Sultana et al. *Advances in Difference Equations* (2018) 2018:316
https://doi.org/10.1186/s13662-018-1763-z

Advances in Difference Equations
a SpringerOpen Journal

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A new non-polynomial spline method for solution of linear and non-linear third order dispersive equations

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Abstract

In this paper, a new three-level implicit method is developed to solve linear and non-linear third order dispersive partial differential equations. The presented method is obtained by using exponential quartic spline to approximate the spatial derivative of third order and finite difference discretization to approximate the first order spatial and temporal derivative. The developed method is tested on four examples and the results are compared with other methods from the literature, which shows the applicability and feasibility of the presented method. Furthermore, the truncation error and stability analysis of the presented method are investigated, and graphical comparison between analytical and approximate solution is also shown for each example.

MSC: 65D07; 65M12

Keywords: Spline function approximation; Third order dispersive equation; Stability analysis; Korteweg-de Vries (KdV) equation; Soliton

1 Introduction

In [4] Boussinesq and Korteweg-de Vries (KdV) equations described the problem of water waves and the long waves in which dispersive effects are present. We use an exponential quartic spline function to develop a numerical method to approximate the solution of third order homogeneous and non-homogeneous linear dispersive equation in one space dimension with $f(x,t)$ as a source term:

$$\frac{\partial y(x,t)}{\partial t}+\mu\frac{\partial^3 y(x,t)}{\partial x^3}=f(x,t),\quad a\le x\le b, t>0, \mu>0, \tag{1.1}$$

with

$$y(x,0)=g_1(x),\quad a\le x\le b, \tag{1.2}$$

and

$$\left.\begin{aligned} y(a,t)&=\gamma_0(t), && t>0,\\ y_x(a,t)&=\gamma_1(t), && t>0,\\ y_{xx}(a,t)&=\gamma_2(t), && t>0, \end{aligned}\right\} \tag{1.3}$$

where $\gamma_0(t)$, $\gamma_1(t)$, $\gamma_2(t)$, and $g_1(x)$ are assumed to be continuous functions, t is time and x is space variable [1, 7].

In [1] theorems for the existence and uniqueness of solution of such dispersive equations are given. In [20, 21], the criteria for deriving stability conditions of difference method were considered for the numerical solution of a third order linear dispersive equation. In [25], the analytical solution was obtained of such equations by using Adomian decomposition method, and in [19], such equations were solved numerically. The authors in [17] solved fourth order parabolic partial differential equation numerically by using parametric septic spline. Djidjeli and Twizell [7] developed numerical method for solution of third order linear dispersive equation with time-dependent boundary conditions.

We have also solved the third order non-linear dispersive equation named as Korteweg-de Vries (KdV) equation:

$$\frac{\partial y(x,t)}{\partial t} + \varepsilon y(x,t)\frac{\partial y(x,t)}{\partial x} + \mu\frac{\partial^3 y(x,t)}{\partial x^3} = 0, \quad a \le x \le b, t > 0, \mu > 0 \tag{1.4}$$

with

$$y(x,0) = g_2(x), \quad a \le x \le b, \tag{1.5}$$

and

$$\left.\begin{array}{ll} y(x,t) = \gamma_3(t), & x \in \partial\Omega, t > 0, \\ y_x(b,t) = \gamma_4(t), & t > 0, \end{array}\right\} \tag{1.6}$$

where $\Omega = [a,b] \subset R$, ε and μ are positive parameters, and $g_2(x)$, $\gamma_3(t)$, $\gamma_4(t)$ are known functions. This equation shows both dispersion and non-linearity [4, 23, 27].

The solution of Eq. (1.4) may exhibit solitons. Solitons are localized waves that propagate without change in their shape and velocity and are stable in mutual interaction just like the phenomenon of totally elastic collision in kinetics. The KdV type equations have been an important class of non-linear evolution equations with numerous applications in physical sciences and engineering fields see [5, 8, 12, 18, 23, 24, 26].

The existence and uniqueness of solutions of the KdV equation for appropriate conditions have shown in [11] . Many well-known numerical methods, such as finite difference scheme, finite element schemes, Fourier spectral methods and mesh-free radial basis functions (RBF), collocation method, multiquadric (MQ), multiquadric quasi-interpolation [3, 5, 10, 13–15, 22, 23, 27, 28], have been used to solve the KdV equation. Authors in [16] also used decomposition method for solution of KdV equation. The numerical solution was presented in [6] for the first and fifth order KdV equations. In [2], authors solved coupled Burgers' equations using non-polynomial spline method. In [9], authors presented non-polynomial spline method for solving the generalized regularized long wave (GRLW) equations.

The purpose of this paper is to present a new method to solve third order linear and non-linear dispersive partial differential equations based on spline function approximation. In our method, the third order spatial and first order temporal derivatives are approximated by exponential quartic spline and finite difference respectively. Besides, we have used the first order central difference discretization to approximate the first order spatial derivative in a non-linear term.

Here, we obtain a derivation of exponential quartic spline and its relations in Sect. 2. In Sect. 3, we present the formulation of our methods along with boundary equations for both the linear and non-linear dispersive equations. In Sect. 4, a class of methods and truncation error are given. Stability analysis is discussed in Sect. 5. Numerical evidences and comparison with other available methods are included in Sect. 6 to show the accuracy of our method. The presented method is tested on four examples. Finally, conclusion is presented in Sect. 7.

2 Exponential quartic spline

Let a set of grid points in the interval $[a,b]$ be such that

$$x_j = a + jh, \quad j = 0(1)n, h = \frac{b-a}{n}. \tag{2.1}$$

We also denote the function value $y(x_j)$ by y_j.

Let $E_j(x,t)$ be the exponential quartic spline at the grid point (x_j,t) given by

$$E_j(x,t) = a_{1j}(t)e^{\tau(x-x_j)} + a_{2j}(t)e^{-\tau(x-x_j)} + a_{3j}(t)(x-x_j)^2 + a_{4j}(t)(x-x_j) + a_{5j}(t) \tag{2.2}$$

for each $j = 0,1,\dots,n$, where a_{1j}, a_{2j}, a_{3j}, a_{4j}, a_{5j} are unknown coefficients and τ is a free parameter. We determine the unknown coefficients in (2.2) from the interpolatory conditions $E_j(x_j,t) = y_j(t)$, $E_j(x_{j+1},t) = y_{j+1}(t)$, $E_j'(x_j,t) = m_j(t)$, $E_j^{(3)}(x_j,t) = T_j(t)$, and $E_j^{(3)}(x_{j+1},t) = T_{j+1}(t)$ as given below:

$$\begin{aligned}
a_{1j}(t) &= \frac{1}{\tau^3}\frac{(T_{j+1} - T_j e^{-\theta})}{(e^{\theta} - e^{-\theta})},\\
a_{2j}(t) &= \frac{1}{\tau^3}\frac{(T_{j+1} - T_j e^{\theta})}{(e^{\theta} - e^{-\theta})},\\
a_{3j}(t) &= \frac{1}{h^2}(y_{j+1} - y_j) - \frac{1}{h}M_j - \frac{1}{h^2}a_{1j}(t)\bigl(e^{\theta} - \theta - 1\bigr) - \frac{a_{2j}(t)}{h^2}\bigl(e^{-\theta} + \theta - 1\bigr),\\
a_{4j}(t) &= m_j - a_{1j}(t)\tau + a_{2j}(t)\tau,\\
a_{5j}(t) &= y_j - a_{1j}(t) - a_{2j}(t).
\end{aligned} \tag{2.3}$$

Applying the continuity conditions of first and second derivatives of $E_j(x,t)$ at the knots, that is, $E_j'(x_j) = E_{j-1}'(x_j)$, $E_j''(x_j) = E_{j-1}''(x_j)$, and using (2.3) yields the following relations:

$$m_j + m_{j-1} = h^2(\alpha_1 T_j + \alpha_1 T_{j-1}) + \frac{2}{h}(y_j - y_{j-1}), \tag{2.4}$$

$$m_j - m_{j-1} = h^2(\beta_1 T_{j+1} + \beta_2 T_j + \beta_3 T_{j-1}) + \frac{1}{h}(y_{j+1} - 2y_j + y_{j-1}), \tag{2.5}$$

where

$$\begin{aligned}
\alpha_1 &= \frac{\theta(e^\theta - e^{-\theta}) - 2(e^\theta + e^{-\theta}) + 4}{\theta^3(e^\theta - e^{-\theta})},\\
\beta_1 &= \frac{\theta^2 - (e^\theta + e^{-\theta}) + 2}{\theta^3(e^\theta - e^{-\theta})},\\
\beta_2 &= \frac{\theta(e^\theta - e^{-\theta}) - \theta^2(e^\theta + e^{-\theta})}{\theta^3(e^\theta - e^{-\theta})},\\
\beta_3 &= \frac{\theta^2 - \theta(e^\theta - e^{-\theta}) + (e^\theta + e^{-\theta}) - 2}{\theta^3(e^\theta - e^{-\theta})}.
\end{aligned} \tag{2.6}$$

Using Eqs. (2.4) and (2.5), we obtain the following method:

$$h^3(pT_{j+1} + qT_j + qT_{j-1} + pT_{j-2}) = -y_{j+1} + 3y_j - 3y_{j-1} + y_{j-2}, \quad j = 2(1)(n-1), \tag{2.7}$$

where the coefficients $p = \beta_1$ and $q = -\alpha_1 + \beta_1 + \beta_2$. As $\tau \to 0$ that is $\theta \to 0$, we have $(p,q) \longrightarrow (-\frac{1}{24}, -\frac{11}{24})$. Now, the operator Λ_x for any function W is supposed to have the following form according to Eq. (2.7):

$$\Lambda_x W_j = pW_{j+1} + qW_j + qW_{j-1} + pW_{j-2}. \tag{2.8}$$

3 Derivation of the method

Let the region $R = [a \le x \le b] \times [t > 0]$ be discretized by a set of points $R_{h,k}$ which are the vertices of grid points (x_j, t_m), where $x_j = jh$, $j = 0(1)n$, $nh = b - a$, and $t_m = mk$, $m = 0, 1, 2, 3, \ldots$. The quantities h in space and k in time directions are mesh sizes.

3.1 Spline solution for linear dispersive equation

In this section we develop an approximation for (1.1) in which the time and space derivatives are replaced by a finite difference and exponential quartic spline respectively. Equation (1.1) is discretized as:

$$\frac{k^{-1}}{2}\delta_t\bigl(1 + \sigma\delta_t^2\bigr)^{-1} y_j^m + \mu T_j^m = f_j^m, \tag{3.1}$$

where $T_j^m = E_\triangle^{(3)}(x_j, t_m)$ is the third order spline derivative at (x_j, t_m) w.r.t. the space variable, $f_j^m = f(x_j, t_m)$, y_j^m is the approximate solution of (1.1) at (x_j, t_m), δ_t is the central difference operator w.r.t. t and σ is a parameter such that finite difference approximation to the time derivative is of $O(k)$ for arbitrary σ.

Operating Λ_x on both sides of (3.1) and after some simplifications, we obtain the following method:

$$\begin{aligned}
&\delta_t\bigl(py_{j+1}^m + qy_j^m + qy_{j-1}^m + py_{j-2}^m\bigr) + \frac{2k\mu}{h^3}\bigl(1 + \sigma\delta_t^2\bigr)\bigl(-y_{j+1}^m + 3y_j^m - 3y_{j-1}^m + y_{j-2}^m\bigr)\\
&\quad = 2k\bigl(1 + \sigma\delta_t^2\bigr)\bigl(pf_{j+1}^m + qf_j^m + qf_{j-1}^m + pf_{j-2}^m\bigr), \quad j = 2(1)(n-1).
\end{aligned} \tag{3.2}$$

The final method (3.2) may be written in the schematic form as follows:

$$\left.\begin{matrix} P & Q & R & S \\ -N & 3N & -3N & N \\ -S & -R & -Q & -P \end{matrix}\right\} y_j^m = \left.\begin{matrix} K_1p & K_1q & K_1q & K_1p \\ K_2p & K_2q & K_2q & K_2p \\ K_1p & K_1q & K_1q & K_1p \end{matrix}\right\} f_j^m,$$

where

$$\left.\begin{array}{l} r = \frac{k}{h^3}, \qquad P = p - 2\sigma r\mu, \qquad Q = q + 6\sigma r\mu, \\ R = q - 6\sigma r\mu, \qquad S = p + 2\sigma r\mu, \\ N = 2r\mu(1-2\sigma), \qquad K_1 = 2\sigma k, \qquad K_2 = 2k(1-2\sigma). \end{array}\right\} \tag{3.3}$$

Relation (3.2) gives $(n-2)$ equations in $(n-1)$ unknowns y_j, $j = 1(1)(n-1)$. We require one more equation at $j = 1$, i.e., at the end of the range of integration in order to have a closed form solution for y_j. We discretize the boundary conditions in (1.3) and develop the following boundary equation of accuracy $O(k + h^2)$:

$$-21y_0^m + 24y_1^m - 3y_2^m - 18h\left(y_0^m\right)' - 6h^2\left(y_0^m\right)'' = 0, \quad j = 1, \tag{3.4}$$

where

$$y_0^m = y(a, t_m), \qquad \left(y_0^m\right)' = \frac{\partial y}{\partial x}(a, t_m), \qquad \left(y_0^m\right)'' = \frac{\partial^2 y}{\partial x^2}(a, t_m).$$

3.2 Spline solution for non-linear dispersive equation

In the similar manner, Eq.(1.4) is discretized as follows:

$$\frac{k^{-1}}{2}\delta_t\left(1 + \sigma\delta_t^2\right)^{-1} y_j^m + \frac{\delta_x}{2h}F_j^m + \mu T_j^m = 0, \tag{3.5}$$

where $F = \frac{\varepsilon}{2}y^2$.

Operating Λ_x on both sides of (3.5) and after some simplifications, we obtain the following method:

$$\begin{aligned} &\delta_t\left(py_{j+1}^m + qy_j^m + qy_{j-1}^m + py_{j-2}^m\right) + \frac{2k\mu}{h^3}\left(1 + \sigma\delta_t^2\right)\left(-y_{j+1}^m + 3y_j^m - 3y_{j-1}^m + y_{j-2}^m\right) \\ &\quad + \frac{k}{h}\left(1 + \sigma\delta_t^2\right)\left(pF_{j+2}^m + qF_{j+1}^m - (p-q)F_j^m + (p-q)F_{j-1}^m - qF_{j-2}^m - pF_{j-3}^m\right) \\ &= 0, \quad j = 3(1)(n-2). \end{aligned} \tag{3.6}$$

The schematic form of method (3.6) is given by

$$\left.\begin{matrix} P & Q & R & S \\ -N & 3N & -3N & N \\ -S & -R & -Q & -P \end{matrix}\right\} y_j^m$$

$$= \left.\begin{matrix} -K_1'p & -K_1'q & K_1'(p-q) & -K_1'(p-q) & K_1'q & K_1'p \\ -K_2'p & -K_2'q & K_2'(p-q) & -K_2'(p-q) & K_2'q & K_2'p \\ -K_1'p & -K_1'q & K_1'(p-q) & -K_1'(p-q) & K_1'q & K_1'p \end{matrix}\right\} F_j^m,$$

where

$$K_1' = \frac{k}{h}\sigma, \qquad K_2' = \frac{k}{h}(1-2\sigma). \tag{3.7}$$

Relation (3.6) gives $(n-4)$ equations in $(n-1)$ unknowns y_j, $j = 1(1)(n-1)$. We require three more equations each at $j = 1, 2, (n-1)$, i.e., at the ends of the range of integration in order to have a closed form solution for y_j. We discretize the boundary conditions in (1.6) and develop the following boundary equation of accuracy $O(k+h^2)$:

$$\left.\begin{array}{ll} -y_0^m + 4y_1^m - 6y_2^m + 4y_3^m - y_4^m = 0, & j = 1, \\ -y_1^m + 4y_2^m - 6y_3^m + 4y_4^m - y_5^m = 0, & j = 2, \\ -\frac{4}{3}y_{n-3}^m + 6y_{n-2}^m - 12y_{n-1}^m + \frac{22}{3}y_n^m - 4h(y_n^m)' = 0, & j = (n-1), \end{array}\right\} \tag{3.8}$$

where

$$y_0^m = y(a, t_m), \qquad y_n^m = y(b, t_m), \qquad \left(y_n^m\right)' = \frac{\partial y}{\partial x}(b, t_m).$$

4 Truncation error and a class of methods

Expanding (3.2) or (3.6) in a Taylor series in terms of $y(x_j, t_m)$ and its derivatives and using (1.1) or (1.4) respectively, we obtain the truncation error as follows:

$$\begin{aligned} TE_j^m = \Bigg[& 2(p+q)kD_t - (p+q)khD_tD_x + \frac{1}{2}(5p+q)kh^2D_tD_x^2 \\ & - \frac{1}{6}(7p+q)kh^3D_tD_x^3 + \left(\frac{1}{3}-\sigma\right)2(p+q)k^3D_t^3 - \left(\frac{1}{3}-\sigma\right)(p+q)k^3hD_t^3D_x \\ & + \frac{1}{2}\left(\frac{1}{3}-\sigma\right)(5p+q)k^3h^2D_t^3D_x^2 - \frac{1}{6}\left(\frac{1}{3}-\sigma\right)(7p+q)k^3h^3D_t^3D_x^3 \\ & - 2(p+q)kD_x^3 + (p+q)khD_x^4 + \frac{1}{2}(5p+q)kh^2D_x^5 + \frac{1}{6}(7p+q)kh^3D_x^6 \\ & - 2(p+q)\sigma k^3D_t^2D_x^3 + (p+q)\sigma k^3hD_t^2D_x^4 - \frac{1}{2}(5p+q)\sigma k^3h^2D_t^2D_x^5 \\ & + \frac{1}{6}(7p+q)\sigma k^3h^3D_t^2D_x^6 - kh^2D_x^3 + \frac{1}{2}kh^3D_x^4 - \sigma k^3h^2D_t^2D_x^3 \\ & + \frac{1}{2}\sigma k^3h^3D_t^2D_x^4 + \cdots\cdots \Bigg] y_j^m, \end{aligned}$$

where $D_t \equiv \frac{\partial}{\partial t}$, $D_x \equiv \frac{\partial}{\partial x}$, $D_t^2 \equiv \frac{\partial^2}{\partial t^2}$, $D_x^2 \equiv \frac{\partial^2}{\partial x^2}$, and so on.

Here, the following class of methods are obtained:

Case 1: If $p + q \neq 0$, then various methods of $O(k+h)$ for arbitrary values of σ are obtained.

Case 2: If $p + q = 0$, then various methods of $O(k+h^2)$ for arbitrary values of σ are obtained.

5 Stability analysis and convergence

Theorem *Methods* (3.3) *and* (3.7) *are conditionally stable for* $\sigma \geq (\frac{1}{2} - \frac{1}{2r})$, *where* $r > 0$, $p + q = 0$, *and* $\phi = \frac{\theta}{2}$.

Proof Here the stability analysis of any one of methods (3.3) or (3.7) will be investigated, and it can be investigated for other method in the same manner. For this, we use the Von Neumann method. Let the solution of (3.3) at the point (x_j, t_m) is

$$y_j^m = \xi^m e^{ji\theta}, \tag{5.1}$$

where $i = \sqrt{-1}$, θ is real and ξ in general is complex.

We get the following equation after putting (5.1) in the homogeneous part of (3.3):

$$U\xi^2 + V\xi + W = 0, \tag{5.2}$$

where

$$\begin{aligned} U &= Pe^{i\theta} + Q + Re^{-i\theta} + Se^{-2i\theta}, \\ V &= N\left(e^{i\theta} + 3 - 3e^{-i\theta} + e^{-2i\theta}\right), \\ W &= -\left(Se^{i\theta} + R + Qe^{-i\theta} + Pe^{-2i\theta}\right). \end{aligned}$$

The necessary and sufficient condition for method (3.3) to be stable is $|\xi| \leq 1$. For this, we obtain the following condition:

$$\begin{aligned} &\left(4N\sin^3\phi\right)/\left((p+q)^2 - 4\left(10pq + 9p^2 + q^2 + 12\sigma^2 r^2\right)\sin^2\phi \right. \\ &\quad + 16\left(4p^2 + 2(3p-q)\sigma r + 8\sigma^2 r^2\right)\sin^4\phi \\ &\quad \left. - 32\left(pq - p^2 - (3p-q)\sigma r - 2\sigma^2 r^2\right)\sin^6\phi\right)^{1/2} \leq 1. \end{aligned}$$

Simplifying and putting $p + q = 0$, we deduce that method (3.3) is conditionally stable for $\sigma \geq (\frac{1}{2} - \frac{1}{2r})$, where $r > 0$ and $\phi = \frac{\theta}{2}$. □

The present method is convergent by Lax theorem as the stability criterion is satisfied.

6 Numerical simulation and comparison

In this section, the presented three-level implicit method based on exponential quartic spline is tested on four examples. The following norms are used in this paper:

$$\begin{aligned} L_\infty &= \max_{1\leq i\leq n}\left|y_{\text{ana}}(i) - y_{\text{app}}(i)\right|, \\ L_2 &= \sqrt{\sum_{i=1}^{n}\left(y_{\text{ana}}(i) - y_{\text{app}}(i)\right)^2}, \\ RMS &= \sqrt{\left(\sum_{i=1}^{n}\left(y_{\text{ana}}(i) - y_{\text{app}}(i)\right)^2\right)\Big/ n}, \end{aligned} \tag{6.1}$$

where y_{ana} is analytical and y_{app} is approximate solution of third order dispersive equation for our method.

Example 1 (Linear homogeneous case) Consider the following linear homogeneous dispersive equation [7]

$$\frac{\partial y}{\partial t} + \mu \frac{\partial^3 y}{\partial x^3} = 0, \quad 0 \le x \le 1, t \ge 0, \mu > 0$$

with

$$y(x,0) = \cos x, \quad 0 \le x \le 1$$

and

$$y(0,t) = \cos \mu t, \qquad \frac{\partial y}{\partial x}(0,t) = -\sin \mu t, \qquad \frac{\partial^2 y}{\partial x^2}(0,t) = -\cos \mu t, \quad t \ge 0.$$

The analytical solution is

$$y(x,t) = \cos(x + \mu t).$$

The computational results of this example for $\mu = 1$ are tabulated in Tables 1 and 2. Table 1 shows L_∞, L_2 and *RMS* errors for $h = \frac{1}{20}, \frac{1}{40}$; $r = \frac{1}{100}, \sqrt{\frac{7}{60}}$; $\sigma = \frac{1}{12}$ and time steps = 50, 100 for different values of parameters p and q. The comparison of L_∞ error between our method and in [7] with $p = 30, 75$; $r = 1$; $h = 0.1$; $\sigma = \frac{1}{12}$; time steps = 100 for $x = 0.1, 0.2, \ldots, 0.9$ is tabulated in Table 2. Also the comparison between analytical and approximate solution for $h = \frac{1}{32}$, $r = \frac{1}{\sqrt{6}}$, and time steps = 100 is shown graphically in Fig. 1.

Example 2 (Linear non-homogeneous case) Consider the following linear non-homogeneous dispersive equation [25]

$$\frac{\partial y}{\partial t} + \mu \frac{\partial^3 y}{\partial x^3} = -\pi^3 \cos(\pi x)\cos t - \sin(\pi x)\sin t, \quad 0 \le x \le 1, t \ge 0, \mu > 0$$

Table 1 L_∞, L_2 and *RMS* errors for Example 1

(p,q,σ)	r	Time steps	$h = \frac{1}{20}$			$h = \frac{1}{40}$		
			L_∞	L_2	*RMS*	L_∞	L_2	*RMS*
$(25,-25,\frac{1}{12})$	$\frac{1}{100}$	50	1.0462(−6)	2.7700(−6)	6.3549(−7)	5.8011(−7)	2.5112(−6)	4.0211(−7)
	$\sqrt{\frac{7}{60}}$		3.5860(−5)	1.0775(−4)	2.4719(−5)	1.9822(−5)	8.7003(−5)	1.3932(−5)
	$\frac{1}{100}$	100	1.0462(−6)	2.7704(−6)	6.3557(−7)	5.8012(−7)	2.5112(−6)	4.0211(−7)
	$\sqrt{\frac{7}{60}}$		3.5989(−5)	1.0815(−4)	2.4810(−5)	1.9823(−5)	8.7044(−5)	1.3938(−5)
$(30,-30,\frac{1}{12})$	$\frac{1}{100}$	50	8.4689(−7)	2.1899(−6)	5.0240(−7)	4.8030(−7)	2.0718(−6)	3.3175(−7)
	$\sqrt{\frac{7}{60}}$		2.9030(−5)	8.7176(−5)	1.9999(−5)	1.6412(−5)	7.2067(−5)	1.1540(−5)
	$\frac{1}{100}$	100	8.4699(−7)	2.1902(−6)	5.0246(−7)	4.8031(−7)	2.0718(−6)	3.3175(−7)
	$\sqrt{\frac{7}{60}}$		2.9134(−5)	8.7499(−5)	2.0074(−5)	1.6420(−5)	7.2101(−5)	1.1545(−5)

Table 2 Comparison of L_∞ error with [7] for Example 1

Time steps	r	h	x	(p,q,σ) $(30,-30,\frac{1}{12})$	(p,q,σ) $(75,-75,\frac{1}{12})$	[7]
100	1	0.1	0.1	2.14(−6)	7.22(−5)	1.70(−3)
			0.2	4.99(−5)	2.49(−5)	5.00(−4)
			0.3	4.50(−5)	2.88(−5)	4.00(−4)
			0.4	8.42(−5)	4.20(−5)	7.00(−4)
			0.5	6.71(−5)	4.00(−5)	8.00(−4)
			0.6	9.30(−5)	4.63(−5)	9.00(−4)
			0.7	5.91(−5)	3.62(−5)	1.00(−3)
			0.8	6.73(−5)	3.35(−5)	1.10(−3)
			0.9	1.28(−5)	1.35(−5)	1.10(−3)

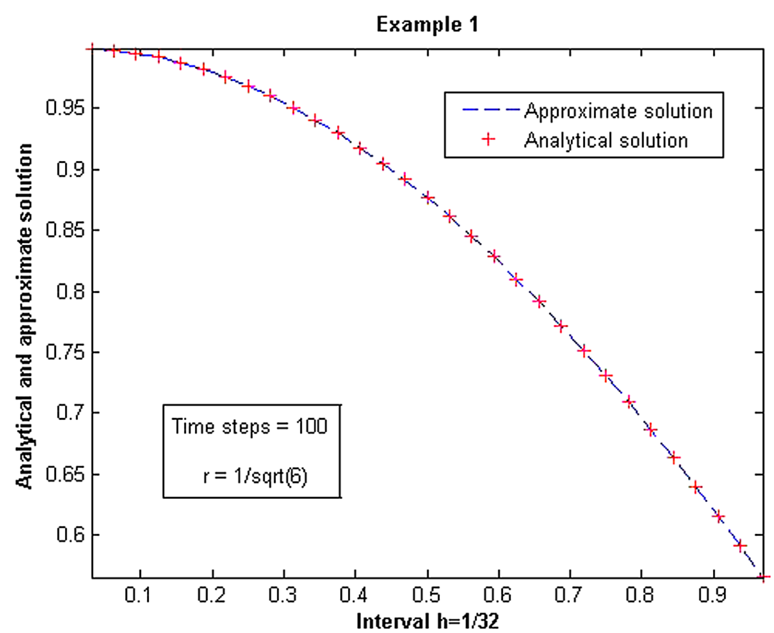

Figure 1 Comparison between analytical and approximate solution

with

$$y(x,0) = \sin(\pi x), \quad 0 \le x \le 1$$

and

$$y(0,t) = 0, \qquad \frac{\partial y}{\partial x}(0,t) = \pi \cos t, \qquad \frac{\partial^2 y}{\partial x^2}(0,t) = 0, \quad t \ge 0.$$

The analytical solution is

$$y(x,t) = \sin(\pi x)\cos t.$$

The computational results of this example for $\mu = 1$ are tabulated in Tables 3 and 4. The L_∞, L_2 and *RMS* errors are tabulated in Table 3 for the same values of parameters as taken in Table 1 of Example 1. Also Table 4 shows L_∞ error with $p = 25, 50$; $r = 1$; $h = 0.05, 0.1$; $\sigma =$

Table 3 L_∞, L_2 and *RMS* errors for Example 2

(p,q,σ)	r	Time steps	$h=\frac{1}{20}$			$h=\frac{1}{40}$		
			L_∞	L_2	*RMS*	L_∞	L_2	*RMS*
$(25,-25,\frac{1}{12})$	$\frac{1}{100}$	50	6.4058(−6)	1.4539(−5)	3.3356(−6)	5.2848(−6)	2.0991(−5)	3.3613(−6)
	$\sqrt{\frac{7}{60}}$		1.4202(−4)	3.5828(−4)	8.2196(−5)	1.8686(−4)	7.2452(−4)	1.1601(−4)
	$\frac{1}{100}$	100	6.4058(−6)	1.4539(−5)	3.3356(−6)	5.2848(−6)	2.0991(−5)	3.3613(−6)
	$\sqrt{\frac{7}{60}}$		1.4202(−4)	3.5828(−4)	8.2196(−5)	1.8686(−4)	7.2452(−4)	1.1601(−4)
$(30,-30,\frac{1}{12})$	$\frac{1}{100}$	50	9.3013(−6)	2.1326(−5)	4.8926(−7)	3.9930(−6)	1.5921(−5)	2.5494(−6)
	$\sqrt{\frac{7}{60}}$		2.4187(−4)	6.1364(−4)	1.4078(−4)	1.4128(−4)	5.4912(−4)	8.7930(−5)
	$\frac{1}{100}$	100	9.3013(−6)	2.1326(−5)	4.8926(−7)	3.9930(−6)	1.5921(−5)	2.5494(−6)
	$\sqrt{\frac{7}{60}}$		2.4187(−4)	6.1364(−4)	1.4078(−4)	1.4128(−4)	5.4912(−4)	8.7930(−5)

Table 4 L_∞ error for Example 2

Time steps	r	h	x	(p,q,σ) $(25,-25,\frac{1}{12})$	(p,q,σ) $(50,-50,\frac{1}{12})$
100	1	0.05	0.1	2.19(−4)	6.94(−4)
			0.3	2.72(−4)	8.63(−4)
			0.5	1.28(−6)	1.58(−6)
			0.7	2.74(−4)	8.66(−4)
			0.9	2.20(−4)	6.96(−4)
		0.1	0.1	1.20(−3)	1.37(−3)
			0.3	2.81(−3)	3.29(−3)
			0.5	6.59(−3)	7.21(−3)
			0.7	1.60(−2)	1.77(−2)
			0.9	1.43(−2)	1.57(−2)

$\frac{1}{12}$; time steps = 100 for $x = 0.1, 0.3, 0.5, 0.7, 0.9$. Figure 2 shows the graphical comparison between analytical and approximate solution for $h = \frac{1}{64}$, $r = \frac{1}{100}$ and time steps = 100.

Example 3 (Non-linear single soliton case) Consider a propagation of single solitary wave of non-linear KdV Eq. (1.4) with $\varepsilon = 6$, $\mu = 1$ [5, 15, 23, 27] and

$$y(x,0) = \frac{\kappa}{2}\sec h^2\left(\frac{\sqrt{\kappa}}{2}x - L\right), \quad a \le x \le b.$$

The analytical solution is

$$y(x,t) = \frac{\kappa}{2}\sec h^2\left(\frac{\sqrt{\kappa}}{2}(x-\kappa t) - L\right).$$

The functions $\gamma_3(t)$ and $\gamma_4(t)$ are extracted from the analytical solution. The computational results are tabulated in Tables 5–7. The L_∞, L_2 and *RMS* errors with $L = 7$; $\kappa = 0.5$; $[a,b] = [0,40]$; $p = 25$; $k = 0.001, 0.0001$; $n = 40, 80, 120, 160, 200$; $\sigma = \frac{1}{12}$ and $t = 1$, and the comparison of L_∞ error with [15, 23, 27] with changes $p = 30$; $k = 0.01, 0.001$; $n = 200$; $t = 1, 2, \ldots, 5$ are tabulated in Table 5 and Table 6, respectively. Table 7 shows L_∞, L_2 and *RMS* errors and comparison with [5] with $L = 10$; $\kappa = 0.14$; $[a,b] = [30,80]$; $p = 100$; $k = 0.001$; $h = 0.2$; $\sigma = \frac{1}{12}$ and $t = 1, 3, 5, 7, 10$. Figure 3 shows the graphical comparison between analytical and approximate solution for $n = 200$, $k = 0.001$, $p = 25$, and $t = 5$.

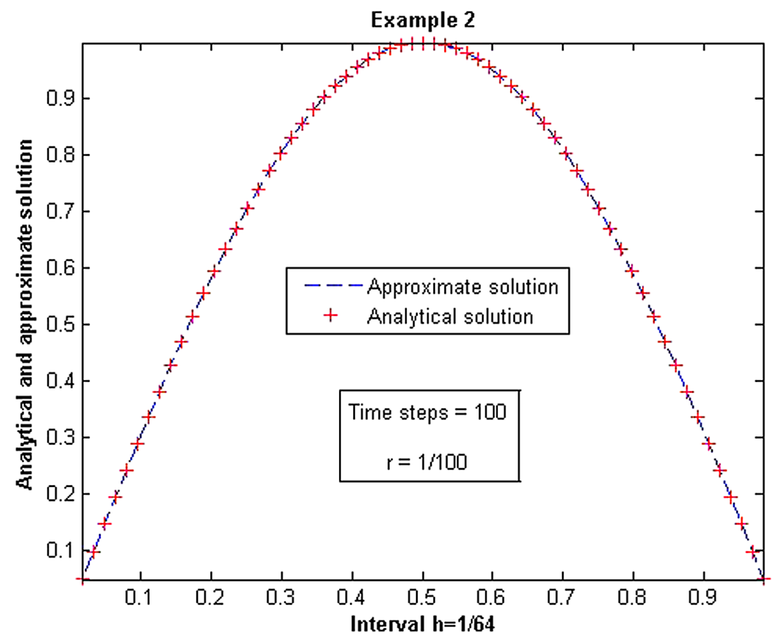

Figure 2 Comparison between analytical and approximate solution

Table 5 L_∞, L_2 and *RMS* errors for $p = 25$, $\sigma = \frac{1}{12}$, $t = 1$, $\kappa = 0.5$, $L = 7$, $[a,b] = [0,40]$ for Example 3

n	$k = 0.001$			$k = 0.0001$		
	L_∞	L_2	*RMS*	L_∞	L_2	*RMS*
40	6.1192(–5)	1.1111(–4)	1.7793(–5)	6.7755(–6)	1.3143(–5)	2.1045(–6)
80	7.3626(–5)	1.7866(–4)	2.0101(–5)	7.3800(–6)	1.7870(–5)	2.0106(–6)
120	6.5572(–5)	1.9993(–4)	1.8327(–5)	6.5602(–6)	1.9994(–5)	1.8328(–6)
160	5.1806(–5)	2.1081(–4)	1.6718(–5)	5.1798(–6)	2.1082(–5)	1.6719(–6)
200	4.8556(–5)	2.5697(–4)	1.8216(–5)	4.8559(–6)	2.5697(–5)	1.8216(–6)

Table 6 Comparison of L_∞ error with [5, 15, 23, 27] for $p = 30$, $\sigma = \frac{1}{12}$, $\kappa = 0.5$, $L = 7$, $n = 200$, $[a,b] = [0,40]$ for Example 3

t	Our method	IMQQI [23]	Our method	[5]	MQQI [27]	MQ [15]	IMQ [15]
	$k = 0.01$		$k = 0.001$				
1	4.2123(–4)	1.6728(–4)	4.2120(–5)	1.8048(–5)	1.5259(–3)	1.7923(–5)	6.9584(–5)
2	4.2238(–4)	2.3758(–4)	4.2235(–5)	3.0373(–5)	2.8677(–3)	3.0151(–5)	1.9553(–4)
3	4.2123(–4)	2.3758(–4)	4.2120(–5)	4.0088(–5)	4.1428(–3)	3.9839(–5)	3.8286(–3)
4	4.2237(–4)	3.1348(–4)	4.2234(–5)	4.8347(–5)	5.3859(–3)	4.7835(–5)	5.9098(–3)
5	4.2126(–4)	3.4136(–4)	4.2122(–5)	5.6090(–5)	6.8141(–3)	5.4599(–5)	8.3667(–3)

Example 4 (Non-linear soliton interaction case) Consider a propagation of two solitary waves of non-linear KdV Eq. (1.4) with $\varepsilon = 6$, $\mu = 1$ [5, 15, 23, 27] and

$$y(x,0) = 12\left[\frac{3 + 4\cos h(2x) + \cos h(4x)}{\{3\cos h(x) + \cos h(3x)\}^2}\right], \quad -5 \le x \le 15.$$

The analytical solution is

$$y(x,t) = 12\left[\frac{3 + 4\cos h(2x - 8t) + \cos h(4x - 64t)}{\{3\cos h(x - 28t) + \cos h(3x - 36t)\}^2}\right].$$

Table 7 L_∞, L_2 and *RMS* errors for $p = 100$, $\sigma = \frac{1}{12}$, $k = 0.001$, $\kappa = 0.14$, $L = 10$, $n = 250$, $[a, b] = [30, 80]$ and comparison of L_∞ error with [5] for Example 3

t	Our method			[5]
	L_∞	L_2	*RMS*	L_∞
1	1.8313(–6)	1.4082(–5)	8.9242(–7)	6.8860(–6)
3	1.8310(–6)	1.4082(–5)	8.9243(–7)	8.5988(–6)
5	1.8308(–6)	1.4082(–5)	8.9244(–7)	8.3958(–6)
7	1.8306(–6)	1.4083(–5)	8.9244(–7)	9.2090(–6)
10	1.8314(–6)	1.4082(–5)	8.9243(–7)	8.5625(–6)

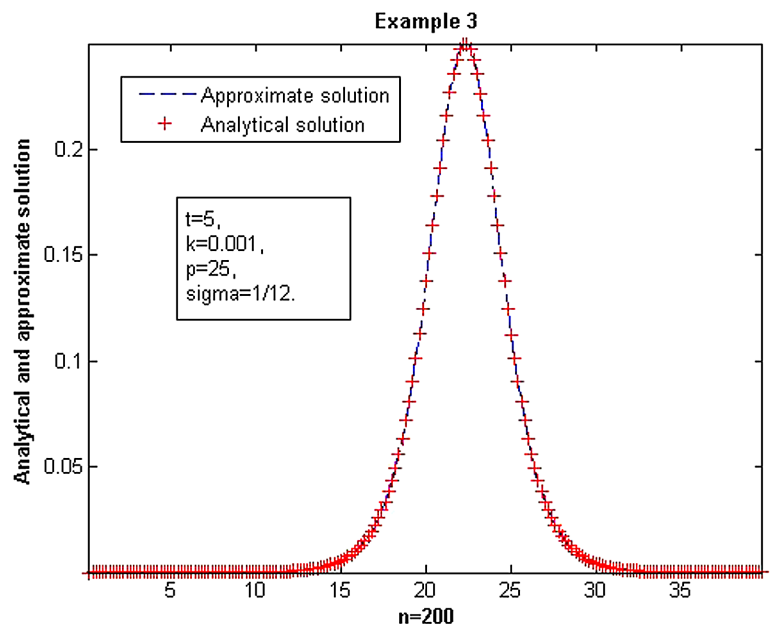

Figure 3 Comparison between analytical and approximate solution

Table 8 Errors for $p = 10$, $\sigma = \frac{1}{12}$, $k = 0.00001$, $h = 0.1$ and comparison of L_∞ error with [5, 15, 27] for Example 4

t	Our method			[5]	MQQI [27]	MQ [15]	IMQ [15]
	L_∞	L_2	*RMS*	L_∞			
0.01	3.9805(–4)	1.6355(–3)	1.1594(–4)	–	7.7405(–3)	9.2114(–4)	2.2071(–2)
0.05	1.7221(–3)	4.4270(–3)	3.1382(–4)	–	6.3762(–2)	2.9608(–2)	7.2316(–2)
0.1	2.8226(–3)	6.8615(–3)	4.8640(–4)	5.6353(–3)	1.6196(–1)	1.2806(–2)	1.0121(–1)
0.2	3.5029(–3)	8.2930(–3)	5.8787(–4)	2.3376(–2)	–	–	–
0.3	3.5726(–3)	8.4542(–3)	5.9930(–4)	5.9437(–2)	–	–	–

Similarly, the functions $\gamma_3(t)$ and $\gamma_4(t)$ are extracted from the analytical solution. The computational results are tabulated in Tables 8–10. Table 8 and Table 9 show the comparison of L_∞, L_2 and *RMS* errors with [5, 15, 23, 27] with $p = 10$; $k = 0.0001, 0.00001$; $n = 200$; $\sigma = \frac{1}{12}$ and $t = 0.01, 0.05, 0.10, 0.15, 0.20, 0.3$. Also, the L_∞, L_2 and *RMS* errors with $p = 100$; $k = 0.0001$; $n = 40, 80, 120, 160, 200$; $\sigma = \frac{1}{12}$ and $t = 1$ are tabulated in Table 10. Figure 4 shows the graphical comparison between analytical and approximate solution for $n = 200$, $k = 0.0001$, $p = 25$, and $t = 3$.

Table 9 Comparison of L_∞, L_2 and *RMS* errors with [23] for $p = 10$, $\sigma = \frac{1}{12}$, $k = 0.0001$, $n = 200$ for Example 4

t	Our method			[23]		
	L_∞	L_2	*RMS*	L_∞	L_2	*RMS*
0.01	4.0120(−3)	1.6335(−4)	1.1579(−3)	4.0579(−3)	9.9105(−3)	6.9903(−4)
0.05	1.7249(−2)	4.4265(−2)	3.1378(−3)	4.1003(−2)	1.0295(−1)	7.2619(−3)
0.10	6.5572(−2)	6.8612(−2)	4.8638(−3)	9.1691(−2)	2.3373(−1)	1.6486(−2)
0.15	5.1806(−2)	7.9154(−2)	5.6111(−3)	1.3257(−1)	3.4201(−1)	2.4124(−2)
0.20	4.8556(−2)	8.2929(−2)	5.8786(−3)	1.6644(−1)	4.3607(−1)	3.0758(−2)

Table 10 L_∞, L_2 and *RMS* errors for $p = 100$, $\sigma = \frac{1}{12}$, $k = 0.0001$, $t = 1$ for Example 4

n	L_∞	L_2	*RMS*
40	5.4535(−2)	2.0145(−4)	3.2258(−2)
80	8.9607(−3)	4.8109(−2)	5.4127(−3)
120	3.5368(−3)	2.3639(−2)	2.1670(−3)
160	2.5034(−3)	1.8749(−2)	1.4869(−3)
200	2.3476(−3)	1.8440(−2)	1.3072(−3)

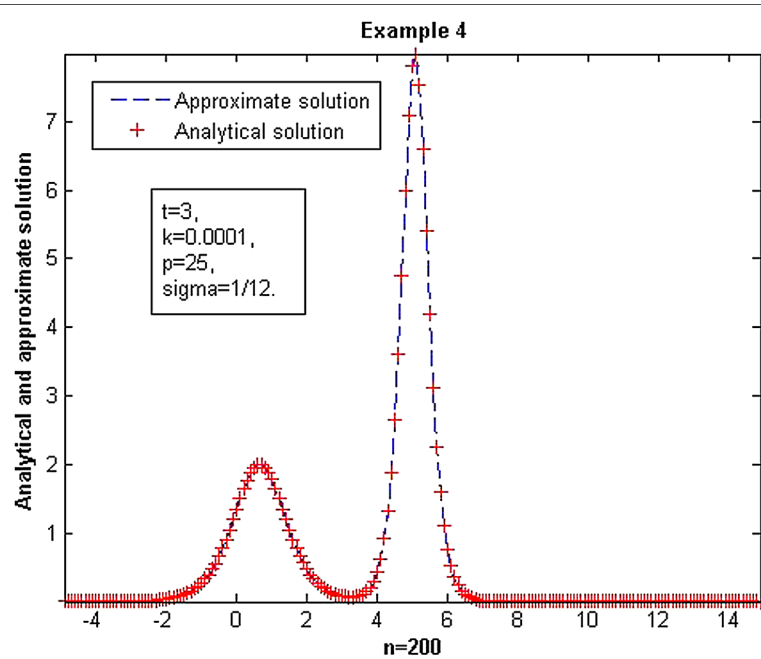

Figure 4 Comparison between analytical and approximate solution

7 Conclusion

The class of a new three-level implicit methods has been obtained using exponential quartic spline for numerical approximation of third order linear and non-linear dispersive partial differential equations and is tested on four examples using MATLAB. The performance of these methods have been examined for different values of parameters. Having compared the solutions with available results in the literature, we found them to be better. The comparison between analytical and approximate solutions is also shown graphically in Figs. 1–4. Tables and figures show the feasibility and applicability of our method.

Funding
Jamia Millia Islamia, New Delhi, India.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors drafted the manuscript, and they read and approved the final version.

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Received: 16 May 2018 Accepted: 15 August 2018 Published online: 10 September 2018

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