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On neutral impulsive stochastic differential equations with Poisson jumps

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Abstract

We study the results of existence and continuous dependence on neutral impulsive stochastic differential equations with Poisson jumps. We have also created some conditions confirming exponential stability.

Keywords: Stochastic differential equations; Contraction mapping; Continuous dependence exponential stability; Poisson process; Impulsive system

1 Introduction

Stochastic differential equations have been investigated as mathematical models to describe the dynamical behavior of real life phenomena. It is essential to take into account the environmental disturbances as well as the time delay while constructing realistic models in the area of engineering, biology, etc. Neutral functional differential equations have been introduced in [11] for the deterministic case. Neutral stochastic functional differential equations (NSFDEs) have been initiated in [12] and their usage in aeroelasticity was pointed out. In the last few decades several studies on quantitative and qualitative properties of NSFDEs were carried out (see [4, 5, 20] and the references therein).

Impulsive differential equations thrive to be a promising area and have gained much attention among the researchers due to their potential application in various fields such as orbital transfer of satellite, dosage supply in pharmacokinetics, etc. It is worth mentioning that many real world systems are subjected to stochastic abrupt changes, and therefore it is necessary to investigate them using impulsive stochastic functional differential equations. Few works have been reported in the study of NSFDEs with impulsive effects, refer to [1, 2, 18].

Moreover, many practical systems (such as sudden price variations (jumps) due to market crashes, earthquakes, hurricanes, epidemics, and so on) may undergo some jump type stochastic perturbations. The sample paths of such systems are not continuous. Therefore, it is more appropriate to consider stochastic processes with jumps to describe such models. In general, these jump models are derived from Poisson random measure. The sample paths of such systems are right continuous and possess left limits. Recently, many researchers have been focusing their attention towards the theory and applications of NSFDEs with Poisson jumps. To be more precise, existence and stability results on NSFDEs with jump process can be found in [3, 4, 6, 8, 14, 17, 19, 21, 23] and the references therein. Particularly, Boufoussi and Hajji [4] investigated successive approximation of NSFDEs with jumps. Subsequently, SDEs with Poisson jumps were established by few authors; for example, Wang et al. [21] studied them under a local non-Lipschitz condition, Cui and Yan [8] investigated them for the case of infinite delay. Chen [6, 7] studied the exponential stability by establishing impulsive integral inequality. Further, we refer [10, 15, 19, 24] to investigate the exponential stability. The purpose of this manuscript is to study the impulsive NSFDEs driven by Poisson jumps.

This paper comprises five sections. Section 1 becomes the introduction. We recollect some basic concepts and preliminaries briefly in Sect. 2. Section 3 focuses on the study of sufficient conditions for the existence and uniqueness of mild solution to NSFDEs with impulses and Poisson process by the contraction mapping principle. The continuous dependence result is proposed in Sect. 4. Section 5 involves the results of exponential stability of mild solution by using impulsive integral inequality.

2 Preliminaries

Let X and Y be the separable Hilbert spaces and $L(Y,X)$ be the space of bounded linear operators from Y into X. Consider a complete probability space $(\Omega, B, \mathbb{P})$ in which B is a complete σ-algebra generated by $\{B_t\}_{t\geq 0}$, an increasing right continuous family. Assume a Y-valued Q-Wiener process $\{W(t): t\geq 0\}$ with respect to $\{B_t\}_{t\geq 0}$. Here Q indicates the trace class covariance and positive self-adjoint operator on Y, that is,

$$E\langle W(t), x\rangle_Y \langle W(s), y\rangle_Y = (t\wedge s)\langle Qx, y\rangle, \quad \text{for all } x, y \in Y.$$

Let $Y_0 = Q^{1/2}(Y)$, which is a Hilbert subspace of Y with $\langle u, v\rangle_{Y_0} = \langle Q^{-1/2}v\rangle_Y$. Let

$$\langle W(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_i}\langle e_i, e\rangle \beta_i(t), \quad e \in Y,$$

where $\{e_i\}_{i\geq 1}$ is a complete orthonormal system which belongs to Y, and $Qe_i = \lambda_i e_i$, $i = 1, 2, \ldots$, where λ_i is a bounded sequence of positive real numbers and $\{B_t\}$ are independent Brownian motions.

Now, consider the impulsive NSFDE driven by Poisson jumps of the form

$$\begin{aligned} d\bigl[x(t) + g(t, x_t)\bigr] &= \bigl[Ax(t) + f(t, x_t)\bigr]\,dt + \sigma(t, x_t)\,dW(t) \\ &\quad + \int_{\mathcal{U}} h(t, x_t, u)\widetilde{N}(dt, du), \quad 0 \leq t \leq T, t \neq t_j, \end{aligned} \tag{2.1}$$

$$\Delta x(t_j) = x(t_j+) - x(t_j-) = I_j\bigl(x(t_j)\bigr), \quad t = t_j, j = 1, 2, \ldots, \tag{2.2}$$

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0, \tag{2.3}$$

where $f, g : [0, +\infty) \times X \to X$, $\sigma : [0,\infty) \times X \to \mathcal{L}_2^0(Y, X)$, $h = [0,\infty) \times X \times \mathcal{U} \to X$, $I_j : X \to X$, and are defined later. The space $\mathcal{L}_2^0(Y, X)$ contains all Q-Hilbert–Schmidt operators from Y into X with the norm $\|\zeta\|^2_{\mathcal{L}_2^0} := tr(\zeta Q\zeta^*)$, where $\zeta \in \mathcal{L}(Y, X)$.

Let $D((-\infty, 0], X)$ be the phase space with $\|\phi\|_t = \sup_{-\infty<\theta<0} |\phi(\theta)|$ and $D^b_{B_0}((-\infty, 0], X)$ indicates the family of almost surely bounded, B_0-measurable square integrable random variables with values in X. Consider the Banach space $\mathcal{B}_T = \mathcal{B}_T((-\infty, T], L_2)$, the family of all B_T-adapted processes $\phi(t,w)$ which are càdlàg (right continuous and left limit exists) in t for a.e., for $w \in \Omega$

$$\|\phi\|_{\mathcal{B}_T} = \Bigl(\sup_{0\le t\le T} E\|\phi\|_t^2\Bigr)^{1/2}, \quad \phi \in \mathcal{B}_T.$$

The counting measure of stationary Poisson process $(p_t)_{t>0}$ is denoted by $N(t,du)$ and $\hat{N}(t,A) = \mathbb{E}(N(t,A)) = t\nu(A)$ for $A \in \mathcal{E}$, where ν is the characteristic measure. The Poisson martingale measure is defined as $\widetilde{N}(t,du) = N(t,du) - t\nu(du)$, generated by p_t.

The impulsive moments t_j satisfy $0 < t_1 < t_2, \ldots, \lim_{j\to\infty} t_j = \infty$, $\Delta x(t_j) = x(t_j^+) - x(t_j^-)$, where $\Delta x(t_j)$ indicates the jump at time t_j in the state x with I_j defining the size of the jump and $x(t_j^-)$ and $x(t_j^+)$ are respectively the left and the right limits at t_j of $x(t)$.

Here $A : D(A) \to X$ is the infinitesimal generator of an analytic semigroup $(S(t))_{t\ge 0}$ of bounded linear operators on Xsatisfying the usual conditions; for details, refer to [16] and [9].

Lemma 2.1 ([16]) *If* $0 \le \alpha \le 1$, *then* X_α *is a Banach space and there exists* $M_\alpha > 0$ *such that*

$$\bigl\|(-A)^\alpha S(t)\bigr\| \le \frac{M_\alpha}{t^\alpha} e^{-\lambda t}, \quad t \ge 0, \textit{ and } \lambda > 0.$$

Lemma 2.2 (Burkholder's inequality [9]) *If* $\phi(t), t \ge 0$ *is an* $\mathcal{L}_2^0$*-valued predictable process and* $W_A^\phi = \int_0^t S(t-s)\phi(s)\,dW(s)$, $t \in [0,T]$. *Then, for any arbitrary* $p > 2$, *there exists a constant* $c(p,T) > 0$ *such that*

$$\mathbb{E}\sup_{t\le T}\bigl|W_A^\phi\bigr|^p \le c(p;T)\sup_{t\le T}\bigl\|S(t)\bigr\|^p \mathbb{E}\int_0^t \bigl\|\phi(s)\bigr\|^p\,ds.$$

Moreover, if $\mathbb{E}\int_0^t \|\phi(s)\|^p\,ds < +\infty$, *then there exists a continuous version of the process* $\{W_A^\phi : t \ge 0\}$. *If* $(S(t))_{t\ge 0}$ *is a contraction semigroup, then the above result is true for* $p \ge 2$.

Lemma 2.3 ([22]) *Let* $E(t) : [-\tau, +\infty) \to [0, +\infty)$ *be a function and if there exists some constant* $\gamma > 0$, $\alpha_j (j = 1,2,3)$ *and* $\beta_i (i = 1,2,3)$ *satisfy*

$$E(t) \le \alpha_1 e^{-\gamma t} \quad \textit{for } t \in [-\tau, 0]$$

and

$$\begin{aligned} E(t) \le{}& \alpha_1 e^{-\gamma t} + \alpha_2 \sup_{\theta\in[-\tau,0]} E(t+\theta) + \alpha_3 \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} E(t+\theta)\,ds \\ &+ \sum_{t_i<t} \beta_i e^{-\gamma(t-s)} E\bigl(t_i^-\bigr) \quad \textit{for } t \ge 0. \end{aligned}$$

If $\alpha_2 + \frac{\alpha_3}{\gamma} + \sum_{i=1}^{+\infty} \beta_i < 1$, *then* $E(t) \le Me^{-\mu t}$ *for* $t \ge -\tau$, $\mu > 0$ *denotes the unique solution to the algebraic equation*: $\alpha_2 + \frac{\alpha_3}{\gamma-\mu} e^{\mu\tau} + \sum_{i=1}^{+\infty} \beta_i = 1$ *and* $M = \max\{\frac{\alpha_1(\gamma-\mu)}{\alpha_3 e^{\mu\tau}}, \alpha_1\}$.

3 Existence and uniqueness

Suppose $0 \in \rho(A)$ and from Lemma 2.1, for the constants $M, M_{1-\beta}$, $\|S(t)\| \le M$ and $\|(-A)^{1-\beta}S(t)\| \le \frac{M_{1-\beta}}{t^{1-\beta}}$ for every $t \in [0,T]$.

Definition 3.1 If $x : [-\tau, T] \to X$ is a stochastic process and

(i) $x(t)$ is measurable and F_t adapted for all $-\tau \le t \le T$;

(ii) $x(t)$ has càdlàg paths almost surely;

(iii) $x(t) = S(t)(\phi(0) + g(0,\phi)) - g(t,x_t) - \int_0^t AS(t-s)g(s,x_s)\,ds + \int_0^t S(t-s)f(s,x_s)\,ds + \int_0^t S(t-s)\sigma(s,x_s)\,dWs + \int_0^t \int_{\mathcal{U}} S(t-s)h(s,x_s,u)\widetilde{N}(ds,du) + \sum_{0<t_k<t} S(t-t_k)I_j(x(t_j))$ if $t \in [0,T]$;

(iv) $x(t) = \phi(t), -\tau \le t \le 0$.

then x is said to be the mild solution of Eqs. (2.1)–(2.3) on $[-\tau, T]$.

Assumptions

(A_1) $f(t,\cdot), \sigma(t,\cdot)$, and $h(t,\cdot)$ satisfy the following Lipschitz conditions for all $t \in [0,T]$ and $x, y \in X$:

(1a) $\|f(t,x_t) - f(t,y_t)\|^2 \le C_f^2 \|x-y\|_t^2$;

(1b) $\|\sigma(t,x_t) - \sigma(t,y_t)\|^2 \le C_\sigma^2 \|x-y\|_t^2$;

(1c) (i) $\int_{\mathcal{U}} \|h(t,x_t,u) - h(t,y_t,u)\|^2 \nu(du) \vee ((\int_{\mathcal{U}} \|h(t,x_t,u) - h(t,y_t,u)\|^4 \nu(du))^{1/2} \le C_h \int_0^t \|x-y\|_t^2$;

(ii) $(\int_{\mathcal{U}} \|h(t,x_t,u)\|^4 \nu(du))^{1/2} \le C_h \|x\|_t^2\,ds$;

for some positive constants C_f, C_σ, C_h. We further assume that, for $t \ge 0$ and $u \in \mathcal{U}$, $f(t,0) \vee \sigma(t,0) \vee h(t,0,u) = \kappa_0$, where $\kappa_0 > 0$ is a constant.

(A_2) The function g is X_β-valued and satisfies

(2a) $\|(-A)^{-\beta}\|C_g < 1$ and $g(t,0) = 0$, where the constants $\frac{1}{2} < \beta < 1, C_g > 0$.

(2b) $\|(-A)^\beta g(t,x_t) - (-A)^\beta g(t,y_t)\|^2 \le C_g^2 \|x-y\|_t^2$ for all $t \in [0,T]$ and $x, y \in X$.

(A_3) The function $(-A)^\beta g$ is continuous in the quadratic mean sense:

$$\lim_{t \to s} \mathbb{E} \left\| (-A)^\beta \big(g(t,x_t) - g(t,x_s)\big) \right\|^2 = 0.$$

(A_4) The function $I_j \in C(X,X)$ for all $x, y \in X$, $\|I_j(x(t_j)) - I_j(y(t_j))\|^2 \le q_j^2 \|x-y\|_t^2$, where q_j is a constant and $j = 1, 2, \ldots$.

Theorem 3.1 *Suppose that* (A_1)–(A_4) *hold. Then, for all* $T > 0$, *system* (2.1)–(2.3) *has a unique mild solution on* $[-\tau, T]$ *provided that*

$$\frac{5M^2 \sum_{j=1}^\infty q_j^2}{(1-k)^2} < 1, \tag{3.1}$$

where $k = C_g \|(-A)^{-\beta}\|$.

Proof Define an operator $\pi : \mathcal{B}_T \to \mathcal{B}_T$ by

$$\begin{aligned}\pi\big(x(t)\big) &= S(t)\big(\phi(0) + g(0,\phi)\big) - g(t,x_t) \\ &\quad - \int_0^t AS(t-s)g(s,x_s)\,ds + \int_0^t S(t-s)f(s,x_s)\,ds\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t S(t-s)\sigma(s,x_s)\,dW(s)\\
&+\int_0^t\int_{\mathcal{U}} S(t-s)h(s,x_s,u)\widetilde{N}(ds,du)\\
&+\sum_{0<t_j<t} S(t-t_j)I_j\bigl(x(t_j)\bigr)\quad \text{for } t\in[0,T]
\end{aligned}
$$

and

$$
\pi\bigl(x(t)\bigr)=\phi(t)\quad \text{for } t\in[-\tau,T].
$$

Now, to prove the existence of mild solutions of (2.1)–(2.3), it is sufficient to show that π has a fixed point.

Step (i): First, we verify that $t\to\pi(x(t))$ is càdlàg on $[0,T]$.

Let $|h|$ be small enough, for $x\in\mathcal{B}_T$ and $0<t<T$, we get

$$
\begin{aligned}
&\bigl\|\pi\bigl(x(t+h)\bigr)-\pi\bigl(x(t)\bigr)\bigr\|^2\\
&\quad\le\biggl\|\bigl[S(t+h)-S(t)\bigr]\bigl[\phi(0)+g(0,\phi)\bigr]-\bigl[g(t+h,x_{t+h})-g(t,x_t)\bigr]\\
&\qquad-\biggl[\int_0^t A\bigl[S(t+h-s)-S(t-s)\bigr]g(s,x_s)\,ds+\int_t^{t+h}AS(t+h-s)g(s,x_s)\,ds\biggr]\\
&\qquad+\int_0^t\bigl[S(t+h-s)-S(t-s)\bigr]f(s,x_s)\,ds+\int_t^{t+h}S(t+h-s)f(s,x_s)\,ds\\
&\qquad+\int_0^t\bigl[S(t+h-s)-S(t-s)\bigr]\sigma(s,x_s)\,dW(s)+\int_t^{t+h}S(t+h-s)\sigma(s,x_s)\,dW(s)\\
&\qquad+\int_0^t\int_{\mathcal{U}}\bigl[S(t+h-s)-S(t-s)\bigr]h(s,x_s,u)\widetilde{N}(ds,du)\\
&\qquad+\int_t^{t+h}\int_{\mathcal{U}}S(t+h-s)h(s,x_s,u)\widetilde{N}(ds,du)\\
&\qquad+\sum_{0<t_j<t}\bigl[S(t+h-t_j)-S(t-t_j)\bigr]I_j\bigl(x(t_j)\bigr)+\sum_{t<t_j<t+h}S(t+h-t_j)I_j\bigl(x(t_j)\bigr)\biggr\|^2\\
&\bigl\|\pi\bigl(x(t+h)\bigr)-\pi\bigl(x(t)\bigr)\bigr\|^2\\
&\quad\le 7\bigl\|S(t+h)-S(t)\bigl[\phi(0)+g(0,\phi)\bigr]\bigr\|^2+7\sum_{j=1}^{6}\bigl\|F_j(t+h)-F_j(t)\bigr\|^2.
\end{aligned}
$$

Then employing the Lebesgue dominated theorem and the strong continuity of $S(t)$ implies that

$$
\lim_{h\to0}\bigl\|S(t+h)-S(t)\bigr\|^2\mathbb{E}\bigl\|\bigl[\phi(0)+g(0,\phi)\bigr]\bigr\|^2\to0.
$$

Next, it is well known that $(-A)^{-\beta}$ is bounded,

$$
\mathbb{E}\bigl\|F_1(t+h)-F_1(t)\bigr\|^2\le\bigl\|(-A)^{-\beta}\bigr\|^2\mathbb{E}\bigl\|(-A)^{\beta}g(t+h,x_{t+h})-(-A)^{\beta}g(t,x_t)\bigr\|^2.
$$

By assumption (A_3), we obtain that $\lim_{h\to 0}\mathbb{E}\|F_1(t+h)-F_1(t)\|^2 \to 0$. Then, for the term F_2, applying (A_1), Hölder's inequality, and the Lebesgue dominated theorem, we obtain

$$\begin{aligned}
\mathbb{E}\big\|F_2(t+h)-F_2(t)\big\|^2 &\le 2\mathbb{E}\bigg\|\int_0^t \big[S(t+h-s)-S(t-s)\big](-A)^{1-\beta}(-A)^{\beta}g(s,x_s)\,ds\bigg\|^2 \\
&\quad + 2\mathbb{E}\bigg\|\int_t^{t+h} S(t+h-s)(-A)^{1-\beta}(-A)^{\beta}g(s,x_s)\,ds\bigg\|^2 \\
&\le 2C_g^2.t\int_0^t \big\|S(t+h-s)-S(t-s)\big\|^2\big\|(-A)^{1-\beta}\big\|^2\mathbb{E}\|x\|_s^2\,ds \\
&\quad + 2C_g^2.h\int_t^{t+h} \big\|S(t+h-s)\big\|^2\big\|(-A)^{1-\beta}\big\|^2\mathbb{E}\|x\|_s^2\,ds \\
&\to 0 \quad \text{as } |h|\to 0.
\end{aligned}$$

A similar computation gives us $\mathbb{E}\|F_3(t+h)-F_3(t)\|^2 \to 0$ as $|h|\to 0$.

Further, using Lemma 2.2 and Hölder's inequality, we get

$$\begin{aligned}
\mathbb{E}\big\|F_4(t+h)-F_4(t)\big\|^2 &\le 2\bigg\|\int_0^t \big[S(t+h-s)-S(t-s)\big]\sigma(s,x_s)\,dW(s)\bigg\|^2 \\
&\quad + 2\bigg\|\int_t^{t+h} S(t+h-s)\sigma(s,x_s)\,dW(s)\bigg\|^2 \\
&\le 2C_pC_\sigma^2\int_0^t \big\|S(t+h-s)-S(t-s)\big\|^2\|x\|_s^2\,ds \\
&\quad + 2C_pC_\sigma^2\int_t^{t+h} \big\|S(t+h-s)\big\|^2\|x\|_s^2\,ds \\
&\to 0 \quad \text{as } |h|\to 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\big\|F_5(t+h)-F_5(t)\big\|^2 &\le 2\mathbb{E}\bigg\|\int_0^t\int_{\mathcal{U}} \big[S(t+h-s)-S(t-s)\big]h(s,x_s,u)\widetilde{N}(ds,du)\bigg\|^2 \\
&\quad + 2\mathbb{E}\bigg\|\int_t^{t+h}\int_{\mathcal{U}} S(t+h-s)h(s,x_s,u)\widetilde{N}(ds,du)\bigg\|^2 \\
&\le 2C_h\bigg[\mathbb{E}\int_0^t\int_{\mathcal{U}}\big\|S(t+h-s)-S(t-s)\big\|^2\|x\|_s^2\nu(du)\,ds \\
&\quad + \mathbb{E}\bigg(\int_0^t\int_{\mathcal{U}}\big\|S(t+h-s)-S(t-s)\big\|^2\|x\|_s^4\nu(du)\,ds\bigg)^{\frac{1}{2}}\bigg] \\
&\quad + 2C_h\bigg[\mathbb{E}\int_t^{t+h}\int_{\mathcal{U}}\big\|S(t+h-s)\big\|^2\|x\|_s^2\nu(du)\,ds \\
&\quad + \mathbb{E}\bigg(\int_t^{t+h}\int_{\mathcal{U}}\big\|S(t+h-s)\big\|^2\|x\|_s^4\nu(du)\,ds\bigg)^{\frac{1}{2}}\bigg] \\
&\to 0 \quad \text{as } |h|\to 0.
\end{aligned}$$

For F_6, using assumptions (A_1) and (A_4), we have

$$\begin{aligned}
\mathbb{E}\|F_6(t+h)-F_6(t)\|^2 &\leq 2\mathbb{E}\bigg\|\sum_{0<t_j<t}\big[S(t+h-t_j)-S(t-t_j)\big](I_j(x(t_j)))\bigg\|^2 \\
&\quad + 2\mathbb{E}\bigg\|\sum_{t<t_j<t+h} S(t+h-t_j)(I_j(x(t_j)))\bigg\|^2 \\
&\leq 2\sum_{0<t_j<t}\mathbb{E}\|S(t+h-t_j)-S(t-t_j)\|^2\big[q_j^2\mathbb{E}\|x(t_j)\|^2\big] \\
&\quad + 2\sum_{t<t_j<t+h}\mathbb{E}\|S(t+h-t_j)\|^2\big[q_j^2\mathbb{E}\|x(t_j)\|^2\big] \\
&\to 0 \quad \text{as } |h|\to 0.
\end{aligned}$$

Hence, the above arguments imply that $t\to\pi(x(t))$ is càdlàg on $[0,T]$ a.s.

Step (ii): We shall verify that $\pi(S_T)\subset\mathcal{B}_T$, let $x\in\mathcal{B}_T$, $t\in[0,T]$.

From Hölder's inequality,

$$\begin{aligned}
\mathbb{E}\|\pi(x(t))\|^2 &\leq 7\mathbb{E}\|S(t)[\phi(0)+g(0,\phi)]\|^2 + 7\mathbb{E}\|g(t,x_t)\|^2 \\
&\quad + 7\mathbb{E}\bigg\|\int_0^t AS(t-s)g(s,x_s)\,dt\bigg\|^2 + 7\mathbb{E}\bigg\|\int_0^t S(t-s)f(s,x_s)\,dt\bigg\|^2 \\
&\quad + 7\mathbb{E}\bigg\|\int_0^t S(t-s)\sigma(s,x_s)\,dW(s)\bigg\|^2 \\
&\quad + 7\mathbb{E}\bigg\|\int_0^t\int_{\mathcal{U}} S(t-s)h(s,x_s,u)\tilde{N}(ds,du)\bigg\|^2 \\
&\quad + 7\bigg\|\sum_{0<t_j<t} S(t-t_j)I_j(x(t_j))\bigg\|^2 \\
&= 7\sum_{i=1}^{7}F_i.
\end{aligned}\tag{3.2}$$

We now estimate $F_i, i=1,2,\ldots,7$. By assumption A_2-$(2a)$, we have

$$\begin{aligned}
F_1 &\leq 2\big[\mathbb{E}\|S(t)\phi(0)\|^2+\mathbb{E}\|S(t)g(0,\phi)\|^2\big] \\
&\leq 2M^2\big[1+C_g^2\|(-A)^{-\beta}\|^2\big]\mathbb{E}\|\phi\|^2.
\end{aligned}$$

Applying Hölder's inequality and A_2-$(2a)$, we have

$$F_2 \leq \|(-A)^{-\beta}\|^2 C_g^2\mathbb{E}\|x\|_t^2$$

and

$$\begin{aligned}
F_3 &\leq \mathbb{E}\int_0^t \|(-A)^{1-\beta}S(t-s)(-A)^{\beta}g(t,x_t)\|^2\,ds \\
&\leq M^2 t\|(-A)^{1-\beta}\|^2 C_g^2\int_0^t\mathbb{E}\|x\|_s^2\,ds.
\end{aligned}$$

By Hölder's inequality and A_1-$(1a)$, we derive that

$$
\begin{aligned}
F_4 &\le 2\mathbb{E}\int_0^t \bigl\|S(t-s)\bigl[f(s,x_s)-f(s,0)\bigr]\bigr\|^2\,ds + 2\mathbb{E}\int_0^t \bigl\|S(t-s)f(s,0)\bigr\|^2\,ds \\
&\le 2M^2tC_f^2\int_0^t \mathbb{E}\|x\|_s^2\,ds + 2tM^2\kappa_0 \\
&\le 2M^2t\biggl[C_f^2\int_0^t \mathbb{E}\|x\|_s^2\,ds + \kappa_0\biggr].
\end{aligned}
$$

On the other hand, applying assumption (A_1)-$(1b)$ and Lemma 2.2, we get, for some positive constant C_p,

$$
\begin{aligned}
F_5 &\le C_p\bigl\|S(t)\bigr\|^2\mathbb{E}\int_0^t \bigl\|\sigma(s,x_s)-\sigma(s,0)+\sigma(s,0)\bigr\|^2\,ds \\
&\le 2C_pM^2\biggl[C_\sigma^2\mathbb{E}\int_0^t \|x\|_s^2\,ds + \kappa_0 t\biggr].
\end{aligned}
$$

Employing assumption (A_1)-$(1c)$ and Lemma 2.2 in [13], we obtain

$$
\begin{aligned}
F_6 &\le M^2\biggl[\mathbb{E}\int_0^t\int_{\mathcal{U}} \bigl\|h(s,x_s,u)\bigr\|^2\nu(du)\,ds + \mathbb{E}\biggl(\int_0^t\int_{\mathcal{U}} \bigl\|h(s,x_s,u)\bigr\|^4\nu(du)\,ds\biggr)^{\frac{1}{2}}\biggr] \\
&\le M^2\biggl[\mathbb{E}\int_0^t\int_{\mathcal{U}} \bigl\|h(s,x_s,u)-h(s,0,u)+h(s,0,u)\bigr\|^2\nu(du)\,ds \\
&\quad + \mathbb{E}\biggl(\int_0^t\int_{\mathcal{U}} \bigl\|h(s,x_s,u)\bigr\|^4\nu(du)\,ds\biggr)^{\frac{1}{2}}\biggr] \\
&\le 2M^2\biggl[C_h^2\int_0^t \mathbb{E}\|x\|_s^2\,ds + \kappa_0 t\biggr] + M^2C_h^2\int_0^t \mathbb{E}\|x\|_s^2\,ds \\
&\le 3M^2C_h^2\int_0^t \mathbb{E}\|x\|_s^2\,ds + 2M^2\kappa_0 t.
\end{aligned}
$$

From Hölder's inequality and assumption (A_4), we have

$$
\begin{aligned}
F_7 &\le 2\mathbb{E}\sum_{j=1}^{\infty}\bigl[\bigl\|S(t-t_j)I_j\bigl(x(t_j)\bigr)-I_j(0)\bigr\|^2 + \bigl\|S(t-t_j)I_j(0)\bigr\|^2\bigr] \\
&\le 2M^2\Biggl[\sum_{j=1}^{\infty} q_j^2\mathbb{E}\|x\|_t^2 + \sum_{j=1}^{\infty} q_j^2\kappa_0\Biggr].
\end{aligned}
$$

From the above estimations, Eq. (3.2) becomes

$$
\begin{aligned}
\mathbb{E}\bigl\|\pi\bigl(x(t)\bigr)\bigr\|^2 \le{}& 14M^2\bigl[1+C_g^2\bigl\|(-A)^{-\beta}\bigr\|^2\bigr]\mathbb{E}\|\phi\|^2 + 7C_g^2\bigl\|(-A)^{-\beta}\bigr\|^2\mathbb{E}\|x\|_t^2 \\
&+ 7M^2tC_g^2\bigl\|(-A)^{1-\beta}\bigr\|^2\int_0^t \mathbb{E}\|x\|_s^2\,ds \\
&+ 14M^2t\biggl[C_f^2\int_0^t \mathbb{E}\|x\|_s^2\,ds + \kappa_0\biggr]
\end{aligned}
$$

$$
\begin{aligned}
&+ 14C_pM^2\left[C_\sigma^2\int_0^t \mathbb{E}\|x\|_s^2\,ds + \kappa_0 t\right] \\
&+ 21M^2C_h^2\int_0^t \mathbb{E}\|x\|_s^2\,ds + 14M^2t\kappa_0 \\
&+ 14M^2\sum_{j=1}^{\infty} q_j^2\bigl[\mathbb{E}\|x\|_t^2 + \kappa_0\bigr] \\
\le{}& R_1 + 7C_g^2\bigl\|(-A)^{-\beta}\bigr\|^2\mathbb{E}\|x\|_t^2 + 7M^2tC_g^2\bigl\|(-A)^{1-\beta}\bigr\|^2\int_0^t \mathbb{E}\|x\|_s^2\,ds \\
&+ 7M^2\bigl[2tC_f^2 + 2C_pC_\sigma^2 + 3C_h^2\bigr]\int_0^t \mathbb{E}\|x\|_s^2\,ds + 14M^2\sum_{j=1}^{\infty} q_j^2\mathbb{E}\|x\|_t^2,
\end{aligned}
$$

where $R_1 = 14M^2[1 + C_g^2\|(-A)^{-\beta}\|^2]\mathbb{E}\|\phi\|^2 + 14M^2[2t + C_pt + \sum_{j=1}^{\infty} q_j^2]\kappa_0$.

We obtain

$$
\begin{aligned}
\mathbb{E}\bigl\|\pi\bigl(x(t)\bigr)\bigr\|^2 \le{}& R_1 + 7\left\{C_g^2\bigl\|(-A)^{-\beta}\bigr\|^2 + 2M^2\sum_{j=1}^{\infty} q_j^2\right\}\mathbb{E}\|x\|_t^2 \\
&+ 7M^2\bigl[tC_g^2\bigl\|(-A)^{1-\beta}\bigr\|^2 + 2tC_f^2 + 2C_pC_\sigma^2 + 3C_h^2\bigr]\int_0^t \mathbb{E}\|x\|_s^2\,ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sup_{0\le s\le T}\mathbb{E}\bigl\|\pi\bigl(x(t)\bigr)\bigr\|^2 &\le R_1 + R_2\sup_{-\tau\le t\le T}\mathbb{E}\|x\|_t^2 + R_3\int_0^t \sup_{-\tau\le s\le T}\mathbb{E}\|x\|_s^2\,ds \\
&\le R_1 + R_2\sup_{-\tau\le t\le T}\mathbb{E}\|x\|_t^2 + R_3t\sup_{-\tau\le t\le T}\mathbb{E}\|x\|_t^2 \\
&\le R_1 + R_4\sup_{-\tau\le t\le T}\mathbb{E}\|x\|_t^2,
\end{aligned}
$$

where

$$
\begin{aligned}
&R_2 = 7\left\{C_g^2\bigl\|(-A)^{-\beta}\bigr\|^2 + 2M^2\sum_{j=1}^{\infty} q_j^2\right\} \\
&R_3 = 7M^2\bigl[tC_g^2\bigl\|(-A)^{1-\beta}\bigr\|^2 + 2tC_f^2 + 2C_pC_\sigma^2 + 3C_h^2\bigr] \\
&R_4 = R_2 + R_3\cdot t.
\end{aligned}
$$

Since $\pi(x) = \phi$ on $[-\tau, 0]$, it follows that

$$
\mathbb{E}\sup_{-\tau\le s\le T}\bigl\|\pi\bigl(x(s)\bigr)\bigr\|^2 < \infty.
$$

This proves the boundedness of $\pi\mathcal{B}_T$.

Step (iii): Next, we will verify that π is a contraction mapping in $\mathcal{B}_{T_1}$ with some $T_1 \le T$ to be specified later.

Let $x, y \in \mathcal{B}_T$. Based on this simple inequality $(x+y+z)^2 \le \frac{1}{k}x^2 + \frac{2}{1-k}y^2 + \frac{2}{1-k}z^2$ and recalling that $k : c_g\|(-A)^{-\beta}\| < 1$, for $t \in [0,T]$,

$$\begin{aligned}
\mathbb{E}\big\|\pi\big(x(t)\big) - \pi\big(y(t)\big)\big\|^2 &\le \frac{1}{k}\mathbb{E}\big\|(-A)^{-\beta}\big\|^2\big\|(-A)^{\beta}g(t,x_t) - g(t,y_t)\big\|^2 \\
&\quad + \frac{5}{1-k}\mathbb{E}\left\|\int_0^t (-A)^{1-\beta}S(t-s)(-A)^{\beta}\big[g(s,x_s) - g(s,y_s)\big]\,ds\right\|^2 \\
&\quad + \frac{5}{1-k}\mathbb{E}\left\|\int_0^t S(t-s)\big[f(s,x_s) - f(s,y_s)\big]\,ds\right\|^2 \\
&\quad + \frac{5}{1-k}\mathbb{E}\left\|\int_0^t S(t-s)\big[\sigma(s,x_s) - \sigma(s,y_s)\big]\,dWs\right\|^2 \\
&\quad + \frac{5}{1-k}\mathbb{E}\left\|\int_0^t\int_{\mathcal{U}} S(t-s)\big[h(s,x_s,u) - h(s,y_s,u)\big]\widetilde{N}(ds,du)\right\|^2 \\
&\quad + \frac{5}{1-k}\mathbb{E}\left\|\sum_{0<t_j<t} S(t-t_j)\big[I_j\big(x(t_j)\big) - I_j\big(y(t_j)\big)\big]\right\|^2.
\end{aligned}$$

By using Holder's inequality, Lemma 2.2 together with assumptions (A_1), (A_2), and (A_4), we get

$$\begin{aligned}
&\mathbb{E}\big\|\pi\big(x(t)\big) - \pi\big(y(t)\big)\big\|^2 \\
&\quad\le k\mathbb{E}\|x-y\|_t^2 \\
&\qquad + \frac{5}{1-k}M_{1-\beta}^2 C_g^2\left(\frac{t^{2\beta-1}}{2\beta-1}\right)\int_0^t \mathbb{E}\|x-y\|_s^2\,ds \\
&\qquad + \frac{5}{1-k}M^2 tC_f^2\int_0^t \mathbb{E}\|x-y\|_s^2\,ds + \frac{5}{1-k}M^2C_\sigma^2C_p\int_0^t \mathbb{E}\|x-y\|_s^2\,ds \\
&\qquad + \frac{5}{1-k}M^2C_h^2\int_0^t \mathbb{E}\|x-y\|_s^2\,ds + \frac{5}{1-k}M^2\sum_{j=1}^{\infty} q_j^2\mathbb{E}\|x-y\|_t^2 \\
&\mathbb{E}\big\|\pi\big(x(t)\big) - \pi\big(y(t)\big)\big\|^2 \\
&\quad\le k\mathbb{E}\|x-y\|_t^2 \\
&\qquad + \frac{5}{1-k}\left[C_g^2M_{1-\beta}^2\left(\frac{t^{2\beta-1}}{2\beta-1}\right) + M^2\big(tC_f^2 + C_pC_\sigma^2 + C_h^2\big)\right]\int_0^t \mathbb{E}\|x-y\|_s^2\,ds \\
&\qquad + \frac{5}{1-k}M^2\sum_{j=1}^{\infty} q_j^2\mathbb{E}\|x-y\|_t^2.
\end{aligned}$$

Hence, $\sup_{s\in[-\tau,T]}\mathbb{E}\|\pi(x(t)) - \pi(y(t))\|^2 \le \gamma(t)\sup_{s\in[-\tau,T]}\mathbb{E}\|x-y\|_s^2$, where $\gamma(t) = k + \frac{5}{1-k}[C_g^2M_{1-\beta}^2(\frac{t^{2\beta}}{2\beta-1}) + M^2t(tC_f^2 + C_pC_\sigma^2 + C_h^2)] + \frac{5M^2}{1-k}\sum_{j=1}^{\infty} q_j^2$. By Eq. (3.1), we have $\gamma(0) = k + \frac{5M^2}{1-k}\sum_{j=1}^{\infty} q_j^2 = \frac{5M^2\sum_{j=1}^{\infty} q_j^2}{(1-k)^2} < 1$. Hence, there exists $0 < T_1 < T$ such that $0 < \gamma(T_1) < 1$ and π is a contraction mapping on $\mathcal{B}_{T_1}$. Therefore it is clear that it has a unique fixed point, which is a mild solution of (2.1)–(2.3). By repeating a similar process the solution can be extended to the entire interval $[-\tau, T]$ in infinitely many steps. This concludes Theorem 3.1. □

4 Stability

Definition 4.1 Let $x, \hat{x}$ be different mild solutions of (2.1)–(2.3) with initial values ϕ_1 and ϕ_2, respectively. If for all $\epsilon > 0$, $\exists \delta > 0$ such that $\mathbb{E}\|x(t) - \hat{x}(t)\|^2 \leq \epsilon$ when $\mathbb{E}\|\phi_1 - \phi_2\|^2 < \delta$ for all $t \in [0, T]$, then $x(t)$ is said to be stable in mean square.

Theorem 4.1 *Assume that any two mild solutions of* (2.1)–(2.3) *are* $x(t)$ *and* $y(t)$ *with initial values* ϕ_1 *and* ϕ_2*, respectively. Suppose that* (A_1)–(A_4) *are satisfied, then the mild solution of* (2.1)–(2.3) *is stable in the quadratic mean.*

Proof For $0 \leq t \leq T$,

$$\begin{aligned}
&\mathbb{E}\big\|x(t) - y(t)\big\|^2 \\
&\quad \leq 7\mathbb{E}\big\|S(t)\big(\big[\phi_1(0) - \phi_2(0)\big] + \big[g(0,\phi_1) - g(0,\phi_2)\big]\big)\big\|^2 + 7\mathbb{E}\big\|g(t,x_t) - g(t,y_t)\big\|^2 \\
&\qquad + 7\mathbb{E}\bigg\|\int_0^t AS(t-s)\big[g(s,x_s) - g(s,y_s)\big]\,ds\bigg\|^2 + 7\mathbb{E}\bigg\|\int_0^t S(t-s)\big[f(s,x_s) - f(s,y_s)\big]\,ds\bigg\|^2 \\
&\qquad + 7\mathbb{E}\bigg\|\int_0^t S(t-s)\big[\sigma(s,x_s) - \sigma(s,y_s)\big]\,dW(s)\bigg\|^2 \\
&\qquad + 7\mathbb{E}\bigg\|\int_0^t \int_{\mathcal{U}} S(t-s)\big[h(s,x_s,u) - h(s,y_s,u)\big]\widetilde{N}(ds,du)\bigg\|^2 \\
&\qquad + 7\mathbb{E}\bigg\|\sum_{0<t_j<t} S(t-t_j)\big[I_j\big(x(t_j)\big)\big]\bigg\|^2.
\end{aligned}$$

By using Hölder's inequality and assumptions (A_1), (A_2), and (A_4), we derive that

$$\begin{aligned}
&\mathbb{E}\big\|x(t) - y(t)\big\|^2 \\
&\quad \leq 7M^2\big[1 + C_g^2\big\|(-A)^{-\beta}\big\|^2\big]\mathbb{E}\|\phi_1 - \phi_2\|^2 \\
&\qquad + 7C_g^2\big\|(-A)^{-\beta}\big\|^2\mathbb{E}\|x - y\|_t^2 \\
&\qquad + 7M_{1-\beta}^2 C_g^2\bigg(\frac{t^{2\beta-1}}{2\beta - 1}\bigg)\int_0^t \mathbb{E}\|x - y\|_s^2\,ds \\
&\qquad + 7M^2 t C_f^2 \int_0^t \mathbb{E}\|x - y\|_s^2\,ds + 7M^2 C_\sigma^2 C_p \int_0^t \mathbb{E}\|x - y\|_s^2\,ds \\
&\qquad + 7M^2 C_h^2 \int_0^t \mathbb{E}\|x - y\|_s^2\,ds + 7M^2 \sum_{j=1}^{\infty} q_j^2 \mathbb{E}\|x - y\|_t^2 \\
&\quad \leq 7M^2\big[1 + C_g^2\big\|(-A)^{-\beta}\big\|^2\big]\mathbb{E}\|\phi_1 - \phi_2\|^2 \\
&\qquad + 7\Bigg[C_g^2\big\|(-A)^{-\beta}\big\|^2 + M^2\sum_{j=1}^{\infty} q_j^2\Bigg]\mathbb{E}\|x - y\|_t^2 \\
&\qquad + 7\bigg[M_{1-\beta}^2 C_g^2\bigg(\frac{t^{2\beta-1}}{2\beta - 1}\bigg) + M^2\big(tC_f^2 + C_p C_\sigma^2 + C_h^2\big)\big)\bigg]\int_0^t \mathbb{E}\|x - y\|_s^2\,ds.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\sup_{t\in[\tau,T]} \mathbb{E}\|x-y\|_t^2 \\
&\quad\leq \frac{7M^2[1+\|(-A)^{-\beta}\|^2C_g^2]}{1-Q}\mathbb{E}\|\phi_1-\phi_2\|^2 \\
&\qquad + \frac{7[M_{1-\beta}^2C_g^2(\frac{t^{2\beta-1}}{2\beta-1})+M^2(tC_f^2+C_pC_\sigma^2+C_h^2)]}{1-Q}\int_0^t \sup_{t\in[\tau,T]} \mathbb{E}\|x-y\|_s^2\,ds,
\end{aligned}$$

where $Q=7[C_g^2\|(-A)^{-\beta}\|^2+M^2\sum_{j=1}^{\infty}q_j^2]$.

By applying Gronwall's inequality, we have

$$\begin{aligned}
\sup_{t\in[\tau,T]} \mathbb{E}\|x-y\|_t^2 &\leq \frac{7M^2[1+\|(-A)^{-\beta}\|^2C_g^2]}{1-Q}\mathbb{E}\|\phi_1-\phi_2\|^2 \\
&\quad\times \exp\biggl(\frac{7[M_{1-\beta}^2C_g^2(\frac{t^{2\beta-1}}{2\beta-1})+M^2(tC_f^2+C_pC_\sigma^2+C_h^2)]}{1-Q}\biggr) \\
&\leq \wp\mathbb{E}\|\phi_1-\phi_2\|^2,
\end{aligned}$$

where $\wp=\frac{7M^2[1+\|(-A)^{-\beta}\|^2C_g^2]}{1-Q}\exp(\frac{7[M_{1-\beta}^2C_g^2(\frac{t^{2\beta-1}}{2\beta-1})+M^2(tC_f^2+C_pC_\sigma^2+C_h^2)]}{1-Q})$.

Now, given $\epsilon>0$, choose $\delta=\frac{\epsilon}{\wp}$ such that $\mathbb{E}\|\phi_1-\phi_2\|^2<\delta$. Then

$$\sup_{t\in[\tau,T]} \mathbb{E}\|x-y\|_t^2 \leq \epsilon.$$

This concludes Theorem 4.1. □

5 Exponential stability

A system is defined to be exponentially stable if the system response decays exponentially towards zero as time approaches infinity.

For example, consider that a system, marble ball in a ladle, when undisturbed will occupy the lowest point in the ladle. But when the ball is subjected to a push, it will exhibit a diminishing sinusoidal oscillation and eventually resettle in the bottom of the ladder. Also, the system is said to be marginally stable when the ball is away from the bottom of the ladle when a constant force equal to its weight is applied. But when the ball is given a big push, it will fall away from the ladle and stop when it reaches the ground. Therefore it is proper to state that the system is exponentially stable for a range of inputs.

Definition 5.1 System (2.1)–(2.3) is said to be exponentially stable in the quadratic mean if there exist positive constant C_1 and $\lambda>0$ such that

$$E\|x(t)\|^2 \leq C_1E\|\varphi\|^2e^{-\lambda(t-t_0)}, \quad t\geq t_0.$$

We assume that $f(t,0)=\sigma(t,0)=h(t,0,u)=0$ for all $t\geq 0, u\in\mathcal{U}$. So that system (2.1)–(2.3) admits a trivial solution. We further need the following assumptions.

(A_5) $\|S(t)\|\leq Me^{-\lambda(t-t_0)}$, $t\geq t_0$, where $M\geq 1$, $\lambda>0$.

(A_6) There exist nonnegative real numbers $E_1, E_2, E_3, E_4 \geq 0$ and continuous functions $\delta_1, \delta_2, \delta_3, \delta_4 : [0, +\infty) \to \mathbb{R}_+$ such that, for all $t \geq 0$ and $x, y \in X$,

$$\begin{aligned}
&\text{(i)} \quad \|f(t, x_t)\|^2 \leq E_1 \|x\|_t^2 + \delta_1(t),\\
&\text{(ii)} \quad \|(-A)^\beta g(t, x_t)\|^2 \leq E_2 \|x\|_t^2 + \delta_2(t),\\
&\text{(iii)} \quad \|\sigma(t, x_t)\|^2 \leq E_3 \|x\|_t^2 + \delta_3(t),\\
&\text{(iv)} \quad \int_{\mathcal{U}} \|h(t, x_t, u)\|^2 \nu(du) \vee \left(\int_{\mathcal{U}} \|h(t, x_t, u)\|^4 \nu(du)\right)^{\frac{1}{2}} \leq E_4 \|x\|_t^2 + \delta_4(t).
\end{aligned}$$

(A_7) There exist nonnegative real numbers $P_j \geq 0, j = 1, 2, 3, 4$, such that $\delta_j(t) \leq P_j e^{-\lambda t}$, $\forall t \geq 0, j = 1, 2, 3, 4$.

Theorem 5.1 *Assume that* (A_4)–(A_7) *and the following inequality holds*:

$$\begin{aligned}
&\frac{6\{[\lambda^{1-2\beta} 2^{2(1-\beta)} M_{1-\beta}^2 M^2 \Gamma(2\beta - 1) E_2/\lambda] + M^2[E_1^2 + C_p E_3^2 + E_4^2]/\lambda^2 + M^2 \sum_{k=1}^{\infty} q_j^2\}}{(1 - k)^2}\\
&\quad < 1, \qquad (5.1)
\end{aligned}$$

where $k = \sqrt{E_2}\|(-A)^{-\beta}\|$. *Then the mild solution of system* (2.1)–(2.3) *is exponentially stable in the mean square moment.*

Proof From inequality (5.1), it is possible to find a small positive quantity ϵ such that

$$k + \frac{6\lambda^{1-2\beta} 2^{2(1-\beta)} M_{1-\beta}^2 M^2 \Gamma(2\beta - 1) E_2}{(\lambda - \epsilon)(1 - k)} + \frac{6M^2[E_1^2 + C_p E_3^2 + E_4^2]}{\lambda(\lambda - \epsilon)(1 - k)} + \frac{6M^2 \sum_{k=1}^{\infty} q_j^2}{(1 - k)} < 1.$$

Let $\eta = \lambda - \epsilon$ and $x(t)$ be the mild solution of (2.1)–(2.3).

For $t \geq 0$,

$$\begin{aligned}
\mathbb{E}\|x(t)\|^2 \leq{}& \frac{1}{k}\mathbb{E}\|g(t, x_t)\|^2 + \frac{6}{1 - k}\mathbb{E}\Bigg\{\big\|S(t)\big[\phi(0) + g(0, \phi)\big]\big\|^2\\
&+ \left\|\int_0^t AS(t - s) g(s, x_s)\, ds\right\|^2 + \left\|\int_0^t S(t - s) f(s, x_s)\, ds\right\|^2\\
&+ \left\|\int_0^t S(t - s)\sigma(s, x_s)\, dW(s)\right\|^2 + \left\|\int_0^t \int_{\mathcal{U}} S(t - s) h(s, x_s, u)\widetilde{N}(ds, du)\right\|^2\\
&+ \left\|\sum_{0<t_j<t} S(t - t_j) I_j\big(x(t_j)\big)\right\|^2\Bigg\}\\
\leq{}& \sum_{j=1}^{7} F_j(t).
\end{aligned}$$

By conditions (A_6) and (A_7), we obtain

$$\begin{aligned} F_1(t) &= \frac{1}{k}\mathbb{E}\big\|(-A)^{-\beta}(-A)^{\beta}g(t,x_t)\big\|^2 \\ &\le \frac{\|(-A)^{-\beta}\|^2}{k}\big[E_2^2\mathbb{E}\|x\|_t^2+\delta_2\big] \\ &\le k\mathbb{E}\|x\|_t^2+K_1e^{-\eta t} \quad \text{where } K_1=\frac{\|(-A)^{-\beta}\|^2P_2}{k}. \end{aligned}$$

Using assumptions (A_5), (A_6), and (A_7), we have

$$\begin{aligned} F_2(t) &\le \frac{12}{1-k}\big[\mathbb{E}\big\|S(t)\phi(0)\big\|^2+\mathbb{E}\big\|S(t)g(0,\phi)\big\|^2\big] \\ &\le \frac{12M^2}{1-k}e^{-2\lambda t}\mathbb{E}\big\|\phi(0)\big\|^2+\frac{12M^2}{1-k}e^{-2\lambda t}\big\|(-A)^{-\beta}\big\|^2\big[E_2\mathbb{E}\|\phi\|^2+P_2\big] \\ &\le K_2e^{-\eta t}, \end{aligned}$$

$$\text{where } K_2=\frac{12M^2}{1-k}\big\{\mathbb{E}\big\|\phi(0)\big\|^2+\big\|(-A)^{-\beta}\big\|^2\big[E_2\mathbb{E}\|\phi\|^2+P_2\big]\big\}.$$

Applying assumptions (A_5), (A_6), and (A_7) together with Lemma 2.1 and Hölder's inequality, we get

$$\begin{aligned} F_3(t) &= \frac{6}{1-k}\mathbb{E}\bigg\|\int_0^t(-A)^{1-\beta}S\Big(\frac{t-s}{2}\Big)S\Big(\frac{t-s}{2}\Big)(-A)^{\beta}g(s,x_s)\,ds\bigg\|^2 \\ &\le \frac{6}{1-k}\int_0^t\frac{M_{1-\beta}^2e^{-\lambda(t-s)}}{(\frac{t-s}{2})^{2(1-\beta)\,ds}}\int_0^t M^2e^{-\lambda(t-s)}\mathbb{E}\big\|(-A)^{\beta}g(s,x_s)\big\|^2\,ds \\ &\le \frac{6\lambda^{1-2\beta}2^{2(1-\beta)}M_{1-\beta}^2M^2\Gamma(2\beta-1)}{1-k}\int_0^t e^{-\lambda(t-s)}\big[E_2\mathbb{E}\|x\|_s^2+\delta_2(s)\big]\,ds \\ &\le \frac{6\lambda^{1-2\beta}2^{2(1-\beta)}M_{1-\beta}^2M^2\Gamma(2\beta-1)E_2}{1-k}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|x\|_s^2\,ds+K_3e^{-\eta t}, \end{aligned}$$

where Γ is the usual gamma function and $K_3=\frac{6\lambda^{1-2\beta}2^{2(1-\beta)}M^2M_{1-\beta}^2\Gamma(2\beta-1)}{1-k}\frac{P_2}{\lambda-\eta}$.

Again, using (A_5)–(A_7) and Hölder's inequality, we get

$$\begin{aligned} F_4(t) &= \frac{6}{1-k}\mathbb{E}\bigg(\int_0^t S(t-s)e^{-\lambda(t-s)}\big\|f(s,x_s)\big\|\,ds\bigg)^2 \\ &\le \frac{6M^2}{1-k}\int_0^t e^{-\lambda(t-s)}\,ds\int_0^t e^{-\lambda(t-s)}\big[E_1^2\mathbb{E}\|x\|_s^2+\delta_1(s)\big]\,ds \\ &\le \frac{6M^2E_1^2}{(1-k)\lambda}\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|x\|_s^2\,ds+K_4e^{-\eta t}, \end{aligned}$$

where $K_4=\frac{6M^2}{\lambda(1-k)}\frac{P_1}{\lambda-\eta}$. Similarly, for the term F_5,

$$\begin{aligned} F_5(t) &\le \frac{6}{1-k}\bigg(\mathbb{E}\bigg\|\int_0^t S(t-s)\sigma(s,x_s)\,dW(s)\bigg\|\bigg)^2 \\ &\le \frac{6M^2}{1-k}C_p\bigg(\int_0^t e^{-\lambda(t-s)}\mathbb{E}\big\|\sigma(s,x_s)\big\|\,ds\bigg)^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{6}{1-k} C_p M^2 \int_0^t e^{-\lambda(t-s)}\,ds \\
&\quad \times \int_0^t e^{-\lambda(t-s)}\,ds\bigl[E_3\mathbb{E}\|x\|_s^2 + \delta_3(s)\bigr]\,ds \\
&\leq \frac{6C_pM^2E_3}{\lambda(1-k)} \int_0^t e^{-\lambda(t-s)}\mathbb{E}\|x\|_s^2\,ds + K_5e^{-\eta t},
\end{aligned}
$$

where $K_5 = \frac{6C_pM^2}{\lambda(1-k)}\frac{P_3}{\lambda-\eta}$. By assumptions ($A_5$)–($A_7$) together with Lemma 2.3, we have

$$
\begin{aligned}
F_6(t) &\leq \frac{6}{1-k}\mathbb{E}\biggl(\biggl\|\int_0^t\int_{\mathcal{U}} S(t-s)h(s,x_s,u)\widetilde{N}(ds,du)\biggr\|\biggr)^2 \\
&\leq \frac{6}{1-k}M^2\biggl(\int_0^t e^{-2\lambda(t-s)}\biggl[\int_{\mathcal{U}}\mathbb{E}\bigl\|h(s,x_s,u)\bigr\|^2\nu(du) \\
&\quad + \biggl(\int_{\mathcal{U}}\mathbb{E}\bigl\|h(s,x_s,u)\bigr\|^4\nu(du)\biggr)^{\frac{1}{2}}\biggr]\,ds\biggr) \\
&\leq \frac{M^2}{\lambda(1-k)}\int_0^t e^{-\lambda(t-s)}\bigl[E_4\mathbb{E}\|x\|_s^2 + \delta_4(s)\bigr]\,ds \\
&\leq \frac{6M^2}{\lambda(1-k)}E_4\int_0^t e^{-\lambda(t-s)}\mathbb{E}\|x\|_s^2\,ds + K_6e^{-\eta t},
\end{aligned}
$$

where $K_6 = \frac{6M^2}{\lambda(1-k)}\frac{P_4}{\lambda-\eta}$. By applying assumption (A_4), one can get

$$
\begin{aligned}
F_7(t) &\leq \frac{6M^2}{1-k}\sum_{j=1}^{\infty} q_j^2 e^{-2\lambda(t-t_j)}\mathbb{E}\bigl\|x(t_j)\bigr\|^2 \\
&\leq \frac{6M^2}{1-k}\sum_{j=1}^{\infty} q_j^2 e^{-\eta(t-t_j)}\mathbb{E}\bigl\|x(t_j)\bigr\|^2.
\end{aligned}
$$

The above inequalities together with Lemma 2.3 imply that

$$
\mathbb{E}\bigl\|x(t)\bigr\|^2 \leq \gamma e^{-\eta t} \quad \text{for } t\in[-\tau,0]
$$

and

$$
\begin{aligned}
\mathbb{E}\bigl\|x(t)\bigr\|^2 &\leq \gamma e^{-\eta t} + k\sup_{-\tau\leq u\leq 0}\mathbb{E}\bigl\|x(t+u)\bigr\|^2 \\
&\quad + \widetilde{k}\int_0^t e^{-\eta(t-s)}\sup_{-\tau\leq u\leq 0}\mathbb{E}\bigl\|x(t+u)\bigr\|^2\,ds \\
&\quad + \sum_{j=1}^{\infty} e^{-\eta(t-t_j)}\mathbb{E}\bigl\|x(t_j)\bigr\|^2 \quad \text{for } t\geq 0.
\end{aligned}
$$

Here $\gamma = \max(\sum_{i=1}^{6}K_i, \sup_{-\tau\leq u\leq 0}\mathbb{E}\|\phi(u)\|^2)$ and

$$
\widetilde{k} = \frac{6\lambda^{1-2\beta}2^{2(1-\beta)}M^2M_{1-\beta}^2\Gamma(2\beta-1)E_2}{1-k} + \frac{6M^2[E_1^2 + C_pE_3^2 + E_4^2]}{\lambda(1-k)}
$$

since $k + \frac{\widetilde{k}}{\eta} + \sum_{i=1}^{\infty} d_i^2 < 1$, and from Lemma 2.3 there exist constants $K > 0$ and $\theta > 0$ such that $\mathbb{E}\|x(t)\|^2 \leq Ke^{-\theta t}$, $\forall t \geq -\tau$. This ensures the exponential stability of the mild solution in mean square. Hence the proof. □

Remark 5.1 If the impulsive term $\Delta(x(t_j)) = I_j(\cdot) = 0$, $j = 1, 2, \ldots$, then (2.1)–(2.3) takes the following form:

$$d\bigl[x(t) + g(t,x_t)\bigr] = \bigl[Ax(t) + f(t,x_t)\bigr]\,dt + \sigma(t,x_t)\,dW(t) + \int_{\mathcal{U}} h(t,x_t,u)\widetilde{N}(dt,du), \quad 0 \leq t \leq T, \tag{5.2}$$

$$x(t) = \phi(t), \quad -\tau \leq t \leq 0, \tag{5.3}$$

where $C = C([-\tau,0];X)$ denotes the family of almost surely bounded and continuous functions ϕ from $[-\tau,0]$ into X and, as usual, with $\|\phi\|_c = \sup_{\theta\in[-\tau,0]}\|\phi(\theta)\|$. Also, if we assume that all the functions are defined the same as earlier, then by the same procedure as in Theorem 5.1, we may deduce the next corollary.

Corollary 5.2 *Suppose that* (A_1)–(A_3) *and* (A_5)–(A_7) *are satisfied, then the mild solution of* (2.1)–(2.3) *is exponentially stable in the mean square moment if the following inequality holds*:

$$\frac{5\{[\lambda^{1-2\beta}2^{2(1-\beta)}M_{1-\beta}^2 M^2\Gamma(2\beta-1)E_2/\lambda] + M^2[E_1^2 + C_pE_3^2 + E_4^2]/\lambda^2\}}{(1-k)^2} < 1. \tag{5.4}$$

6 Conclusion

In this article, the existence and uniqueness results for neutral impulsive stochastic functional differential equations with Poisson jumps have been derived using fixed point approach. Also, sufficient conditions are derived for the continuous dependence of solutions on the initial value by means of the corollary of Bihari's inequality. Finally, the exponential stability of mild solutions for neutral impulsive stochastic functional differential equations with Poisson jumps is investigated based on the impulsive integral inequality. This will motivate the future research work such as the study of controllability and stability in distribution for neutral impulsive stochastic functional differential equations with Poisson jumps.

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