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# Nonlocal symmetries of Frobenius sinh-Gordon systems

Huijuan Zhou[1], Chuanzhong Li[1*], Xinyue Li[2] and Fushan Li[3]

*Correspondence:
lichuanzhong@nbu.edu.cn
[1] Department of Mathematics, Ningbo University, Ningbo, China
Full list of author information is available at the end of the article

**Abstract**

In this paper, we consider a weakly coupled sinh-Gordon equation which takes values in a commutative Frobenius subalgebra of $gl(2, \mathbb{C})$. Then we construct some nonlocal symmetries of the Frobenius sinh-Gordon system using its Bäcklund transformation and infinitesimal transformations. Based on the nonlocal symmetries, we show some conserved densities of the Frobenius sinh-Gordon system. Using these symmetries, we also construct some new coupled integro-differential systems.

**Keywords:** Bäcklund transformations; Frobenius sinh-Gordon equation; Nonlocal symmetry

## 1 Introduction

The sinh-Gordon equation and sine-Gordon equation are important integrable equations and they describe many interesting phenomena including dynamics of coupled pendulums, Josephson junction arrays [1], nonlinear excitations in complex systems in physics, and living cellular structures [2]. These two models have a transformation which links them together. In [3], Grauel studied the Painlevé property and Bäcklund transformation of sinh-Gordon equation.

As we know, Lie symmetries are very important in finding solutions of integrable equations [4–13], particularly the residual symmetries and nonlocal symmetries [14–16]. In [17], nonlocal symmetries of the $(1 + 1)$-dimensional sinh-Gordon equation are obtained. Making advantages of the consistent conditions introduced when solving the nonlocal symmetries, some new nonlocal conservation laws of the sinh-Gordon equation related to the nonlocal symmetries are obtained. Some new finite and infinite dimensional nonlinear systems are constructed by taking the nonlocal symmetries as symmetry constraint conditions imposed on the Bäcklund transformations.

Nonlocal symmetries were first studied rigorously early in 1980 [18] in which a satisfactory geometric formulation was developed, and later a series of works [19, 20] appeared. A constructive method for deriving nonlocal symmetries of differential equations based on the Lie–Bäcklund theory of groups was developed in [21]. Systematic procedures were presented for finding nonlocally related partial differential equations and their many local and nonlocal conservation laws and nonlocal symmetries in [22]. Nonlocal symmetries are of interest because they are associated with the existence of linearizing transformations, Bäcklund transformations, and Darboux transformations. Applying the infinitesimal transformation on the nonlinear system and its lax pair simultaneously, some useful nonlocal symmetries involving the eigenfunction can be obtained. These nonlocal symmetries are also known as the eigenfunction symmetries [23, 24], and they have been recently studied to construct explicit solutions [25].

In [26], from the algebraic reductions from the Lie algebra $gl(n,\mathbb{C})$ to its commutative subalgebra $Z_n$, we construct the general $Z_n$-sine-Gordon and $Z_n$-sinh-Gordon systems which contain many multi-component sine-Gordon type and sinh-Gordon type equations. Meanwhile, we give the Bäcklund transformations of the $Z_n$-sine-Gordon and $Z_n$-sinh-Gordon equations which can generate new solutions from seed solutions. A natural question is what is the nonlocal symmetry of them, particularly the $Z_2$-sinh-Gordon equation (also named as Frobenius sinh-Gordon equation in this paper). In this paper, we will answer this question in detail. This paper is arranged as follows. In Sect. 2, we recall some basic facts about the Frobenius sinh-Gordon equations and their Bäcklund transformations. In Sect. 3, we construct some coupled integro-differential systems using the nonlocal symmetries.

## 2 The Frobenius sinh-Gordon equation and its Bäcklund transformation

In this section, we recall the Frobenius sinh-Gordon equation which was constructed firstly in our recent paper [26]. The Frobenius sinh-Gordon equation was constructed in the commutative algebra $Z_2 = \mathbb{C}[\Gamma]/(\Gamma^2)$ and $\Gamma = (\delta_{i,j+1})_{ij} \in gl(2,\mathbb{C})$. In this section, we will use a similar method in the last section to consider the Bäcklund transformation of the Frobenius sinh-Gordon equation. Based on the well-known sinh-Gordon equation

$$u_{xt} = \sinh u, \tag{1}$$

the following equation in $Z_2$ is the Frobenius sinh-Gordon equation:

$$\begin{cases} u_{xt} = \sinh u, \\ v_{xt} = v\cosh u. \end{cases} \tag{2}$$

The Frobenius sinh-Gordon equation has the following Bäcklund transformation [26]:

$$\begin{cases} (\frac{u'+u}{2})_x = a\sinh\frac{u'-u}{2}, \\ (\frac{v'+v}{2})_x = \frac{v'-v}{2}\cosh\frac{u'-u}{2}, \\ (\frac{u'-u}{2})_t = \frac{1}{a}\sinh\frac{u'+u}{2}, \\ (\frac{v'-v}{2})_t = \frac{1}{a}\frac{v'+v}{2}\cosh\frac{u'+u}{2}. \end{cases} \tag{3}$$

## 3 Nonlocal symmetries of the Frobenius sinh-Gordon equation

Suppose that the above Frobenius sinh-Gordon equation (2) and the Bäcklund transformation (3) are invariant up to an infinitesimal transformation

$$u \to u + \epsilon\tau^u, \qquad v \to v + \epsilon\tau^v, \tag{4}$$

$$u' \to u' + \epsilon\tau^{u'}, \qquad v' \to v' + \epsilon\tau^{v'}, \tag{5}$$

$$a \to a - 2\epsilon\delta, \tag{6}$$

we can derive the following identities:

$$\tau^u_{xt} - n^2\cosh(u)\tau^u = 0, \tag{7}$$

$$\tau^{u'}_{xt} - n^2\cosh(u')\tau^{u'} = 0, \tag{8}$$

$$\tau^u_x - \tau^{u'}_x - 2\delta n\sinh\left(\frac{u}{2}+\frac{u'}{2}\right) - \frac{a}{2}n\cosh\left(\frac{u}{2}+\frac{u'}{2}\right)\left(\tau^{u'}+\tau^u\right) = 0, \tag{9}$$

$$\tau^u_t - \tau^{u'}_t - \frac{8}{a^2}\delta n\sinh\left(\frac{u}{2}-\frac{u'}{2}\right) + \frac{2}{a}n\cosh\left(\frac{u}{2}-\frac{u'}{2}\right)\left(\tau^u-\tau^{u'}\right) = 0, \tag{10}$$

$$\tau^v_{xt} - n^2\cosh(u)\tau^v + n^2 v\sinh(u)\tau^u = 0, \tag{11}$$

$$\tau^{v'}_{xt} - n^2\cosh(u')\tau^{v'} + n^2 v'\sinh(u')\tau^{u'} = 0, \tag{12}$$

$$\begin{aligned}&\tau^v_x - \tau^{v'}_x - 2\delta n\left(\frac{v}{2}+\frac{v'}{2}\right)\cosh\left(\frac{u}{2}+\frac{u'}{2}\right) + \frac{a}{2}n\left(\frac{v}{2}+\frac{v'}{2}\right)\sinh\left(\frac{u}{2}+\frac{u'}{2}\right)\left(\tau^{u'}+\tau^u\right)\\ &\quad - \frac{a}{2}n\cosh\left(\frac{u}{2}+\frac{u'}{2}\right)\left(\tau^{v'}+\tau^v\right) = 0,\end{aligned} \tag{13}$$

$$\begin{aligned}&\tau^v_t - \tau^{v'}_t - \frac{8}{a^2}\delta n\left(\frac{v}{2}-\frac{v'}{2}\right)\cosh\left(\frac{u}{2}-\frac{u'}{2}\right) + \frac{2}{a}n\cosh\left(\frac{u}{2}-\frac{u'}{2}\right)\left(\tau^v-\tau^{v'}\right)\\ &\quad - \frac{2}{a}n\left(\frac{v}{2}-\frac{v'}{2}\right)\sin\left(\frac{u}{2}-\frac{u'}{2}\right)\left(\tau^u-\tau^{u'}\right) = 0.\end{aligned} \tag{14}$$

Similar to [17], we can derive the following three symmetries.

*I*: If $\delta = 0$, $\tau^{u'} = \tau^{v'} = 0$, then the Frobenius sinh-Gordon equation has a nonlocal symmetry with

$$\tau^u = e^{ap}, \qquad \tau^v = aqe^{ap}, \tag{15}$$

where

$$p_x = \cosh\left(\frac{u}{2}-\frac{u'}{2}\right), \qquad p_t = \frac{1}{a^2}\cosh\left(\frac{u}{2}+\frac{u'}{2}\right), \tag{16}$$

$$q_x = \left(\frac{v}{2}-\frac{v'}{2}\right)\sinh\left(\frac{u}{2}-\frac{u'}{2}\right), \qquad q_t = \frac{1}{a^2}\left(\frac{v}{2}+\frac{v'}{2}\right)\sinh\left(\frac{u}{2}+\frac{u'}{2}\right). \tag{17}$$

*II*: If $\delta = \frac{1}{2n}$, $\tau^{u'} = \tau^{v'} = 0$, then the Frobenius sinh-Gordon equation has a nonlocal symmetry with

$$\tau^u = re^{ap}, \qquad \tau^v = se^{ap} + raqe^{ap}, \tag{18}$$

where

$$r_x = e^{-ap}\sinh\left(\frac{u}{2}-\frac{u'}{2}\right), \qquad r_t = -\frac{1}{a^2}e^{-ap}\sinh\left(\frac{u}{2}+\frac{u'}{2}\right), \tag{19}$$

$$s_x = e^{-ap}\left(\frac{v}{2}-\frac{v'}{2}\right)\sinh\left(\frac{u}{2}-\frac{u'}{2}\right) - aqe^{-ap}\sinh\left(\frac{u}{2}-\frac{u'}{2}\right), \tag{20}$$

$$s_t = -\frac{1}{a^2}e^{-ap}\left(\frac{v}{2}+\frac{v'}{2}\right)\sinh\left(\frac{u}{2}+\frac{u'}{2}\right) + \frac{q}{a}e^{-ap}\sinh\left(\frac{u}{2}+\frac{u'}{2}\right). \tag{21}$$

*III*: If $\delta = 0$, $\tau^{u'} = u'_x$, $\tau^{v'} = v'_x$, then the Frobenius sinh-Gordon equation has a nonlocal symmetry with

$$\tau^u = u'_x - nafe^{-\frac{a}{2}np}, \qquad \tau^v = v'_x - nage^{-\frac{a}{2}np} + \frac{n^2a^2}{2}qfe^{-\frac{a}{2}np}, \tag{22}$$

where

$$f_x = u'_x \cos\left(\frac{u}{2} + \frac{u'}{2}\right)e^{\frac{a}{2}np}, \qquad f_t = \frac{a}{2}nu'_{xt}e^{\frac{a}{2}np}, \tag{23}$$

$$g_x = \left[v'_x - u'_x\left(\frac{v}{2} + \frac{v'}{2}\right)\right]\cos\left(\frac{u}{2} + \frac{u'}{2}\right)e^{\frac{a}{2}np} + \frac{a}{2}nqu'_x \cos\left(\frac{u}{2} + \frac{u'}{2}\right)e^{\frac{a}{2}np}, \tag{24}$$

$$g_t = \frac{a}{2}nv'_{xt}e^{\frac{a}{2}np} + \frac{a^2}{4}n^2qu'_{xt}e^{\frac{a}{2}np}. \tag{25}$$

It is evident that symmetries of the Frobenius sinh-Gordon equation (2) obtained above are really nonlocal as they depend on the function $u'$, $v'$, which is related to the function $u$, $v$ through the Bäcklund transformation (3).

Integrating with respect to $x$ and $t$ will lead to

$$\tau^u = -4e^{ap}r\delta - 2ae^{ap}\int p_x\tau^{u'}e^{-ap}\,dx - \tau^{u'} + e^{ap}G_0(t), \tag{26}$$

$$\begin{aligned}\tau^v ={}& -4aqe^{ap}r\delta - 4e^{ap}s\delta - 2a^2qe^{ap}\int p_x\tau^{u'}e^{-ap}\,dx - 2ae^{ap}\int q_x\tau^{u'}e^{-ap}\,dx\\ &- 2ae^{ap}\int p_x\tau^{v'}e^{-ap}\,dx + 2a^2e^{ap}\int p_x\tau^{u'}qe^{-ap}\,dx - \tau^{v'}\\ &+ aqe^{ap}G_0(t) + e^{ap}G_1(t),\end{aligned} \tag{27}$$

and

$$\tau^u = \frac{4w}{a^2}e^{\frac{h}{a}}\delta + \frac{2}{a}e^{\frac{h}{a}}\int h_x\tau^{u'}e^{-\frac{h}{a}}\,dx - \tau^{u'} + e^{\frac{h}{a}}G_3(x), \tag{28}$$

$$\begin{aligned}\tau^v ={}& \frac{4\bar{w}}{a^2}e^{\frac{h}{a}}\delta + \frac{4w}{a^2}\frac{\bar{h}}{a}e^{\frac{h}{a}}\delta + \frac{2}{a^2}\bar{h}e^{\frac{h}{a}}\int h_x\tau^{u'}e^{-\frac{h}{a}}\,dx + \frac{2}{a}e^{\frac{h}{a}}\int \bar{h}_x\tau^{u'}e^{-\frac{h}{a}}\,dx\\ &+ \frac{2}{a}e^{\frac{h}{a}}\int h_x\tau^{v'}e^{-\frac{h}{a}}\,dx - \frac{2}{a^2}e^{\frac{h}{a}}\int h_x\tau^{u'}\bar{h}e^{-\frac{h}{a}}\,dx - \tau^{v'} + \frac{\bar{h}}{a}e^{\frac{h}{a}}G_3(t) + e^{\frac{h}{a}}G_4(x),\end{aligned} \tag{29}$$

where $p$, $q$, $r$, $s$, $h$, $w$, $\bar{h}$, $\bar{w}$ satisfy

$$\begin{aligned}&p_x = \cosh\left(\frac{u}{2} - \frac{u'}{2}\right), \qquad q_x = \left(\frac{v}{2} - \frac{v'}{2}\right)\sinh\left(\frac{u}{2} - \frac{u'}{2}\right),\\ &r_x = \sinh\left(\frac{u}{2} - \frac{u'}{2}\right)e^{-ap}, \qquad s_x = \left(\frac{v}{2} - \frac{v'}{2}\right)\sinh\left(\frac{u}{2} - \frac{u'}{2}\right)e^{-ap} - aq\sinh\left(\frac{u}{2} - \frac{u'}{2}\right)e^{-ap},\end{aligned} \tag{30}$$

$$\begin{aligned}&h_t = \cosh\left(\frac{u}{2} + \frac{u'}{2}\right), \qquad \bar{h}_t = \left(\frac{v}{2} + \frac{v'}{2}\right)\sinh\left(\frac{u}{2} + \frac{u'}{2}\right),\\ &w_t = \sinh\left(\frac{u}{2} + \frac{u'}{2}\right)e^{\frac{h}{a}}, \qquad \bar{w}_t = \left(\frac{v}{2} + \frac{v'}{2}\right)\sinh\left(\frac{u}{2} + \frac{u'}{2}\right)e^{\frac{h}{a}} + \frac{\bar{h}}{a}\sinh\left(\frac{u}{2} + \frac{u'}{2}\right)e^{\frac{h}{a}}.\end{aligned} \tag{31}$$

$G_0(t)$, $G_0(t)$, $G_3(x)$, $G_4(x)$ are arbitrary integration functions. The following conditions should be satisfied:

$$h = a^2 p, \qquad w = -a^2 r, \qquad \bar{h} = a^2 q, \qquad \bar{w} = -a^2 s, \tag{32}$$

$$\int p_x \tau^{u'} e^{-ap}\,dx + \int p_x \tau^{u'} e^{-ap}\,dt + \frac{1}{a}\tau^{u'} e^{-ap} = 0, \tag{33}$$

$$\begin{aligned} &\int q_x \tau^{u'} e^{-ap}\,dx + \int p_x \tau^{u'} e^{-ap}\,dx + \int p_x \tau^{v'} e^{-ap}\,dx - a\int p_x q \tau^{v'} e^{-ap}\,dx \\ &\quad + \int q_x \tau^{u'} e^{-ap}\,dt + \int p_x \tau^{v'} e^{-ap}\,dt - a\int p_x q \tau^{u'} e^{-ap}\,dt + \frac{1}{a}\tau^{v'} e^{-ap} - q\tau^{u'} e^{-ap} \\ &= 0. \end{aligned} \tag{34}$$

Then we can get the following corresponding conserved density and flux:

$$\rho_1 = a^2 \cosh\left(\frac{u}{2} - \frac{u'}{2}\right), \qquad J_1 = -\cosh\left(\frac{u}{2} + \frac{u'}{2}\right), \tag{35}$$

$$\bar{\rho}_1 = a^2 \left(\frac{v}{2} - \frac{v'}{2}\right)\cosh\left(\frac{u}{2} - \frac{u'}{2}\right), \qquad \bar{J}_1 = -\left(\frac{v}{2} + \frac{v'}{2}\right)\cosh\left(\frac{u}{2} + \frac{u'}{2}\right), \tag{36}$$

$$\rho_2 = \frac{a^2}{e^{ap}} \sinh\left(\frac{u}{2} - \frac{u'}{2}\right), \qquad J_2 = \frac{1}{e^{ap}} \sinh\left(\frac{u}{2} + \frac{u'}{2}\right), \tag{37}$$

$$\bar{\rho}_2 = \frac{a^2}{e^{ap}} \left(\frac{v}{2} - \frac{v'}{2}\right)\sinh\left(\frac{u}{2} - \frac{u'}{2}\right) - \frac{a^3 q}{e^{ap}} \sinh\left(\frac{u}{2} - \frac{u'}{2}\right), \tag{38}$$

$$\bar{J}_2 = \frac{1}{\tau^u} \left(\frac{v}{2} - \frac{v'}{2}\right)\sinh\left(\frac{u}{2} - \frac{u'}{2}\right) - \frac{aq}{e^{ap}} \sinh\left(\frac{u}{2} - \frac{u'}{2}\right), \tag{39}$$

$$\rho_3 = -\frac{a}{2}\frac{\tau^{u'}}{e^{ap}}, \qquad J_3 = -\frac{\tau^{u'}}{2e^{ap}} \cosh\left(\frac{u}{2} + \frac{u'}{2}\right), \tag{40}$$

$$\bar{\rho}_3 = -\frac{a}{2}\frac{\tau^{v'}}{e^{ap}} + \frac{a}{2}\frac{aq\tau^{u'}}{e^{ap}}, \tag{41}$$

$$\bar{J}_3 = \left[\frac{aq\tau^{u'}}{2e^{ap}} - \frac{\tau^{u'}}{2e^{ap}}\left(\frac{v}{2} + \frac{v'}{2}\right) - \frac{\tau^{v'}}{2e^{ap}}\right]\cosh\left(\frac{u}{2} + \frac{u'}{2}\right). \tag{42}$$

These conservation laws of the Frobenius sinh-Gordon equation satisfy the identity

$$\partial_t \rho_i = \partial_x J_i, \qquad \partial_t \bar{\rho}_i = \partial_x \bar{J}_i. \tag{43}$$

## 4 Coupled integro-differential systems

From the nonlocal symmetry (16) and (17), we can construct the coupled integro-differential integrable systems with respect to the variable $x$:

$$u_x = \sum_{i=1}^{m} b_i \exp\left(a\int \cosh\left(\frac{u}{2} - \frac{u_i}{2}\right)dx\right), \tag{44}$$

$$u_x + u_{ix} = 2a_i \sinh\left(\frac{u}{2} - \frac{u_i}{2}\right), \quad i = 1, 2, \ldots, m, \tag{45}$$

$$v_x = \sum_{i=1}^{m} b_i a\left[\int \left(\frac{v}{2} - \frac{v_i}{2}\right)\cosh\left(\frac{u}{2} - \frac{u_i}{2}\right)dx\right]\exp\left(a\int \cosh\left(\frac{u}{2} - \frac{u_i}{2}\right)dx\right), \tag{46}$$

$$v_x + v_{ix} = 2a_i\left(\frac{v}{2} - \frac{v_i}{2}\right)\sinh\left(\frac{u}{2} - \frac{u_i}{2}\right), \quad i = 1, 2, \ldots, m. \tag{47}$$

Similarly, from the nonlocal symmetry, we can construct the coupled integro-differential integrable systems with respect to the variable $t$:

$$u_t = \sum_{i=1}^{m} c_i \exp\left(\frac{1}{a}\int \cosh\left(\frac{u}{2} + \frac{u_i}{2}\right) dx\right), \tag{48}$$

$$u_t - u_{it} = \frac{2}{a_i}\sinh\left(\frac{u}{2} + \frac{u_i}{2}\right), \quad i = 1, 2, \ldots, m, \tag{49}$$

$$v_t = \sum_{i=1}^{m} c_i \frac{1}{a}\left[\int \left(\frac{v}{2} + \frac{v_i}{2}\right)\cosh\left(\frac{u}{2} + \frac{u_i}{2}\right) dx\right]\exp\left(\frac{1}{a}\int \cosh\left(\frac{u}{2} + \frac{u_i}{2}\right) dx\right), \tag{50}$$

$$v_t - v_{it} = \frac{2}{a_i}\left(\frac{v}{2} + \frac{v_i}{2}\right)\sinh\left(\frac{u}{2} + \frac{u_i}{2}\right), \quad i = 1, 2, \ldots, m. \tag{51}$$

Of course, these coupled integro-differential systems are integrable systems which might be taken into our detailed study in the future.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
CL contributed to the idea. Other authors contributed to the calculation of this paper. The authors read and approved the final manuscript.

**Author details**
[1]Department of Mathematics, Ningbo University, Ningbo, China. [2]College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, China. [3]School of Mathematical Sciences, Qufu Normal University, Qufu, China.

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