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Outer synchronization of a class of mixed delayed complex networks based on pinning control

Lin Shi¹, Guiling Chen^{1*} and Shouming Zhong²

*Correspondence:

glchen2016@163.com

¹School of Mathematics, Southwest Jiaotong University, Chengdu, P.R. China

Full list of author information is available at the end of the article

Abstract

In this paper, the problem on outer synchronization is investigated for a class of mixed delayed complex networks by using the pinning control strategy. Together with some Lyapunov–Krasovskii functional and effective mathematical techniques, several conditions are derived to guarantee a class of complex networks with mixed delays to be outer synchronization. By proposing a novel functional condition which has not been proposed so far, further improved synchronization criteria are proposed. Finally, two examples are given to illustrate the effectiveness of the results.

Keywords: Synchronization; Complex networks; LMI approach; Time-varying delays; Pinning control

1 Introduction

Complex networks [1, 2] lie everywhere in our daily life, such as the Internet, world wide web, communication networks, social networks, genetic regulatory networks, power grid networks, and so on. In 1990, controlling chaos was proposed, and it was shown that one can convert a chaotic attractor to any one of a large number of possible attracting time-periodic motions via making only small time-dependent perturbations of an available system parameter [1]. Synchronization of complex networks has become an active field of research [3–10], because the synchronization mechanism can better explain many natural phenomena, including the synchronous information exchange in the Internet and world wide web, and the synchronous transfer of digital or analog signals in communication networks. In 1998, focusing on the transition from a regular lattice to a random graph, Watts and Strogatz introduced an interesting model, i.e., the small-world network [4].

From the control strategy point of view, all kinds of control approaches have been proposed in the field of the synchronization of complex networks. These approaches can be divided into two classes in general. One is based on the continuous updating feedback signals, for instance time-delay feedback control [11], adaptive control [12, 13], and nonlinear feedback control [14]. The other kind is based on the discrete signals updated at instant times, such as sampled-data control [15] and impulsive control [16]. The traditional method to synchronize a complex network is to add a controller to each of the network nodes to tame the dynamics to approach a desired synchronization trajectory. However, a complex network is normally composed of a large number of high-dimensional nodes, and

it is expensive and infeasible to control all the neurons. Motivated by this practical consideration, the idea of controlling a small portion of nodes, named pinning control, was introduced in [17]. Pinning control, as a feasible strategy, could drive networks of coupled oscillators onto some desired trajectory. Generally speaking, we apply pinning control by adding a feedback control input on a fraction of networks nodes. Though we just exert the direct control on such pinned nodes, it can be propagated to the rest of the oscillators through the coupling among the nodes. Many pinning algorithms have been reported for the synchronization of dynamical networks (see [18–23]). [18] investigated the effects of control strength on nonlinearly coupled systems in the process of synchronization. Wu and Fu [21] studied the cluster mixed synchronization of complex networks with community structure and nonidentical nodes by using some linear pinning control schemes. It turned out that the pinning control method reduces the control cost to a certain extent by reducing the amount of controllers added to the nodes.

Up to now, all sorts of different synchronization phenomena, such as complete synchronization [24, 25], generalized synchronization [26], phase synchronization [27], cluster synchronization [28], exponential synchronization [29] etc., have been reported in the literature. Complete synchronization means that the coupled chaotic systems remain in step with each other in the course of time. In coupled systems with identical elements (i.e., each component having the same dynamics and parameter set), we can observe complete synchronization. Complete synchronization can be regarded as inner synchronization. Except the aforementioned inner synchronization behavior, there exists another synchronization phenomenon named outer synchronization, which occurs between two or more coupled networks [30]. In reality, many practical systems, such as the different species development in balance [31], the groups of *Drosophila* clock neurons [32], and the spread of diseases, such as SARS and bird flu, between two communities [33], can be used to illustrate the outer synchronization phenomenon between two networks. Due to its theoretical and practical importance, outer synchronization between two dynamical networks has drawn much attention in recent years [34–38]. In [34], the outer synchronization was studied between two delay-coupled complex dynamical networks with nonidentical topological structures and a noise perturbation. Zheng et al. [35] investigated the adaptive synchronization between two nonlinearly delay-coupled complex networks with the bidirectional actions and nonidentical topological structures. Outer synchronization between drive and response networks via adaptive impulsive pinning control was investigated in [36]. The problem of pinning outer synchronization was considered between two delayed complex networks with nonlinear coupling [37]. In [38], an aperiodically adaptive intermittent control scheme combined with the pinning strategy was proposed for outer synchronization between two general complex delayed dynamical networks.

Motivated by the above discussions, in this paper, we study the outer synchronization of complex dynamical networks with mixed time delayed coupling. Combining the pinning control method and the linear matrix inequality technique, some sufficient conditions for synchronization of complex networks with time-varying delays are derived. By proposing a novel activation functional condition, further improved outer synchronization criteria are proposed. Finally, two examples are given to illustrate the effectiveness of the proposed methods.

This paper is organized as follows. The network model is introduced and some necessary lemmas are given in Sect. 2. Section 3 discusses the outer synchronization of the complex

dynamical networks with mixed delayed coupling by the pinning control method. Corresponding synchronization criteria for guaranteeing synchronization are obtained. The theoretical results are verified numerically by several representative examples in Sect. 4. Finally, this paper is concluded in Sect. 5.

Notation Throughout this paper, \mathbb{R}^n denotes the n -dimensional Euclidean space and $\mathbb{R}^{n \times n}$ is the set of all $n \times n$ real matrices. For symmetric matrices X and Y , the notation $X > Y$ ($X \geq Y$) means that the matrix $X - Y$ is positive definite (nonnegative). $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} = \begin{bmatrix} X & Y \\ * & Z \end{bmatrix}$ with $*$ denoting the symmetric term in a symmetric matrix.

2 Preliminaries

Consider a complex dynamical network consisting of N identical nodes with linearly coupling network with mixed delays:

$$\dot{x}_i(t) = Ax_i(t) + Bf(x_i(t)) + Cf(x_i(t - \tau(t))) + D \int_{t-\sigma}^t f(x_i(s)) ds + \sum_{j=1}^N h_{ij} \Gamma x_j(t), \tag{1}$$

where $i = 1, 2, \dots, N$, $x_i(t) = (x_{i1}(t), \dots, x_{in}(t))^T \in \mathbb{R}^n$ is the state vector of node i , $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is continuously differentiable, $f(x_i(t)) = [f_1(x_{i1}(t)), \dots, f_n(x_{in}(t))]^T$. Here $A = \text{diag}\{a_1, a_2, \dots, a_n\} < 0$. The matrices $B = (b_{ij})_{n \times n}$ and $C = (c_{ij})_{n \times n}$ are weight and delayed weight matrices, respectively. The matrix $D = (d_{ij})_{n \times n}$ is a distributed delayed weight matrix. σ is a distributed delay. The matrix $\Gamma = \text{diag}\{\gamma_1, \gamma_2, \dots, \gamma_n\} \in \mathbb{R}^{n \times n}$ is an inner coupling matrix of complex networks and it is a positive definite matrix. $H = (h_{ij})_{N \times N}$ is an outer coupling matrix representing the complex network topology defined as follows: If there is a connection between nodes i and j ($j \neq i$), then $h_{ij} \neq 0$; otherwise, $h_{ij} = 0$ ($j \neq i$) and the diagonal elements of matrix H are defined by $h_{ii} = -\sum_{j=1, j \neq i}^N h_{ij}$.

Assumption 1 Here $\tau(t)$ denotes the interval time-varying delay satisfying

$$0 \leq \tau_0 \leq \tau(t) \leq \tau_m, \quad \mu_0 \leq \dot{\tau}(t) \leq \mu_m < +\infty, \quad \text{and} \quad \bar{\tau} = \tau_m - \tau_0. \tag{2}$$

Assumption 2

$$l_i^- \leq \frac{f_i(x) - f_i(y)}{x - y} \leq l_i^+, \quad \forall x, y \in \mathbb{R}, x \neq y, i = 1, 2, \dots, n, \tag{3}$$

where $l_i^-, l_i^+, i = 1, 2, \dots, n$, are constants and $L_1 = \text{diag}(l_1^-, l_2^-, \dots, l_n^-)$, $L_2 = \text{diag}(l_1^+, l_2^+, \dots, l_n^+)$.

Remark 1 The constants l_i^- and l_i^+ in Assumption 2 are allowed to be positive, negative, or zero. This assumption is weaker than Assumption 2 in [20], which is the special case of the function satisfying condition (3).

We refer to model (1) as the drive-coupled complex dynamical network. Correspondingly, the response complex network with controller $u_i(t)$, $i = 1, 2, \dots, N$, can be written

as

$$\begin{aligned} \dot{y}_i(t) &= Ay_i(t) + Bf(y_i(t)) + Cf(y_i(t - \tau(t))) \\ &\quad + D \int_{t-\sigma}^t f(y_i(s)) ds + \sum_{j=1}^N h_{ij} \Gamma y_j(t) + u_i(t), \end{aligned} \tag{4}$$

$$u_i(t) = \begin{cases} -d(y_i(t) - x_i(t)), & i = 1, 2, \dots, l; \\ 0, & i = l + 1, \dots, N, \end{cases} \tag{5}$$

where $d = \text{diag}(d_1, d_2, \dots, d_n) > 0$ is a positive definite feedback gain matrix, and $1 \leq l \leq N$.

Definition 2.1 The drive-response networks (1) and (4) are said to achieve outer synchronization if

$$\lim_{t \rightarrow \infty} \|y_i(t) - x_i(t)\| = 0, \quad i = 1, 2, \dots, N.$$

We define outer synchronization errors between the drive network (1) and the response network (4) as $e_i(t) = y_i(t) - x_i(t)$.

Then, the following error dynamical network can be obtained:

$$\begin{aligned} \dot{e}_i(t) &= Ae_i(t) + Bg(e_i(t)) + Cg(e_i(t - \tau(t))) \\ &\quad + D \int_{t-\sigma}^t g(e_i(s)) ds + \sum_{j=1}^N h_{ij} \Gamma e_j(t) - de_i(t), \end{aligned} \tag{6}$$

where $i = 1, 2, \dots, l, g(e_i(t)) = f(y_i(t)) - f(x_i(t)), g(e_i(t)) = [g_1(e_{i1}(t)), \dots, g_n(e_{in}(t))]$. According to inequality (3), one can obtain

$$l_i^- \leq \frac{g_i(x)}{x} \leq l_i^+, \quad g_i(0) = 0, \quad \forall x \in R, i = 1, 2, \dots, n. \tag{7}$$

Lemma 2.1 (Jensen’s inequality) *For any constant matrix $X \in \mathfrak{R}^{n \times n}, X = X^T > 0$, two scalars $h_2 \geq h_1 > 0$ such that the following integration is well defined, then*

$$-(h_2 - h_1) \int_{t-h_2}^{t-h_1} x^T(s) X x(s) ds \leq - \left(\int_{t-h_2}^{t-h_1} x(s) ds \right)^T X \left(\int_{t-h_2}^{t-h_1} x(s) ds \right).$$

Lemma 2.2 *For any constant matrix $Z \in \mathfrak{R}^{n \times n}, Z = Z^T > 0$, two scalars $\tau_m \geq \tau_0 > 0$ such that the following integration is well defined, then*

$$\begin{aligned} & - \int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t \dot{x}^T(s) Z \dot{x}(s) ds \\ & \leq \frac{2(1 - \bar{\tau})}{\tau_m + \tau_0} x^T(t) Z x(t) - \frac{2(1 - \bar{\tau})}{\tau_m^2 - \tau_0^2} \int_{t-\tau_m}^{t-\tau_0} x^T(s) ds Z \int_{t-\tau_m}^{t-\tau_0} x(s) ds. \end{aligned}$$

Proof

$$\begin{bmatrix} \dot{x}^T(s) Z \dot{x}(s) & \dot{x}^T(s) \\ \dot{x}(s) & Z^{-1} \end{bmatrix} \geq 0.$$

Integration of the above inequality from $t + \theta$ to t , where $-\tau_m \leq \theta \leq -\tau_0$, yields

$$\begin{bmatrix} \int_{t+\theta}^t \dot{x}^T(s)Z\dot{x}(s) ds & \int_{t+\theta}^t \dot{x}^T(s) ds \\ \int_{t+\theta}^t \dot{x}(s) ds & -\theta Z^{-1} \end{bmatrix} \geq 0.$$

Integration from $-\tau_m$ to $-\tau_0$ yields

$$\begin{bmatrix} \int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t \dot{x}^T(s)Z\dot{x}(s) ds d\theta & \int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t \dot{x}^T(s) ds d\theta \\ \int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t \dot{x}(s) ds d\theta & \int_{-\tau_m}^{-\tau_0} -\theta Z^{-1} d\theta \end{bmatrix} \geq 0.$$

According to Schur complements,

$$\begin{aligned} & - \int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t \dot{x}^T(s)Z\dot{x}(s) ds d\theta \\ & \leq - \frac{2}{\tau_m^2 - \tau_0^2} \left(\int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t \dot{x}^T(s) ds d\theta \right) Z \left(\int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t \dot{x}(s) ds d\theta \right) \\ & = - \frac{2}{\tau_m^2 - \tau_0^2} \left[(\tau_m - \tau_0)x^T(t) - \int_{t-\tau_m}^{t-\tau_0} x^T(s) ds \right] Z \left[(\tau_m - \tau_0)x(t) - \int_{t-\tau_m}^{t-\tau_0} x(s) ds \right] \\ & = - \frac{2(\tau_m - \tau_0)^2}{\tau_m^2 - \tau_0^2} x^T(t)Zx(t) + \frac{4}{\tau_m + \tau_0} x^T(t)Z \int_{t-\tau_m}^{t-\tau_0} x(s) ds \\ & \quad - \frac{2}{\tau_m^2 - \tau_0^2} \int_{t-\tau_m}^{t-\tau_0} x^T(s) ds Z \int_{t-\tau_m}^{t-\tau_0} x(s) ds. \end{aligned}$$

Applying the inequality $2X^T PY \leq X^T PX + Y^T PY$, one can obtain

$$\begin{aligned} & - \int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t \dot{x}^T(s)Z\dot{x}(s) ds d\theta \\ & \leq - \frac{2(\tau_m - \tau_0)}{\tau_m + \tau_0} x^T(t)Zx(t) - \frac{2}{\tau_m^2 - \tau_0^2} \int_{t-\tau_m}^{t-\tau_0} x^T(s) ds Z \int_{t-\tau_m}^{t-\tau_0} x(s) ds \\ & \quad + \frac{2}{\tau_m + \tau_0} x^T(t)Zx(t) + \frac{2}{\tau_m + \tau_0} \int_{t-\tau_m}^{t-\tau_0} x^T(s) ds Z \int_{t-\tau_m}^{t-\tau_0} x(s) ds \\ & = \frac{2(1 - \bar{\tau})}{\tau_m + \tau_0} x^T(t)Zx(t) - \frac{2(1 - \bar{\tau})}{\tau_m^2 - \tau_0^2} \int_{t-\tau_m}^{t-\tau_0} x^T(s) ds Z \int_{t-\tau_m}^{t-\tau_0} x(s) ds. \quad \square \end{aligned}$$

3 Main results

In this section, by using a Lyapunov–Krasovskii functional, new outer synchronization criteria for mixed delayed complex networks will be proposed.

Theorem 3.1 *Under Assumptions 1 and 2, for given matrices $L_1 = \text{diag}\{l_1^-, l_2^-, \dots, l_n^-\}$, $L_2 = \text{diag}\{l_1^+, l_2^+, \dots, l_n^+\}$, and scalars $\tau_m, \tau_0, \mu_0, \mu_m$, the complex networks (1) and (4) can achieve outer synchronization if there exist positive definite matrices $P, Q, R_1, R_2, X_1, X_2, Y, Z$, positive diagonal matrices $W_i = \text{diag}\{w_{1i}, \dots, w_{ni}\}, i = 1, 2, \dots, 8, \Lambda = \text{diag}\{\lambda_1, \lambda_2, \dots, \lambda_n\}$, and $\Delta = \text{diag}\{\delta_1, \delta_2, \dots, \delta_n\}$ such that the following LMIs hold:*

$$F = \begin{pmatrix} 2(p_k + l_k^+ \delta_k - l_k^- \lambda_k)H - w_k^8 I_N & (\lambda_k - \theta_k)\gamma_k H & w_k^5 \gamma_k H \\ * & -w_k^7 I_N & 0 \\ * & * & -w_k^6 I_N \end{pmatrix} < 0 \tag{8}$$

and

$$E = \begin{pmatrix} E_{11} & 0 & 0 & 0 & E_{15} & 0 & E_{17} & 0 & 0 & 0 & E_{1,11} & E_{1,12} \\ * & E_{22} & 0 & 0 & E_{25} & E_{26} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & E_{33} & 0 & 0 & 0 & 0 & E_{38} & 0 & 0 & 0 & 0 \\ * & * & * & E_{44} & 0 & 0 & E_{47} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & E_{55} & 0 & E_{57} & 0 & 0 & 0 & E_{5,11} & E_{5,12} \\ * & * & * & * & * & E_{66} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & E_{77} & 0 & 0 & 0 & 0 & E_{7,12} \\ * & * & * & * & * & * & * & E_{88} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & E_{99} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & E_{10,10} & 0 & E_{10,12} \\ * & * & * & * & * & * & * & * & * & * & E_{11,11} & 0 \\ * & * & * & * & * & * & * & * & * & * & * & E_{12,12} \end{pmatrix} < 0, \tag{9}$$

where

$$\begin{aligned} E_{11} &= 2P(A - d) + Q + 2(L_2\Delta - L_1\Lambda)(A - d) + \tau_0^2 X_1 + \bar{\tau}^2 X_2 \\ &\quad + \frac{2(1 - \bar{\tau})}{\tau_m + \tau_0} Z - 2L_1 W_1 L_2 + W_8, \\ E_{15} &= PB + (\Lambda - \Delta)(A - d) + (L_2\Delta - L_1\Lambda)B + L_1 W_1 + L_2 W_1, \\ E_{17} &= PC + (L_2\Delta - L_1\Lambda)C, \quad E_{1,11} = PD + (L_2\Delta - L_1\Lambda)D, \quad E_{1,12} = (A - d)W_5, \\ E_{22} &= R_2 - 2L_1 W_2 L_2, \quad E_{26} = L_2 W_2 + L_1 W_1, \\ E_{33} &= -Q - R_1 - 2L_1 W_4 L_2, \quad E_{38} = L_1 W_4 + L_2 W_4, \\ E_{44} &= (1 - \mu_0)R_1 - (1 - \mu_m)R_2 - 2L_1 W_3 L_2, \quad E_{47} = L_1 W_3 + L_2 W_3, \\ E_{55} &= 2(\Lambda - \Delta)B + \sigma^2 Y - 2W_1 + W_7, \quad E_{57} = (\Lambda - \Delta)C, \quad E_{5,11} = (\Lambda - \Delta)D, \\ E_{5,12} &= BW_5, \quad E_{66} = -2W_2, \quad E_{77} = -2W_3, \quad E_{7,12} = -CW_5, \\ E_{88} &= -2W_4, \quad E_{99} = -X_1, \\ E_{10,10} &= -X_2 - \frac{2(1 - \bar{\tau})}{\tau_m^2 - \tau_0^2} Z, \quad E_{10,12} = DW_5, \\ E_{11,11} &= -\sigma Y, \quad \text{and} \quad E_{12,12} = \frac{\tau_m^2 - \tau_0^2}{2} Z - 2W_5 + W_6. \end{aligned}$$

Proof Construct a Lyapunov functional as follows:

$$V(e_t) = \sum_{i=1}^7 V_i(e_t),$$

where

$$V_1(e_t) = \sum_{i=1}^N e_i^T(t) P e_i(t),$$

$$\begin{aligned}
 V_2(e_t) &= \sum_{i=1}^N \int_{t-\tau_m}^t e_i^T(s) Q e_i(s) ds, \\
 V_3(e_t) &= 2 \sum_{i=1}^N \sum_{j=1}^n \lambda_i \int_0^{e_{ij}(t)} (g_j(s) - l_j^* s) ds + 2 \sum_{i=1}^N \sum_{j=1}^n \delta_i \int_0^{e_{ij}(t)} (l_j^* s - g_j(s)) ds, \\
 V_4(e_t) &= \sum_{i=1}^N \int_{t-\tau_m}^{t-\tau(t)} e_i^T(s) R_1 e_i(s) ds + \sum_{i=1}^N \int_{t-\tau(t)}^{t-\tau_0} e_i^T(s) R_2 e_i(s) ds, \\
 V_5(e_t) &= \tau_0 \sum_{i=1}^N \int_{-\tau_0}^0 \int_{t+\theta}^t e_i^T(s) X_1 e_i(s) ds d\theta + \bar{\tau} \sum_{i=1}^N \int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t e_i^T(s) X_2 e_i(s) ds d\theta, \\
 V_6(e_t) &= \sigma \sum_{i=1}^N \int_{-\sigma}^0 \int_{t+\theta}^t g^T(e_i(s)) Y g(e_i(s)) ds d\theta,
 \end{aligned}$$

and

$$V_7(e_t) = \sum_{i=1}^N \int_{-\tau_m}^{-\tau_0} \int_{\theta}^0 \int_{t+\lambda}^t \dot{e}_i^T(s) Z \dot{e}_i(s) ds d\lambda d\theta.$$

The time derivative of $V(e_t)$ along the trajectory of system (6) is given by

$$\begin{aligned}
 \dot{V}_1(e_t) &= 2 \sum_{i=1}^N e_i^T(t) P \dot{e}_i(t) \\
 &= 2 \sum_{i=1}^N e_i^T(t) P \left[A e_i(t) + B g(e_i(t)) + C g(e_i(t - \tau(t))) + D \int_{t-\sigma}^t g(e_i(s)) ds - d e_i(t) \right] \\
 &\quad + 2 \sum_{i=1}^N e_i^T(t) P \sum_{j=1}^N h_{ij} \Gamma e_j(t),
 \end{aligned} \tag{10}$$

$$\dot{V}_2(e_t) = \sum_{i=1}^N e_i^T(t) Q e_i(t) - \sum_{i=1}^N e_i^T(t - \tau_m) Q e_i(t - \tau_m), \tag{11}$$

$$\begin{aligned}
 \dot{V}_3(e_t) &= 2 \sum_{i=1}^N \left[g^T(e_i(t)) (\Lambda - \Delta) + e_i^T(t) (L_2 \Delta - L_1 \Lambda) \right] \dot{e}_i(t) \\
 &= 2 \sum_{i=1}^N g^T(e_i(t)) (\Lambda - \Delta) \left[(A - d) e_i(t) + B g(e_i(t)) + C g(e_i(t - \tau(t))) \right. \\
 &\quad \left. + D \int_{t-\sigma}^t g(e_i(s)) ds \right] \\
 &\quad + 2 \sum_{i=1}^N e_i^T(t) (L_2 \Delta - L_1 \Lambda) \left[(A - d) e_i(t) + B g(e_i(t)) + C g(e_i(t - \tau(t))) \right. \\
 &\quad \left. + D \int_{t-\sigma}^t g(e_i(s)) ds \right] + 2 \sum_{i=1}^N g^T(e_i(t)) (\Lambda - \Delta) \sum_{j=1}^N h_{ij} \Gamma e_j(t) \\
 &\quad + 2 \sum_{i=1}^N e_i^T(t) (L_2 \Delta - L_1 \Lambda) \sum_{j=1}^N h_{ij} \Gamma e_j(t),
 \end{aligned} \tag{12}$$

$$\begin{aligned}
 \dot{V}_4(e_t) &= \sum_{i=1}^N [(1 - \dot{\tau}(t))e_i^T(t - \tau(t))R_1e_i(t - \tau(t)) - e_i^T(t - \tau_m)R_1e_i(t - \tau_m) \\
 &\quad + e_i^T(t - \tau_0)R_2e_i(t - \tau_0) - (1 - \dot{\tau}(t))e_i^T(t - \tau(t))R_2e_i(t - \tau(t))] \\
 &\leq \sum_{i=1}^N [(1 - \mu_0)e_i^T(t - \tau(t))R_1e_i(t - \tau(t)) - e_i^T(t - \tau_m)R_1e_i(t - \tau_m) \\
 &\quad + e_i^T(t - \tau_0)R_2e_i(t - \tau_0) - (1 - \mu_m)e_i^T(t - \tau(t))R_2e_i(t - \tau(t))]. \tag{13}
 \end{aligned}$$

By employing Lemma 2.1, one can obtain

$$\begin{aligned}
 \dot{V}_5(e_t) &= \tau_0^2 \sum_{i=1}^N e_i^T(t)X_1e_i(t) - \tau_0 \sum_{i=1}^N \int_{t-\tau_0}^t e_i^T(s)X_1e_i(s) ds \\
 &\quad + \sum_{i=1}^N \bar{\tau}^2 e_i^T(t)X_2e_i(t) - \sum_{i=1}^N \bar{\tau} \int_{t-\tau_m}^{t-\tau_0} e_i^T(s)X_2e_i(s) ds \\
 &\leq \tau_0^2 \sum_{i=1}^N e_i^T(t)X_1e_i(t) - \sum_{i=1}^N \left(\int_{t-\tau_0}^t e_i^T(s) ds \right) X_1 \left(\int_{t-\tau_0}^t e_i(s) ds \right) \\
 &\quad + \bar{\tau}^2 \sum_{i=1}^N e_i^T(t)X_2e_i(t) - \sum_{i=1}^N \int_{t-\tau_m}^{t-\tau_0} e_i^T(s) ds X_2 \int_{t-\tau_m}^{t-\tau_0} e_i(s) ds, \tag{14}
 \end{aligned}$$

$$\begin{aligned}
 \dot{V}_6(e_t) &= \sigma^2 \sum_{i=1}^N g^T(e_i(t))Yg(e_i(t)) - \sigma \sum_{i=1}^N \int_{t-\sigma}^t g^T(e_i(s))Yg(e_i(s)) ds \\
 &\leq \sigma^2 \sum_{i=1}^N g^T(e_i(t))Yg(e_i(t)) - \sum_{i=1}^N \int_{t-\sigma}^t g^T(e_i(s)) ds Y \int_{t-\sigma}^t g(e_i(s)) ds. \tag{15}
 \end{aligned}$$

According to Lemma 2.2, one can obtain

$$\begin{aligned}
 \dot{V}_7(e_t) &= \frac{\tau_m^2 - \tau_0^2}{2} \sum_{i=1}^N \dot{e}_i^T(t)Z\dot{e}_i(t) - \sum_{i=1}^N \int_{-\tau_m}^{-\tau_0} \int_{t+\theta}^t \dot{e}_i^T(s)Z\dot{e}_i(s) ds d\theta \\
 &\leq \frac{\tau_m^2 - \tau_0^2}{2} \sum_{i=1}^N \dot{e}_i^T(t)Z\dot{e}_i(t) + \frac{2(1 - \bar{\tau})}{\tau_m + \tau_0} \sum_{i=1}^N e_i^T(t)Ze_i(t) \\
 &\quad - \frac{2(1 - \bar{\tau})}{\tau_m^2 - \tau_0^2} \sum_{i=1}^N \int_{t-\tau_m}^{t-\tau_0} e_i^T(s) ds Z \int_{t-\tau_m}^{t-\tau_0} e_i(s) ds. \tag{16}
 \end{aligned}$$

According to (7), there exist positive diagonal matrices $W_i = \text{diag}(w_{1i}, \dots, w_{ni})$, $i = 1, 2, \dots, 8$, such that the following inequalities hold:

$$\begin{aligned}
 0 &\leq -2 \sum_{j=1}^n \sum_{i=1}^n w_{i1} [g_i(e_{ij}(t)) - l_i^- e_{ij}(t)] [g_i(e_{ij}(t)) - l_i^+ e_{ij}(t)] \\
 &\quad - 2 \sum_{j=1}^n \sum_{i=1}^n w_{i2} [g_i(e_{ij}(t - \tau_0)) - l_i^- e_{ij}(t - \tau_0)] [g_i(e_{ij}(t - \tau_0)) - l_i^+ e_{ij}(t - \tau_0)]
 \end{aligned}$$

$$\begin{aligned}
 & -2 \sum_{j=1}^n \sum_{i=1}^n w_{i3} [g_i(e_{ij}(t - \tau(t))) - l_i^- e_{ij}(t - \tau(t))] [g_i(e_{ij}(t - \tau_0)) - l_i^+ e_{ij}(t - \tau(t))] \\
 & -2 \sum_{j=1}^n \sum_{i=1}^n w_{i4} [g_i(e_{ij}(t - \tau_m)) - l_i^- e_{ij}(t - \tau_m)] [g_i(e_{ij}(t - \tau_m)) - l_i^+ e_{ij}(t - \tau_m)], \tag{17}
 \end{aligned}$$

and

$$\begin{aligned}
 0 = & 2 \sum_{i=1}^N \dot{e}_i^T(t) W_5 \left[-\dot{e}_i(t) + (A - d)e_i(t) + Bg(e_i(t)) + Cg(e_i(t - \tau(t))) \right. \\
 & \left. + D \int_{t-\sigma}^t g(e_i(s)) ds + \sum_{j=1}^N h_{ij} \Gamma e_j(t) \right]. \tag{18}
 \end{aligned}$$

Let $\tilde{e}_k(t) = [\tilde{e}_{1k}, \tilde{e}_{2k}, \dots, \tilde{e}_{Nk}]$, $i = 1, 2, \dots, n$. Then

$$\begin{aligned}
 0 = & \sum_{i=1}^N [\dot{e}_i^T(t) W_6 \dot{e}_i(t) + g^T(e_i(t)) W_7 g(e_i(t)) + e_i^T(t) W_8 e_i(t)] \\
 & - \sum_{i=1}^N [w_k^6 \dot{\tilde{e}}_k^T(t) \dot{\tilde{e}}_k(t) + w_k^7 g^T(\tilde{e}_k(t)) g(\tilde{e}_k(t)) + w_k^8 \tilde{e}_k^T(t) \tilde{e}_k(t)]. \tag{19}
 \end{aligned}$$

Combining with the terms in (10)–(19), we can get

$$\dot{V}(e_i) \leq \sum_{i=1}^N \xi_i^T(t) E \xi_i(t) + \sum_{k=1}^n \zeta_k^T(t) F \zeta_k(t),$$

where

$$\begin{aligned}
 \xi_i^T(t) = & [e_i^T(t) \quad e_i^T(t - \tau_0) \quad e_i^T(t - \tau_m) \quad e_i^T(t - \tau(t)) \quad g^T(e_i(t)) \quad g^T(e_i(t - \tau_0)) \\
 & g^T(e_i(t - \tau(t))) \quad g^T(e_i(t - \tau_m)) \quad \int_{t-\tau_0}^t e_i^T(s) ds \quad \int_{t-\tau_m}^{t-\tau_0} e_i^T(s) ds \\
 & \int_{t-\sigma}^t g^T(e_i(s)) ds \quad \dot{e}^T(t)], \\
 \zeta_k^T(t) = & [\tilde{e}_k^T(t) \quad g^T(\tilde{e}_k(t)) \quad \dot{\tilde{e}}_k^T(t)].
 \end{aligned}$$

Based on LMIs (8) and (9), one can guarantee $\dot{V}(e_i) < 0$ to be true. Based on the theory of Lyapunov–Krasovskii stability theorem, the controlled networks (1) and (4) can achieve the desired synchronization and the proof is completed. \square

Next, we derive an improved result for the dynamical networks (1) and (4).

Theorem 3.2 *Under Assumptions 1 and 2, for given matrices $L_1 = \text{diag}(l_1^-, l_2^-, \dots, l_n^-)$, $L_2 = \text{diag}(l_1^+, l_2^+, \dots, l_n^+)$, and scalars τ_m, τ_0 , the complex networks (1) and (4) can achieve outer synchronization if there exist symmetric positive definite matrices $P, Q, R_1, R_2, X_1, X_2, Y, Z$, positive diagonal matrices $w_i = \text{diag}(k_{1i}, \dots, w_{ni})$, $i = 5, 6, 7, 8$, $K_i = \text{diag}(k_{1i}, \dots, k_{ni})$, $i = 1, 2, \dots, 8$, $\Lambda = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, and $\Delta = \text{diag}(\delta_1, \delta_2, \dots, \delta_n)$ such that the following LMIs*

hold:

$$F = \begin{pmatrix} 2(p_k + l_k^+ \delta_k - l_k^- \lambda_k)H - w_k^8 I_N & (\lambda_k - \theta_k)\gamma_k H & w_k^5 \gamma_k H \\ * & -w_k^7 I_N & 0 \\ * & * & -w_k^6 I_N \end{pmatrix} < 0, \tag{20}$$

$$\Pi = (\Pi_{ij})_{n \times n} < 0, \quad i, j = 1, \dots, 12, \tag{21}$$

and

$$\Omega = (\Omega_{ij})_{n \times n} < 0, \quad i, j = 1, \dots, 12, \tag{22}$$

where

$$\begin{aligned} \Pi_{11} &= 2P(A - d) + Q + 2(L_2\Delta - L_1\Lambda)(A - d) + \tau_0^2 X_1 + \bar{\tau}^2 X_2 \\ &\quad + \frac{2(1 - \bar{\tau})}{\tau_m + \tau_0} Z - L_1 K_1 (L_1 + L_2) + W_8, \\ \Pi_{15} &= PB + (\Lambda - \Delta)(A - d) + (L_2\Delta - L_1\Lambda)B + L_1 K_1 + \frac{L_1 + L_2}{2} K_1, \\ \Pi_{17} &= E_{17}, \quad \Pi_{1,11} = E_{1,11}, \quad \Pi_{1,12} = E_{1,12} \\ \Pi_{22} &= R_2 - L_1 K_2 (L_1 + L_2), \quad \Pi_{26} = \frac{L_1 + L_2}{2} K_2 + L_1 K_1, \\ \Pi_{33} &= -Q - R_1 - L_1 K_4 (L_1 + L_2), \quad \Pi_{38} = L_1 K_4 + \frac{L_1 + L_2}{2} K_4, \\ \Pi_{44} &= (1 - \mu_0)R_1 - (1 - \mu_m)R_2 - L_1 K_3 (L_1 + L_2), \quad \Pi_{47} = L_1 K_3 + \frac{L_1 + L_2}{2} K_3, \\ \Pi_{55} &= 2(\Lambda - \Delta)B + \sigma^2 Y - 2K_1 + W_7, \quad \Pi_{57} = E_{57}, \quad \Pi_{5,11} = E_{57}, \\ \Pi_{5,12} &= E_{5,12}, \quad \Pi_{66} = -2K_2, \quad \Pi_{77} = -2K_3, \quad \Pi_{7,12} = -CW_5, \\ \Pi_{88} &= -2K_4, \quad \Pi_{99} = -X_1, \\ \Pi_{10,10} &= -X_2 - \frac{2(1 - \bar{\tau})}{\tau_m^2 - \tau_0^2} Z, \quad \Pi_{10,12} = DW_5, \\ \Pi_{11,11} &= -\sigma Y, \quad \Pi_{12,12} = \frac{\tau_m^2 - \tau_0^2}{2} Z - 2W_5 + W_6, \\ \Omega_{11} &= 2P(A - d) + Q + 2(L_2\Delta - L_1\Lambda)(A - d) + \tau_0^2 X_1 + \bar{\tau}^2 X_2 \\ &\quad + \frac{2(1 - \bar{\tau})}{\tau_m + \tau_0} Z - (L_1 + L_2)K_5 L_2 + W_8, \\ \Omega_{15} &= PB + (\Lambda - \Delta)(A - d) + (L_2\Delta - L_1\Lambda)B + \frac{L_1 + L_2}{2} K_5 + L_2 K_5, \\ \Omega_{17} &= E_{17}, \quad \Omega_{1,11} = E_{1,11}, \quad \Omega_{1,12} = E_{1,12} \\ \Omega_{22} &= R_2 - (L_1 + L_2)K_6 L_2, \quad \Omega_{26} = L_2 K_6 + \frac{L_1 + L_2}{2} K_5, \\ \Omega_{33} &= -Q - R_1 - (L_1 + L_2)K_7 L_2, \quad \Omega_{38} = \frac{L_1 + L_2}{2} K_7 + L_2 K_7, \\ \Omega_{44} &= (1 - \mu_0)R_1 - (1 - \mu_m)R_2 - (L_1 + L_2)K_6 L_2, \quad \Omega_{47} = \frac{L_1 + L_2}{2} K_6 + L_2 K_6, \end{aligned}$$

$$\begin{aligned} \Omega_{55} &= 2(\Lambda - \Delta)B + \sigma^2 Y - 2K_4 + W_7, & \Omega_{57} &= E_{57}, & \Omega_{5,11} &= E_{57}, \\ \Omega_{5,12} &= E_{5,12}, & \Omega_{66} &= -2K_5, & \Omega_{77} &= -2K_6, & \Omega_{7,12} &= -CW_5, \\ \Omega_{88} &= -2W_7, & \Omega_{99} &= -X_1, \\ \Omega_{10,10} &= -X_2 - \frac{2(1 - \bar{\tau})}{\tau_m^2 - \tau_0^2} Z, & \Omega_{10,12} &= DW_5, \\ \Omega_{11,11} &= -\sigma Y, & \text{and } \Omega_{12,12} &= \frac{\tau_m^2 - \tau_0^2}{2} Z - 2W_5 + W_6. \end{aligned}$$

Proof For positive definite matrices $P, Q, R_1, R_2, X_1, X_2, Y, Z$, let us consider the same Lyapunov functional $V(e_t) = \sum_{i=1}^7 V_i(e_t)$ proposed in Theorem 3.1.

Here, we consider inequality (17) to two cases.

Case 1:

$$l_i^- \leq \frac{g_i(x)}{x} \leq \frac{l_i^- + l_i^+}{2}.$$

This condition is equivalent to

$$[g_i(x) - l_i^- x] \left[g_i(x) - \frac{l_i^- + l_i^+}{2} x \right] < 0, \quad i = 1, \dots, n. \tag{23}$$

For any positive diagonal matrices $K_i = \text{diag}(k_{i1}, \dots, k_{in}), i = 1, 2, 3, 4$, the following inequality holds:

$$\begin{aligned} 0 &\leq -2 \sum_{j=1}^n \sum_{i=1}^n k_{1i} [g_i(e_{ij}(t)) - l_i^- e_{ij}(t)] \left[g_i(e_{ij}(t)) - \frac{l_i^- + l_i^+}{2} e_{ij}(t) \right] \\ &\quad - 2 \sum_{j=1}^n \sum_{i=1}^n k_{2i} [g_i(e_{ij}(t - \tau_0)) - l_i^- e_{ij}(t - \tau_0)] \left[g_i(e_{ij}(t - \tau_0)) - \frac{l_i^- + l_i^+}{2} e_{ij}(t - \tau_0) \right] \\ &\quad - 2 \sum_{j=1}^n \sum_{i=1}^n k_{3i} [g_i(e_{ij}(t - \tau(t))) - l_i^- e_{ij}(t - \tau(t))] \\ &\quad \times \left[g_i(e_{ij}(t - \tau(t))) - \frac{l_i^- + l_i^+}{2} e_{ij}(t - \tau(t)) \right] \\ &\quad - 2 \sum_{j=1}^n \sum_{i=1}^n k_{4i} [g_i(e_{ij}(t - \tau_m)) - l_i^- e_{ij}(t - \tau_m)] \\ &\quad \times \left[g_i(e_{ij}(t - \tau_m)) - \frac{l_i^- + l_i^+}{2} e_{ij}(t - \tau_m) \right]. \end{aligned} \tag{24}$$

Then, from the proof of Theorem 3.1, when $l_i^- \leq \frac{g_i(x)}{x} \leq \frac{l_i^- + l_i^+}{2}$, an upper bound of $\dot{V}(e_t)$ can be obtained

$$\dot{V}(e_t) \leq \sum_{i=1}^N \xi_i^T(t) \Pi \xi_i(t) + \sum_{k=1}^n \zeta_k^T(t) F \zeta_k(t) < 0. \tag{25}$$

Based on the theory of Lyapunov–Krasovskii stability, (20) and (21), the controlled networks (1) can achieve synchronization when $l_i^- \leq \frac{g_i(x)}{x} \leq \frac{l_i^- + l_i^+}{2}$.

Case 2:

$$\frac{l_i^- + l_i^+}{2} \leq \frac{g_i(x)}{x} \leq k_i^+.$$

This condition is equivalent to

$$\left[g_i(x) - \frac{l_i^- + l_i^+}{2} x \right] [g_i(x) - l_i^+ x] < 0, \quad i = 1, \dots, n. \tag{26}$$

For any positive diagonal matrices $K_i = \text{diag}(k_{i1}, \dots, k_{in})$, $i = 5, 6, 7, 8$, the following inequality holds:

$$\begin{aligned} 0 \leq & -2 \sum_{i=1}^N \sum_{j=1}^n k_{5i} \left[g_i(e_{ij}(t)) - \frac{l_i^- + l_i^+}{2} e_{ij}(t) \right] [g_i(e_{ij}(t)) - l_i^+ e_{ij}(t)] \\ & - 2 \sum_{i=1}^N \sum_{j=1}^n k_{6i} \left[g_i(e_{ij}(t - \tau_0)) - \frac{l_i^- + l_i^+}{2} e_{ij}(t - \tau_0) \right] [g_i(e_{ij}(t - \tau_0)) - l_i^+ e_{ij}(t - \tau_0)] \\ & - 2 \sum_{i=1}^N \sum_{j=1}^n k_{7i} \left[g_i(e_{ij}(t - \tau(t))) - \frac{l_i^- + l_i^+}{2} e_{ij}(t - \tau(t)) \right] \\ & \times [g_i(e_{ij}(t - \tau(t))) - l_i^+ e_{ij}(t - \tau(t))] \\ & - 2 \sum_{i=1}^N \sum_{j=1}^n k_{8i} \left[g_i(e_{ij}(t - \tau_m)) - \frac{l_i^- + l_i^+}{2} e_{ij}(t - \tau_m) \right] \\ & \times [g_i(e_{ij}(t - \tau_m)) - l_i^+ e_{ij}(t - \tau_m)]. \end{aligned} \tag{27}$$

Then, from the proof of Theorem 3.1, when $l_i^- \leq \frac{g_i(x)}{x} \leq \frac{l_i^- + l_i^+}{2}$, an upper bound of $\dot{V}(e_t)$ can be obtained

$$\dot{V}(e_t) \leq \sum_{i=1}^N \xi_i^T(t) \Omega \xi_i(t) + \sum_{k=1}^n \zeta_k^T(t) F \zeta_k(t) < 0. \tag{28}$$

Based on the theory of Lyapunov–Krasovskii stability, (20) and (22), the controlled networks (1) can achieve synchronization when $\frac{l_i^- + l_i^+}{2} \leq \frac{g_i(x)}{x} \leq l_i^+$. □

Remark 2 In Theorem 3.2, by choosing (x, y) in (3) as $(e_i(t - \tau_0), e_i(t - \tau(t)))$ and $(e_i(t - \tau(t)), e_i(t - \tau_m))$ at each subinterval $l_i^- \leq \frac{g_i(x)}{x} \leq \frac{l_i^- + l_i^+}{2}$ and $\frac{l_i^- + l_i^+}{2} \leq \frac{g_i(x)}{x} \leq k_i^+$, respectively, more information on cross terms among the states $g_i(e_i(t - \tau_0)), g_i(e_i(t - \tau(t))), g_i(e_i(t - \tau_m)), e_i(t - \tau_0), e_i(t - \tau(t))$, and $e_i(t - \tau_m)$ will be utilized, which may lead to less conservative stability criteria.

Remark 3 In [39], they only investigated the pinning synchronization in neural networks with discrete time-varying delay. In this paper, we consider a complex network with discrete and distributed delays. For the coupling matrix H , we need $h_{ij} \neq 0$, but in [14] they need $h_{ij} > 0$ or $h_{ij} \geq 0$. So our results improve and generalize the works in [14, 39].

4 Illustrative example

In this section, a numerical example is given to demonstrate the effectiveness of the proposed methods.

Example 1 Consider a two-dimensional network with both discrete and distributed delays:

$$\dot{x}_i(t) = Ax_i(t) + Bf(x_i(t)) + Cf(x_i(t - \tau(t))) + D \int_{t-\sigma}^t f(x_i(s)) ds, \tag{29}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^T$ is the state vector of the networks, $f(x_i(t)) = (\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^T$ is the activation functions vector, $\tau(t) = |\sin(t)|$. Taking

$$A = \begin{pmatrix} -1.5 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -2.5 & -0.3 \\ 6 & 5 \end{pmatrix},$$

$$C = \begin{pmatrix} -3 & 2 \\ 0.3 & -8 \end{pmatrix}, \quad D = \begin{pmatrix} -0.5 & -1 \\ -3.6 & -0.5 \end{pmatrix},$$

the dynamical behavior of (29) with the initial condition $x_1(0) = [0.2, 0.5]^T$, $x_2(0) = [-0.6, 0.1]^T$, Fig. 1 displays the chaotic attractor.

In this simulation, we consider the two-dimensional error system

$$\dot{e}_i(t) = Ae_i(t) + Bg(e_i(t)) + Cg(e_i(t - \tau(t))) + D \int_{t-\sigma}^t g(e_i(s)) ds + \sum_{j=1}^{20} h_{ij} \Gamma e_j(t) - de_i(t), \tag{30}$$

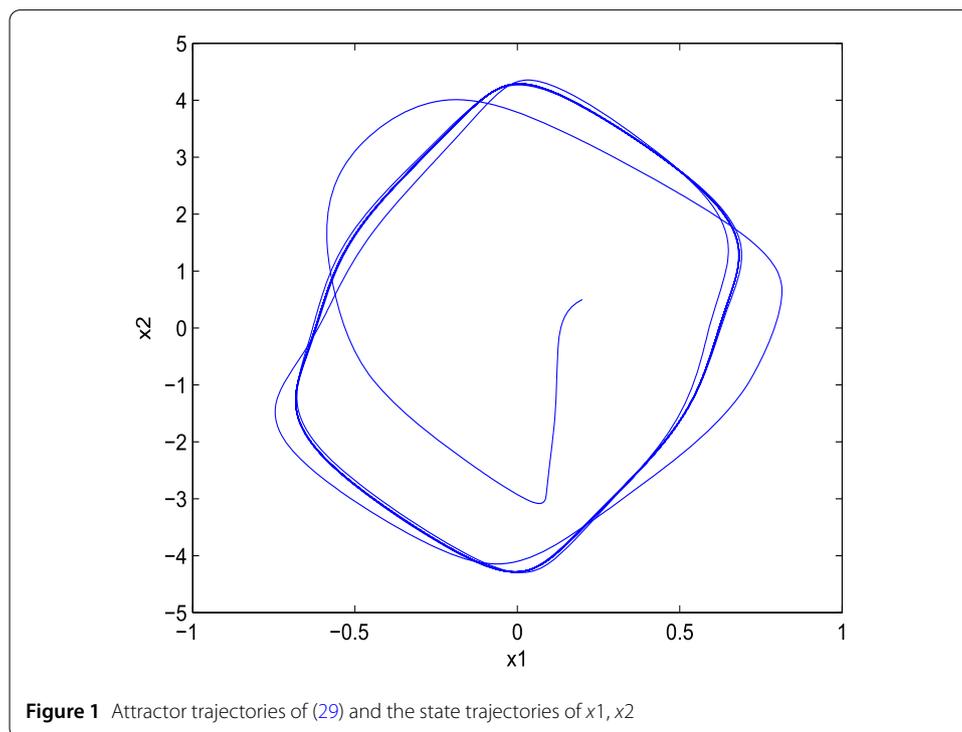


Figure 1 Attractor trajectories of (29) and the state trajectories of x_1, x_2

where $e_i(t) = (e_{i1}(t), e_{i2}(t))^T$ is the state vector of the i th neural network. Here $i = 1, 2, \dots, 10$, which means we just choose 10 nodes to control. We set the parameters to be

$$H = \begin{pmatrix} H_{11} & H_{11} \\ H_{11} & H_{11} \end{pmatrix}, \quad H_{11} = \begin{pmatrix} -9 & 1 & 1 & 1 & 1 \\ 1 & -9 & 1 & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & -9 & 1 \\ 1 & 1 & \dots & 1 & -9 \end{pmatrix}_{10 \times 10},$$

$$\Gamma = I, \quad \sigma = 0.5, \quad g(x_i(t)) = (\tanh(x_{i1}(t)), \tanh(x_{i2}(t)))^T, \quad d = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Here, we just choose the control gains d_i to be 2. If it is selected as too high value, it is easy to synchronize the systems. Figure 2 shows the drive system $x_i(t)$ state orbits which are unstable.

The evolution of the synchronization errors $e_i(t)$ ($i = 1, 2, \dots, 20$) is depicted in Fig. 3. It shows that the error system can be outer synchronization under pinning controller.

Then by resorting to LMI in Matlab Toolbox, Theorem 3.1 can guarantee the outer synchronization of the dynamical networks, and we can derive the feasible solution to LMIs in (8) and (9) as follows:

$$P = \begin{pmatrix} 0.3345 & 0.0331 \\ 0.0331 & 0.1210 \end{pmatrix}, \quad Q = \begin{pmatrix} 3.7550 & 0.0007 \\ 0.0007 & 3.7551 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} 0.7754 & -0.0676 \\ -0.0676 & 0.8167 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -0.1286 & 0.0064 \\ 0.0064 & -0.1433 \end{pmatrix},$$

$$X_1 = \begin{pmatrix} 0.1541 & -0.0063 \\ -0.0063 & 0.1414 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0.1566 & -0.0078 \\ -0.0078 & 0.1407 \end{pmatrix},$$

$$Y = \begin{pmatrix} 0.7120 & 0.0843 \\ 0.0843 & 0.7254 \end{pmatrix}, \quad Z = \begin{pmatrix} 0.2000 & -0.0011 \\ -0.0011 & 0.1971 \end{pmatrix},$$

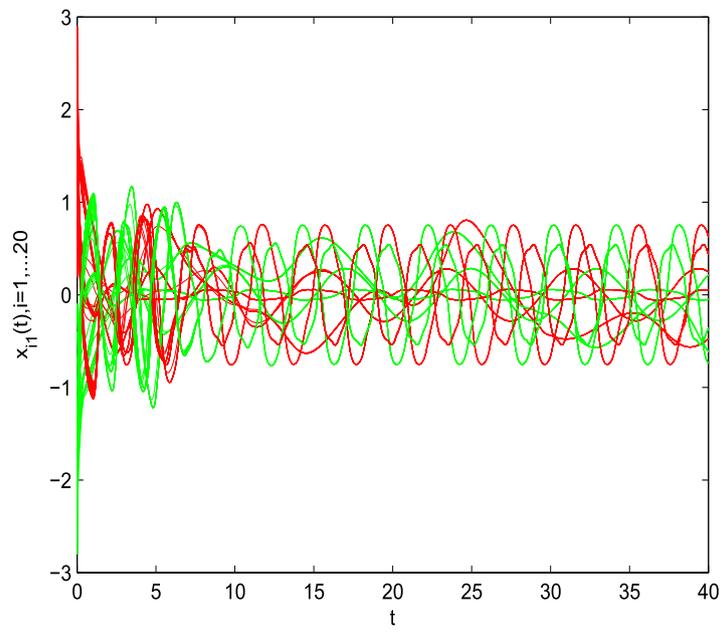
$$W_1 = \begin{pmatrix} 0.3272 & -0.0058 \\ -0.0058 & 0.3363 \end{pmatrix}, \quad W_2 = \begin{pmatrix} 0.7126 & 0.2715 \\ 0.2715 & 0.6910 \end{pmatrix},$$

$$W_3 = \begin{pmatrix} 0.8899 & -0.0479 \\ -0.0479 & 0.9368 \end{pmatrix}, \quad W_4 = \begin{pmatrix} 2.3118 & 0.0008 \\ 0.0008 & 2.3119 \end{pmatrix},$$

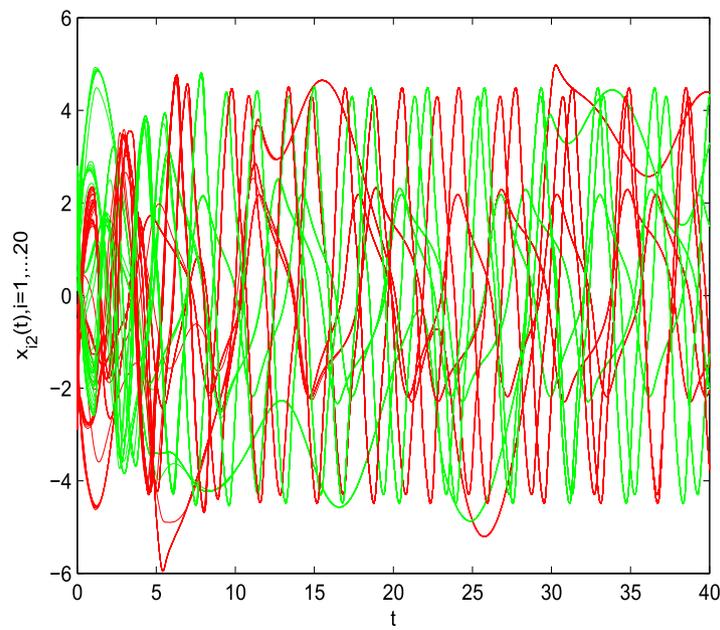
$$W_5 = \begin{pmatrix} 0.4599 & -0.0685 \\ -0.0685 & 0.1909 \end{pmatrix}, \quad W_6 = \begin{pmatrix} -0.4803 & 0.0001 \\ 0.0001 & -0.4813 \end{pmatrix},$$

$$W_7 = \begin{pmatrix} 0.2768 & 0.3642 \\ 0.3642 & 0.3453 \end{pmatrix}, \quad W_8 = \begin{pmatrix} 0.4511 & -0.0316 \\ -0.0316 & 0.3875 \end{pmatrix},$$

$$\Lambda = \begin{pmatrix} -0.6757 & -0.2486 \\ -0.2486 & -0.6525 \end{pmatrix}, \quad \Delta = \begin{pmatrix} -0.5372 & -0.1231 \\ -0.1231 & -0.3499 \end{pmatrix}.$$

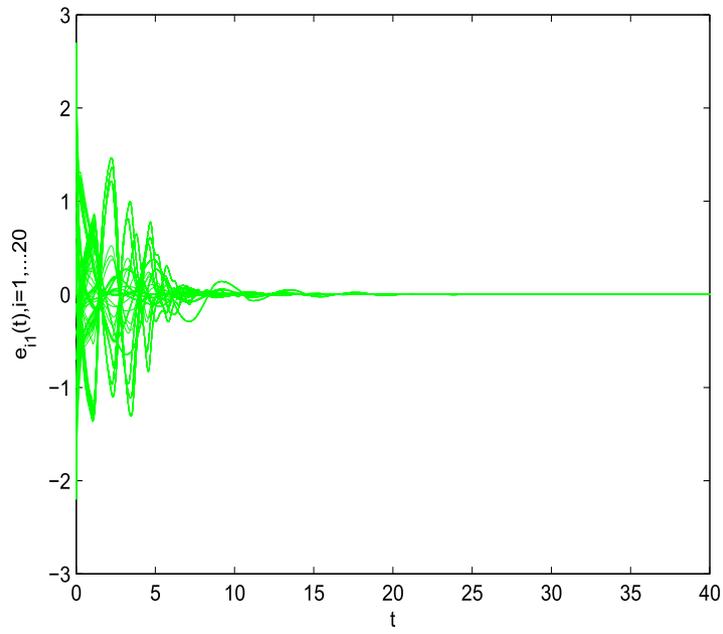


(a)

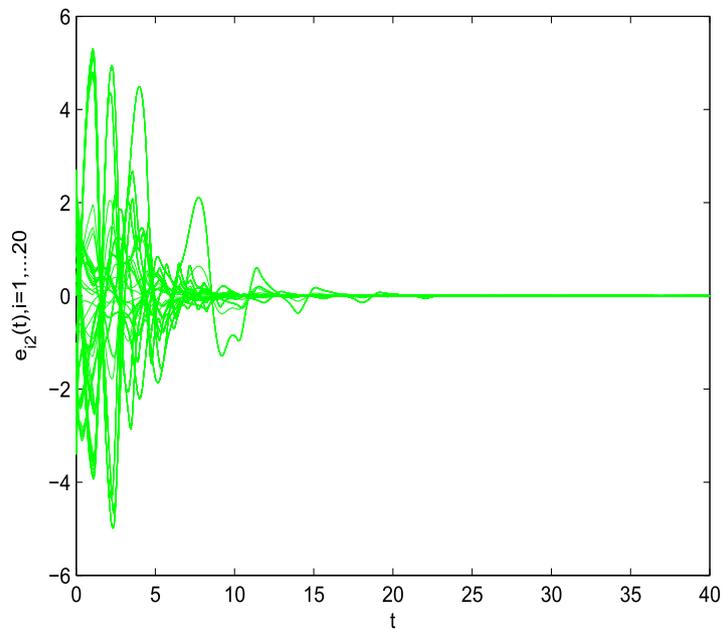


(b)

Figure 2 The state trajectories of x_{i1}, x_{i2} in system (29)



(a)



(b)

Figure 3 The evolution of $e_1(t), e_2(t)$ in system (30) under pinning control

From these simulations, one can conclude that using the proposed method in this paper, outer synchronization of systems (1) and (4) can be achieved.

5 Conclusion

In Theorem 3.1, by choosing a new Lyapunov functional, some new synchronization condition for complex networks with mixed time-varying delays has been proposed. Based on the results of Theorem 3.1, by constructing new inequalities of activation functions, further improved stability criteria are proposed in Theorem 3.2. Finally, a numerical example has been given to illustrate the effectiveness of the proposed methods. It should be mentioned that the given method can be further studied and can be extended to fractional-order systems and discrete complex systems.

Acknowledgements

The authors thank the anonymous referee for careful reading and some helpful comments.

Funding

The work is supported by the Fundamental Research Funds for the Central Universities (Grant No. 2682017CX068 and 2682017CX059), and the National Natural Science Foundation of China (Grant No. 11701475).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

Author details

¹School of Mathematics, Southwest Jiaotong University, Chengdu, P.R. China. ²School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, P.R. China.

Publisher's Note

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Received: 28 January 2018 Accepted: 3 August 2018 Published online: 18 September 2018

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