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Sums of finite products of Legendre and Laguerre polynomials

Taekyun Kim[1], Dae San Kim[2], Dmitry V. Dolgy[3] and Jin-Woo Park[4*]

*Correspondence:
a0417001@knu.ac.kr
[4]Department of Mathematics Education, Daegu University, Gyeongsan-si, Republic of Korea
Full list of author information is available at the end of the article

Abstract

In this paper, we study sums of finite products of Legendre and Laguerre polynomials and derive Fourier series expansions of functions associated with them. From these Fourier series expansions, we are going to express those sums of finite products as linear combinations of Bernoulli polynomials. Further, by using a method other than Fourier series expansions, we will be able to express those sums in terms of Euler polynomials.

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1 Introduction and preliminaries

The Coulomb potential can be written as a series in Legendre polynomials $P_n(x)$ $(n \geq 0)$. They satisfy the orthogonality relation

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{n,m},$$

and are solutions to the Legendre equation

$$\left(1-x^2\right)P_n''(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0.$$

We let the reader refer to [1] for further applications of Legendre polynomials.

The Laguerre polynomials $L_n(x)$ have important applications to the solution of Schrödinger's equation for the hydrogen atom. They are orthogonal over the interval $[0,\infty)$ with the weight function e^{-x}, namely

$$\int_{0}^{\infty} L_n(x)L_m(x)e^{-x}\,dx = \delta_{n,m},$$

and are solution to the Laguerre equation

$$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0.$$

The Legendre polynomials $P_n(x)$ and Laguerre polynomials $L_n(x)$ are respectively defined by the recurrence relations as in the following (see [2–4]):

$$\begin{aligned} &(n+2)P_{n+2}(x) = (2n+3)xP_{n+1}(x) - (n+1)P_n(x) \quad (n \geq 0), \\ &P_0(x) = 1, \qquad P_1(x) = x, \end{aligned} \tag{1.1}$$

$$\begin{aligned} &(n+2)L_{n+2}(x) = (2n+3-x)L_{n+1}(x) - (n+1)L_n(x) \quad (n \geq 0), \\ &L_0(x) = 1, \qquad L_1(x) = -x+1. \end{aligned} \tag{1.2}$$

Both $P_n(x)$ and $L_n(x)$ are polynomials of degree n with rational coefficients.

From (1.1) and (1.2), it can be easily seen that the generating functions for $P_n(x)$ and $L_n(x)$ are given by (see [2–4])

$$F(t,x) = \left(1 - 2xt + t^2\right)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1.3}$$

$$G(t,x) = (1-t)^{-1}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)t^n. \tag{1.4}$$

As is well known, the *Bernoulli polynomials* $B_n(x)$ are given by

$$\frac{t}{e^t - 1}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}. \tag{1.5}$$

For any real number x, we let

$$\langle x \rangle = x - [x] \in [0,1) \tag{1.6}$$

denote the fractional part of x, where $[x]$ is the greatest integer $\leq x$.

We recall here that

(a) for $m \geq 2$,

$$B_m\left(\langle x \rangle\right) = -m! \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi inx}}{(2\pi in)^m}; \tag{1.7}$$

(b)

$$-\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{e^{2\pi inx}}{2\pi in} = \begin{cases} B_1(\langle x \rangle), & \text{for } x \in \mathbb{R} - \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z}. \end{cases} \tag{1.8}$$

For any integers m, r with $m, r \geq 1$, we put

$$\alpha_{m,r}(x) = \sum_{i_1+i_2+\cdots+i_{2r+1}=m} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2r+1}}(x), \tag{1.9}$$

where the sum runs over all nonnegative integers $i_1, i_2, \ldots, i_{2r+1}$ with $i_1 + i_2 + \cdots + i_{2r+1} = m$.

Then we will consider the function $\alpha_{m,r}(\langle x\rangle)$ and derive its Fourier series expansions. As an immediate corollary to these Fourier series expansions, we will be able to express $\alpha_{m,r}(x)$ as a linear combination of Bernoulli polynomials $B_n(x)$. We state this here as Theorem A.

Theorem A *For any integers m, r with $m, r \geq 1$, we let*

$$\Delta_{m,r} = \frac{1}{(2r-1)!!2^{m+r}} \sum_{k=0}^{[\frac{m-1}{2}]} (-1)^k \binom{m+r}{k} \binom{2m+2r-2k}{m+r} (m+r-2k)_r.$$

Then we have the identity

$$\sum_{i_1+\cdots+i_{2r+1}=m} P_{i_1}(x) \cdots P_{i_{2r+1}}(x) = \frac{1}{2r-1} \sum_{j=0}^{m} \frac{(2r+2j-3)!!}{(2r-3)!!j!} \Delta_{m-j+1,r+j-1} B_j(x). \tag{1.10}$$

Here $(x)_r = x(x-1)\cdots(x-r+1)$ for $r \geq 1$, $(x)_0 = 1$, $(2n-1)!! = (2n-1)(2n-3)\cdots 1$ for $n \geq 1$, and $(-1)!! = 1$.

Also, for any integers m, r with $m \geq 1$, $r \geq 0$, we put

$$\beta_{m,r}(x) = \sum_{i_1+\cdots+i_{r+1}=m} L_{i_1}\left(\frac{x}{r+1}\right) L_{i_2}\left(\frac{x}{r+1}\right) \cdots L_{i_{r+1}}\left(\frac{x}{r+1}\right), \tag{1.11}$$

where the sum is over all nonnegative integers $i_1, i_2, \ldots, i_{r+1}$ with $i_1 + \cdots + i_{r+1} = m$.

Then we will study the function $\beta_{m,r}(\langle x\rangle)$ and obtain its Fourier series expansions. Again, as a corollary to these, we can express $\beta_{m,r}(x)$ in terms of Bernoulli polynomials. Here our result is as follows.

Theorem B *For any integers m, r with $m \geq 1$, $r \geq 0$, we let*

$$\Omega_{m,r} = (-1)^m \sum_{k=0}^{m-1} \frac{(-1)^k}{(m-k)!} \binom{m+r}{k}.$$

Then the following identity holds:

$$\begin{aligned} &\sum_{i_1+\cdots+i_{r+1}=m} L_{i_1}\left(\frac{x}{r+1}\right) L_{i_2}\left(\frac{x}{r+1}\right) \cdots L_{i_{r+1}}\left(\frac{x}{r+1}\right) \\ &\quad = -\sum_{j=0}^{m} \frac{(-1)^j}{j!} \Omega_{m-j+1,r+j-1} B_j(x). \end{aligned} \tag{1.12}$$

Here we note that neither $P_n(x)$ nor $L_n(x)$ is an Appell polynomial, whereas all our related results, except [5], have been only about Appell polynomials (see [6–9]).

Assume that the polynomials $p_n(x)$, $q_n(x)$, and $r_n(x)$ have degree n. The linearization problem in general consists in determining the coefficients $c_{nm}(k)$ in the expansion of the product of two polynomials $q_n(x)$ and $r_m(x)$ in terms of an arbitrary polynomial sequence

$\{p_k(x)\}_{k\geq 0}$:

$$q_n(x)r_m(x) = \sum_{k=0}^{n+m} c_{nm}(k)p_k(x).$$

Thus our results in Theorems A and B can be viewed as generalizations of the linearization problem.

Our study of sums of finite products of special polynomials in this paper can be further justified by the following. Let us put

$$\gamma_m(x) = \sum_{k=1}^{m-1} \frac{1}{k(m-k)} B_k(x)B_{m-k}(x) \quad (m \geq 2). \tag{1.13}$$

Just as we see in (1.10) and (1.12), $\gamma_m(x)$ can be expressed in terms of Bernoulli polynomials from the Fourier series expansions of $\gamma_m(\langle x\rangle)$. From these expansions and after some simple modification, we can obtain the famous Faber–Pandharipande–Zagier identity (see [10]) and a variant of Miki's identity (see [11–14]).

The papers [15, 16] are excellent sources for operational techniques. Finally, for some of the recent results, we let the reader refer to the papers [5–9, 17–19].

2 Fourier series expansions for functions associated with Legendre polynomials

We start with the following lemma which will play an important role in this section.

Lemma 2.1 *Let n, r be integers with $n, r \geq 0$. Then we have the identity*

$$\sum_{i_1+i_2+\cdots+i_{2r+1}=n} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2r+1}}(x) = \frac{1}{(2r-1)!!} P_{n+r}^{(r)}(x), \tag{2.1}$$

where the sum is over all nonnegative integers $i_1, i_2, \ldots, i_{2r+1}$ with $i_1 + i_2 + \cdots + i_{2r+1} = n$.

Proof By differentiating (1.3) r times, we obtain

$$\frac{\partial^r F(t,x)}{\partial x^r} = (2r-1)!!t^r\left(1-2xt+t^2\right)^{-r-\frac{1}{2}}, \tag{2.2}$$

$$\frac{\partial^r F(t,x)}{\partial x^r} = \sum_{n=r}^{\infty} P_n^{(r)}(x)t^n = \sum_{n=0}^{\infty} P_{n+r}^{(r)}(x)t^{n+r}. \tag{2.3}$$

From (2.2) and (2.3), we have

$$\frac{(2r-1)!!}{(1-2xt+t^2)^{r+\frac{1}{2}}} = \sum_{n=0}^{\infty} P_{n+r}^{(r)}(x)t^n. \tag{2.4}$$

On the other hand, from (1.3) and (2.2) we observe that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\Bigg(\sum_{i_1+i_2+\cdots+i_{2r+1}=n} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2r+1}}(x)\Bigg)t^n \\
&\quad = \left(\sum_{n=0}^{\infty} P_n(x)t^n\right)^{2r+1} \\
&\quad = \left(1-2xt+t^2\right)^{-r-\frac{1}{2}} \\
&\quad = \frac{1}{(2r-1)!!}\sum_{n=0}^{\infty} P_{n+r}^{(r)}(x)t^n. \qquad (2.5)
\end{aligned}$$

Comparing both sides of (2.5), we get the desired result. □

It is known that the Legendre polynomials $P_n(x)$ are given by (see [2, 3])

$$P_n(x) = {}_2F_1\left(\begin{matrix}-n,n+1\\1\end{matrix}\,\middle|\,\frac{1-x}{2}\right) = \frac{1}{2^n}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}x^{n-2k}, \qquad (2.6)$$

where ${}_2F_1\left(\begin{smallmatrix}a,b\\c\end{smallmatrix}\middle|z\right) = \sum_{n=0}^{\infty}\frac{\langle a\rangle_n\langle b\rangle_n}{\langle c\rangle_n}\frac{z^n}{n!}$ is the Gauss hypergeometric function with $\langle x\rangle_n$ denoting the rising factorial polynomial defined by

$$\langle x\rangle_n = x(x+1)\cdots(x+n-1) \quad (n\geq 1), \qquad \langle x\rangle_0 = 1.$$

The rth derivative of (2.6) is given by

$$P_n^{(r)}(x) = \frac{1}{2^n}\sum_{k=0}^{[\frac{n-r}{2}]}(-1)^k\binom{n}{k}\binom{2n-2k}{n}(n-2k)_r x^{n-2k-r} \quad (0\leq r\leq n). \qquad (2.7)$$

Combining (2.1) and (2.7), we get the following lemma.

Lemma 2.2 *For integers n, r with $n,r\geq 0$, we have the following identity*:

$$\begin{aligned}
&\sum_{i_1+i_2+\cdots+i_{2r+1}=n} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2r+1}}(x) \\
&\quad = \frac{1}{(2r-1)!!2^{n+r}}\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n+r}{k}\binom{2n+2r-2k}{n+r}(n+r-2k)_r x^{n-2k}. \qquad (2.8)
\end{aligned}$$

For integers m, r with $m,r\geq 1$ as in (1.9), we let

$$\alpha_{m,r}(x) = \sum_{i_1+i_2+\cdots+i_{2r+1}=m} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2r+1}}(x).$$

Then we will consider the function

$$\alpha_{m,r}\big(\langle x\rangle\big) = \sum_{i_1+i_2+\cdots+i_{2r+1}=m} P_{i_1}\big(\langle x\rangle\big)P_{i_2}\big(\langle x\rangle\big)\cdots P_{i_{2r+1}}\big(\langle x\rangle\big), \qquad (2.9)$$

defined on $\mathbb{R}$, which is periodic with period 1. The Fourier series of $\alpha_{m,r}(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} A_n^{(m,r)} e^{2\pi inx}, \tag{2.10}$$

where

$$\begin{aligned} A_n^{(m,r)} &= \int_0^1 \alpha_{m,r}\big(\langle x\rangle\big)e^{-2\pi inx}\,dx \\ &= \int_0^1 \alpha_{m,r}\big(\langle x\rangle\big)e^{-2\pi inx}\,dx. \end{aligned} \tag{2.11}$$

For integers $m, r \geq 1$, we put

$$\begin{aligned} \Delta_{m,r} &= \alpha_{m,r}(1) - \alpha_{m,r}(0) \\ &= \sum_{i_1+\cdots+i_{2r+1}=m} \big(P_{i_1}(1)P_{i_2}(1)\cdots P_{i_{2r+1}}(1) - P_{i_1}(0)P_{i_2}(0)\cdots P_{i_{2r+1}}(0)\big). \end{aligned} \tag{2.12}$$

Now, from (2.8), we see that

$$\Delta_{m,r} = \frac{1}{(2r-1)!!2^{m+r}} \sum_{k=0}^{[\frac{m-1}{2}]} (-1)^k \binom{m+r}{k}\binom{2m+2r-2k}{m+r}(m+r-2k)_r. \tag{2.13}$$

Here we note that

$$\alpha_{m,r}(0) = \begin{cases} \frac{(-1)^{\frac{m}{2}}}{(2r-1)!!2^{m+r}}\binom{m+r}{\frac{m}{2}}\binom{m+2r}{m+r}r!, & \text{if } m \text{ even}, \\ 0, & \text{if } m \text{ odd}. \end{cases} \tag{2.14}$$

From (2.1), we observe that

$$\begin{aligned} \frac{d}{dx}\alpha_{m,r} &= \frac{d}{dx}\left(\frac{1}{(2r-1)!!}P_{m+r}^{(r)}(x)\right) \\ &= \frac{1}{(2r-1)!!}P_{m+r}^{(r+1)}(x) \\ &= (2r+1)\alpha_{m-1,r+1}(x). \end{aligned}$$

Thus we have shown that

$$\frac{d}{dx}\alpha_{m,r}(x) = (2r+1)\alpha_{m-1,r+1}(x). \tag{2.15}$$

Replacing m by $m+1$, r by $r-1$, from (2.15) we get

$$\frac{d}{dx}\left(\frac{1}{2r-1}\alpha_{m+1,r-1}(x)\right) = \alpha_{m,r}(x), \tag{2.16}$$

$$\int_0^1 \alpha_{m,r}(x)\,dx = \frac{1}{2r-1}\Delta_{m+1,r-1}, \tag{2.17}$$

$$\alpha_{m,r}(0) = \alpha_{m,r}(1) \quad \iff \quad \Delta_{m,r} = 0. \tag{2.18}$$

We are now ready to determine the Fourier coefficients $A_n^{(m,r)}$.

Case 1: $n \neq 0$.

$$\begin{aligned}
A_n^{(m,r)} &= \int_0^1 \alpha_{m,r}(x) e^{-2\pi i n x}\,dx \\
&= -\frac{1}{2\pi i n}\left[\alpha_{m,r}(x)e^{-2\pi i n x}\right]_0^1 + \frac{1}{2\pi i n}\int_0^1 \left(\frac{d}{dx}\alpha_{m,r}(x)\right)e^{-2\pi i n x}\,dx \\
&= -\frac{1}{2\pi i n}\bigl(\alpha_{m,r}(1) - \alpha_{m,r}(0)\bigr) + \frac{2r+1}{2\pi i n}\int_0^1 \alpha_{m-1,r+1}(x) e^{-2\pi i n x}\,dx \\
&= \frac{2r+1}{2\pi i n}A_n^{(m-1,r+1)} - \frac{1}{2\pi i n}\Delta_{m,r} \\
&= \frac{2r+1}{2\pi i n}\left(\frac{2r+3}{2\pi i n}A_n^{(m-2,r+2)} - \frac{1}{2\pi i n}\Delta_{m-1,r+1}\right) - \frac{1}{2\pi i n}\Delta_{m,r} \\
&= \frac{(2r+3)!!}{(2\pi i n)^2(2r-1)!!}A_n^{(m-2,r+2)} - \sum_{j=1}^{2}\frac{(2r+2j-3)!!}{(2\pi i n)^j(2r-1)!!}\Delta_{m-j+1,r+j-1} \\
&= \cdots \\
&= \frac{(2r+2m-1)!!}{(2\pi i n)^m(2r-1)!!}A_n^{(0,r+m)} - \sum_{j=1}^{m}\frac{(2r+2j-3)!!}{(2\pi i n)^j(2r-1)!!}\Delta_{m-j+1,r+j-1} \\
&= -\frac{1}{2r-1}\sum_{j=1}^{m}\frac{(2r+2j-3)!!}{(2\pi i n)^j(2r-3)!!}\Delta_{m-j+1,r+j-1}.
\end{aligned} \tag{2.19}$$

Case 2: $n = 0$.

$$A_0^{(m,r)} = \int_0^1 \alpha_{m,r}(x)\,dx = \frac{1}{2r-1}\Delta_{m+1,r-1}. \tag{2.20}$$

Now, from (1.7), (1.8), (2.10), (2.11), (2.19), and (2.20), we have the following Fourier series expansion of $\alpha_{m,r}(\langle x\rangle)$:

$$\begin{aligned}
&\frac{1}{2r-1}\Delta_{m+1,r-1} \\
&\quad - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\frac{1}{2r-1}\sum_{j=1}^{m}\frac{(2r+2j-3)!!}{(2\pi i n)^j(2r-3)!!}\Delta_{m-j+1,r+j-1}\right)e^{2\pi i n x} \\
&= \frac{1}{2r-1}\Delta_{m+1,r-1} \\
&\quad + \frac{1}{2r-1}\sum_{j=1}^{m}\frac{(2r+2j-3)!!}{(2r-3)!!j!}\Delta_{m-j+1,r+j-1}\left(-j!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi i n x}}{(2\pi i n)^j}\right) \\
&= \frac{1}{2r-1}\Delta_{m+1,r-1} + \frac{1}{2r-1}\sum_{j=2}^{m}\frac{(2r+2j-3)!!}{(2r-3)!!j!}\Delta_{m-j+1,r+j-1}B_j\bigl(\langle x\rangle\bigr) \\
&\quad + \Delta_{m,r}\times\begin{cases} B_1(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ 0, & \text{for } x \in \mathbb{Z} \end{cases}
\end{aligned}$$

$$= \frac{1}{2r-1}\sum_{\substack{j=0\\j\neq 1}}^{m}\frac{(2r+2j-3)!!}{(2r-3)!!j!}\Delta_{m-j+1,r+j-1}B_j\big(\langle x\rangle\big)$$
$$+\ \Delta_{m,r}\times\begin{cases}B_1(\langle x\rangle), & \text{for } x\notin\mathbb{Z},\\ 0, & \text{for } x\in\mathbb{Z}.\end{cases}\tag{2.21}$$

$\alpha_{m,r}(\langle x\rangle)$ $(m,r\geq 1)$ is piecewise C^∞. Moreover, $\alpha_{m,r}(\langle x\rangle)$ is continuous for those positive integers m, r with $\Delta_{m,r}=0$, and discontinuous with jump discontinuities at integers for the positive integers m, r with $\Delta_{m,r}\neq 0$. Thus, for $\Delta_{m,r}=0$, the Fourier series of $\alpha_{m,r}(\langle x\rangle)$ converges uniformly to $\alpha_{m,r}(\langle x\rangle)$, whereas, for $\Delta_{m,r}\neq 0$, the Fourier series of $\alpha_{m,r}(\langle x\rangle)$ converges pointwise to $\alpha_{m,r}(\langle x\rangle)$ for $x\notin\mathbb{Z}$ and converges to

$$\frac{1}{2}\big(\alpha_{m,r}(0)+\alpha_{m,r}(1)\big)=\alpha_{m,r}(0)+\frac{1}{2}\Delta_{m,r}$$
$$=\begin{cases}\frac{(-1)^{\frac{m}{2}}}{(2r-1)!!2^{m+r}}\binom{m+r}{\frac{m}{2}}\binom{m+2r}{m+r}r!+\frac{1}{2}\Delta_{m,r}, & \text{if } m \text{ even},\\ \frac{1}{2}\Delta_{m,r}, & \text{if } m \text{ odd},\end{cases}\tag{2.22}$$

for $x\in\mathbb{Z}$ (see (2.14)).

From these observations together with (2.21) and (2.22), we obtain the next two theorems.

Theorem 2.3 *For any integers m, r with $m,r\geq 1$, we let*

$$\Delta_{m,r}=\frac{1}{(2r-1)!!2^{m+r}}\sum_{k=0}^{[\frac{m-1}{2}]}(-1)^k\binom{m+r}{k}\binom{2m+2r-2k}{m+r}(m+r-2k)_r.$$

Assume that $\Delta_{m,r}=0$ for some positive integers m, r. Then we have the following:

(a)

$$\sum_{i_1+i_2+\cdots+i_{2r+1}=m}P_{i_1}\big(\langle x\rangle\big)P_{i_2}\big(\langle x\rangle\big)\cdots P_{i_{2r+1}}\big(\langle x\rangle\big)$$

has the Fourier series expansion

$$\sum_{i_1+i_2+\cdots+i_{2r+1}=m}P_{i_1}\big(\langle x\rangle\big)P_{i_2}\big(\langle x\rangle\big)\cdots P_{i_{2r+1}}\big(\langle x\rangle\big)$$
$$=\frac{1}{2r-1}\Delta_{m+1,r-1}-\frac{1}{2r-1}\sum_{\substack{n=-\infty\\n\neq 0}}^{\infty}\left(\sum_{j=1}^{m}\frac{(2r+2j-3)!!}{(2\pi in)^j(2r-3)!!}\Delta_{m-j+1,r+j-1}\right)e^{2\pi inx}$$

for all $x\in\mathbb{R}$, where the convergence is uniform.

(b)

$$\sum_{i_1+i_2+\cdots+i_{2r+1}=m}P_{i_1}\big(\langle x\rangle\big)P_{i_2}\big(\langle x\rangle\big)\cdots P_{i_{2r+1}}\big(\langle x\rangle\big)$$
$$=\frac{1}{2r-1}\sum_{\substack{j=0\\j\neq 1}}^{m}\frac{(2r+2j-3)!!}{(2r-3)!!j!}\Delta_{m-j+1,r+j-1}B_j\big(\langle x\rangle\big)$$

for all $x\in\mathbb{R}$.

Theorem 2.4 *For any integers m, r with $m,r \geq 1$, we let*

$$\Delta_{m,r} = \frac{1}{(2r-1)!!2^{m+r}} \sum_{k=0}^{[\frac{m-1}{2}]} (-1)^k \binom{m+r}{k} \binom{2m+2r-2k}{m+r} (m+r-2k)_r.$$

Assume that $\Delta_{m,r} \neq 0$ for some positive integers m, r. Then we have the following:

(a)

$$\begin{aligned} &\frac{1}{2r-1}\Delta_{m+1,r-1} - \frac{1}{2r-1} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\sum_{j=1}^{m} \frac{(2r+2j-3)!!}{(2\pi in)^j (2r-3)!!} \Delta_{m-j+1,r+j-1} \right) e^{2\pi inx} \\ &= \begin{cases} \sum_{i_1+\cdots+i_{2r+1}=m} P_{i_1}(\langle x\rangle) P_{i_2}(\langle x\rangle) \cdots P_{i_{2r+1}}(\langle x\rangle), & \text{for } x \notin \mathbb{Z}, \\ \frac{(-1)^{\frac{m}{2}}}{(2r-1)!!2^{m+r}} \binom{m+r}{\frac{m}{2}} \binom{m+2r}{m+r} r! + \frac{1}{2}\Delta_{m,r}, & \text{for } x \in \mathbb{Z} \text{ and } m \text{ even}, \\ \frac{1}{2}\Delta_{m,r}, & \text{for } x \in \mathbb{Z} \text{ and } m \text{ odd}. \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} &\frac{1}{2r-1} \sum_{j=0}^{m} \frac{(2r+2j-3)!!}{(2r-3)!!j!} \Delta_{m-j+1,r+j-1} B_j(\langle x\rangle) \\ &= \sum_{i_1+i_2+\cdots+i_{2r+1}=m} P_{i_1}(\langle x\rangle) P_{i_2}(\langle x\rangle) \cdots P_{i_{2r+1}}(\langle x\rangle), \quad \text{for } x \notin \mathbb{Z}; \\ &\frac{1}{2r-1} \sum_{\substack{j=0 \\ j\neq 1}}^{m} \frac{(2r+2j-3)!!}{(2r-3)!!j!} \Delta_{m-j+1,r+j-1} B_j(\langle x\rangle) \\ &= \begin{cases} \frac{(-1)^{\frac{m}{2}}}{(2r-1)!!2^{m+r}} \binom{m+r}{\frac{m}{2}} \binom{m+2r}{m+r} r! + \frac{1}{2}\Delta_{m,r}, & \text{if } m \text{ even}, \\ \frac{1}{2}\Delta_{m,r}, & \text{if } m \text{ odd}, \end{cases} \end{aligned}$$

for $x \in \mathbb{Z}$.

Finally, we observe that the statement in Theorem A follows immediately from Theorems 2.3 and 2.4.

3 Fourier series expansions for functions associated with Laguerre polynomials

The following lemma is needed for our discussion in this section.

Lemma 3.1 *Let n, r be integers with $n,r \geq 0$. Then we have the identity*

$$\sum_{i_1+i_2+\cdots+i_{r+1}=n} L_{i_1}\left(\frac{x}{r+1}\right) L_{i_2}\left(\frac{x}{r+1}\right) \cdots L_{i_{r+1}}\left(\frac{x}{r+1}\right) = (-1)^r L_{n+r}^{(r)}(x), \tag{3.1}$$

where the sum runs over all nonnegative integers $i_1, i_2, \ldots, i_{r+1}$ with $i_1 + i_2 + \cdots + i_{r+1} = n$.

Proof Differentiating (1.4) r times, we get

$$\frac{\partial^r G(t,x)}{\partial x^r} = (-1)^r t^r (1-t)^{-r-1} \exp\left(-\frac{xt}{1-t}\right), \tag{3.2}$$

$$\frac{\partial^r G(t,x)}{\partial x^r} = \sum_{n=r}^{\infty} L_n^{(r)}(x)t^n = \sum_{n=0}^{\infty} L_{n+r}^{(r)}(x)t^{n+r}. \tag{3.3}$$

Equating (3.2) and (3.3) gives us

$$(-1)^r(1-t)^{-r-1}\exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_{n+r}^{(r)}t^n. \tag{3.4}$$

On the other hand, from (1.4) and (3.4), we note that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left(\sum_{i_1+i_2+\cdots+i_{r+1}=n} L_{i_1}\left(\frac{x}{r+1}\right)L_{i_2}\left(\frac{x}{r+1}\right)\cdots L_{i_{r+1}}\left(\frac{x}{r+1}\right)\right)t^n \\
&\quad = \left(\sum_{n=0}^{\infty} L_n\left(\frac{x}{r+1}\right)t^n\right)^{r+1} \\
&\quad = \left((1-t)^{-1}\exp\left(-\frac{xt}{(r+1)(1-t)}\right)\right)^{r+1} \\
&\quad = (-1)^r\sum_{n=0}^{\infty} L_{n+r}^{(r)}t^n.
\end{aligned} \tag{3.5}$$

Comparing both sides of (3.5) yields the desired result. □

It is known that the Laguerre polynomials $L_n(x)$ are given by (see [2, 3])

$$L_n(x) = {}_1F_1\left(\begin{smallmatrix}-n\\1\end{smallmatrix}\middle| x\right) = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{k!}x^k, \tag{3.6}$$

where ${}_1F_1\left(\begin{smallmatrix}a\\b\end{smallmatrix}\middle| z\right) = \sum_{n=0}^{\infty}\frac{\langle a\rangle_n}{\langle b\rangle_n}\frac{z^n}{n!}$ is the hypergeometric function.

The rth derivative of (3.6) is given by

$$\begin{aligned}
L_n^{(r)}(x) &= \sum_{k=r}^{n}(-1)^k\frac{1}{k!}\binom{n}{k}(k)_r x^{k-r} \\
&= \sum_{k=r}^{n}(-1)^k\frac{1}{(k-r)!}\binom{n}{k}x^{k-r} \\
&= \sum_{k=0}^{n-r}(-1)^{k+r}\frac{1}{k!}\binom{n}{k+r}x^k \quad (0 \le r \le n).
\end{aligned} \tag{3.7}$$

From (3.1) and (3.7), we obtain the following result.

Lemma 3.2 *For integers n, r with $n, r \ge 0$, we have the following identity:*

$$\begin{aligned}
&\sum_{i_1+i_2+\cdots+i_{r+1}=n} L_{i_1}\left(\frac{x}{r+1}\right)L_{i_2}\left(\frac{x}{r+1}\right)\cdots L_{i_{r+1}}\left(\frac{x}{r+1}\right) \\
&\quad = \sum_{k=0}^{n}\frac{(-1)^k}{k!}\binom{n+r}{k+r}x^k.
\end{aligned} \tag{3.8}$$

For integers m, r with $m \geq 1$, $r \geq 0$ as in (1.11), we put

$$\beta_{m,r}(x) = \sum_{i_1+i_2+\cdots+i_{r+1}=m} L_{i_1}\left(\frac{x}{r+1}\right)L_{i_2}\left(\frac{x}{r+1}\right)\cdots L_{i_{r+1}}\left(\frac{x}{r+1}\right).$$

Then we will consider the function

$$\beta_{m,r}\bigl(\langle x\rangle\bigr) = \sum_{i_1+i_2+\cdots+i_{r+1}=m} L_{i_1}\left(\frac{\langle x\rangle}{r+1}\right)L_{i_2}\left(\frac{\langle x\rangle}{r+1}\right)\cdots L_{i_{r+1}}\left(\frac{\langle x\rangle}{r+1}\right), \tag{3.9}$$

defined on $\mathbb{R}$, which is periodic with period 1.

The Fourier series of $\beta_{m,r}(\langle x\rangle)$ is

$$\sum_{n=-\infty}^{\infty} B_n^{(m,r)} e^{2\pi inx}, \tag{3.10}$$

where

$$\begin{aligned} B_n^{(m,r)} &= \int_0^1 \beta_{m,r}\bigl(\langle x\rangle\bigr)e^{-2\pi inx}\,dx \\ &= \int_0^1 \beta_{m,r}(x)e^{-2\pi inx}\,dx. \end{aligned} \tag{3.11}$$

For integers m, r with $m \geq 1$, $r \geq 0$, we set

$$\begin{aligned} \Omega_{m,r} &= \beta_{m,r}(1) - \beta_{m,r}(0) \\ &= \sum_{i_1+i_2+\cdots+i_{r+1}=m} \biggl(L_{i_1}\left(\frac{1}{r+1}\right)L_{i_2}\left(\frac{1}{r+1}\right)\cdots L_{i_{r+1}}\left(\frac{1}{r+1}\right) \\ &\quad - L_{i_1}(0)L_{i_2}(0)\cdots L_{i_{r+1}}(0)\biggr). \end{aligned} \tag{3.12}$$

Now, from (3.8), we see that

$$\Omega_{m,r} = (-1)^m \sum_{k=0}^{m-1} \frac{(-1)^k}{(m-k)!}\binom{m+r}{k}. \tag{3.13}$$

From (3.1), we note that

$$\begin{aligned} \frac{d}{dx}\beta_{m,r}(x) &= \frac{d}{dx}\bigl((-1)^r L_{m+r}^{(r)}(x)\bigr) \\ &= (-1)^r L_{m+r}^{(r+1)}(x) \\ &= -\beta_{m-1,r+1}(x). \end{aligned}$$

Thus we have shown that

$$\frac{d}{dx}\beta_{m,r}(x) = -\beta_{m-1,r+1}(x). \tag{3.14}$$

Replacing m by $m+1$, r by $r-1$, from (3.14), we get

$$\frac{d}{dx}\big(-\beta_{m+1,r-1}(x)\big) = \beta_{m,r}(x), \tag{3.15}$$

$$\int_0^1 \beta_{m,r}(x)\,dx = -\Omega_{m+1,r-1}, \tag{3.16}$$

$$\beta_{m,r}(0) = \beta_{m,r}(1) \quad \Longleftrightarrow \quad \Omega_{m,r} = 0. \tag{3.17}$$

We are now going to determine the Fourier coefficients.

Case 1: $n \neq 0$.

$$\begin{aligned}
B_n^{(m,r)} &= \int_0^1 \beta_{m,r}(x) e^{-2\pi inx}\,dx \\
&= -\frac{1}{2\pi in}\big[\beta_{m,r}(x)e^{-2\pi inx}\big]_0^1 + \frac{1}{2\pi in}\int_0^1 \left(\frac{d}{dx}\beta_{m,r}(x)\right) e^{-2\pi inx}\,dx \\
&= -\frac{1}{2\pi in}\big(\beta_{m,r}(1) - \beta_{m,r}(0)\big) - \frac{1}{2\pi in}\int_0^1 \beta_{m-1,r+1}(x) e^{-2\pi inx}\,dx \\
&= -\frac{1}{2\pi in} B_n^{(m-1,r+1)} - \frac{1}{2\pi in}\Omega_{m,r} \\
&= -\frac{1}{2\pi in}\left(-\frac{1}{2\pi in} B_n^{(m-2,r+2)} - \frac{1}{2\pi in}\Omega_{m-1,r+1}\right) - \frac{1}{2\pi in}\Omega_{m,r} \\
&= \left(-\frac{1}{2\pi in}\right)^2 B_n^{(m-2,r+2)} + \sum_{j=1}^{2}\left(-\frac{1}{2\pi in}\right)^j \Omega_{m-j+1,r+j-1} \\
&= \cdots \\
&= \left(-\frac{1}{2\pi in}\right)^m B_n^{(0,r+m)} + \sum_{j=1}^{m}\left(-\frac{1}{2\pi in}\right)^j \Omega_{m-j+1,r+j-1} \\
&= \sum_{j=1}^{m}\left(-\frac{1}{2\pi in}\right)^j \Omega_{m-j+1,r+j-1}.
\end{aligned} \tag{3.18}$$

Case 2: $n = 0$.

$$B_0^{(m,r)} = \int_0^1 \beta_{m,r}(x)\,dx = -\Omega_{m+1,r-1}. \tag{3.19}$$

Now, from (1.7), (1.8), (3.10), (3.11), (3.18), and (3.19), the Fourier series expansion of $\beta_{m,r}(\langle x\rangle)$ is given by

$$\begin{aligned}
&-\Omega_{m+1,r-1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(\sum_{j=1}^{m}\left(-\frac{1}{2\pi in}\right)^j \Omega_{m-j+1,r+j-1}\right) e^{2\pi inx} \\
&\quad = -\Omega_{m+1,r-1} - \sum_{j=1}^{m}\frac{(-1)^j}{j!}\Omega_{m-j+1,r+j-1}\left(-j!\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{e^{2\pi inx}}{(2\pi in)^j}\right) \\
&\quad = -\Omega_{m+1,r-1} - \sum_{j=2}^{m}\frac{(-1)^j}{j!}\Omega_{m-j+1,r+j-1}B_j\big(\langle x\rangle\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \Omega_{m,r} \times \begin{cases} B_1(\langle x\rangle), & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z} \end{cases} \\
&= -\sum_{\substack{j=0 \\ j\neq 1}}^{m} \frac{(-1)^j}{j!} \Omega_{m-j+1,r+j-1} B_j\bigl(\langle x\rangle\bigr) + \Omega_{m,r} \times \begin{cases} B_1(\langle x\rangle), & \text{if } x \notin \mathbb{Z}, \\ 0, & \text{if } x \in \mathbb{Z}. \end{cases}
\end{aligned} \tag{3.20}
$$

$\beta_{m,r}(\langle x\rangle)$ ($m \geq 1$, $r \geq 0$) is piecewise C^{∞}. In addition, $\beta_{m,r}(\langle x\rangle)$ is continuous for those integers m, r ($m \geq 1$, $r \geq 0$) with $\Omega_{m,r} = 0$, and discontinuous with jump discontinuities at integers for those integers m, r ($m \geq 1$, $r \geq 0$) with $\Omega_{m,r} \neq 0$. Hence, for $\Omega_{m,r} = 0$, the Fourier series of $\beta_{m,r}(\langle x\rangle)$ converges uniformly to $\beta_{m,r}(\langle x\rangle)$. Whereas, for $\Omega_{m,r} \neq 0$, the Fourier series of $\beta_{m,r}(\langle x\rangle)$ converges pointwise to $\beta_{m,r}(\langle x\rangle)$ for $x \notin \mathbb{Z}$ and converges to

$$
\frac{1}{2}\bigl(\beta_{m,r}(0) + \beta_{m,r}(1)\bigr) = \beta_{m,r}(0) + \frac{1}{2}\Omega_{m,r} = \binom{m+r}{r} + \frac{1}{2}\Omega_{m,r} \tag{3.21}
$$

for $x \in \mathbb{Z}$ (see (3.8)).

From these observations together with (3.20) and (3.21), we have the next two theorems.

Theorem 3.3 *For any integers m, r with $m \geq 1$, $r \geq 0$, we let*

$$
\Omega_{m,r} = (-1)^m \sum_{k=0}^{m-1} \frac{(-1)^k}{(m-k)!} \binom{m+r}{k}.
$$

Assume that $\Omega_{m,r} = 0$ for some integers m, r with $m \geq 1$, $r \geq 0$. Then we have the following:

(a)

$$
\sum_{i_1+i_2+\cdots+i_{r+1}=m} L_{i_1}\left(\frac{\langle x\rangle}{r+1}\right) L_{i_2}\left(\frac{\langle x\rangle}{r+1}\right) \cdots L_{i_{r+1}}\left(\frac{\langle x\rangle}{r+1}\right)
$$

has the Fourier series expansion

$$
\begin{aligned}
&\sum_{i_1+i_2+\cdots+i_{r+1}=m} L_{i_1}\left(\frac{\langle x\rangle}{r+1}\right) L_{i_2}\left(\frac{\langle x\rangle}{r+1}\right) \cdots L_{i_{r+1}}\left(\frac{\langle x\rangle}{r+1}\right) \\
&\quad = -\Omega_{m+1,r-1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\sum_{j=1}^{m} \left(-\frac{1}{2\pi in}\right)^j \Omega_{m-j+1,r+j-1} \right) e^{2\pi inx}
\end{aligned}
$$

for all $x \in \mathbb{R}$, where the convergence is uniform.

(b)

$$
\begin{aligned}
&\sum_{i_1+i_2+\cdots+i_{r+1}=m} L_{i_1}\left(\frac{\langle x\rangle}{r+1}\right) L_{i_2}\left(\frac{\langle x\rangle}{r+1}\right) \cdots L_{i_{r+1}}\left(\frac{\langle x\rangle}{r+1}\right) \\
&\quad = -\sum_{\substack{j=0 \\ j\neq 1}}^{m} \frac{(-1)^j}{j!} \Omega_{m-j+1,r+j-1} B_j\bigl(\langle x\rangle\bigr)
\end{aligned}
$$

for all $x \in \mathbb{R}$.

Theorem 3.4 *For any integers m, r with $m \geq 1$, $r \geq 0$, we let*

$$\Omega_{m,r} = (-1)^m \sum_{k=0}^{m-1} \frac{(-1)^k}{(m-k)!} \binom{m+r}{k}.$$

Assume that $\Omega_{m,r} \neq 0$ for some integers m, r with $m \geq 1$, $r \geq 0$. Then we have the following:

(a)

$$-\Omega_{m+1,r-1} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \left(\sum_{j=1}^{m} \left(-\frac{1}{2\pi in} \right)^j \Omega_{m-j+1,r+j-1} \right) e^{2\pi inx}$$
$$= \begin{cases} \sum_{i_1+i_2+\cdots+i_{r+1}=m} L_{i_1}(\frac{\langle x\rangle}{r+1}) L_{i_2}(\frac{\langle x\rangle}{r+1}) \cdots L_{i_{r+1}}(\frac{\langle x\rangle}{r+1}), & \text{for } x \notin \mathbb{Z}, \\ \binom{m+r}{r} + \frac{1}{2}\Omega_{m,r}, & \text{for } x \in \mathbb{Z}. \end{cases}$$

(b)

$$-\sum_{j=0}^{m} \frac{(-1)^j}{j!} \Omega_{m-j+1,r+j-1} B_j(\langle x\rangle)$$
$$= \sum_{i_1+i_2+\cdots+i_{r+1}=m} L_{i_1}\left(\frac{\langle x\rangle}{r+1}\right) L_{i_2}\left(\frac{\langle x\rangle}{r+1}\right) \cdots L_{i_{r+1}}\left(\frac{\langle x\rangle}{r+1}\right), \quad \text{for } x \notin \mathbb{Z};$$
$$-\sum_{\substack{j=0 \\ j\neq 1}}^{m} L_{i_1}\left(\frac{\langle x\rangle}{r+1}\right) L_{i_2}\left(\frac{\langle x\rangle}{r+1}\right) \cdots L_{i_{r+1}}\left(\frac{\langle x\rangle}{r+1}\right)$$
$$= \binom{m+r}{r} + \frac{1}{2}\Omega_{m,r}, \quad \text{for } x \in \mathbb{Z}.$$

Finally, we see that the statement in Theorem B follows from Theorems 3.3 and 3.4.

4 Expressions in terms of Euler polynomials

For any polynomial $p(x) \in \mathbb{C}[x]$ with degree m, we know that

$$p(x) = \sum_{k=0}^{m} b_k E_k(x), \tag{4.1}$$

where $E_k(x)$ are the Euler polynomials given by $\frac{2}{e^t+1} e^{xt} = \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!}$, and

$$b_k = \frac{1}{2k!} \bigl(p^{(k)}(1) + p^{(k)}(0) \bigr), \quad k = 0, 1, \ldots, m. \tag{4.2}$$

Applying (4.1) and (4.2) to $p(x) = \alpha_{m,r}(x)$ and from (2.15), we see that

$$\alpha_{m,r}^{(k)}(x) = \frac{(2r+2k-1)!!}{(2r-1)!!} \alpha_{m-k,r+k}(x). \tag{4.3}$$

Hence, from (4.3), we have

$$\begin{aligned} b_k &= \frac{1}{2k!}\left(\alpha_{m,r}^{(k)}(1) + \alpha_{m,r}^{(k)}(0)\right) \\ &= \frac{(2r+2k-1)!!}{2k!(2r-1)!!}\left(\alpha_{m-k,r+k}(1) + \alpha_{m-k,r+k}(0)\right) \\ &= \frac{(2r+2k-1)!!}{k!(2r-1)!!}\left(\alpha_{m-k,r+k}(0) + \frac{1}{2}\Delta_{m-k,r+k}\right). \end{aligned} \tag{4.4}$$

Now, from (2.14), (4.1), and (4.4), we get the following theorem.

Theorem 4.1 *For any integers m, r with $m, r \geq 1$, we have the following*:

$$\sum_{i_1+i_2+\cdots+i_{2r+1}=m} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2r+1}}(x) = \sum_{k=0}^{m} b_k E_k(x),$$

where

$$b_k = \begin{cases} \frac{(-1)^{\frac{m-k}{2}}(r+k)!}{k!(2r-1)!!2^{m+r}}\binom{m+r}{\frac{m-k}{2}}\binom{m+2r+k}{m+r} + \frac{(2r+2k-1)!!}{2k!(2r-1)!!}\Delta_{m-k,r+k}, & \text{for } m-k \text{ even}, \\ \frac{(2r+2k-1)!!}{2k!(2r-1)!!}\Delta_{m-k,r+k}, & \text{for } m-k \text{ odd}, \end{cases}$$

and $\Delta_{m,r}$ is given by (2.13).

Proceeding analogously to the above discussion, we get the next result, the details of which are left to the reader.

Theorem 4.2 *For any integers m, r with $m \geq 1$, $r \geq 0$, we have the following identity*:

$$\begin{aligned} &\sum_{i_1+i_2+\cdots+i_{r+1}=m} L_{i_1}\left(\frac{x}{r+1}\right)L_{i_2}\left(\frac{x}{r+1}\right)\cdots L_{i_{r+1}}\left(\frac{x}{r+1}\right) \\ &\quad = \sum_{k=0}^{m} \frac{(-1)^k}{k!}\left(\binom{m+r}{r+k} + \frac{1}{2}\Omega_{m-k,r+k}\right)E_k(x), \end{aligned}$$

where $\Omega_{m,r}$ is given by (3.13).

5 Results and discussion

In this paper, we investigated sums of finite products of Legendre and Laguerre polynomials and derived Fourier series expansions of functions associated with those polynomials. From these Fourier series expansions, we were able to express those sums of finite products as linear combinations of Bernoulli polynomials. Further, by using a different method, we expressed those sums of finite products as linear combinations of Euler polynomials as well. It is expected that we will be able to express those sums of finite products as linear combinations of some classical orthogonal polynomials. Here we note that the Bernoulli and Euler polynomials are not orthogonal polynomials but Appell polynomials.

6 Conclusion

In this paper, we considered the Fourier series expansions for functions associated with Legendre and Laguerre polynomials. In addition, by using a method other than Fourier series expansions, we were able to express those sums in terms of Euler polynomials. It is noteworthy that all the other previous related papers, except [5, 20–22], were associated with Appell polynomials, while the present one is about classical orthogonal polynomials.

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Competing interests

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Authors' contributions

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Author details

[1]Department of Mathematics, Kwangwoon University, Seoul, Republic of Korea. [2]Department of Mathematics, Sogang University, Seoul, Republic of Korea. [3]Hanrimwon, Kwangwoon University, Seoul, Republic of Korea. [4]Department of Mathematics Education, Daegu University, Gyeongsan-si, Republic of Korea.

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