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Stability and Hopf bifurcation analysis in a fractional-order delayed paddy ecosystem

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Abstract

By introducing a delayed fractional-order differential equation model, we deal with the dynamics of the stability and Hopf bifurcation of a paddy ecosystem with three main components: rice, weeds, and inorganic fertilizer. In the system, there exists an equilibrium for rice and weeds extinction and an equilibrium for rice extinction or weeds extinction. We obtain sufficient conditions for the stability and Hopf bifurcation by analyzing their characteristic equation. Some numerical simulations validate our theoretical results.

Keywords: Paddy ecosystem; Fractional order; Delay; Stability; Hopf bifurcation

1 Introduction

Rice is one of the major grain crops in the world. China is the largest rice producer and consumer country in the world, where over 60% of the population is staple food for rice. Throughout the world rice producing countries, it is a major research topic to improve rice yield and quality. Obviously, there are a lot of factors affecting the production of rice, such as weed, insect, microorganism, inorganic fertilizer, light intensity, moisture, and so on. These factors interact and transform each other to form a complex nonlinear relationship. It is a common research method to analyze the interaction of all factors in a population system by using mathematical models [1–11]. As far as we know, there are only a few mathematical models that have been established for paddy ecosystems [12–14].

A differential equation model of a paddy ecosystem in fallow season was proposed by Xiang et al. [14]. They revealed the interaction between weeds and inorganic fertilizer and found that in the system, there exists a stable node, an unstable saddle point, or a saddle-node point. By considering the effects of herbivores on the paddy ecosystem in fallow season, Xiang, Wu, and Zhou found that the content of inorganic fertilizer is improved by putting some herbivores into the paddy ecosystem in fallow season. They also found that the system can exhibit Hopf bifurcation phenomenon and gave the critical value of Hopf bifurcation by taking a system parameter as the bifurcation parameter [13]. Wang et al. [12] further studied the interaction of rice, weeds, and inorganic fertilizer in a paddy ecosystem. They discussed the existence and stability of equilibria in a paddy ecosystem. They also found that there exist Hopf bifurcations in such a system.

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The three models mentioned have been restricted to integer-order (delay) differential equations [12–14]. In recent more than 20 years, the research of fractional-order differential equations has been the concern of many scholars. According to the study and numerical experiments in different fields such as physical, mechanical, and engineering problems, many phenomena can be described more successfully by using factional-order differential equation models. In view of this, some scholars have used fractional differential equations to study the interaction relationship of biological populations [15–20]. Recently, some researchers have also concerned about the existence of Hopf bifurcation of fractional-order models [21–27]. Abdelouahab et al. [21] obtained the Hopf bifurcation conditions of a three-dimensional fractional-order system without time delay, and Li et al. [23] obtained the Hopf bifurcation conditions of a four-dimensional fractional-order system without time delay. For general delayed fractional-order systems, the Hopf bifurcation conditions were proposed by Xiao et al. [26] in 2017.

It is of practical significance to study whether there exists a Hopf bifurcation in a paddy ecosystem. If a Hopf bifurcation exists in a paddy ecosystem, the stability of the system will be destroyed. An unstable paddy ecosystem brings difficulties and uncertainties to management of rice production. Therefore, we want to delay or eliminate the Hopf bifurcation by using the existence conditions of Hopf bifurcation. On the other hand, in case the paddy ecosystem has come up with a Hopf bifurcation, we should try to harvest rice at the peak of its biomass to increase rice yield.

In this paper, we establish a factional-order differential equation model with delay for the interaction among the main components of a paddy ecosystem. We give a detailed stability analysis of the system equilibria and study the existence of Hopf bifurcation by using the Hopf bifurcation conditions proposed by Xiao et al. [26].

2 Preliminaries

Considering a general delayed fractional-order system

$$D^\alpha Y(t) = F\big(Y(t), Y(t - \tau)\big) \tag{1}$$

the time delay $\tau > 0$, where $Y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T \in \mathbb{R}^n$, and D^α is the Caputo fractional derivative defined as

$$D^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t (t - s)^{m-\alpha-1} f^{(m)}(s)\, ds,$$

where $\Gamma(q) = \int_0^\infty e^{-t} t^{q-1}\, dt$ is the gamma function, $m \in \mathbb{N}$, and $m - 1 < \alpha < m$. When $\alpha = m$, $D^\alpha f(t) = f^{(m)}(t)$. In this paper, we suppose $0 < \alpha \leq 1$.

The equilibrium Y^* of system (1) is the solution to equation $F(Y, Y) = 0$.

The corresponding linearized system of (1) at an equilibrium Y^* is of the form

$$D^\alpha X(t) = AX(t) + BX(t - \tau). \tag{2}$$

The characteristic equation of system (2) is

$$\Delta(\lambda) = \det\big(\lambda^\alpha E - A - Be^{-\lambda\tau}\big) = 0.$$

If $\tau = 0$, then system (2) is simplified as

$$D^\alpha X(t) = MX(t), \tag{3}$$

where the coefficient matrix $M = A + B$.

Based on the characteristic equation $\Delta(\lambda) = 0$ and the coefficient matrix M, we have the following stability result on the delayed fractional-order system (2) [28].

Lemma 1 *If $\alpha \in (0,1)$, then all the eigenvalues λ of M satisfy $|\arg(\lambda)| > \pi/2$, and the characteristic equation $\Delta(\lambda) = 0$ has no purely imaginary roots for any $\tau > 0$, then the zero solution of system (2) is Lyapunov globally asymptotically stable.*

The Hopf bifurcation conditions were proposed in [26] for the general delayed fractional-order system (1). If the following conditions hold, then system (1) undergoes a Hopf bifurcation at the equilibrium Y^* when $\tau = \tau_0$.

(1) All the eigenvalues of the coefficient matrix M of the linearized system of (1) satisfy $|\arg(\lambda)| > \alpha\pi/2$.

(2) The characteristic equation $\Delta(\lambda) = 0$ of the linearized system of (1) has a pair of purely imaginary roots $\pm i\omega_0$ when $\tau = \tau_0$.

(3) $\frac{d\,\mathrm{Re}(\lambda(\tau))}{d\tau}\big|_{\tau=\tau_0} > 0$, where $\mathrm{Re}(\cdot)$ denotes the real part of a complex number.

3 Fractional-order model of a paddy ecosystem

Wang et al. [12] have considered the following paddy ecosystem with three main components, rice, weeds, and inorganic fertilizer:

$$\begin{cases} \dot{r}(t) = c_1 s_1 u(t) r(t) - d_1 r(t), \\ \dot{p}(t) = c_2 s_2 u(t) p(t) - d_2 p(t), \\ \dot{u}(t) = b + d_1 r(t - \tau) + d_2 p(t - \tau) - s_1 u(t) r(t) - s_2 u(t) p(t) - d_3 u(t), \end{cases} \tag{4}$$

where $r(t)$ and $p(t)$ denote the rice and weeds biomasses per unit area at time t, respectively, and $u(t)$ denotes the inorganic fertilizer content per unit area at time t. The system can reflect the interactions among rice, weeds, and inorganic fertilizer. The first two equations in system (4) indicate that the growth of rice $r(t)$ and weeds $p(t)$ are affected by soil fertility $u(t)$, light and other factors s_i, and there is natural death $d_1 r(t)$ and $d_2 p(t)$ for the rice and weeds. The coefficients c_1 and c_2 represent rice and weeds utilization rate of inorganic fertilizer, light energy, and other factors, respectively. The third equation in system (4) shows that the inorganic fertilizer in soil partly comes from fertilization b and partly comes from organic fertilizer such as decaying leaves of rice and weeds, $d_1 r(t - \tau)$ and $d_2 p(t - \tau)$, which can be transformed to inorganic fertilizer after some time τ by microbial. Natural loss $d_3 u(t)$ also reduces the content of inorganic fertilizers in soil.

Using the Caputo fractional-order derivative of order $\alpha \in (0,1)$, a fractional-order delayed paddy ecosystem is established as follows:

$$\begin{cases} D^\alpha r(t) = c_1 s_1 u(t) r(t) - d_1 r(t), \\ D^\alpha p(t) = c_2 s_2 u(t) p(t) - d_2 p(t), \\ D^\alpha u(t) = b + c_3 d_1 r(t - \tau) + c_4 d_2 p(t - \tau) - s_1 u(t) r(t) \\ \qquad\qquad - s_2 u(t) p(t) - d_3 u(t), \end{cases} \tag{5}$$

where c_3 and c_4 are the conversion rates from organic fertilizer $d_1 r(t - \tau)$ and $d_2 p(t - \tau)$ to inorganic fertilizer $u(t)$, respectively. The meaning of other symbols in system (5) are consistent with system (4). Similarly, the parameters in system (5) are nonnegative and satisfy the following conditions: $0 < c_i < 1$, $b \geq 0$, $\tau \geq 0$, $s_i > 0$, and $d_i > 0$. We also introduce the following notation [12]:

$$\theta_1 = \frac{d_1}{c_1 s_1}, \qquad \theta_2 = \frac{d_2}{c_2 s_2},$$

where θ_1 is called the relative mortality of rice, and θ_2 is called the relative mortality of weeds.

4 The stability of equilibria and Hopf bifurcation

Similarly to [12], system (5) always has an equilibrium for rice and weeds extinction

$$\left(r_1^*, p_1^*, u_1^*\right) = \left(0, 0, \frac{b}{d_3}\right).$$

If $b/d_3 > \theta_1$, then system (5) has an equilibrium for weeds extinction

$$\left(r_2^*, p_2^*, u_2^*\right) = \left(\frac{b - d_3 \theta_1}{s_1 \theta_1 (1 - c_1 c_3)}, 0, \theta_1\right).$$

If $b/d_3 > \theta_2$, then system (5) still has an equilibrium for rice extinction

$$\left(r_3^*, p_3^*, u_3^*\right) = \left(0, \frac{b - d_3 \theta_2}{s_2 \theta_2 (1 - c_2 c_4)}, \theta_2\right).$$

For an equilibrium (r^*, p^*, u^*) of system (5), we make a coordinate transformation $x = r - r^*$, $y = p - p^*$, $z = u - u^*$; then system (5) can be converted to

$$\begin{cases} D^\alpha x(t) = c_1 s_1 (u^* - \theta_1) x(t) + c_1 s_1 r^* z(t) + c_1 s_1 x(t) z(t), \\ D^\alpha y(t) = c_2 s_2 (u^* - \theta_2) y(t) + c_2 s_2 p^* z(t) + c_2 s_2 y(t) z(t), \\ D^\alpha z(t) = c_3 d_1 x(t - \tau) - s_1 u^* x(t) + c_4 d_2 y(t - \tau) - s_2 u^* y(t) \\ \qquad - (s_1 r^* + s_2 p^* + d_3) z(t) - s_1 x(t) z(t) - s_2 y(t) z(t). \end{cases} \tag{6}$$

Obviously, the linearized system of (6) is

$$\begin{cases} D^\alpha x(t) = c_1 s_1 (u^* - \theta_1) x(t) + c_1 s_1 r^* z(t), \\ D^\alpha y(t) = c_2 s_2 (u^* - \theta_2) y(t) + c_2 s_2 p^* z(t), \\ D^\alpha z(t) = c_3 d_1 x(t - \tau) - s_1 u^* x(t) + c_4 d_2 y(t - \tau) \\ \qquad - s_2 u^* y(t) - (s_1 r^* + s_2 p^* + d_3) z(t). \end{cases} \tag{7}$$

Its characteristic equation is

$$\Delta(\lambda) = \begin{vmatrix} \lambda^\alpha - c_1 s_1 (u^* - \theta_1) & 0 & -c_1 s_1 r^* \\ 0 & \lambda^\alpha - c_2 s_2 (u^* - \theta_2) & -c_2 s_2 p^* \\ s_1 u^* - c_3 d_1 e^{-\lambda \tau} & s_2 u^* - c_4 d_2 e^{-\lambda \tau} & \lambda^\alpha + s_1 r^* + s_2 p^* + d_3 \end{vmatrix} = 0. \tag{8}$$

When the time delay $\tau = 0$, the coefficient matrix of system (7) is

$$
M = \begin{pmatrix} c_1 s_1 (u^* - \theta_1) & 0 & c_1 s_1 r^* \\ 0 & c_2 s_2 (u^* - \theta_2) & c_2 s_2 p^* \\ c_3 d_1 - s_1 u^* & c_4 d_2 - s_2 u^* & -s_1 r^* - s_2 p^* - d_3 \end{pmatrix}.
$$

(9)

Next, we consider the stability of the three equilibria of system (5).

Case (I) for the equilibrium (r_1^*, p_1^*, u_1^*). At this case, we have the following conclusion of the stability of the equilibrium.

Theorem 1 *If $b/d_3 < \min\{\theta_1, \theta_2\}$, then the equilibrium for rice and weeds extinction of system (5) (r_1^*, p_1^*, u_1^*) is locally asymptotically stable. Otherwise, if $b/d_3 > \min\{\theta_1, \theta_2\}$, then the equilibrium (r_1^*, p_1^*, u_1^*) is unstable.*

Proof At the equilibrium (r_1^*, p_1^*, u_1^*), by (8) the characteristic equation of the linearized system is

$$
\Delta(\lambda) = \begin{vmatrix} \lambda^\alpha - c_1 s_1 (u_1^* - \theta_1) & 0 & 0 \\ 0 & \lambda^\alpha - c_2 s_2 (u_1^* - \theta_2) & 0 \\ s_1 u_1^* - c_3 d_1 e^{-\lambda \tau} & s_2 u_1^* - c_4 d_2 e^{-\lambda \tau} & \lambda^\alpha + d_3 \end{vmatrix} = 0.
$$

So the eigenvalues satisfy $\lambda_1^\alpha = -d_3$, $\lambda_2^\alpha = c_1 s_1 (b/d_3 - \theta_1)$, and $\lambda_3^\alpha = c_2 s_2 (b/d_3 - \theta_2)$. Therefore the characteristic equation $\Delta(\lambda) = 0$ has no purely imaginary roots for any $\tau > 0$.

Similarly, the eigenvalues of the coefficient matrix M are $\lambda_1 = -d_3 < 0$, $\lambda_2 = c_1 s_1 (b/d_3 - \theta_1)$, and $\lambda_3 = c_2 s_2 (b/d_3 - \theta_2)$.

If $b/d_3 < \min\{\theta_1, \theta_2\}$, then the eigenvalues of the matrix M $\lambda_2 < 0$ and $\lambda_3 < 0$. By Lemma 1 the equilibrium $(0, 0, 0)$ of system (7) is Lyapunov globally asymptotically stable. Therefore, the equilibrium (r_1^*, p_1^*, u_1^*) of system (5) is locally asymptotically stable.

If $b/d_3 > \min\{\theta_1, \theta_2\}$, then at least one of the eigenvalues λ_2 and λ_3 of the matrix M is positive. Therefore the equilibrium (r_1^*, p_1^*, u_1^*) is unstable under this condition. □

Case (II) for the equilibrium (r_2^*, p_2^*, u_2^*).

To discuss the stability of the other two equilibria, we introduce the polynomial of degree 4 with real coefficients $a = (1, a_1, a_2, a_3, a_4)$

$$
f_a(\xi) = \xi^4 + a_1 \xi^3 + a_2 \xi^2 + a_3 \xi + a_4
$$

(10)

and the cubic polynomial equation

$$
v^3 - a_2 v^2 + (a_1 a_3 - 4 a_4) v + 4 a_2 a_4 - a_1^2 a_4 - a_3^2 = 0.
$$

(11)

If $a_1 > 0$, $a_2 < 0$, $a_3 > 0$, and $a_4 > 0$, then $4 a_2 a_4 - a_1^2 a_4 - a_3^2 < 0$. Hence equation (11) must have a positive real root, denoted by v_a. It is noted that equation (11) can also be expressed in the form

$$
\left(\frac{a_1 v}{2} - a_3 \right)^2 - 4 \left(\frac{a_1^2}{4} - a_2 + v \right) \left(\frac{v^2}{4} - a_4 \right) = 0.
$$

(12)

Therefore from equation (12) it follows that the positive real root v_a must satisfy $\frac{v_a^2}{4} - a_4 \geq 0$. Let

$$M_a = \sqrt{\frac{a_1^2}{4} - a_2 + v_a}, \qquad N_a = \sqrt{\frac{v_a^2}{4} - a_4} \tag{13}$$

and

$$\Delta_a = \frac{a_1^2}{2} - a_2 - v_a - a_1 M_a + 4\,\mathrm{sgn}\!\left(\frac{a_1 v_a}{2} - a_3\right) N_a, \tag{14}$$

where $\mathrm{sgn}(\cdot)$ is the sign function.

Lemma 2 *Suppose that $a_i > 0$ ($i = 1, 3, 4$). If $a_2 < 0$ and $\Delta_a \geq 0$, then there are only two positive real roots of the equation $f_a(\xi) = \xi^4 + a_1\xi^3 + a_2\xi^2 + a_3\xi + a_4 = 0$. If $a_2 \geq 0$, or $a_2 < 0$ and $\Delta_a < 0$, then the equation $f_a(\xi) = 0$ has no positive real root.*

Proof The polynomial $f_a(\xi)$ can be decomposed into the product of two quadratic polynomials

$$f_1(\xi) = \xi^2 + \left(\frac{a_1}{2} - M_a\right)\xi + \frac{v_a}{2} - \mathrm{sgn}\!\left(\frac{a_1 v_a}{2} - a_3\right) N_a$$

and

$$f_2(\xi) = \xi^2 + \left(\frac{a_1}{2} + M_a\right)\xi + \frac{v_a}{2} + \mathrm{sgn}\!\left(\frac{a_1 v_a}{2} - a_3\right) N_a.$$

The discriminants of the polynomials $f_1(\xi)$ and $f_2(\xi)$ are Δ_a and

$$\Delta_1 = \frac{a_1^2}{2} - a_2 - v_a + a_1 M_a - 4\,\mathrm{sgn}\!\left(\frac{a_1 v_a}{2} - a_3\right) N_a,$$

respectively. If $\Delta_a \geq 0$, then the equation $f_1(\xi) = 0$ has two real roots ξ_{a1} and ξ_{a2}. From

$$\xi_{a1} + \xi_{a2} = -\frac{a_1}{2} + M_a > 0 \quad \text{and} \quad \xi_{a1}\xi_{a2} = \frac{v_a}{2} - \mathrm{sgn}\!\left(\frac{a_1 v_a}{2} - a_3\right) N_a > 0$$

we haven $\xi_{a1} > 0$ and $\xi_{a2} > 0$.

If $\Delta_1 \geq 0$, then the equation $f_2(\xi) = 0$ has two real roots ξ_{a3} and ξ_{a4}. From

$$\xi_{a3} + \xi_{a4} = -\frac{a_1}{2} - M_a < 0 \quad \text{and} \quad \xi_{a3}\xi_{a4} = \frac{v_a}{2} + \mathrm{sgn}\!\left(\frac{a_1 v_a}{2} - a_3\right) N_a > 0$$

we have $\xi_{a3} < 0$ and $\xi_{a4} < 0$.

Therefore equation $f_a(\xi) = 0$ has only two positive real roots.

Otherwise, if $a_i > 0$ ($i = 1, 3, 4$) and $a_2 \geq 0$, it is obvious that the equation $f_a(\xi) = 0$ has no positive real root. If $a_i > 0$ ($i = 1, 3, 4$), $a_2 < 0$, and $\Delta_a < 0$, then the equation $f_1(\xi) = 0$ has no real root, and so the equation $f_a(\xi) = 0$ has no positive real root. □

Let the coefficients of polynomial (10) be as follows:

$$a_1 = 2\left(s_1 r_2^* + d_3\right)\cos\frac{\alpha\pi}{2}, \qquad a_2 = \left(s_1 r_2^* + d_3\right)^2 + 2 s_1 r_2^* d_1 \cos(\alpha\pi), \tag{15}$$

$$a_3 = 2 s_1 r_2^* d_1\left(s_1 r_2^* + d_3\right)\cos\frac{\alpha\pi}{2}, \qquad a_4 = s_1^2 r_2^{*2} d_1^2\left(1 - c_1^2 c_3^2\right). \tag{16}$$

Since $0 < \alpha < 1$, we have that $a_1 > 0$, $a_3 > 0$, and $a_4 > 0$.

The conclusion of the stability of the equilibrium (r_2^*, p_2^*, u_2^*) is as follows.

Theorem 2 *Suppose that $b/d_3 > \theta_1$.*
  (I) *If $\theta_1 > \theta_2$, then the equilibrium (r_2^*, p_2^*, u_2^*) is unstable.*
 (II) *If $\theta_1 < \theta_2$ and $a_2 \geq 0$, or $\theta_1 < \theta_2$, $a_2 < 0$, and $\Delta_a < 0$, then the equilibrium (r_2^*, p_2^*, u_2^*) is locally asymptotically stable for $\tau \geq 0$.*
(III) *If $\theta_1 < \theta_2$, $a_2 < 0$, and $\Delta_a > 0$, then there exists a positive number τ_a such that when $\tau \in [0, \tau_a)$, the equilibrium (r_2^*, p_2^*, u_2^*) is locally asymptotically stable; when $\tau > \tau_a$, the equilibrium (r_2^*, p_2^*, u_2^*) is unstable; and a Hopf bifurcation emerges at $\tau = \tau_a$.*

Proof By (8) the characteristic equation of the linearized system is

$$\left(\lambda^\alpha - c_2 s_2(\theta_1 - \theta_2)\right)\left(\lambda^{2\alpha} + \left(s_1 r_2^* + d_3\right)\lambda^\alpha + s_1 r_2^* d_1\left(1 - c_1 c_3 e^{-\lambda\tau}\right)\right) = 0.$$

It has one real eigenvalue satisfying $\lambda_1^\alpha = c_2 s_2(\theta_1 - \theta_2)$, the real part of which cannot be zero. Its other eigenvalues are the roots of the equation

$$\lambda^{2\alpha} + \left(s_1 r_2^* + d_3\right)\lambda^\alpha + s_1 r_2^* d_1\left(1 - c_1 c_3 e^{-\lambda\tau}\right) = 0. \tag{17}$$

By (9) the characteristic equation of the coefficient matrix M is

$$\left(\lambda - c_2 s_2(\theta_1 - \theta_2)\right)\left(\lambda^2 + \left(s_1 r_2^* + d_3\right)\lambda + s_1 r_2^* d_1(1 - c_1 c_3)\right) = 0.$$

It has one real eigenvalue $\lambda_1 = c_2 s_2(\theta_1 - \theta_2)$. Its other eigenvalues are

$$\lambda_{2,3} = \frac{1}{2}\left(-\left(s_1 r_2^* + d_3\right) \pm \sqrt{\left(s_1 r_2^* + d_3\right)^2 - 4 s_1 r_2^* d_1(1 - c_1 c_3)}\right).$$

Obviously, the real parts of $\lambda_{2,3}$ are less than zero.

(I) If $\theta_1 > \theta_2$, then the eigenvalue of the coefficient matrix M $\lambda_1 > 0$. It indicates that the equilibrium (r_2^*, p_2^*, u_2^*) is unstable.

(II) If $\theta_1 < \theta_2$, then the real eigenvalue of the coefficient matrix M $\lambda_1 < 0$. So the eigenvalues λ_j of M satisfy $|\arg(\lambda_j)| > \frac{\pi}{2} > \frac{\alpha\pi}{2}$ $(j = 1, 2, 3)$.

Assume that equation (17) has a purely imaginary root $\lambda = i\xi = \xi\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ $(\xi > 0)$. Substituting it into (17) gives

$$\xi^{2\alpha}(\cos\alpha\pi + i\sin\alpha\pi) + \left(s_1 r_2^* + d_3\right)\xi^\alpha\left(\cos\frac{\alpha\pi}{2} + i\sin\frac{\alpha\pi}{2}\right)$$

$$+ s_1 r_2^* d_1(1 - c_1 c_3 \cos\tau\xi + i c_1 c_3 \sin\tau\xi) = 0.$$

Separating its real and imaginary parts yields

$$\xi^{2\alpha}\cos\alpha\pi + \xi^{\alpha}\left(s_1 r_2^* + d_3\right)\cos\frac{\alpha\pi}{2} + s_1 r_2^* d_1(1 - c_1 c_3 \cos\tau\xi) = 0$$

and

$$\xi^{2\alpha}\sin\alpha\pi + \xi^{\alpha}\left(s_1 r_2^* + d_3\right)\sin\frac{\alpha\pi}{2} + c_1 c_3 s_1 r_2^* d_1 \sin\tau\xi = 0.$$

So we have

$$\cos\tau\xi = \frac{\xi^{2\alpha}\cos\alpha\pi + \xi^{\alpha}(s_1 r_2^* + d_3)\cos\frac{\alpha\pi}{2} + s_1 r_2^* d_1}{c_1 c_3 s_1 r_2^* d_1}, \tag{18}$$

$$\sin\tau\xi = -\frac{\xi^{2\alpha}\sin\alpha\pi + \xi^{\alpha}(s_1 r_2^* + d_3)\sin\frac{\alpha\pi}{2}}{c_1 c_3 s_1 r_2^* d_1}. \tag{19}$$

Since $\sin^2\tau\xi + \cos^2\tau\xi = 1$, we have

$$\left(\xi^{2\alpha}\cos\alpha\pi + \xi^{\alpha}\left(s_1 r_2^* + d_3\right)\cos\frac{\alpha\pi}{2} + s_1 r_2^* da_1\right)^2$$
$$+ \left(\xi^{2\alpha}\sin\alpha\pi + \xi^{\alpha}\left(s_1 r_2^* + d_3\right)\sin\frac{\alpha\pi}{2}\right)^2 = c_1^2 c_3^2 s_1^2 r_2^{*2} d_1^2,$$

that is,

$$f_a\left(\xi^{\alpha}\right) = \xi^{4\alpha} + a_1\xi^{3\alpha} + a_2\xi^{2\alpha} + a_3\xi^{\alpha} + a_4 = 0, \tag{20}$$

where a_i ($i = 1, 2, 3, 4$) are given in (15) and (16); note that $a_1, a_3, a_4 > 0$. If $a_2 \geq 0$, or $a_2 < 0$ and $\Delta_a < 0$, then the equation $f_a(\xi) = 0$ has no positive real root by Lemma 2. This leads to that equation (20) has no any positive real number ξ. Therefore the real parts of any roots of (17) must be negative for any $\tau > 0$. This shows that the equilibrium (r_2^*, p_2^*, u_2^*) is locally asymptotically stable for any $\tau \geq 0$.

(III) If $\theta_1 < \theta_2$, $a_2 < 0$, and $\Delta_a > 0$, then the equation $f_a(\xi) = 0$ has two unequal positive real roots by Lemma 2; denote the larger root by ξ_+. So equation (20) has a positive real root ξ_a satisfying $\xi_a^{\alpha} = \xi_+$.

Notice that $\sin\tau_a\xi_a < 0$ from (19). If $\cos\tau_a\xi_a > 0$, then from (19) we have

$$\tau_a = \frac{2\pi}{\xi_a} - \frac{1}{\xi_a}\arcsin\frac{\xi_a^{2\alpha}\sin\alpha\pi + \xi_a^{\alpha}(s_1 r_2^* + d_3)\sin\frac{\alpha\pi}{2}}{c_1 c_3 s_1 r_2^* d_1}. \tag{21}$$

If $\cos\tau_a\xi_a < 0$, then we have

$$\tau_a = \frac{\pi}{\xi_a} + \frac{1}{\xi_a}\arcsin\frac{\xi_a^{2\alpha}\sin\alpha\pi + \xi_a^{\alpha}(s_1 r_2^* + d_3)\sin\frac{\alpha\pi}{2}}{c_1 c_3 s_1 r_2^* d_1}. \tag{22}$$

Next, we verify the transversal condition. Taking the derivative of λ with respect to τ in (17), we have

$$\frac{d\lambda}{d\tau} = -\frac{c_1 c_3 s_1 r_2^* d_1 \lambda e^{-\lambda\tau}}{2\alpha\lambda^{2\alpha-1} + \alpha(s_1 r_2^* + d_3)\lambda^{\alpha-1} + c_1 c_3 s_1 r_2^* d_1 \tau e^{-\lambda\tau}}.$$

So we obtain

$$
\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=i\xi_a} = -\left[\frac{2\alpha\lambda^{2\alpha-2}e^{\lambda\tau} + \alpha(s_1 r_2^* + d_3)\lambda^{\alpha-2}e^{\lambda\tau}}{c_1 c_3 s_1 r_2^* d_1} + \frac{\tau}{\lambda}\right]\Bigg|_{\lambda=i\xi_a}
$$

$$
= -\frac{\alpha\xi_a^{\alpha-2}}{c_1 c_3 s_1 r_2^* d_1}\left[2\xi_a^{\alpha}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{2\alpha-2}e^{i\xi_a\tau_a}\right.
$$

$$
\left. + \left(s_1 r_2^* + d_3\right)\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^{\alpha-2}e^{i\xi_a\tau_a}\right] - \frac{\tau_a}{i\xi_a}.
$$

Thus, its real part is

$$
\mathrm{Re}\left\{\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=i\xi_a}\right\} = -\frac{\alpha\xi_a^{\alpha-2}}{c_1 c_3 s_1 r_2^* d_1}\left[2\xi_a^{\alpha}\left(\cos(\alpha-1)\pi\cos\xi_a\tau_a - \sin(\alpha-1)\pi\sin\xi_a\tau_a\right)\right.
$$

$$
\left. + \left(s_1 r_2^* + d_3\right)\left(\cos\frac{(\alpha-2)\pi}{2}\cos\xi_a\tau_a - \sin\frac{(\alpha-2)\pi}{2}\sin\xi_a\tau_a\right)\right]
$$

$$
= \frac{\alpha\xi_a^{\alpha-2}}{c_1 c_3 s_1 r_2^* d_1}\left[\cos\xi_a\tau_a\left(2\xi_a^{\alpha}\cos\alpha\pi + \left(s_1 r_2^* + d_3\right)\cos\frac{\alpha\pi}{2}\right)\right.
$$

$$
\left. - \sin\xi_a\tau_a\left(2\xi_a^{\alpha}\sin\alpha\pi + \left(s_1 r_2^* + d_3\right)\sin\frac{\alpha\pi}{2}\right)\right].
$$

Substituting (18) and (19) into this expression, we obtain

$$
\mathrm{Re}\left\{\left.\left(\frac{d\lambda}{d\tau}\right)^{-1}\right|_{\lambda=i\xi_a}\right\} = \frac{\alpha\xi_a^{\alpha-2}}{(c_1 c_3 s_1 r_2^* d_1)^2}\left[2\xi_a^{3\alpha} + 3\left(s_1 r_2^* + d_3\right)\cos\frac{\alpha\pi}{2}\xi_a^{2\alpha}\right.
$$

$$
\left. + \left(s_1 r_2^* + d_3\right)^2\xi_a^{\alpha} + s_1 r_2^* d_1\left(2\xi_a^{\alpha}\cos\alpha\pi + \left(s_1 r_2^* + d_3\right)\cos\frac{\alpha\pi}{2}\right)\right]
$$

$$
= \frac{\alpha\xi_a^{\alpha-2}}{2(c_1 c_3 s_1 r_2^* d_1)^2}\left(4\xi_a^{3\alpha} + 3a_1\xi_a^{2\alpha} + 2a_2\xi_a^{\alpha} + a_3\right)
$$

$$
= \frac{\alpha\xi_a^{\alpha-2}f'(\xi_+)}{2(c_1 c_3 s_1 r_2^* d_1)^2}.
$$

Since ξ_+ is the larger single root of the equation $f_a(\xi) = 0$ and the highest order power coefficient of the polynomial is positive, we have $f'(\xi_+) > 0$. Therefore the transversal condition is satisfied, and thus a Hopf bifurcation occurs at $\tau = \tau_a$. $\qquad\square$

Remark 1 From (15) we know that if $(s_1 r_2^* + d_3)^2 \geq 2s_1 r_2^* d_1$, then $a_2 \geq 0$. When $(s_1 r_2^* + d_3)^2 < 2s_1 r_2^* d_1$, we let

$$
\alpha_a = 1 - \frac{1}{\pi}\arccos\frac{(s_1 r_2^* + d_3)^2}{2s_1 r_2^* d_1}. \tag{23}
$$

Obviously, if $0 < \alpha < \alpha_a$, then $a_2 > 0$. Otherwise, if $\alpha_a < \alpha < 1$, then $a_2 < 0$.

Case (III) for the equilibrium (r_3^*, p_3^*, u_3^*).

To discuss the stability of the equilibrium (r_3^*, p_3^*, u_3^*), similarly to case (II), we introduce a polynomial of degree 4 with real coefficients $q = (1, q_1, q_2, q_3, q_4)$ as follows:

$$f_q(\xi) = \xi^4 + q_1\xi^3 + q_2\xi^2 + q_3\xi + q_4, \tag{24}$$

where

$$q_1 = 2\left(s_2 p_3^* + d_3\right)\cos\frac{\alpha\pi}{2}, \qquad q_2 = \left(s_2 p_3^* + d_3\right)^2 + 2s_2 p_3^* d_2 \cos\alpha\pi, \tag{25}$$

$$q_3 = 2s_2 p_3^* d_2\left(s_2 p_3^* + d_3\right)\cos\frac{\alpha\pi}{2}, \qquad q_4 = s_2^2 p_3^{*2} d_2^2\left(1 - c_2^2 c_4^2\right). \tag{26}$$

Since $0 < \alpha < 1$, it is obvious that $q_1 > 0$, $q_3 > 0$, and $q_4 > 0$.

Similarly to case (II), we have the following stability conclusion of the equilibrium (r_3^*, p_3^*, u_3^*).

Theorem 3 *Suppose that $b/d_3 > \theta_2$.*
  (I) *If $\theta_1 < \theta_2$, then the equilibrium (r_3^*, p_3^*, u_3^*) is unstable.*
  (II) *If $\theta_1 > \theta_2$ and $q_2 \geq 0$, or $\theta_1 > \theta_2$, $q_2 < 0$, and $\Delta_q < 0$, then the equilibrium (r_3^*, p_3^*, u_3^*) is locally asymptotically stable for $\tau \geq 0$.*
  (III) *If $\theta_1 > \theta_2$, $q_2 < 0$, and $\Delta_q > 0$, then there exists a positive number τ_q such that the equilibrium (r_3^*, p_3^*, u_3^*) is locally asymptotically stable for $\tau \in [0, \tau_q)$ and unstable when $\tau > \tau_q$. A Hopf bifurcation emerges at the equilibrium (r_3^*, p_3^*, u_3^*) when $\tau = \tau_q$.*

The proof of Theorem 3 is similar to Theorem 2 and is omitted here.

Remark 2 From (25) we know that if $(s_2 p_3^* + d_3)^2 \geq 2s_2 p_3^* d_2$, then $q_2 \geq 0$. When $(s_2 p_3^* + d_3)^2 < 2s_2 p_3^* d_2$, we let

$$\alpha_q = 1 - \frac{1}{\pi}\arccos\left(\frac{(s_2 p_3^* + d_3)^2}{2s_2 p_3^* d_2}\right). \tag{27}$$

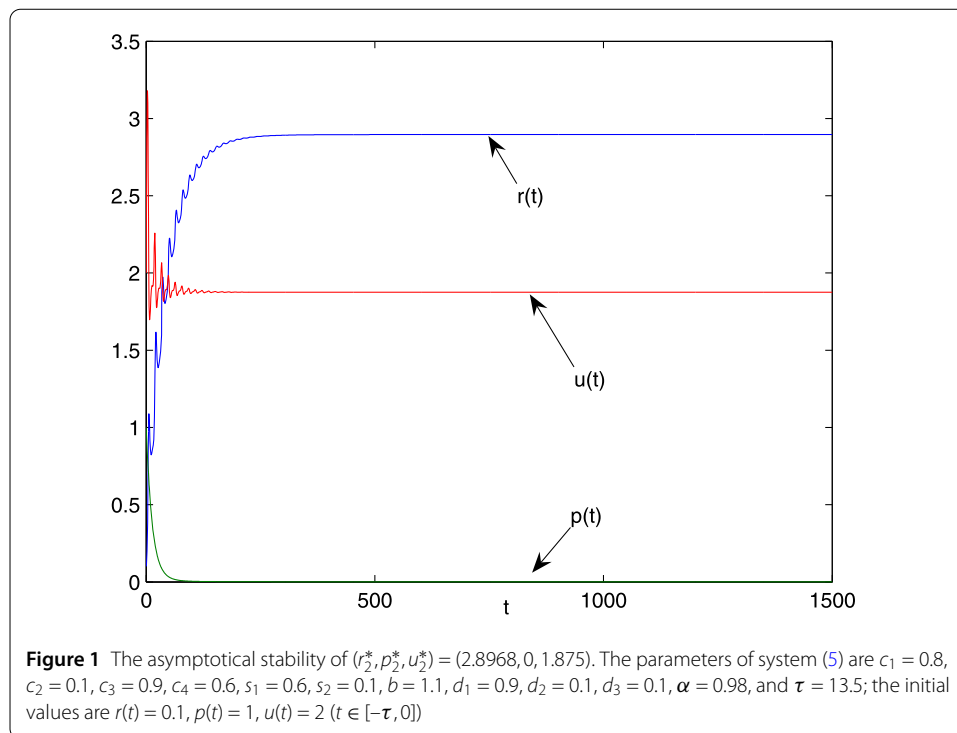Obviously, if $0 < \alpha < \alpha_q$, then $q_2 > 0$. Otherwise, if $\alpha_q < \alpha < 1$, then $a_q < 0$.

5 Examples

In this section, we give two examples to confirm our theoretical results obtained in Sect. 4 and use the predictor–corrector scheme to calculate their numerical solutions [29]. In system (5), we let $\alpha = 0.98$, $c_1 = 0.8$, $c_2 = 0.1$, $c_3 = 0.9$, $c_4 = 0.6$, $s_1 = 0.6$, $s_2 = 0.1$, $d_1 = 0.9$, $d_2 = 0.1$, and $d_3 = 0.1$.

First, we take $b = 1.1$. System (5) has three equilibria: the equilibrium for rice and weeds extinction $(0, 0, 11)$, the equilibrium for weeds extinction $(2.8968, 0, 1.875)$, and the equilibrium for rice extinction $(0, 0.1064, 10)$. By computing we have $b/d_3 = 11$, $\theta_1 = 1.875$, $\theta_2 = 10$, and $a_2 \approx 0.2562 > 0$. So the inequality $b/d_3 > \theta_2 > \theta_1$ holds. By Theorem 2 the equilibrium $(2.8968, 0, 1.875)$ is asymptotically stable for any $\tau \geq 0$ as illustrated in Fig. 1 (where $\tau = 13.5$). By Theorems 1 and 3 the equilibria $(0, 0, 11)$ and $(0, 0.1064, 10)$ are unstable.
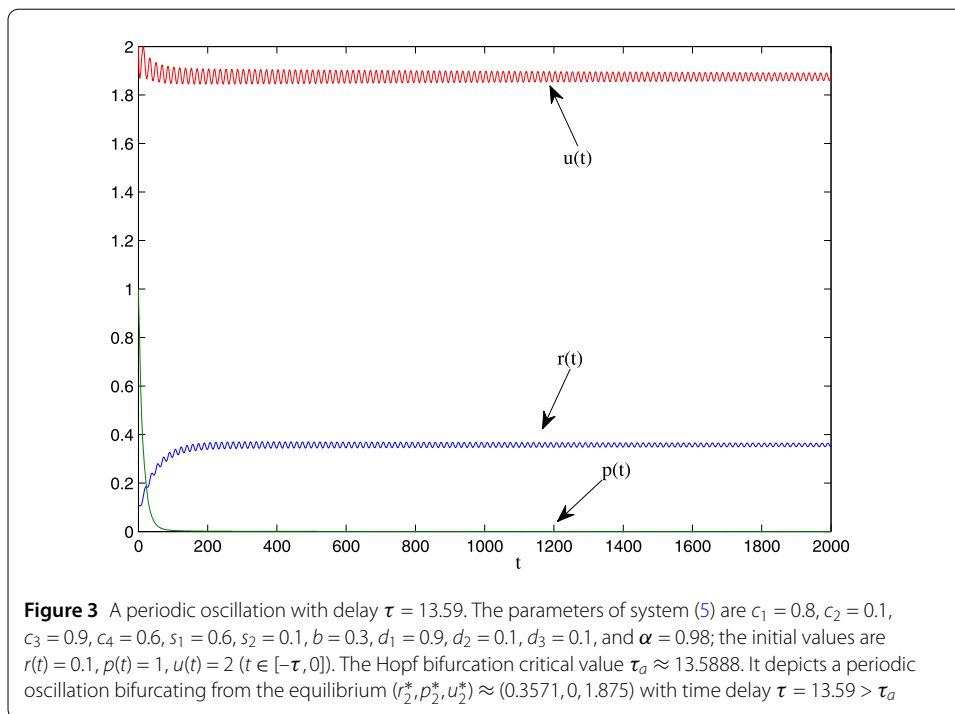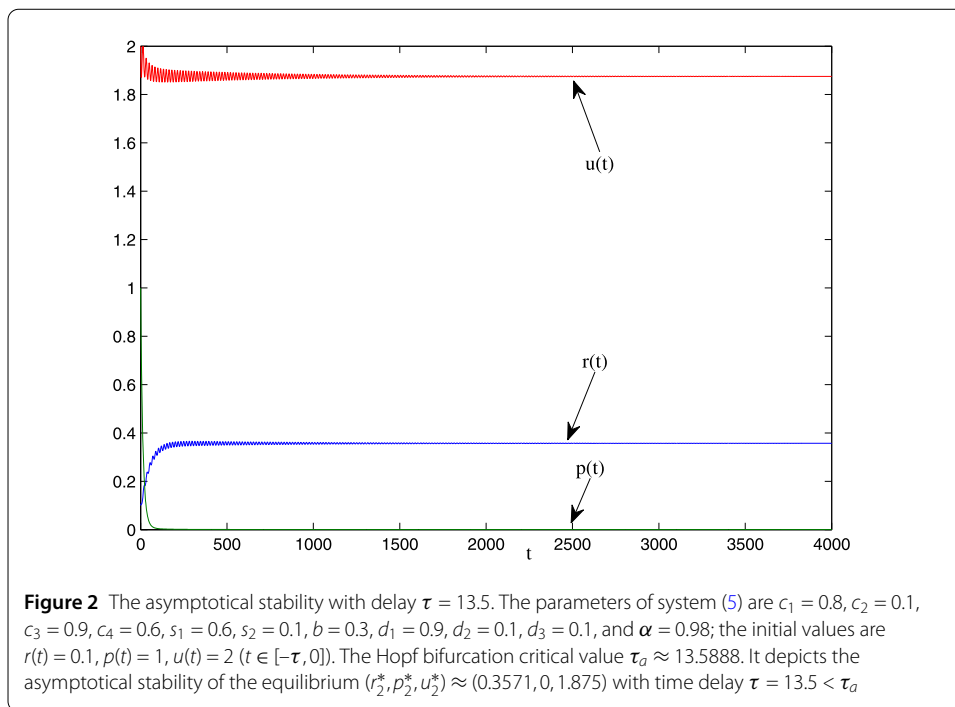
In succession, we take $b = 0.3$ again. System (5) has two equilibria: the equilibrium for rice and weeds extinction $(0, 0, 3)$ and the equilibrium for weeds extinction

Figure 1 The asymptotical stability of $(r_2^*, p_2^*, u_2^*) = (2.8968, 0, 1.875)$. The parameters of system (5) are $c_1 = 0.8$, $c_2 = 0.1$, $c_3 = 0.9$, $c_4 = 0.6$, $s_1 = 0.6$, $s_2 = 0.1$, $b = 1.1$, $d_1 = 0.9$, $d_2 = 0.1$, $d_3 = 0.1$, $\alpha = 0.98$, and $\tau = 13.5$; the initial values are $r(t) = 0.1$, $p(t) = 1$, $u(t) = 2$ ($t \in [-\tau, 0]$)

$(0.3571, 0, 1.875)$. Because $b/d_3 = 3$, $\theta_1 = 1.875$, and $\theta_2 = 10$, the inequality $\theta_2 > b/d_3 > \theta_1$ holds. By Theorem 1 the equilibrium $(0, 0, 3)$ is unstable. From (15) and (16) we have $a_1 \approx 0.0197$, $a_2 \approx -0.2862$, $a_3 \approx 0.0038$, $a_4 \approx 0.0179$. Equation (11) has a positive real root $v_a \approx 0.2677$. From (13) we obtain $M_a \approx 0.7443$ and $N_a \approx 7.8279$. Because the discriminant $\Delta_a \approx 0.000866 > 0$ and $a_2 < 0$, the equation $f_a(\xi) = 0$ has two unequal positive real roots by Lemma 2, and the larger root $\xi_+ \approx 0.3819$. So equation (20) has a positive real root ξ_a satisfying $\xi_a^\alpha = \xi_+$. Substituting ξ_a into (18), we get $\cos \tau \xi_a \approx 0.3677 > 0$. So we obtain the Hopf bifurcation critical value $\tau_a \approx 13.5888$ by using (21). Therefore, by Theorem 2 the equilibrium $(0.3571, 0, 1.875)$ is asymptotically stable when $\tau \in [0, 13.5888)$ as illustrated in Fig. 2 (where $\tau = 13.5$); Otherwise, the equilibrium $(0.3571, 0, 1.875)$ is unstable, and a Hopf bifurcation emerges at $\tau \approx 13.5888$ (see Fig. 3, where $\tau = 13.59$).

6 Conclusions

We have proposed a delayed fractional-order differential equation model that reflects the interaction among rice, weeds, and inorganic fertilizer in a paddy ecosystem. If $\alpha = 1$ and $c_3 = c_4 = 1$, then system (5) degenerates into system (4), which was studied in [12]. The equilibria and their existence conditions of system (5) are the same as those of system (4), where those conditions are related to the relative mortality of rice and weeds, θ_1 and θ_2, and to the ratio of fertilizer supply and loss b/d_3, but not to other parameters. Under the condition $b/d_3 < \min\{\theta_1, \theta_2\}$, there is a unique stable equilibrium $(0, 0, u_1^*)$ in each of the two systems. If $b/d_3 > \max\{\theta_1, \theta_2\}$, then each of the two systems has three equilibria: the equilibrium for rice and weeds extinction $(0, 0, u_1^*)$, the equilibrium for weeds extinction $(r_2^*, 0, u_2^*)$, and the equilibrium for rice extinction $(0, p_3^*, u_3^*)$, where the equilibrium $(0, 0, u_1^*)$ is unstable, and $(r_2^*, 0, u_2^*)$ is also unstable when $\theta_1 > \theta_2$, or $(0, p_3^*, u_3^*)$ is unstable when $\theta_1 < \theta_2$. Under the condition $\theta_1 < b/d_3 < \theta_2$, there exist two equilibria $(0, 0, u_1^*)$ and $(r_2^*, 0, u_2^*)$. Under the condition $\theta_2 < b/d_3 < \theta_1$, there exist two equilibria $(0, 0, u_1^*)$ and $(0, p_3^*, u_3^*)$.

Figure 2 The asymptotical stability with delay $\tau = 13.5$. The parameters of system (5) are $c_1 = 0.8$, $c_2 = 0.1$, $c_3 = 0.9$, $c_4 = 0.6$, $s_1 = 0.6$, $s_2 = 0.1$, $b = 0.3$, $d_1 = 0.9$, $d_2 = 0.1$, $d_3 = 0.1$, and $\alpha = 0.98$; the initial values are $r(t) = 0.1$, $p(t) = 1$, $u(t) = 2$ ($t \in [-\tau, 0]$). The Hopf bifurcation critical value $\tau_a \approx 13.5888$. It depicts the asymptotical stability of the equilibrium $(r_2^*, p_2^*, u_2^*) \approx (0.3571, 0, 1.875)$ with time delay $\tau = 13.5 < \tau_a$

Figure 3 A periodic oscillation with delay $\tau = 13.59$. The parameters of system (5) are $c_1 = 0.8$, $c_2 = 0.1$, $c_3 = 0.9$, $c_4 = 0.6$, $s_1 = 0.6$, $s_2 = 0.1$, $b = 0.3$, $d_1 = 0.9$, $d_2 = 0.1$, $d_3 = 0.1$, and $\alpha = 0.98$; the initial values are $r(t) = 0.1$, $p(t) = 1$, $u(t) = 2$ ($t \in [-\tau, 0]$). The Hopf bifurcation critical value $\tau_a \approx 13.5888$. It depicts a periodic oscillation bifurcating from the equilibrium $(r_2^*, p_2^*, u_2^*) \approx (0.3571, 0, 1.875)$ with time delay $\tau = 13.59 > \tau_a$

We also generalize the conditions of stabilities of equilibria and Hopf bifurcation obtained by Wang et al. [12]. If we take $\alpha = 1$ and $c_3 = c_4 = 1$, then we have $a_1 = a_3 = 0$, $a_2 = (s_1 r_2^* + d_3)^2 - 2s_1 r_2^* d_1$, and $a_4 = s_1^2 r_2^{*2} d_1^2 (1 - c_1^2)$ from (15) and (16). Equation (11) has a positive root $v_a = 2\sqrt{a_4}$. So we obtain $\Delta_a = -a_2 - 2\sqrt{a_4}$ from (14). If $\Delta_a < 0$, then we have

$$\left(s_1 r_2^* + d_3\right)^2 > 2s_1 r_2^* d_1 \left(1 - \sqrt{1 - c_1^2}\right).$$

It is condition (5) of Theorem 2 in [12]. Similarly, from $\Delta_a > 0$ we can obtain condition (6) in [12]. Moreover, substituting $\alpha = 1$ and $c_3 = 1$ into the Hopf bifurcation critical value formulas (21) and (22), we can obtain formulas (12) and (13) in [12], respectively.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final version of the manuscript.

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