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Razumikhin-type theorems for impulsive differential equations with piecewise constant argument of generalized type

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Abstract

In this paper, we focus on developing Razumikhin technique for stability analysis of impulsive differential equations with piecewise constant argument. Based on the Lyapunov–Razumikhin method and impulsive control theory, we obtain some Razumikhin-type theorems on uniform stability, uniform asymptotic stability, and global exponential stability, which are rarely reported in the literature. The significance and novelty of the results lie in that the stability criteria admit the existence of piecewise constant argument and impulses, which may be either slight at infinity or persistently large. Examples are given to illustrate the effectiveness and advantage of the theoretical results.

MSC: 34K20; 34A37

Keywords: Impulsive differential equations; Piecewise constant argument of generalized type; Lyapunov–Razumikhin method; Asymptotic stability; Exponential stability

1 Introduction

Qualitative theories of impulsive differential equations (IDEs) have been investigated by many researchers in the past three decades due to their potential applications in many fields such as biology, engineering, economics, physics, and so on. Among these theories, the stability problem is of great importance. By now a large number of results on stability problem for various IDE have been obtained by some classical methods and techniques; see [1–9] and the references therein.

In the 1980s, differential equations with piecewise constant argument (DEPCA) that contain deviation of arguments were initially proposed for investigation by Cooke, Wiener, Busenberg, and Shah [10–12]. Later, many interesting results have been obtained and applied efficiently to approximation of solutions and various models in biology, electronics, and mechanics [13–17]. Such equations represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations. Akhmet [18–20] generalized the concept of DEPCA by considering arbitrary piecewise constant functions as arguments; the proposed approach overcomes the limitations in the previously used method of study, namely reduction to discrete equations. Afterward, the results of the theory have been further developed [21, 22] and applied for qualitative analysis and control problem of real models, for example, in neural network models with or without impulsive perturbations [23–33], which have great significance in solving engineering and electronic problems.

Razumikhin technique was originally proposed by Razumikhin [34, 35] for delay differential equations (DDE) and was generalized by other researchers to functional differential equations (FDE) and impulsive functional differential equations (IFDE) [36–44]. The idea of Razumikhin technique is to build a relationship between history and current states using Lyapunov functions, so it is usually called the Lyapunov–Razumikhin method. This method avoids the construction of complicated Lyapunov functionals and provides a technically efficient way to study stability problems for delayed systems or impulsive delayed systems. Considering that DEPCA is a delayed-type system, Akhmet et al. [41] investigated the stability of DEPCA and established some Razumikhin-type theorems on uniform stability and asymptotical stability and applied the results to a logistic equation, whereas impulsive perturbations were not taken into consideration. To the best of our knowledge, there have been few results on stability analysis obtained by the Lyapunov–Razumikhin method for impulsive DEPCA.

Motivated by this discussion, in this paper, we develop the Lyapunov–Razumikhin method for stability of impulsive differential equations with piecewise constant argument and establish some Razumikhin-type theorems on uniform stability, uniform asymptotic stability, and global exponential stability, which are rarely reported in the literature. To overcome the difficulties created by piecewise constant argument and impulses, which may be persistently large [44], as we will see, more complicated and interesting analysis is demanded, in which a (persistent) impulsive control plays an important role to achieve stability. This paper is organized as follows. In Sect. 2, we introduce some basic notations, lemmas, and definitions. In Sect. 3, we present the main theoretical results. In Sect. 4, we give some practical examples to illustrate the effectiveness and novelty of our results. Finally, we conclude the paper in Sect. 5.

2 Preliminaries

Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ the set of positive real numbers, $\mathbb{Z}_+$ the set of nonnegative integers, and $\mathbb{R}^n$ the n-dimensional real space equipped with the Euclidean norm $|\cdot|$. Fix a real-valued sequence $\{\theta_k\}$ such that $0 = \theta_0 < \theta_1 < \cdots < \theta_k < \cdots$ with $\theta_k \to \infty$ as $k \to \infty$.

We use the following sets of functions:

$\Omega_1 = \{\varphi(s) \in C(\mathbb{R}_+, \mathbb{R}_+), \text{strictly increasing}, \varphi(0) = 0, 0 < \varphi(s) < s, s > 0\}$

$\Omega_2 = \{\varphi(s) \in C(\mathbb{R}_+, \mathbb{R}_+), \text{strictly increasing}, \varphi(0) = 0\}$

$\Omega_3 = \{\varphi(s) \in C(\mathbb{R}_+, \mathbb{R}_+), \text{strictly increasing}, \varphi(0) = 0, \varphi(s) > s, s > 0\}$

Consider the following system with impulses and piecewise constant argument:

$$\begin{cases} x'(t) = f(t, x(t), x(\beta(t))), & t \geq t_0 \geq 0, t \neq \theta_k, \\ \Delta x|_{t=\theta_k} = x(\theta_k) - x(\theta_k^-) = I_k(\theta_k, x(\theta_k^-)), & k \in \mathbb{Z}_+ - \{0\}, \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $x(\theta_k^-) = \lim_{t\to\theta_k^-} x(t)$, and $\beta(t) = \theta_k$ for $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}_+$, is the so-called piecewise constant argument.

We need the following assumptions [41]:

(A_1) $f(t,x,y):\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ is piecewise continuous with respect to t and is right-continuous at the possible discontinuous points θ_k, $k\in\mathbb{Z}_+-\{0\}$; $f(t,0,0)=0$ for all $t\geq 0$, and f satisfies the Lipschitz condition

$$\bigl|f(t,x_1,y_1)-f(t,x_2,y_2)\bigr|\leq l\bigl(|x_1-x_2|+|y_1-y_2|\bigr)$$

for all $t\in\mathbb{R}_+$ and $x_1,x_2,y_1,y_2\in\mathbb{R}^n$, where $l>0$ is a constant;

(A_2) $I_k(t,x):\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}^n$ is continuous with respect to t and x, $I_k(\theta_k,0)=0$, and $x_1\neq x_2$ implies $x_1+I_k(\theta_k,x_1)\neq x_2+I_k(\theta_k,x_2)$ for all $k\in\mathbb{Z}_+-\{0\}$.

(A_3) there exists a positive constant θ such that $\theta_{k+1}-\theta_k\leq\theta$ for all $k\in\mathbb{Z}_+$;

(A_4) $l\theta[1+(1+l\theta)e^{l\theta}]<1$;

(A_5) $3l\theta e^{l\theta}<1$.

Notation 1 ([41]) $K(l)=\frac{1}{1-l\theta[1+(1+l\theta)e^{l\theta}]}$.

Lemma 1 ([41]) *Under assumptions* (A_1)–(A_5),

$$\bigl|x\bigl(\beta(t)\bigr)\bigr|\leq K(l)\bigl|x(t)\bigr|$$

for all $t\geq 0$.

Definition 1 A function $x(t)$ is called a solution of (1) on $[t_0,\infty)$ if

(i) $x(t)$ is continuous on each $[\theta_k,\theta_{k+1})\subseteq[t_0,\infty)$ and is right-continuous at $t=\theta_k$, $k\in\mathbb{Z}_+$;

(ii) the derivative $x'(t)$ exists for $t\in[t_0,\infty)$ with the possible exception of the points θ_k, $k\in\mathbb{Z}_+$, where the right-hand derivatives exist;

(iii) system (1) is satisfied by $x(t)$ on $[t_0,\infty)$.

We give the following statement assertion on the existence and uniqueness of solutions of the initial value problem (1).

Theorem 1 *Assume that conditions* (A_1)–(A_5) *are fulfilled. Then, for every* $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}^n$, *there exists a unique solution* $x(t)=x(t,t_0,x_0)$ *of* (1) *on* $[t_0,\infty)$ *such that* $x(t_0)=x_0$.

Proof Existence. Without loss of generality, we assume that $\theta_k\leq t_0<\theta_{k+1}$ for some $k\in\mathbb{Z}_+$. Define the norm $\|x(t)\|=\max_{[\theta_k,\theta_{k+1}]}|x(t)|$, take $x_0(t)=x_0$, $t\in[\theta_k,\theta_{k+1}]$, and the sequence

$$x_{m+1}=x_0+\int_{t_0}^t f\bigl(s,x_m(s),x_m(\theta_k)\bigr)\,ds,\qquad t\in[\theta_k,\theta_{k+1}],m\geq 0.$$

We easily obtain that

$$\bigl\|x_{m+1}(t)-x_m(t)\bigr\|\leq(2l\theta)^{m+1}|x_0|.$$

Thus, (A_5) implies that the sequence $\{x_m(t)\}$ uniformly converges to a unique function $x(t)=x(t,t_0,x_0)$, and it is exactly the unique solution of the integral equation

$$x(t)=x_0+\int_{t_0}^t f\bigl(s,x(s),x(\theta_k)\bigr)\,ds$$

on $[\theta_k, \theta_{k+1})$, which is equivalent to (1) on $[\theta_k, \theta_{k+1})$, and $x(\theta_{k+1}^-)$ exists. Moreover, (A_2) implies that $x(\theta_{k+1}) = x(\theta_{k+1}^-) + I_{k+1}(\theta_{k+1}, x(\theta_{k+1}^-))$ also exists. Taking $x(\theta_{k+1})$ as the new initial value, by the same arguments as before we can get the solution $x(t)$ of (1) on $[\theta_{k+1}, \theta_{k+2})$. Since $\theta_k \to \infty$ as $k \to \infty$, induction completes the proof.

Uniqueness. Denote by $x_j(t) = x(t, t_0, x_0^j)$, $x_j(t_0) = x_0^j$, $j = 1, 2$, solutions of (1), where $\theta_k \le t_0 < \theta_{k+1}$. We will show that, for each $t \in [\theta_k, \theta_{k+1})$, $x_0^1 \neq x_0^2$ implies $x_1(t) \neq x_2(t)$. We have

$$x_1(t) - x_2(t) = x_0^1 - x_0^2 + \int_{t_0}^{t} \bigl[f\bigl(s, x_1(s), x_1(\theta_k)\bigr) - f\bigl(s, x_2(s), x_2(\theta_k)\bigr)\bigr]\,ds.$$

Hence

$$\bigl|x_1(t) - x_2(t)\bigr| \le \bigl|x_0^1 - x_0^2\bigr| + l\theta\bigl|x_1(\theta_k) - x_2(\theta_k)\bigr| + l\left|\int_{t_0}^{t} \bigl|x_1(s) - x_2(s)\bigr|\,ds\right|.$$

The Gronwall–Bellman lemma yields that

$$\bigl|x_1(t) - x_2(t)\bigr| \le \bigl(\bigl|x_0^1 - x_0^2\bigr| + l\theta\bigl|x_1(\theta_k) - x_2(\theta_k)\bigr|\bigr)e^{l\theta}.$$

Particularly,

$$\bigl|x_1(\theta_k) - x_2(\theta_k)\bigr| \le \bigl(\bigl|x_0^1 - x_0^2\bigr| + l\theta\bigl|x_1(\theta_k) - x_2(\theta_k)\bigr|\bigr)e^{l\theta}. \tag{2}$$

Thus

$$\bigl|x_1(t) - x_2(t)\bigr| \le e^{l\theta}\left[1 + \frac{l\theta e^{l\theta}}{1 - l\theta e^{l\theta}}\right]\bigl|x_0^1 - x_0^2\bigr|. \tag{3}$$

Assume on the contrary that there exists $t \in [\theta_k, \theta_{k+1})$ such that $x_1(t) = x_2(t)$. Then

$$x_0^1 - x_0^2 = \int_{t_0}^{t} \bigl[f\bigl(s, x_2(s), x_2(\theta_k)\bigr) - f\bigl(s, x_1(s), x_1(\theta_k)\bigr)\bigr]\,ds.$$

Inequalities (2)–(3) and (A_5) imply that

$$\begin{aligned}
\bigl|x_0^1 - x_0^2\bigr| &= \left|\int_{t_0}^{t} \bigl|f\bigl(s, x_2(s), x_2(\theta_k)\bigr) - f\bigl(s, x_1(s), x_1(\theta_k)\bigr)\bigr|\,ds\right| \\
&\le \frac{2l\theta e^{l\theta}}{1 - l\theta e^{l\theta}}\bigl|x_0^1 - x_0^2\bigr| \\
&< \bigl|x_0^1 - x_0^2\bigr|,
\end{aligned}$$

a contradiction. Especially, if $x_1(\theta_{k+1}^-) \neq x_2(\theta_{k+1}^-)$, then (A_2) implies that $x_1(\theta_{k+1}) \neq x_2(\theta_{k+1})$, and by the same arguments as before we can conclude that also $x_1(t) \neq x_2(t)$ on $[\theta_{k+1}, \theta_{k+2})$, and induction completes the proof of uniqueness. Thus, the proof of Theorem 1 is complete. □

Remark 1 From the proof of Theorem 1 we can see that every solution of system (1) exists uniquely and is piecewise continuous on $[t_0, \infty)$. Moreover, every solution $x(t)$ is right-continuous at the possible discontinuous points θ_k, and $x(\theta_k^-)$ exists. In addition, system (1) obviously has the zero solution.

Definition 2 A function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is said to belong to the class ν_0 if

(i) V is continuous on $[\theta_k, \theta_{k+1}) \times \mathbb{R}^n$, $k \in \mathbb{Z}_+$ and $V(t, 0) \equiv 0$ for all $t \in \mathbb{R}_+$;

(ii) V is continuously differentiable on $[\theta_k, \theta_{k+1}) \times \mathbb{R}^n$, $k \in \mathbb{Z}_+$ and for each $x \in \mathbb{R}^n$, its right-hand derivative exists at $t = \theta_k, k \in \mathbb{Z}_+$.

Definition 3 Given a piecewise continuously differentiable Lyapunov function $V(t, x) \in \nu_0$, the upper right-hand derivative of V with respect to system (1) is defined by

$$D^+ V(t, x, y) = \limsup_{h \to 0^+} \frac{1}{h} \bigl[V\bigl(t + h, x + hf(t, x, y)\bigr) - V(t, x) \bigr]$$

for all $t \in \mathbb{R}_+$ and $x, y \in \mathbb{R}^n$. In particular, for $t \neq \theta_k$ and $x, y \in \mathbb{R}^n$, we have

$$D^+ V(t, x, y) = \frac{\partial V(t, x)}{\partial t} + \operatorname{grad}_x^T V(t, x) f(t, x, y).$$

Besides, several basic definitions such as uniform stability, uniform asymptotical stability, and global weak exponential stability are the same as those in [1, 7], and so we omit them.

3 Main results

In this section, under the same assumptions, we obtain the stability of the zero solution of system (1) based on the Lyapunov–Razumikhin method. Firstly, we present some uniform stability results.

Theorem 2 *Assume that there exist functions $V \in \nu_0$, $u, v \in \Omega_2$, such that*

(i) $u(|x|) \leq V(t, x) \leq v(|x|)$, $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$;

(ii) *for all $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}_+$, and $x, y \in \mathbb{R}^n$, $V(\beta(t), y) \leq V(t, x)$ implies that*

$$D^+ V(t, x, y) \leq 0;$$

(iii) *for all $k \in \mathbb{Z}_+ - \{0\}$ and $x \in \mathbb{R}^n$, we have $V(\theta_k, x + I_k(\theta_k, x)) \leq (1 + b_k) V(\theta_k^-, x)$, where $b_k \geq 0$ with $\sum_{k=1}^{\infty} b_k < \infty$.*

Then the zero solution of (1) is uniformly stable.

Theorem 3 *Assume that there exist functions $V \in \nu_0$, $u, v \in \Omega_2$, $\psi \in \Omega_1$, and $W \in \Omega_2$ such that*

(i) $u(|x|) \leq V(t, x) \leq v(|x|)$, $(t, x) \in [t_0, \infty) \times \mathbb{R}^n$;

(ii) *for all $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}_+$, and $x, y \in \mathbb{R}^n$, $V(\beta(t), y) \leq \psi^{-1}(V(t, x))$ implies that*

$$D^+ V(t, x, y) \leq g(t) W\bigl(V(t, x)\bigr),$$

where $g : [t_0, \infty) \to \mathbb{R}_+$ is locally integrable;

(iii) *for all $k \in \mathbb{Z}_+ - \{0\}$ and $x \in \mathbb{R}^n$, $V(\theta_k, x + I_k(\theta_k, x)) \leq \psi(V(\theta_k^-, x))$;*

(iv) *for all $k \in \mathbb{Z}_+$, $\inf_{\mu \in \mathbb{R}_+} \int_{\psi(\mu)}^{\mu} \frac{ds}{W(s)} > \int_{\theta_k}^{\theta_{k+1}} g(s)\, ds$.*

Then the zero solution of (1) is uniformly stable.

Remark 2 We may observe that the two theorems generalize the existing corresponding results, and their proofs can be formulated by combining the corresponding theorems in [38] and [41], and so we omit them.

Theorem 4 *Assume that there exist functions $V \in \nu_0$, $u, \nu \in \Omega_2$, $\psi \in \Omega_3$, and $W \in \Omega_2$ such that*

(i) $u(|x|) \leq V(t,x) \leq \nu(|x|)$, $(t,x) \in [t_0,\infty) \times \mathbb{R}^n$;

(ii) *for all $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}_+$, and $x, y \in \mathbb{R}^n$, $V(\beta(t), y) < \psi(V(t,x))$ implies that*

$$D^+ V(t,x,y) \leq -g(t) W\big(V(t,x)\big),$$

where $g : [t_0,\infty) \to \mathbb{R}_+$ is locally integrable;

(iii) *for all $k \in \mathbb{Z}_+ - \{0\}$ and $x \in \mathbb{R}^n$, $V(\theta_k, x + I_k(\theta_k, x)) \leq \psi(V(\theta_k^-, x))$;*

(iv) *for all $k \in \mathbb{Z}_+$, $\sup_{\nu \in \mathbb{R}_+} \int_\nu^{\psi(\nu)} \frac{ds}{W(s)} < \int_{\theta_k}^{\theta_{k+1}} g(s)\,ds$.*

Then the zero solution of (1) *is uniformly stable.*

Proof For given $\varepsilon > 0$, we may choose $\delta > 0$ such that $\psi(\nu(\delta)) < u(\varepsilon)$. For any $t_0 \geq 0$ and $|x_0| < \delta$, we shall show that $|x(t)| < \varepsilon$, $t \geq t_0$. To show that this δ is the needed one, we consider two cases where $t_0 = \theta_i$ for some $i \in \mathbb{Z}_+$ and another one where $t_0 \neq \theta_j$ for all $j \in \mathbb{Z}_+$.

First, let $t_0 = \theta_{m-1}$ for some $m \in \mathbb{Z}_+ - \{0\}$. For convenience, we take $V(t) = V(t, x(t))$, $D^+V(t) = D^+V(t, x(t), x(\beta(t)))$. We first claim that

$$V(t) \leq \nu(\delta), \quad t \in [t_0, \theta_m). \tag{4}$$

Clearly, $V(t_0) \leq \nu(\delta)$. If (4) does not hold, then there exist points t_1 and t_2, $t_0 \leq t_1 < t_2 < \theta_m$, such that $V(t_1) = \nu(\delta)$ and $V(t) > \nu(\delta)$ for $t \in (t_1, t_2]$. Applying the mean-value theorem, we get

$$\frac{V(t_2) - V(t_1)}{t_2 - t_1} = D^+V(\hat{t}) > 0 \tag{5}$$

for some $\hat{t} \in (t_1, t_2)$. Since $\psi(V(\hat{t})) > V(\hat{t}) > \nu(\delta) \geq V(t_0) = V(\beta(\hat{t}))$, it follows from condition (ii) that $D^+V(\hat{t}) \leq -g(\hat{t})W(V(\hat{t})) < 0$, which contradicts (5), and so (4) holds. From (4) and condition (iii) we obtain

$$V(\theta_m) \leq \psi\big(V\big(\theta_m^-\big)\big) \leq \psi\big(\nu(\delta)\big).$$

Using the same argument as before, we can prove that

$$V(t) \leq \psi\big(\nu(\delta)\big), \quad t \in [\theta_m, \theta_{m+1}). \tag{6}$$

We now further claim that

$$V\big(\theta_{m+1}^-\big) \leq \nu(\delta). \tag{7}$$

Suppose not, that is, $V(\theta_{m+1}^-) > \nu(\delta)$. There are two cases: (a) $V(t) > \nu(\delta)$ for all $t \in [\theta_m, \theta_{m+1})$, and (b) there exists $t \in [\theta_m, \theta_{m+1})$ such that $V(t) \leq \nu(\delta)$. For case (a), we have $\psi(V(t)) > \psi(\nu(\delta)) \geq V(\beta(t))$ for all $t \in [\theta_m, \theta_{m+1})$. By condition (ii) we get

$$D^+V(t) \leq -g(t)W\big(V(t)\big), \quad t \in [\theta_m, \theta_{m+1}).$$

Integrating this inequality yields

$$\int_{\theta_m}^{\theta_{m+1}} g(s)\,ds \le \int_{V(\theta_{m+1}^-)}^{V(\theta_m)} \frac{ds}{W(s)} \le \int_{\nu(\delta)}^{\psi(\nu(\delta))} \frac{ds}{W(s)} \le \sup_{\mu\in\mathbb{R}_+} \int_{\mu}^{\psi(\mu)} \frac{1}{W(s)}\,ds,$$

which is a contradiction with condition (iv). For case (b), we set

$$\bar{t} = \sup\bigl\{t \in [\theta_m,\theta_{m+1}) | V(t) \le \nu(\delta)\bigr\}.$$

Obviously, $V(\bar{t}) = \nu(\delta)$ for $\bar{t} < \theta_{m+1}$. Then there exists $\tilde{t} \in (\bar{t},\theta_{m+1})$ such that $D^+V(\tilde{t}) > 0$, whereas since $\psi(V(\tilde{t})) > \psi(\nu(\delta)) \ge V(\beta(\tilde{t}))$, condition (ii) implies that $D^+V(\tilde{t}) < 0$, a contradiction. By now, we get the following statement:

$$V(t) \le \begin{cases} \nu(\delta), & t \in [t_0,\theta_m), \\ \psi(\nu(\delta)), & t \in [\theta_m,\theta_{m+1}), \\ \nu(\delta), & t \to \theta_{m+1}^-. \end{cases}$$

By the same argument as in the proofs of (6) and (7), in general, we can deduce that

$$V(t) \le \begin{cases} \nu(\delta), & t \in [t_0,\theta_m), \\ \psi(\nu(\delta)), & t \in [\theta_{m+j-1},\theta_{m+j}), \\ \nu(\delta), & t \to \theta_{m+j}^-, j \in \mathbb{Z}_+ - \{0\}. \end{cases} \tag{8}$$

Hence from condition (i) and (8) we have

$$u\bigl(|x(t)|\bigr) \le V(t) \le \psi\bigl(\nu(\delta)\bigr) < u(\varepsilon), \quad t \ge t_0,$$

which implies that $|x(t)| < \varepsilon$, $t \ge t_0$.

Now, let $t_0 \ge 0$ with $t_0 \ne \theta_i$ for any $i \in \mathbb{Z}_+$. By the idea in [41] and Lemma 1 we take $\delta_1 = \frac{\delta}{K(l)}$, where δ satisfies $\psi(\nu(\delta)) < u(\varepsilon)$. Then $|x_0| < \delta_1$ implies $|x(t)| < \varepsilon$, $t \ge t_0$. We see that the evaluation of δ_1 is independent of t_0. The proof of Theorem 4 is complete. □

Remark 3 It should be noted that Theorem 4 allows for significant increases in V at impulse times, which may be persistently large ($\psi(s) > s$ for $s > 0$) and appropriately controlled by the length of impulsive intervals. So, the length of impulsive intervals cannot be too small; in other words, the disturbed impulses cannot happen too frequently, with the same idea as that in [44]; yet the impulse conditions presented in Theorem 4 are more general than that in [44] and can be verified more easily and conveniently.

Now, we present uniform asymptotical stability and exponential stability results.

Theorem 5 *Assume that there exist functions $V \in \nu_0$, $u, \nu \in \Omega_2$, $\psi \in \Omega_3$, and $W \in \Omega_2$ such that*

(i) $u(|x|) \le V(t,x) \le \nu(|x|)$, $(t,x) \in [t_0,\infty) \times \mathbb{R}^n$;

(ii) *for all $k \in \mathbb{Z}_+ - \{0\}$ and $x \in \mathbb{R}^n$, $V(\theta_k, x + I_k(\theta_k,x)) \le (1+b_k)V(\theta_k^-,x)$, where $b_k \ge 0$ with $\bar{M} = \sum_{k=1}^{\infty} b_k < \infty$.*

(iii) *for all* $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}_+$ *and* $x, y \in \mathbb{R}^n$, $V(\beta(t), y) < \psi(V(t,x))$ *implies that*

$$D^+ V(t,x,y) \leq -W(|x|),$$

where $\psi(s) > Ms$ *for* $s > 0$ *with* $M = \prod_{k=1}^{\infty}(1+b_k)$.

Then the zero solution of (1) *is uniformly asymptotically stable.*

Proof Clearly, the conditions of Theorem 5 imply the uniform stability by Theorem 2.

First, let $t_0 = \theta_{m-1}$ for some $m \in \mathbb{Z}_+ - \{0\}$. We take $V(t) = V(t, x(t))$, $D^+V(t) = D^+V(t, x(t), x(\beta(t)))$ for convenience. For given $\rho > 0$, we choose $\delta > 0$ such that $Mv(\delta) = u(\rho)$, and $|x(t_0)| < \delta$ implies that, for $t \geq t_0$,

$$V(t) \leq Mv(\delta) \quad \text{and} \quad \bigl|x(t)\bigr| < \rho.$$

In what follows, we show that, for arbitrary ε, $0 < \varepsilon < \rho$, there exists $T = T(\varepsilon) > 0$ such that $|x(t)| \leq \varepsilon$ for $t \geq t_0 + T$ if $|x(t_0)| < \delta$.

Obviously, there exists a number $a > 0$ such that $\psi(s) - Ms > a$ for $M^{-1}u(\varepsilon) \leq s \leq Mv(\delta)$. Let $N = N(\varepsilon)$ be the smallest positive integer such that $M^{-1}(u(\varepsilon) + Na) \geq Mv(\delta)$. Choose $t_k = k(\frac{Mv(\delta)(1+\bar{M})}{\gamma} + \theta) + \theta_{m-1}$, $k = 1, 2, \ldots, N$, where

$$\gamma = \inf_{v^{-1}(M^{-1}u(\varepsilon)) \leq s \leq \rho} W(s).$$

We will prove that

$$V(t) \leq u(\varepsilon) + (N-k)a \quad \text{for } t \geq t_k, k = 0, 1, \ldots, N. \tag{9}$$

We have $V(t) \leq Mv(\delta) \leq M^{-1}(u(\varepsilon) + Na) \leq u(\varepsilon) + Na$ for $t \geq t_0 = \theta_{m-1}$. Hence (9) holds for $k = 0$. Now, suppose that (9) holds for some $0 < k < N$. Let us show that

$$V(t) \leq u(\varepsilon) + (N-k-1)a \quad \text{for } t \geq t_{k+1}. \tag{10}$$

Let $J_k = [\beta(t_k) + \theta, t_{k+1}]$. We first claim that there exists $t^* \in J_k$ such that

$$V\bigl(t^*\bigr) \leq M^{-1}\bigl[u(\varepsilon) + (N-k-1)a\bigr]. \tag{11}$$

Otherwise, $V(t) > M^{-1}[u(\varepsilon) + (N-k-1)a]$ for all $t \in J_k$. On the other side, $V(t) \leq u(\varepsilon) + (N-k)a$ for $t \geq t_k$ implies that

$$V\bigl(\beta(t)\bigr) \leq u(\varepsilon) + (N-k)a \quad \text{for } t \geq \beta(t_k) + \theta.$$

Hence, for $t \in J_k$,

$$\psi\bigl(V(t)\bigr) > MV(t) + a \geq u(\varepsilon) + (N-k)a \geq V\bigl(\beta(t)\bigr).$$

It follows from hypothesis (iii) and the definition of γ that

$$D^+V(t) \leq -W\bigl(|x|\bigr) \leq -\gamma \quad \text{for all } t \neq \theta_i \text{ in } J_k.$$

We get

$$\begin{aligned} V(t_{k+1}) &\leq V\big(\beta(t_k)+\theta\big) - \gamma\big(t_{k+1}-\beta(t_k)-\theta\big) + \sum_{\beta(t_k)+\theta<\theta_i\leq t_{k+1}} \big[V(\theta_i)-V\big(\theta_i^-\big)\big] \\ &\leq M\nu(\delta) - \gamma(t_{k+1}-t_k-\theta) + \sum_{i=1}^{\infty} b_i V\big(\theta_i^-\big) \\ &\leq M\nu(\delta) + M\nu(\delta)\bar{M} - M\nu(\delta)(1+\bar{M}) = 0, \end{aligned}$$

a contradiction, and so (11) holds.

Let $q = \min\{i \in \mathbb{Z}_+ : \theta_i > t^*\}$. We claim that

$$V(t) \leq M^{-1}\big[u(\varepsilon) + (N-k-1)a\big], \quad t^* \leq t < \theta_q. \tag{12}$$

If (12) does not hold, then there exists $\bar{t} \in (t^*, \theta_q)$ such that

$$V(\bar{t}) > M^{-1}\big[u(\varepsilon) + (N-k-1)a\big] \geq V\big(t^*\big).$$

Thus, there exists $\tilde{t} \in (t^*, \bar{t})$ such that $\tilde{t} \neq \theta_i$, $D^+V(\tilde{t}) > 0$, and $V(\tilde{t}) > M^{-1}[u(\varepsilon)+(N-k-1)a]$. However,

$$\psi\big(V(\tilde{t})\big) > MV(\tilde{t}) + a > u(\varepsilon) + (N-k)a \geq V\big(\beta(\tilde{t})\big)$$

implies that $D^+V(\tilde{t}) \leq -\gamma < 0$, a contradiction, so (12) holds.

From (12) and (ii) we have

$$V(\theta_q) \leq (1+b_q)V\big(\theta_q^-\big) \leq (1+b_q)M^{-1}\big[u(\varepsilon) + (N-k-1)a\big].$$

Therefore, for all $t \in [t^*, \theta_q]$,

$$V(t) \leq (1+b_q)M^{-1}\big[u(\varepsilon) + (N-k-1)a\big].$$

Similarly, we can show that, for all $t \in [\theta_q, \theta_{q+1}]$,

$$V(t) \leq (1+b_q)(1+b_{q+1})M^{-1}\big[u(\varepsilon) + (N-k-1)a\big],$$

and by a simple induction we conclude that, for $t \in [\theta_{q+i}, \theta_{q+i+1}]$, $i = 0, 1, 2, \ldots$,

$$V(t) \leq (1+b_q)(1+b_{q+1}) \cdots (1+b_{q+i+1})M^{-1}\big[u(\varepsilon) + (N-k-1)a\big].$$

Thus $V(t) \leq u(\varepsilon) + (N-k-1)a$ for $t \geq t^*$, and so (10) holds.

For $k = N$, we get

$$V(t) \leq u(\varepsilon), \quad t \geq t_N = N\left[\frac{M\nu(\delta)(1+\bar{M})}{\gamma} + \theta\right] + \theta_{m-1}.$$

Hence $|x(t)| \leq \varepsilon$ for $t \geq t_0 + T$, where $T = N[\frac{M\nu(\delta)(1+\bar{M})}{\gamma} + \theta]$, proving the uniform asymptotic stability for $t_0 = \theta_{m-1}$, $m \in \mathbb{Z}_+ - \{0\}$.

Now, let $t_0 \neq \theta_i$ for any $i \in \mathbb{Z}_+$. Similar to the arguments in Theorem 4, taking $\delta_1 = \frac{\delta}{K(\rho)}$, we obtain that $|x(t_0)| < \delta_1$ implies $|x(t)| < \varepsilon$, $t \geq t_0 + T$. The proof of Theorem 5 is complete. □

Theorem 6 *Assume that there exist functions $V \in \nu_0$, $u, v, W \in \Omega_2$ and constants $\eta > 0$, $\mu > 1$ such that*

(i) $u(|x|) \leq V(t,x) \leq v(|x|)$, $(t,x) \in [t_0, \infty) \times \mathbb{R}^n$;

(ii) *for all $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}_+$ and $x, y \in \mathbb{R}^n$, $e^{\eta\beta(t)} V(\beta(t), y) < \mu e^{\eta t} V(t,x)$ implies that*

$$D^+ V(t,x,y) \leq -g(t) W\bigl(V(t,x)\bigr),$$

where $g : [t_0, \infty) \to \mathbb{R}_+$ is locally integrable;

(iii) *for all $k \in \mathbb{Z}_+ - \{0\}$ and $x \in \mathbb{R}^n$, $V(\theta_k, x + I_k(\theta_k, x)) \leq \mu V(\theta_k^-, x)$;*

(iv) $\inf_{t \in \mathbb{R}_+} g(t) \inf_{t \in \mathbb{R}_+} \frac{W(t)}{t} > \frac{1}{\tau} \ln \mu$, $\tau = \inf_{k \in \mathbb{Z}_+} \{\theta_{k+1} - \theta_k\}$.

Then the zero solution of (1) is globally weakly exponentially stable.

Proof Set

$$h := \inf_{t \in \mathbb{R}_+} g(t) \inf_{t \in \mathbb{R}_+} \frac{W(t)}{t}. \tag{13}$$

Given a constant ε such that

$$0 < \varepsilon < \min\left\{h - \frac{\ln \mu}{\tau}, \eta\right\}, \tag{14}$$

for any $t_0 \geq 0$, we shall prove that $u(|x(t)|) \leq \phi(|x_0|) e^{-\varepsilon(t-t_0)}$, $t \geq t_0$, where $\phi(\mathbb{R}_+, \mathbb{R}_+) \in \Omega_2$. We also consider two cases where $t_0 = \theta_i$ for some $i \in \mathbb{Z}_+$ and where $t_0 \neq \theta_j$ for all $j \in \mathbb{Z}_+$.

First, let $t_0 = \theta_{m-1}$ for some $m \in \mathbb{Z}_+ - \{0\}$. We take $V(t) = V(t, x(t))$, $D^+ V(t) = D^+ V(t, x(t), x(\beta(t)))$ for convenience and define $\Phi(t) = V(t) e^{\varepsilon(t-t_0)}$, $t \geq t_0$. We will further show that

$$\Phi(t) \leq v\bigl(|x_0|\bigr), \quad t \in [t_0, \theta_m). \tag{15}$$

Clearly, $\Phi(t_0) = V(t_0) \leq v(|x_0|)$. If (15) does not hold, then there exist t_1 and $t_2, t_0 \leq t_1 < t_2 < \theta_m$, such that $\Phi(t_1) = v(|x_0|)$ and $\Phi(t) > v(|x_0|)$ for $t \in (t_1, t_2]$. Applying the mean-value theorem, we get

$$\frac{\Phi(t_2) - \Phi(t_1)}{t_2 - t_1} = D^+ \Phi(\hat{t}) > 0 \tag{16}$$

for some $\hat{t} \in (t_1, t_2)$, and $\Phi(\hat{t}) > v(|x_0|) \geq \Phi(t_0) = \Phi(\beta(\hat{t}))$. From the definition of Φ, together with (14), we have

$$e^{\eta\beta(\hat{t})} V\bigl(\beta(\hat{t})\bigr) < e^{\eta\hat{t}} V(\hat{t}) < \mu e^{\eta\hat{t}} V(\hat{t}).$$

It follows from condition (ii), (13), and (14) that

$$\begin{aligned} D^+\Phi(\hat{t}) &= \frac{D^+V(\hat{t})}{V(\hat{t})} e^{\varepsilon(\hat{t}-t_0)} V(\hat{t}) + V(\hat{t})\varepsilon e^{\varepsilon(\hat{t}-t_0)} \\ &\le \Phi(\hat{t})\left[\varepsilon - g(\hat{t})\frac{W(V(\hat{t}))}{V(\hat{t})}\right] \\ &\le \Phi(\hat{t})(\varepsilon - h) < 0, \end{aligned}$$

which contradicts (16), and so (15) holds. From (15) and condition (iii) we obtain

$$\Phi(\theta_m) \le \mu\Phi(\theta_m^-) \le \mu\nu(|x_0|).$$

Using the same argument as before, we can prove that

$$\Phi(t) \le \mu\nu(|x_0|), \quad t \in [\theta_m, \theta_{m+1}). \tag{17}$$

We now further claim that

$$\Phi(\theta_{m+1}^-) \le \nu(|x_0|). \tag{18}$$

Suppose not, that is, $\Phi(\theta_{m+1}^-) > \nu(|x_0|)$. There are two cases: (a) $\Phi(t) > \nu(|x_0|)$ for all $t \in [\theta_m, \theta_{m+1})$, and (b) there exists some $t \in [\theta_m, \theta_{m+1})$ such that $\Phi(t) \le \nu(|x_0|)$. In case (a), we have $\mu\Phi(t) > \mu\nu(|x_0|) \ge \Phi(\beta(t))$ for all $t \in [\theta_m, \theta_{m+1})$, which implies that $\mu e^{\eta t}V(t) > e^{\eta\beta(t)}V(\beta(t))$, $t \in [\theta_m, \theta_{m+1})$. By condition (ii) we get

$$\begin{aligned} D^+\Phi(t) &= D^+V(t)e^{\varepsilon(t-t_0)} + V(t)\varepsilon e^{\varepsilon(t-t_0)} \\ &\le \Phi(t)\left[\varepsilon - g(t)\frac{W(V(t))}{V(t)}\right] \\ &\le \Phi(t)(\varepsilon - h), \quad t \in [\theta_m, \theta_{m+1}). \end{aligned}$$

Integrating this inequality yields that

$$(h-\varepsilon)\tau \le \int_{\Phi(\theta_{m+1}^-)}^{\Phi(\theta_m)} \frac{ds}{s} \le \int_{\nu(|x_0|)}^{\mu\nu(|x_0|)} \frac{ds}{s} = \ln\mu,$$

which is a contradiction with (14). For case (b), we set

$$\bar{t} = \sup\{t \in [\theta_m, \theta_{m+1}) | \Phi(t) \le \nu(|x_0|)\}.$$

Obviously, $\bar{t} < \theta_{m+1}$ and $\Phi(\bar{t}) = \nu(|x_0|)$. Then there exists $\tilde{t} \in (\bar{t}, \theta_{m+1})$ such that $D^+\Phi(\tilde{t}) > 0$, whereas for all $t \in [\bar{t}, \theta_{m+1})$, we have $\mu\Phi(t) \ge \mu\nu(|x_0|) \ge \Phi(\beta(t))$, which implies $\mu V(t)e^{\eta t} \ge V(\beta(t))e^{\eta\beta(t)}$, and by condition (ii) we get $D^+\Phi(t) < 0$, $t \in [\bar{t}, \theta_{m+1})$, which is also a contradiction. By now, we get the following statement:

$$\Phi(t) \le \begin{cases} \nu(|x_0|), & t \in [t_0, \theta_m), \\ \mu\nu(|x_0|), & t \in [\theta_m, \theta_{m+1}), \\ \nu(|x_0|), & t \to \theta_{m+1}^-. \end{cases}$$

By the same argument as in the proofs of (17) and (18), in general, we can deduce that

$$\Phi(t) \leq \begin{cases} \nu(|x_0|), & t \in [t_0, \theta_m), \\ \mu\nu(|x_0|), & t \in [\theta_{m+j-1}, \theta_{m+j}), \\ \nu(|x_0|), & t \to \theta_{m+j}^-, j \in \mathbb{Z}_+ - \{0\}. \end{cases} \tag{19}$$

Hence we have $\Phi(t) \leq \mu\nu(|x_0|)$ for all $t \geq t_0$, which implies that

$$u(|x(t)|) \leq V(t) = \Phi(t)e^{-\varepsilon(t-t_0)} \leq \mu\nu(|x_0|)e^{-\varepsilon(t-t_0)}, \quad t \geq t_0.$$

Now, let $t_0 \neq \theta_i$ for any $i \in \mathbb{Z}_+$. Similarly to the arguments in Theorem 4, we can obtain

$$u(|x(t)|) \leq \mu\nu(K(l)|x_0|)e^{-\varepsilon(t-t_0)}, \quad t \geq t_0.$$

The proof of Theorem 6 is complete. □

4 Examples

In this section, we give some examples to illustrate the theoretical results obtained in the previous section.

Example 1 Consider the system

$$\begin{cases} x'(t) = -ax(t) + bx(\beta(t)), & t \geq 0, \\ x(\theta_k) = px(\theta_k^-)), & k \in \mathbb{Z}_+ - \{0\}, \end{cases} \tag{20}$$

where $a > 0$, $p > 1$, $b \in \mathbb{R}$ are some constants, satisfying $a - |b|p > 0$, $\theta_{k+1} - \theta_k > \frac{\ln p}{a-|b|p}$.

Let $V(t,x) = |x|$, $\psi(s) = ps$, $W(s) = s$. Then

$$V(\theta_k, x + I_k(\theta_k, x)) = |x(\theta_k)| = p|x(\theta_k^-)| = \psi(V(\theta_k^-, x)),$$

and for all $t \neq \theta_k$, $V(\beta(t), x(\beta(t))) < pV(t,x)$, which means that $|x(\beta(t))| < p|x(t)|$ implies

$$V'(t,x)|_{(20)} \leq -a|x(t)| + |b||x(\beta(t))| \leq (-a + |b|p)|x(t)| = -g(t)W(V(t,x)),$$

where $g(t) = a - |b|p > 0$. For all $k \in \mathbb{Z}_+$,

$$\sup_{\nu \in \mathbb{R}_+} \int_\nu^{p\nu} \frac{ds}{W(s)} = \ln p = (a - |b|p)\frac{\ln p}{a - |b|p} < \int_{\theta_k}^{\theta_{k+1}} g(s)\, ds.$$

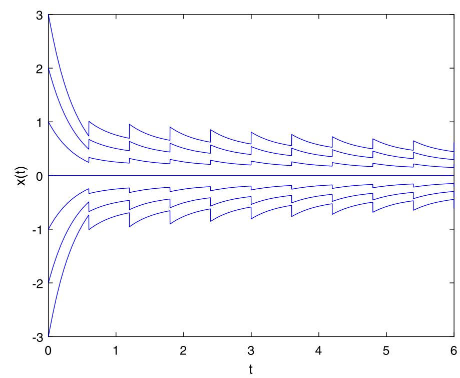
Therefore by Theorem 4 the zero solution of system (20) is uniformly stable.

In particular, let $a = e^{0.3} + 1$, $p = e^{0.3}$, $b = 1$, and $\theta_k = 0.6k$. Then the simulation of system (20) is shown in Fig. 1, which illustrates the stability of zero solution.

Example 2 Consider the system

$$\begin{cases} x'(t) = -a(t)x(t) + b(t)x(\beta(t)), & t \geq 0, \\ x(\theta_k) = x(\theta_k^-)) + I_k(\theta_k, x(\theta_k^-)), & k \in \mathbb{Z}_+ - \{0\}, \end{cases} \tag{21}$$

Figure 1 Simulation of system (20)



where $a(t)$ and $b(t)$ are continuous functions on $[0,+\infty)$ satisfying $a(t) \geq a > 0$ and $|b(t)| \leq b$, $I_k(t,x) \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ satisfy $|x + I_k(t,x)| \leq (1+b_k)|x|$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, with $b_k \geq 0$ and $\sum_{k=1}^{\infty} b_k < \infty$. Also suppose that $a - bM > 0$ with $M = \prod_{k=1}^{\infty}(1+b_k)$.

Let $V(t,x) = |x|$, $\psi(s) = qMs$, where $q > 1$ is such that $a - bqM > 0$. Then

$$V\bigl(\theta_k, x + I_k(\theta_k,x)\bigr) = \bigl|x(\theta_k)\bigr| \leq (1+b_k)\bigl|x\bigl(\theta_k^-\bigr)\bigr| = (1+b_k)V\bigl(\theta_k^-,x\bigr),$$

and for all $t \neq \theta_k$, $V(\beta(t),x(\beta(t))) < \psi(V(t,x))$, which means $|x(\beta(t))| < qM|x(t)|$, implies that

$$V'(t,x)|_{(21)} \leq -a(t)\bigl|x(t)\bigr| + \bigl|b(t)\bigr|\bigl|x\bigl(\beta(t)\bigr)\bigr| \leq -(a - bqM)\bigl|x(t)\bigr|.$$

Therefore by Theorem 5 the zero solution of system (21) is uniformly asymptotically stable.

Example 3 Consider the system

$$\begin{cases} x'(t) = -a(t)x(t) + b(t)x(\beta(t)), & t \geq 0, \\ x(\theta_k) = px(\theta_k^-)), & k \in \mathbb{Z}_+ - \{0\}, \end{cases} \tag{22}$$

where $a(t)$ and $b(t)$ are continuous functions on $[0,+\infty)$, $p > 1$ is a given constant, and there exist constants $\gamma > 0$ and $\eta > 0$ such that $e^{\beta(t)}(a(t) - \gamma) \geq p|b(t)|e^{\eta t}$ for all $t \geq 0$. Also, suppose that $\gamma(\theta_{k+1} - \theta_k) > \ln p$ for all $k \in \mathbb{Z}_+$.

Let $V(t,x) = |x|$, $W(t) = \gamma t$, and $g(t) \equiv 1$. Then

$$V\bigl(\theta_k, x + I_k(\theta_k,x)\bigr) = p\bigl|x\bigl(\theta_k^-\bigr)\bigr| = pV\bigl(\theta_k^-,x\bigr),$$

and for all $t \neq \theta_k$, $e^{\eta\beta(t)}V(\beta(t),x(\beta(t))) < pe^{\eta t}V(t,x)$, which means $e^{\eta\beta(t)}|x(\beta(t))| < pe^{\eta t}|x(t)|$, implies that

$$V'(t,x)|_{(22)} \leq -a(t)\bigl|x(t)\bigr| + \bigl|b(t)\bigr|\bigl|x\bigl(\beta(t)\bigr)\bigr| \leq -\gamma\bigl|x(t)\bigr| = -g(t)W\bigl(V(t,x)\bigr).$$

Also,

$$\inf_{t\in\mathbb{R}_+} g(t) \inf_{t\in\mathbb{R}_+} \frac{W(t)}{t} = \gamma > \frac{\ln p}{\inf_{k\in\mathbb{Z}_+}\{\theta_{k+1} - \theta_k\}}.$$

Figure 2 Simulation of system (22) with stable parameters

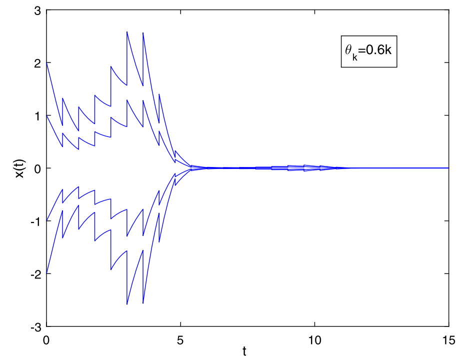
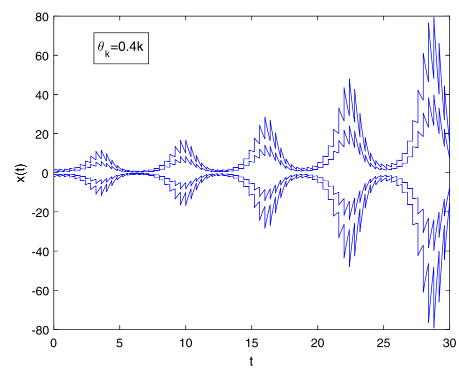


Figure 3 Simulation of system (22) with unstable parameters



Therefore by Theorem 6 the zero solution of system (22) is globally weakly exponentially stable.

In particular, let $p = e^{0.5}$, $\gamma = 1$, $\eta = 0.2$, $b(t) = \sin t$, $\theta_k = 0.6k$, and $a(t) = 1 + |\sin t|e^{0.2t+0.5-0.6k}$, $t \in [\theta_k, \theta_{k+1})$, $k \in \mathbb{Z}_+$. Then the simulation of system (22) is shown in Fig. 2, which illustrates the global exponential stability of zero solution. In addition, if we let $\theta_k = 0.4k$ and other parameters be fixed, then it will go against the required conditions. In this case, Fig. 3 tells us that the system becomes unstable, which shows the sharpness of our results.

Remark 4 On one hand, the stability of the systems in the examples may not be obtained by the results in existing references due to the existence of both impulse and piecewise constant argument. So the results in this paper are more general than those in the references. On the other hand, it should be noted that in the three systems, the sequence $\{\theta_k\}$ needs to satisfy conditions (A3), (A4), and (A5). I particular, in Examples 1 and 3, considering the presence of persisting impulses, we give the lower bound of θ (e.g., $\theta_{k+1} - \theta_k > \frac{\ln p}{a-|b|p}$ in Example 1), and, in fact, the range of the parameter θ is constructed for the stability of zero solution in Examples 1 and 3.

5 Conclusion

In this paper, we have derived several stability theorems for nonlinear systems with impulses and piecewise constant argument by employing the- Lyapunov-Razumikhin method and impulsive control theory. Examples are also given to show the effectiveness and novelty of the results. The theoretical results obtained can be applied to study the stability problem of many nonlinear impulsive models with piecewise constant argument such as neural networks, population models, and other biological models. However, our results are based on the fact that the piecewise constant argument in systems is of retarded type, so it is interesting to develop the Lyapunov–Razumikhin technique to impulsive systems with piecewise constant argument of advanced type or hybrid type, which requires further research in the future.

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Abbreviations
IDE, impulsive differential equation; DEPCA, differential equation with piecewise constant argument; DDE, delay differential equation; FDE, functional differential equation; IFDE, impulsive functional differential equation.

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Authors' contributions
The main idea of this paper was proposed by QX. QX prepared the manuscript initially and performed all the steps of the proofs in this research. All authors read and approved the final manuscript.

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