$u_h^L - U_h^L = \frac{\mathcal{E}(G)}{x_{k+1} - x_k}\left(-\mathcal{G}(x; x_{k+1}) + \mathcal{G}(x; x_k)\right). \tag{3.33}$$

Obviously,

$$\begin{aligned} &\left\|\mathcal{G}(x; x_{k+1}) - \mathcal{G}(x; x_k)\right\|_{L^\infty(a,b)} \\ &\quad = \max\left\{\mathcal{G}(x_k; x_{k+1}) - \mathcal{G}(x_k; x_k), \mathcal{G}(x_{k+1}; x_{k+1}) - \mathcal{G}(x_{k+1}; x_k)\right\} \\ &\quad \le x_{k+1} - x_k. \end{aligned} \tag{3.34}$$

Thus, we obtain

$$\left\| u_h^L - U_h^L \right\|_{L^\infty(a,b)} \le \left| \mathcal{E}(G) \right|.$$

Using (3.28), (3.34) and Theorem (3.2), we get

$$\begin{aligned} \|U_h - u\|_{L^\infty(a,b)} &\le \|u_h - u\|_{L^\infty(a,b)} + \|U_h - u_h\|_{L^\infty(a,b)} \\ &\le \|u_h - u\|_{L^\infty(a,b)} + \left\| U_h^L - u_h^L \right\|_{L^\infty(a,b)} + \left\| U_h^J - u_h^J \right\|_{L^\infty(a,b)} \\ &\le Ch^2 \left\| u'' \right\|_{L^\infty(\Omega_1 \cup \Omega_2)} + 2\left| \mathcal{E}(G) \right|, \end{aligned} \tag{3.35}$$

which completes the proof. □

4 Augmented method for discontinuous coefficients and error estimates

In continuous cases, the augmented variable g needs to be chosen such that the interface condition (2.6) is satisfied. In the augmented method, the numerical augmented variable G should be chosen such that

$$\left[\!\left[\frac{U_h}{\beta} \right]\!\right]_\alpha = \frac{U_h(\alpha^+)}{\beta_2} - \frac{U_h(\alpha^-)}{\beta_1} = w, \tag{4.1}$$

where U_h (see (3.23)) is the solution of method for (3.9) with g replaced by G. Note that (4.1) is a discretization of the augmented equation (2.6).

The matrix-vector form of (3.9) and (4.1) can be written as (see [8])

$$\begin{bmatrix} A & B \\ E & H \end{bmatrix} \begin{bmatrix} U_h \\ G \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}. \tag{4.2}$$

Eliminating U_h from (4.2), we get the Schur complement system for G

$$\left(H - EA^{-1}B\right)G = F_2 - EA^{-1}F_1. \tag{4.3}$$

To simplify the notation, we denote the Schur complement system by

$$SG = F, \quad \text{where } S = H - EA^{-1}B, F = F_2 - EA^{-1}F_1.$$

For one-dimensional problems, the Schur complement S is a number. Note that the matrices such as A, B, E, H are not formed explicitly in implementations and are only used for theoretical analysis.

Define the truncation error of the augmented equation by

$$T := S\left(\mathcal{E}(G)\right) = S(G - g) = SG - Sg. \tag{4.4}$$

Then we have the following estimation of the truncation error.

Lemma 4.1 *We have*

$$|T| = S\big(\mathcal{E}(G)\big) = S(G-g) \le C\left(\frac{\beta_1+\beta_2}{\beta_1\beta_2}\right)h^2\|u''\|_{L^\infty(\Omega_1\cup\Omega_2)}, \tag{4.5}$$

where the constant $C > 0$ is independent of h, α and β.

Proof First, we have

$$\begin{aligned} T = S\big(\mathcal{E}(G)\big) = SG - Sg &= \left(\frac{u_h(\alpha^+)}{\beta_2} - \frac{u_h(\alpha^-)}{\beta_1}\right) - w \\ &= \left(\frac{u_h(\alpha^+)}{\beta_2} - \frac{u_h(\alpha^-)}{\beta_1}\right) - \left(\frac{u(\alpha^+)}{\beta_2} - \frac{u(\alpha^-)}{\beta_1}\right) \\ &= \left(\frac{u_h(\alpha^+)}{\beta_2} - \frac{u(\alpha^+)}{\beta_2}\right) - \left(\frac{u_h(\alpha^-)}{\beta_1} - \frac{u(\alpha^-)}{\beta_1}\right). \end{aligned} \tag{4.6}$$

Then, using Theorem 3.2, we conclude

$$|T| = \big|S\big(\mathcal{E}(G)\big)\big| = \big|S(G-g)\big| \le C\left(\frac{\beta_1+\beta_2}{\beta_1\beta_2}\right)h^2\|u''\|_{L^\infty(\Omega_1\cup\Omega_2)}, \tag{4.7}$$

which completes the proof of this lemma. □

Given a guess augmented variable G^0, the residual of the Schur complement is defined as

$$\begin{aligned} &R_{\mathrm{es}}\big(G^0\big) = F - SG^0 \quad \text{(matrix vector form)} \quad \text{or} \\ &R_{\mathrm{es}}\big(G^0\big) = w - \left[\!\left[\frac{u_h^{G^0}}{\beta}\right]\!\right]_\alpha \quad \text{(function form)}, \end{aligned} \tag{4.8}$$

where $u_h^{G^0}$ is the solution of the method (3.9) with the augmented variable given as $g = G^0$. Thus, we have

$$S = R_{\mathrm{es}}(0) - R_{\mathrm{es}}(1)$$

Next, we show that the Schur complement satisfying $|S|$ is independent of the mesh size h and $|S| \neq 0$.

Lemma 4.2 *We have*

$$S = \frac{\beta_1 k_1 + \beta_2 k_2}{\beta_1\beta_2}, \tag{4.9}$$

where $k_1 = (b-\alpha)/(b-a) > 0$, $k_2 = (\alpha - a)/(b-a)$. It is obvious that $k_1 > 0$, $k_2 > 0$ and $k_1 + k_2 = 1$.

Proof Let $u_h^0 = u_h^{L,0} + u_h^{J,0}$ be the solution of the method (3.9) with the augmented variable given as $g = 0$. And let $u_h^1 = u_h^{L,1} + u_h^{J,1}$ be the solution of the method (3.9) with the augmented variable given as $g = 1$. From (4.8), we have

$$\begin{aligned} S = R_{\mathrm{es}}(0) - R_{\mathrm{es}}(1) &= \left(w - \left[\!\left[\frac{u_h^0}{\beta}\right]\!\right]_\alpha\right) - \left(w - \left[\!\left[\frac{u_h^1}{\beta}\right]\!\right]_\alpha\right) \\ &= \left[\!\left[\frac{u_h^1 - u_h^0}{\beta}\right]\!\right]_\alpha \\ &= \left[\!\left[\frac{(u_h^{L,1} - u_h^{L,0}) + (u_h^{J,1} - u_h^{J,0})}{\beta}\right]\!\right]_\alpha. \end{aligned} \tag{4.10}$$

From (3.4) and (3.5), we find

$$u_h^{J,1} - u_h^{J,0} = \chi(x)u_e(x) - I_h^L\big(\chi(x)u_e(x)\big) \quad \text{with } u_e(x) = 1. \tag{4.11}$$

In other words, the function $u_h^{J,1} - u_h^{J,0}$ is obtained by (3.4) and (3.5) by setting $q = 0$ and $g = 1$. From (3.9), we get

$$a_h\big(u_h^{L,1} - u_h^{L,0}, v_h\big) = -a_h\big(u_h^{J,1} - u_h^{J,0}, v_h\big), \quad \forall v_h \in V_h^L. \tag{4.12}$$

Combining (4.11) and (4.12), we conclude that $u_h^1 - u_h^0$ is the solution of the method for the interface problem (2.5) with $f = 0$, $q = 0$ and $g = 1$, i.e.,

$$\begin{aligned} &-\psi(x)'' = 0, \quad x \in (a,\alpha) \cup (\alpha,b), \\ &[\![\psi]\!]_\alpha = 1, \qquad [\![\psi']\!]_\alpha = 0, \\ &\psi(a) = \psi(b) = 0. \end{aligned} \tag{4.13}$$

It is easy to check that the exact solution is

$$\psi(x) = \begin{cases} (a-x)/(b-a), & x \in [a,\alpha), \\ (b-x)/(b-a), & x \in (\alpha,b]. \end{cases} \tag{4.14}$$

Using the result (3.21), which is proved in Theorem 3.2, we have

$$u_h^{L,1} - u_h^{L,0} = I_h^L \psi. \tag{4.15}$$

Using the fact $[\![u_h^{J,1} - u_h^{J,0}]\!]_\alpha = 1$, $[\![(u_h^{J,1} - u_h^{J,0})']\!]_\alpha = 0$ and ψ is piecewise linear, we conclude

$$u_h^1 - u_h^0 = \psi. \tag{4.16}$$

Combining (4.10), (4.14) and (4.16), we have

$$S = \left[\!\left[\frac{\psi}{\beta}\right]\!\right]_\alpha = \frac{1}{b-a}\left(\frac{b-\alpha}{\beta_2} + \frac{\alpha - a}{\beta_1}\right), \tag{4.17}$$

which completes the proof of this lemma. □

Theorem 4.3 *Let G and U_h be the solution of the augmented method* (3.9) *and* (4.1). *There exists a constant $C > 0$ independent of h, α and β such that*

$$|G-g| \le C\frac{1}{\min\{k_1,k_2\}}h^2\left\|u''\right\|_{L^\infty(\Omega_1\cup\Omega_2)}, \tag{4.18}$$

and

$$\|U_h-u\|_{L^\infty(a,b)} \le C\frac{1}{\min\{k_1,k_2\}}h^2\left\|u''\right\|_{L^\infty(\Omega_1\cup\Omega_2)}, \tag{4.19}$$

where $k_1 = (b-\alpha)/(b-a) > 0$, $k_2 = (\alpha-a)/(b-a)$.

Proof Using Lemma 4.2 and Lemma 4.1, we have

$$\left|\mathcal{E}(G)\right| = |G-g| = \left|S^{-1}T\right| \le C\frac{\beta_1+\beta_2}{k_1\beta_1+k_2\beta_2}h^2\left\|u''\right\|_{L^\infty(\Omega_1\cup\Omega_2)}. \tag{4.20}$$

Note that $k_1 > 0$, $k_2 > 0$, $k_1 + k_2 = 1$ and $\beta(x) > 0$. If $\beta_2 \ge \beta_1$, then we get

$$\frac{\beta_1+\beta_2}{k_1\beta_1+k_2\beta_2} \le \frac{2\beta_2}{\beta_2 k_2} = \frac{2}{k_2}. \tag{4.21}$$

If $\beta_1 > \beta_2$, then we have

$$\frac{\beta_1+\beta_2}{k_1\beta_1+k_2\beta_2} \le \frac{2\beta_1}{\beta_1 k_1} = \frac{2}{k_1}. \tag{4.22}$$

Thus,

$$\left|\mathcal{E}(G)\right| \le C\frac{1}{\min\{k_1,k_2\}}h^2\left\|u''\right\|_{L^\infty(\Omega_1\cup\Omega_2)}, \tag{4.23}$$

which, together with Theorem 3.3 completes the proof of this theorem. □

Remark 4.4 The numerical solution of the original interface problem (2.1)–(2.3) is $\widetilde{U}_h = U_h/\beta$. We have

$$\begin{aligned}\|\widetilde{U}_h-\widetilde{u}\|_{L^\infty(a,b)} &\le C\frac{1}{\min\{\beta_1,\beta_2\}}\frac{1}{\min\{k_1,k_2\}}h^2\left\|u''\right\|_{L^\infty(\Omega_1\cup\Omega_2)}\\ &\le C\frac{1}{\min\{\beta_1,\beta_2\}}\frac{1}{\min\{k_1,k_2\}}h^2\|f\|_{L^\infty(\Omega_1\cup\Omega_2)},\end{aligned} \tag{4.24}$$

where we have used the fact $-u'' = f$ in the last inequality.

5 Numerical examples

In this section, we present numerical examples to confirm our theoretical results. In the following examples, we consider an unit interval $[0,1]$ with uniform partitions. The interface point is given as $\alpha = \pi/10$.

Let $e_h = \widetilde{u} - \widetilde{U}_h$. The method achieves convergence order r if we can show $\|e_h\|_{L^\infty} \approx Ch^r$. The convergence order r is the slope of the line $y = rx + \log C$ with $y = \log\|e_h\|_{L^\infty}$ and

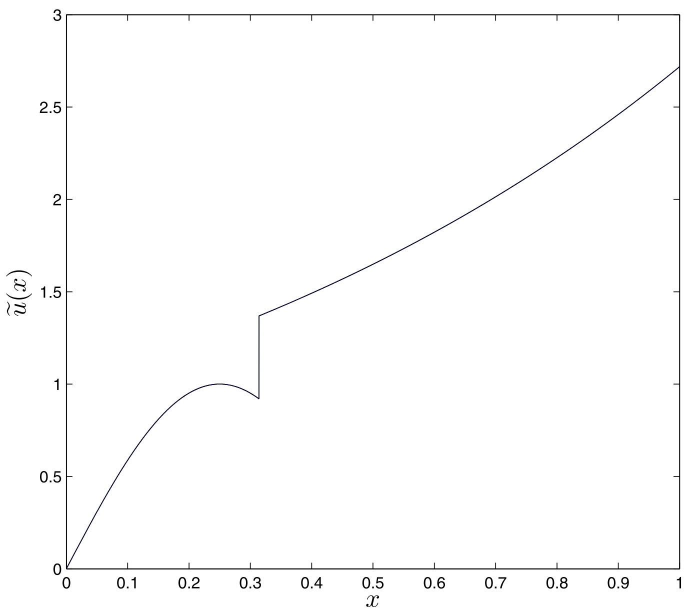

Figure 1 The exact solution of Example 1

$x = \log h$. For the interface problem, the constant C depends on the relative location of the interface and the mesh. Thus, we compute the slope of least squares fitting which can be regarded as the average convergence order.

Example 1 The analytic solution $\widetilde{u}(x)$ is given as

$$\widetilde{u}(x) = \begin{cases} \sin(2\pi x), & 0 \le x < \alpha, \\ e^x, & \alpha < x \le 1. \end{cases}$$

Thus, the corresponding jump conditions are

$$[\![\widetilde{u}]\!]_\alpha = e^\alpha - \sin(2\pi\alpha), \qquad [\![\beta\widetilde{u}']\!]_\alpha = \beta_2 e^\alpha - 2\pi\beta_1\cos(2\pi\alpha).$$

We consider $\beta_1 = 1$, $\beta_2 = 5$ in this example. The exact solution is shown in Fig. 1. Numerical results are reported in Fig. 2. We can see that the augmented method achieves optimal convergence.

Example 2 The analytic solution $\widetilde{u}(x)$ is given as

$$\widetilde{u}(x) = \begin{cases} \sin(2\pi x)/\beta_1, & 0 \le x < \alpha, \\ \sin(2\pi x)/\beta_2 - (1/\beta_2 - 1/\beta_1)\sin(2\pi\alpha), & \alpha < x \le 1. \end{cases}$$

Thus, we have

$$[\![\widetilde{u}]\!]_\alpha = 0, \qquad [\![\beta\widetilde{u}']\!]_\alpha = 0,$$

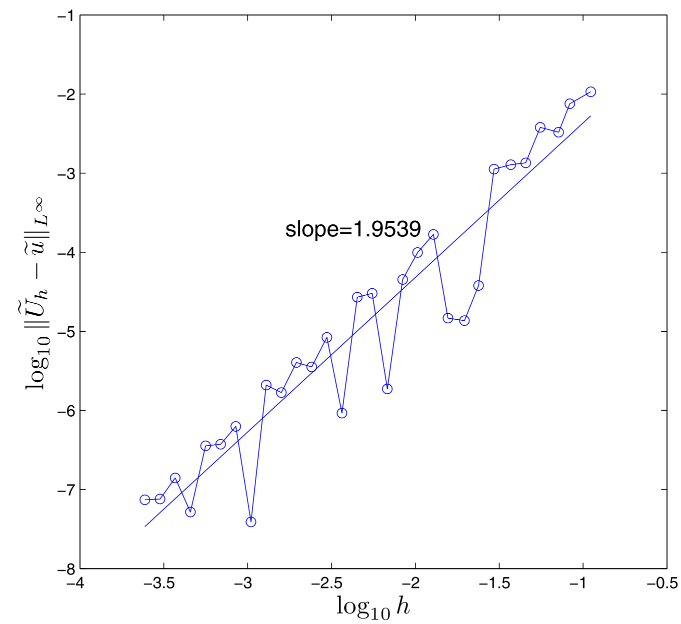

Figure 2 Numerical results obtained by the augmented method for Example 1 with $\beta_1 = 1$ and $\beta_2 = 5$

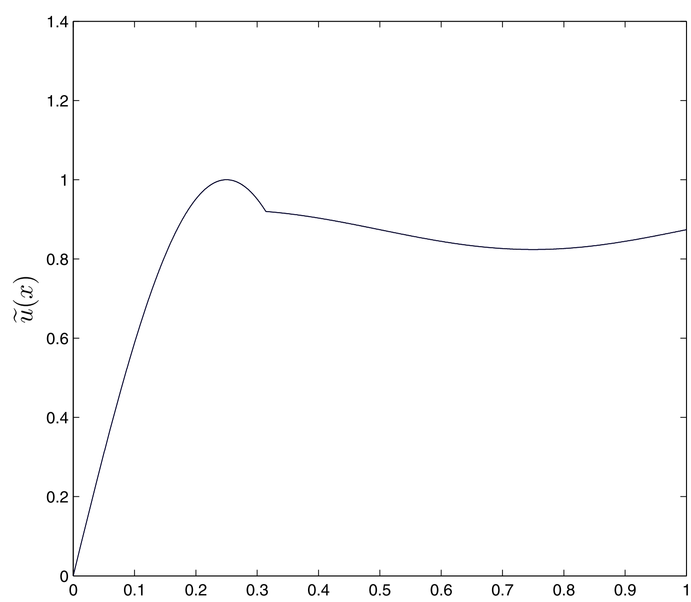

Figure 3 The exact solution of Example 2

which are homogeneous jump conditions. We consider $\beta_1 = 1$, $\beta_2 = 20$ in this example. The corresponding source function is

$$f(x) = 4\pi^2 \sin(2\pi x), \qquad 0 < x < 1.$$

The exact solution is shown in Fig. 3. Similar results are reported in Fig. 4.

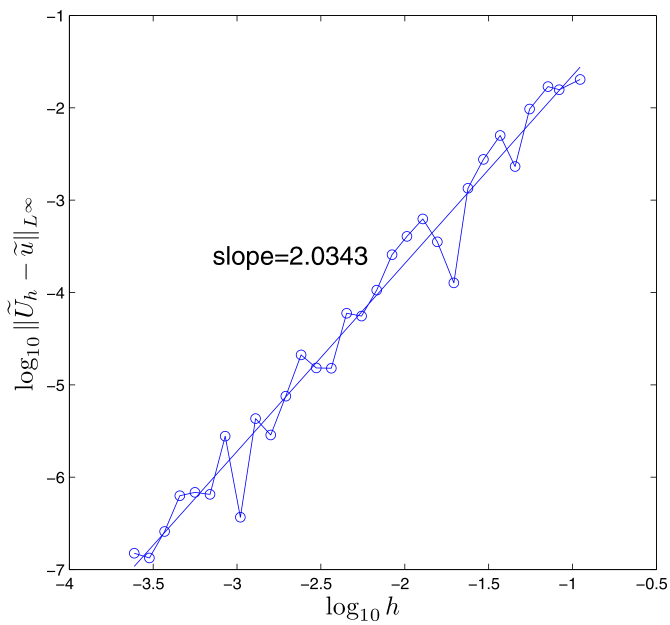

Figure 4 Numerical results obtained by the augmented method for Example 2 with $\beta_1 = 1$ and $\beta_2 = 20$

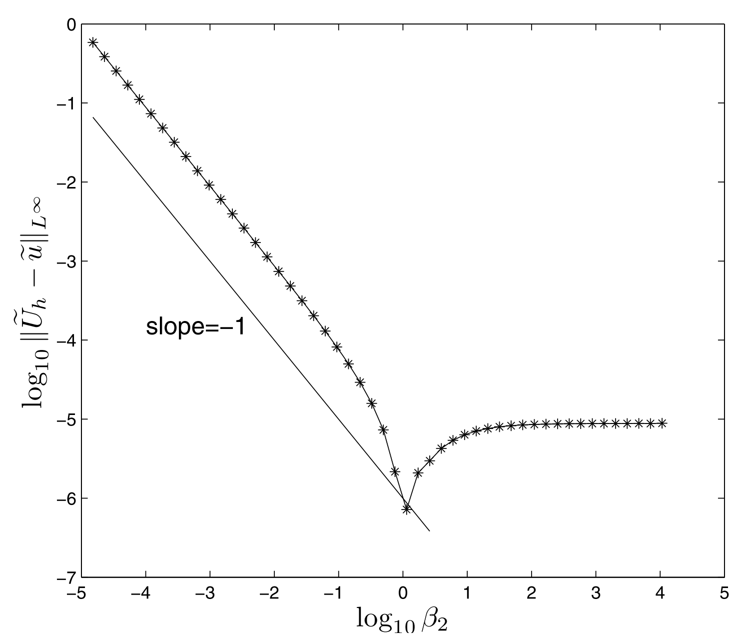

Figure 5 Numerical results obtained by the augmented method for Example 2 with $\beta_1 = 1$, $N = 512$

Next, we consider the dependence of the discontinuous coefficient β. We solve the problem with $\beta_1 = 1$ and varying β_2. Errors are obtained by the augmented method with $N = 512$. Note that $\|f\|_{L^\infty(\Omega_1\cup\Omega_2)} = 4\pi^2$ is independent of β in this example. If $\beta_2 \le 1$, then the error should be $O(\beta_2^{-1})$ according to the estimate (4.24). If $\beta_2 \ge 1$, then the error should be $O(1)$. Numerical results reported in Fig. 5 confirm the theoretical results. The estimate (4.24) is sharp.

6 Conclusion

This article gives rigorous error estimates for the augmented method for one-dimensional interface problems. The influence of the discontinuous coefficient and the location of the interface is considered in the error estimation. Numerical results show that the estimate is sharp. In future work, we will extend the results to high-dimensional interface problems.

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