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Periodic solutions for complex-valued neural networks of neutral type by combining graph theory with coincidence degree theory

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Abstract

In this paper, by combining graph theory with coincidence degree theory as well as Lyapunov functional method, sufficient conditions to guarantee the existence and global exponential stability of periodic solutions of the complex-valued neural networks of neutral type are established. In our results, the assumption on the boundedness for the activation function in (Gao and Du in Discrete Dyn. Nat. Soc. 2016:Article ID 1267954, 2016) is removed and the other inequality conditions in (Gao and Du in Discrete Dyn. Nat. Soc. 2016:Article ID 1267954, 2016) are replaced with new inequalities.

Keywords: The periodic solutions; Exponential stability; Complex-valued neural networks of neutral type

1 Introduction

Because in a lot of practical applications complex signals often occur and the complex-valued neural networks are preferable and practical, hence, up to now, there has been increasing research interest in the stability of equilibrium point and periodic solutions of complex-valued neural networks, for example, see [2–17] and the references therein.

On the other hand, time delays have been extensively studied in last decades due to their potential existence in many fields [12, 13, 18–20]. Up to now, the dynamical behaviors of neural networks of neutral type have been extensively investigated and a lot of interesting results on the global asymptotic stability and global exponential stability of equilibrium point and periodic solutions for neural networks of neutral type have been published, for example, see [18, 21–28] and the references therein.

But so far, very few studies have been reported on the dynamical behaviors of complex-valued neural networks of neutral type with time delays [1, 29]. This motivates us to carry out a study on dynamical behaviors of complex-valued neural networks of neutral type. Recently, in [1], the authors discussed the existence and exponential stability of periodic solutions for the following delayed complex-valued neural networks of neutral type:

$$\frac{d[K_n z_n(t)]}{dt} = -d_n(t)z_n(t) + \sum_{j=1}^{l} b_{nj}(t)F_j\bigl(z_j(t)\bigr) + \sum_{j=1}^{l} e_{nj}(t)G_j\bigl(z_j\bigl(t-\tau_{nj}(t)\bigr)\bigr) + P_n(t), \qquad (1)$$

where $n \in \mathbf{L} = \{1, 2, \ldots, l\}$, l is a positive integer,

$$K_n z_n(t) = K_n u_n(t) + iK_n v_n(t),$$
$$K_n u_n(t) = u_n(t) - c_n u_n(t - \tau),$$
$$K_n v_n(t) = v_n(t) - c_n v_n(t - \tau),$$

$\tau, c_n \in R$ with $|c_n| \neq 1$, $d_n(t) \geq 0$ is the self feedback connection weight, and $e_{nj}(t)$, $b_{nj}(t)$ are complex-valued connection weights, $F_j(z_j)$ and $G_j(z_j) : C \to C$ are the activation functions of the neurons. $P_n(t) \in C$ is the external input, $\tau_{nj} \geq 0$ corresponds to the transmission delays with $\tau'_{nj} \leq \sigma < 1$, $\tau_{nj} \leq \sigma^*$.

In [1], first, by means of using coincidence degree theory and the a priori estimate method of periodic solutions, under the assumptions that the activation functions were bounded, a sufficient condition on the existence of periodic solutions was established for system (1). Then, by constructing an appropriate Lyapunov functional, a sufficient condition to guarantee the global exponential stability of periodic solutions to system (1) was obtained.

In recent years, graph theory has been applied to studying global asymptotic stability of discrete-time Cohen–Grossberg neural networks with finite and infinite delays [30] and the existence and global stability of periodic solutions for coupled networks [31–35]. Some sufficient conditions on the existence and global stability of equilibrium point and periodic solutions for some neural networks and coupled networks have been established [30–36].

Recently, without applying the a priori estimate method of periodic solutions, we have established some criteria to guarantee the existence of periodic solutions for neural networks with time delays by combining coincidence degree theory with Lyapunov functional method or linear matrix inequality method [14, 15, 37].

However, so far, the results on the existence and global exponential stability of periodic solutions for delayed complex-valued neural networks of neutral type have not been obtained by combining coincidence degree theory with graph theory as well as Lyapunov functional method.

The objective of this paper is to establish new criteria to guarantee the existence and global exponential stability of periodic solutions of system (1) by removing the assumption on the boundedness for the activation functions in [1] and replacing inequality conditions in [1] with new inequality conditions, by combining coincidence degree theory with graph theory as well as Lyapunov functional method.

Thus the contribution of our paper lies in the following two aspects: (1) Combination of coincidence degree theory with graph theory as well as Lyapunov functional is introduced to study the existence and exponential stability of periodic solutions for delayed complex-valued neural networks of neutral type; (2) Novel sufficient conditions to guarantee the existence and global exponential stability of periodic solutions for system (1) are derived by removing the limitation on the boundedness for the activation functions in [1] and replacing inequality conditions in [1] with new inequality conditions.

The paper is organized as follows. Some preliminaries and lemmas are introduced in Sect. 2. In Sect. 3, a sufficient condition is derived to guarantee the existence of periodic solutions of system (1). In Sect. 4, a sufficient condition is established to guarantee exponential stability of periodic solutions for system (1). In Sect. 5, an example is stated to prove the effectiveness of our theoretical results.

2 Preliminaries

Let $z_n(t) = u_n(t) + iv_n(t)$, $F_j(z_j(t)) = F_j^R(u_j(t), v_j(t)) + iF_j^I(u_j(t), v_j(t))$, $G_j(z_j(t - \tau_{nj}(t))) = G_j^R(u_j(t - \tau_{nj}(t)), v_j(t - \tau_{nj}(t))) + iG_j^I(u_j(t - \tau_{nj}), v_j(t - \tau_{nj}(t))) = G_j^R(u_j^t, v_j^t) + iG_j^I(u_j^t, v_j^t)$, $u_j(t - \tau_{nj}(t)) = u_j^t$, $v_j(t - \tau_{nj}(t)) = v_j^t$, $b_{nj}(t) = b_{nj}^R(t) + ib_{nj}^I(t)$, $e_{nj}(t) = e_{nj}^R(t) + ie_{nj}^I(t)$, $P_n(t) = P_n^R(t) + iP_n^I(t)$. After separating each state variable, the connection weight, the activation function, and the external input into its real and imaginary parts, then system (1) can be rewritten as follows:

$$\begin{aligned}
\frac{d[K_n u_n(t)]}{dt} &= -d_n(t)u_n(t) + \sum_{j=1}^{l} b_{nj}^R F_j^R\bigl(u_j(t), v_j(t)\bigr) - \sum_{j=1}^{l} b_{nj}^I F_j^I\bigl(u_j(t), v_j(t)\bigr) \\
&\quad + \sum_{j=1}^{l} e_{nj}^R(t) G_j^R\bigl(u_j^t, v_j^t\bigr) - \sum_{j=1}^{l} e_{nj}^I(t) G_j^I\bigl(u_j^t, v_j^t\bigr) + P_n^R(t), \\
\frac{d[K_n v_n(t)]}{dt} &= -d_n(t)v_n(t) + \sum_{j=1}^{l} b_{nj}^R F_j^I\bigl(u_j(t), v_j(t)\bigr) + \sum_{j=1}^{l} b_{nj}^I F_j^R\bigl(u_j(t), v_j(t)\bigr) \\
&\quad + \sum_{j=1}^{l} e_{nj}^R(t) G_j^I\bigl(u_j^t, v_j^t\bigr) + \sum_{j=1}^{l} e_{nj}^I(t) G_j^R\bigl(u_j^t, v_j^t\bigr) + P_n^I(t).
\end{aligned} \tag{2}$$

The initial values of system (2) are

$$u_n(s) = \phi_n(s), \qquad v_n(s) = \psi_n(s), \quad s \in [-\sigma, 0], \sigma = \max\Bigl\{\tau, \max_{t\in[0,\omega], 1\le j\le l}\bigl\{\tau_{nj}(t)\bigr\}\Bigr\}.$$

Let $|\cdot|$ be the Euclidean norm for R and $\mathbf{L} = \{1, 2, \dots, l\}$. We introduce the notations as follows:

(1) $\underline{F} = \min_{t\in[0,\omega]}\{|F(x)|\}$, $\overline{F} = \max_{t\in[0,\omega]}\{|F(t)|\}$, where $F(t)$ is a continuous ω-periodic function with $\omega > 0$;

(2)

$$\begin{aligned}
A_{mj\delta} &= \overline{b_{nj}^R}\bigl[l_j^R + l_j^I + \bigl|F_j^R(0,0)\bigr|\delta^2\bigr] + \overline{b_{nj}^I}\bigl[k_j^R + k_j^I + \bigl|F_j^I(0,0)\bigr|\delta^2\bigr] \\
&\quad + \overline{e_{nj}^R}\bigl[q_j^R + q_j^I + \bigl|G_j^R(0,0)\bigr|\delta^2\bigr] + \overline{e_{nj}^I}\bigl[p_j^R + p_j^I + \bigl|G_j^I(0,0)\bigr|\delta^2\bigr] + \delta^2\overline{P_n^R}, \\
A_{nj} &= \overline{b_{nj}^R}\bigl(l_j^R + l_j^I\bigr) + \overline{b_{nj}^I}\bigl(k_j^R + k_j^I\bigr) + \overline{e_{nj}^R}\bigl(q_j^R + q_j^I\bigr) + \overline{e_{nj}^I}\bigl(p_j^R + p_j^I\bigr), \\
B_{mj\delta} &= \overline{b_{nj}^R}\bigl(l_j^R + l_j^I + \delta^2\bigl|F_j^R(0,0)\bigr|\bigr) + \overline{b_{nj}^I}\bigl(k_j^R + k_j^I + \delta^2\bigl|F_j^I(0,0)\bigr|\bigr) \\
&\quad + \overline{e_{nj}^R}\bigl(q_j^R + q_j^I + \delta^2\bigl|G_j^R(0,0)\bigr|\bigr) + \overline{e_{nj}^I}\bigl(p_j^R + p_j^I + \delta^2\bigl|F_j^I(0,0)\bigr|\bigr), \\
B_{nj} &= \overline{b_{nj}^R}\bigl(l_j^R + l_j^I\bigr) + \overline{b_{nj}^I}\bigl(k_j^R + k_j^I\bigr) + \overline{e_{nj}^R}\bigl(q_j^R + q_j^I\bigr) + \overline{e_{nj}^I}\bigl(p_j^R + p_j^I\bigr), \\
A_{nj\delta}^* &= \overline{b_{nj}^R}\bigl[k_j^R + k_j^I + \bigl|F_j^I(0,0)\bigr|\delta^2\bigr] + \overline{b_{nj}^I}\bigl[l_j^R + l_j^I + \bigl|F_j^R(0,0)\bigr|\delta^2\bigr] \\
&\quad + \overline{e_{nj}^R}\bigl[p_j^R + p_j^I + \bigl|G_j^I(0,0)\bigr|\delta^2\bigr] + \overline{e_{nj}^I}\bigl[q_j^R + q_j^I + \bigl|G_j^R(0,0)\bigr|\delta^2\bigr] + \delta^2\overline{P_n^I}, \\
A_{nj}^* &= \overline{b_{nj}^R}\bigl(k_j^R + k_j^I\bigr) + \overline{b_{nj}^I}\bigl(l_j^R + l_j^I\bigr) + \overline{e_{nj}^R}\bigl(p_j^R + p_j^I\bigr) + \overline{e_{nj}^I}\bigl(q_j^R + q_j^I\bigr), \\
B_{nj\delta}^* &= \overline{b_{nj}^R}\bigl(k_j^R + k_j^I + \delta^2\bigl|F_j^I(0,0)\bigr|\bigr) + \overline{b_{nj}^I}\bigl(l_j^R + l_j^I + \delta^2\bigl|F_j^R(0,0)\bigr|\bigr) \\
&\quad + \overline{e_{nj}^R}\bigl(p_j^R + p_j^I + \delta^2\bigl|G_j^I(0,0)\bigr|\bigr) + \overline{e_{nj}^I}\bigl(q_j^R + q_j^I + \delta^2\bigl|F_j^R(0,0)\bigr|\bigr),
\end{aligned}$$

$$
\begin{aligned}
&B_{nj}^{*}=\overline{b_{nj}^{R}}\left(k_j^R+k_j^I\right)+\overline{b_{nj}^{I}}\left(l_j^R+l_j^I\right)+\overline{e_{nj}^{R}}\left(q_j^I+q_j^R\right)+\overline{e_{nj}^{I}}\left(q_j^R+q_j^I\right),\\
&E_{nj}=\overline{e_{nj}^{R}}q_j^R+\overline{e_{nj}^{I}}p_j^R+\overline{e_{nj}^{R}}p_j^I+\overline{e_{nj}^{I}}q_j^I,\qquad F_{nj}=\overline{e_{nj}^{R}}q_j^I+\overline{e_{nj}^{I}}p_j^I+\overline{e_{nj}^{R}}p_j^R+\overline{e_{nj}^{I}}q_j^R,\\
&U_{nj}=\overline{b_{nj}^{R}}\left(l_j^R+k_j^I\right)+\overline{b_{nj}^{I}}\left(k_j^R+l_j^I\right),\qquad V_{nj}=\overline{b_{nj}^{R}}\left(l_j^I+k_j^R\right)+\overline{b_{nj}^{I}}\left(k_j^I+l_j^R\right).
\end{aligned}
$$

Throughout this paper, we always assume that

(h_1) There exist positive constants l_n^R, l_n^I, k_n^R, k_n^I, q_n^R, q_n^I, p_n^R, p_n^I such that, for $\forall(x_1,y_1),(x_2,y_2)\in R\times R$, $n\in\mathbf{L}$,

$$
\begin{aligned}
&\left|F_n^R(x_1,y_1)-F_n^R(x_2,y_2)\right|\le l_n^R|x_1-x_2|+l_n^I|y_1-y_2|,\\
&\left|F_n^I(x_1,y_1)-F_n^I(x_2,y_2)\right|\le k_n^R|x_1-x_2|+k_n^I|y_1-y_2|,\\
&\left|G_n^R(x_1,y_1)-G_n^R(x_2,y_2)\right|\le q_n^R|x_1-x_2|+q_n^I|y_1-y_2|,\\
&\left|G_n^I(x_1,y_1)-G_n^I(x_2,y_2)\right|\le p_n^R|x_1-x_2|+p_n^I|y_1-y_2|.
\end{aligned}
$$

(h_2) $d_n(t)$, $b_{nj}^R(t)$, $b_{nj}^I(t)$, $e_{nj}^R(t)$, $e_{nj}^I(t)$, $P_n^R(t)$, $P_n^I(t)$ ($n\in\mathbf{L}$) are all continuous ω-periodic functions.

(h_3) $(1+|c_n|)(U_{nj}+\frac{E_{nj}}{1-\sigma})<\frac{2\underline{d_j}}{l}-\frac{|c_j|}{l}\overline{d_j}-A_{jn}-|c_j|B_{jn}$.

(h_4) $(1+|c_n|)(V_{nj}+\frac{F_{nj}}{1-\sigma})<\frac{2\underline{d_j}}{l}-\frac{|c_j|}{l}\overline{d_j}-A_{jn}^*-|c_j|B_{jn}^*$.

Lemma 2.1 (Gaines and Mawhin [38]) *Suppose that X^* and Z^* are two Banach spaces and $L^*:D(L^*)\subset X^*\to Z^*$ is a Fredholm operator with index zero. Moreover, $\Omega\subset X^*$ is an open bounded set and $N^*:\overline{\Omega}\to Z^*$ is L^*-compact on Ω. If the following conditions hold*:

(a) $L^*u\neq\lambda N^*u$, $\forall u\in\partial\Omega\cap D(L^*)$, $\forall\lambda\in(0,1)$,

(b) $QN^*u\neq0$, $\forall u\in\partial\Omega\cap\operatorname{Ker}L^*$,

(c) $\deg_B(J^*QN^*,\Omega\cap\operatorname{Ker}L^*,0)\neq0$,

where $J^*:\operatorname{Im}Q\to\operatorname{Ker}L^*$ is an isomorphism, then the equation $L^*u=N^*u$ has a solution on $\overline{\Omega}\cap D(L^*)$.

Definition 2.1 (Graph theory [39]) A directed graph $g=(U,K)$ contains a set $U=\{1,2,\dots,n\}$ of vertices and a set K of arcs (i,j) leading from initial vertex i to terminal vertex j. A subgraph Γ of g is said to be spanning if Γ and g have the same vertex set. A subgraph Γ is unicyclic if it is a disjoint union of rooted trees whose roots form a directed cycle. For a weighted digraph g with l vertices, we define the weight matrix $B=(b_{ij}^*)_{n\times n}$ whose entry $b_{ij}^*>0$ is equal to the weight of arc (j,i) if it exists, and 0 otherwise. A digraph g is strongly connected if, for any pair of distinct vertices, there exists a directed path from one to the other. The Laplacian matrix of (g,B) is defined as $L=(q_{ij})_{l\times l}$, where $q_{ij}=-b_{ij}^*$ for $i\neq j$ and $q_{ij}=\sum_{k\neq i}b_{ik}^*$ for $i=j$.

Lemma 2.2 ([39]) *Suppose that $l\ge2$ and c_k^* denotes the cofactor of the kth diagonal element of the Laplacian matrix of (g,B). Then*

$$
\sum_{k,h=1}^{l}c_k^*b_{kh}^*G_{kh}(x_k,x_h)=\sum_{Q\in\Omega}W(Q)\sum_{(k,h)\in K(C_Q)}G_{hk}(u_h,u_k),
$$

where $G_{kh}(u_k, u_h)$ is an arbitrary function, Q is the set of all spanning unicyclic graphs of (g, B), $W(Q)$ is the weight of Q, C_Q denotes the directed cycle of Q, and $K(C_Q)$ is the set of arcs in C_Q. In particular, if (g, B) is strongly connected, then $c_k^* > 0$ for $1 \leq k \leq l$.

Remark 1 If $(u_1(t), u_2(t), \ldots, u_l(t), v_1(t), v_2(t), \ldots, v_l(t))^T$ is an ω periodic solution of system (2), then $(z_1(t), z_2(t), \ldots, z_l(t))^T$, where $z_n(t) = u_n^R(t) + iv_n^I(t)$, $n = 1, 2, \ldots, l$, must be an ω periodic solution to system (1). Thus, in order to show the existence of periodic solutions of system (1), we only need to show the existence of periodic solutions of system (2). For proving the global exponential stability of periodic solutions of system (1), we only need to prove the global exponential stability of periodic solutions of system (2).

Lemma 2.3 (Lemma 2.1 [40]) *If $|c_n| < 1$, $n = 1, 2, \ldots, l$, then the inverse of difference operator B_n denoted by B_n^{-1} exists and*

$$\|B_n^{-1}\| \leq \frac{1}{1 - |c_n|}.$$

3 The existence of periodic solutions

Lemma 3.1 *For any $\lambda \in (0, 1)$, we are concerned with the following system*:

$$\begin{cases} \dfrac{d[K_n u_n(t)]}{dt} = \lambda \Bigl\{ -d_n(t)u_n(t) + \sum_{j=1}^{l} b_{nj}^R(t)F_j^R\bigl(u_j(t), v_j(t)\bigr) \\ \qquad - \sum_{j=1}^{l} b_{nj}^I(t)F_j^I\bigl(u_j(t), v_j(t)\bigr) \\ \qquad + \sum_{j=1}^{l} e_{nj}^R(t)G_j^R\bigl(u_j^t, v_j^t\bigr) - \sum_{j=1}^{l} e_{nj}^I(t)G_j^I\bigl(u_j^t, v_j^t\bigr) + P_n^R(t) \Bigr\}, \\ \dfrac{d[K_n v_n(t)]}{dt} = \lambda \Bigl\{ -d_n(t)v_n(t) + \sum_{j=1}^{l} b_{nj}^R(t)F_j^I\bigl(u_j(t), v_j(t)\bigr) \\ \qquad + \sum_{j=1}^{l} b_{nj}^I(t)F_j^R\bigl(u_j(t), v_j(t)\bigr) \\ \qquad + \sum_{j=1}^{l} e_{nj}^R(t)G_j^I\bigl(u_j^t, v_j^t\bigr) + \sum_{j=1}^{l} e_{nj}^I(t)G_j^R\bigl(u_j^t, v_j^t\bigr) + P_n^I(t) \Bigr\}. \end{cases} \tag{3}$$

Then the periodic solutions of system (3) are bounded and the boundary must be independent of the choice of λ under assumptions (h_1)–(h_4) if the periodic solutions of system (3) exist. Namely, there exists a positive constant M such that

$$\bigl\|\bigl(u(t), v(t)\bigr)^T\bigr\| = \bigl\|\bigl(u_1(t), u_2(t), \ldots, u_l(t), v_1(t), v_2(t), \ldots, v_l(t)\bigr)^T\bigr\| \leq M,$$

where

$$\bigl\|\bigl(u(t), v(t)\bigr)^T\bigr\| = \sum_{m=1}^{l} \max_{t\in[0,\omega]} \bigl(\bigl|u_m(t)\bigr| + \bigl|v_m(t)\bigr|\bigr).$$

Proof From (h_3) and (h_4), it follows that there exists a positive number δ such that

(h_5) $(1+|c_n|)(U_{nj}+\frac{E_{nj}}{1-\sigma}) < \frac{2\underline{d_j}}{l} - \frac{|c_j|}{l}\overline{d_j} - A_{jn\delta} - |c_j|B_{jn\delta} - l\delta^2\overline{P_n^R} - \delta$.

(h_6) $(1+|c_n|)(V_{nj}+\frac{F_{nj}}{1-\sigma}) < \frac{2\underline{d_j}}{l} - \frac{|c_j|}{l}\overline{d_j} - A^*_{jn\delta} - |c_j|B^*_{jn\delta} - l\delta^2\overline{P_n^I} - \delta$.

Suppose that $(u(t),v(t))^T = (u_1(t),u_2(t),\dots,u_l(t),v_1(t),v_2(t),\dots,v_l(t))^T$ is one ω-periodic solution of system (3) for some $\lambda \in (0,1)$. Let $V_n(t) = V_{1n}(t) + V_{2n}(t)$,

$$V_{1n}(t) = \big[K_n u_n(t)\big]^2 + \big[K_n v_n(t)\big]^2,$$

where

$$\begin{aligned} V_{2n}(t) = \lambda\Bigg\{ &|c_n|\int_{t-\tau}^{t}\left(\sum_{j=1}^{l} B_{nj\delta} + \delta^2\overline{P_n^R}\right)u_n^2(s)\,ds \\ &+ \frac{(1+|c_n|)}{1-\sigma}\sum_{j=1}^{l} E_{nj}\int_{t-\tau_{nj}(t)}^{t} u_j^2(s)\,ds \\ &+ \frac{(1+|c_n|)}{1-\sigma}\sum_{j=1}^{l} F_{nj}\int_{t-\tau_{nj}(t)}^{t} v_j^2(s)\,ds \\ &+ |c_n|\int_{t-\tau}^{t}\left(\sum_{j=1}^{l} B^*_{nj\delta} + \delta^2\overline{P_n^I}\right)v_n^2(s)\,ds\Bigg\}. \end{aligned}$$

Then we have, along with the solutions of system (3),

$$\begin{aligned} \frac{dV_{1n}(t)}{dt} = {}& \lambda\big[u_n(t) - c_n u_n(t-\tau)\big]\Bigg(-d_n(t)u_n(t) + \sum_{j=1}^{l} b_{nj}^R(t)F_j^R\big(u_j(t),v_j(t)\big) \\ &- \sum_{j=1}^{l} b_{nj}^I(t)F_j^I\big(u_j(t),v_j(t)\big) + \sum_{j=1}^{l} e_{nj}^R(t)G_j^R\big(u_j^t,v_j^t\big) \\ &- \sum_{j=1}^{l} e_{nj}^I(t)G_j^I\big(u_j^t,v_j^t\big) + P_n^R(t)\Bigg) \\ &+ \big[v_n(t) - c_n v_n(t-\tau)\big]\Bigg(-d_n(t)v_n(t) + \sum_{j=1}^{l} b_{nj}^R(t)F_j^I\big(u_j(t),v_j(t)\big) \\ &+ \sum_{j=1}^{l} b_{nj}^I(t)F_j^R\big(u_j(t),v_j(t)\big) + \sum_{j=1}^{l} e_{nj}^R(t)G_j^I\big(u_j^t,v_j^t\big) \\ &+ \sum_{j=1}^{l} e_{nj}^I(t)G_j^R\big(u_j^t,v_j^t\big) + P_n^I(t)\Bigg) \\ \le{}& \lambda\Bigg\{\big(-2\underline{d_n} + |c_n|\overline{d_n}\big)u_n^2(t) + |c_n|\overline{d_n}u_n^2(t-\tau) \\ &+ 2\big[\big|u_n(t)\big| + |c_n|\big|u_n(t-\tau)\big|\big]\Bigg(\sum_{j=1}^{l}\overline{b_{nj}^R}\big[l_j^R\big|u_j(t)\big| + l_j^I\big|v_j(t)\big| + \big|F_j^R(0,0)\big|\big] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{l}\overline{b_{nj}^{I}}\bigl[\bigl|F_j^I(0,0)\bigr|+k_j^R\bigl|u_j(t)\bigr|+k_j^I\bigl|v_j(t)\bigr|\bigr]\\
&+\sum_{j=1}^{l}\overline{e_{nj}^{R}}\bigl[G_j^R\bigl|u_j^t\bigr|+q_j^I\bigl|v_j^t\bigr|+\bigl|G_j^R(0,0)\bigr|\bigr]\\
&+\sum_{j=1}^{l}\overline{e_{nj}^{I}}\bigl[p_j^R\bigl|u_j^t\bigr|+p_j^I\bigl|v_j^t\bigr|+\bigl|G_j^I(0,0)\bigr|\bigr]+\overline{P_n^R}\Biggr)\\
&+\bigl(-2\underline{d_n}+|c_n|\overline{d_n}\bigr)v_n^2(t)+|c_n|\overline{d_n}v_n^2(t-\tau)+2\bigl[\bigl|v_n(t)\bigr|+|c_n|\bigl|v_n(t-\tau)\bigr|\bigr]\\
&\times\Biggl(\sum_{j=1}^{l}\overline{b_{nj}^{R}}\bigl[k_j^R\bigl|u_j(t)\bigr|+k_j^I\bigl|v_j(t)\bigr|+\bigl|F_j^I(0,0)\bigr|\bigr]\\
&+\sum_{j=1}^{l}\overline{b_{nj}^{I}}\bigl[\bigl|F_j^R(0,0)\bigr|+l_j^R\bigl|u_j(t)\bigr|+l_j^I\bigl|v_j(t)\bigr|\bigr]\\
&+\sum_{j=1}^{l}\overline{e_{nj}^{R}}\bigl[p_j^R\bigl|u_j^t\bigr|+p_j^I\bigl|v_j^t\bigr|+\bigl|G_j^I(0,0)\bigr|\bigr]\\
&+\sum_{j=1}^{l}\overline{e_{nj}^{I}}\bigl[q_j^R\bigl|u_j^t\bigr|+q_j^I\bigl|v_j^t\bigr|+\bigl|G_j^R(0,0)\bigr|\bigr]+\overline{P_n^I}\Biggr)\Biggr\}. \qquad (4)
\end{aligned}
$$

From (4), by using the inequality $2|ab|\leq a^2+b^2$ $(a,b=u_n(t),u_n(t-\tau),v_n(t),v_n(t-\tau),u_j(t),v_j(t),u_j^t,v_j^t)$, $2u_n(t)|F_j^R(0,0)|\leq|F_j^R(0,0)|[\delta^2u_n^2(t)+\frac{1}{\delta^2}]$, $2u_n(t)|F_j^I(0,0)|\leq|F_j^I(0,0)|[\delta^2u_n^2(t)+\frac{1}{\delta^2}]$, $2u_n(t)|G_j^I(0,0)|\leq|G_j^I(0,0)|[\delta^2u_n^2(t)+\frac{1}{\delta^2}]$, $2u_n(t)|G_j^R(0,0)|\leq|G_j^R(0,0)|[\delta^2u_n^2(t)+\frac{1}{\delta^2}]$, $2u_n(t)\overline{P_n^R}\leq\overline{P_n^R}[u_n^2(t)\delta^2+\frac{1}{\delta^2}]$, it follows that

$$
\begin{aligned}
\frac{dV_{1n}(t)}{dt}\leq{}&\lambda\Biggl\{\Biggl(-2\underline{d_n}+|c_n|\overline{d_n}+\sum_{j=1}^{l}A_{nj\delta}\Biggr)u_n^2(t)+|c_n|\Biggl(\sum_{j=1}^{l}B_{nj\delta}+\delta^2\overline{P_n^R}\Biggr)u_n^2(t-\tau)\\
&+\sum_{j=1}^{l}\bigl(1+|c_n|\bigr)\bigl(\overline{b_{nj}^{R}}l_j^R+\overline{b_{nj}^{I}}k_j^R\bigr)u_j^2(t)+\bigl(1+|c_n|\bigr)\sum_{j=1}^{l}\bigl(\overline{b_{nj}^{R}}l_j^I+\overline{b_{nj}^{I}}k_j^I\bigr)v_j^2(t)\\
&+\bigl(1+|c_n|\bigr)\sum_{j=1}^{l}\bigl(\overline{e_{nj}^{R}}q_j^R+\overline{e_{nj}^{I}}p_j^R\bigr)\bigl(u_j^t\bigr)^2+\bigl(1+|c_n|\bigr)\sum_{j=1}^{l}\bigl(\overline{e_{nj}^{R}}q_j^I+\overline{e_{nj}^{I}}p_j^I\bigr)\bigl(v_j^t\bigr)^2\\
&+\Biggl(-2\underline{d_n}+|c_n|\overline{d_n}+\sum_{j=1}^{l}A_{nj\delta}^{*}\Biggr)v_n^2(t)+|c_n|\Biggl(\sum_{j=1}^{l}B_{nj\delta}^{*}+\delta^2\overline{P_n^I}\Biggr)v_n^2(t-\tau)\\
&+\sum_{j=1}^{l}\bigl(1+|c_n|\bigr)\bigl(\overline{b_{nj}^{R}}k_j^R+\overline{b_{nj}^{I}}l_j^R\bigr)v_j^2(t)+\bigl(1+|c_n|\bigr)\sum_{j=1}^{l}\bigl(\overline{b_{nj}^{R}}k_j^I+\overline{b_{nj}^{I}}l_j^I\bigr)u_j^2(t)\\
&+\bigl(1+|c_n|\bigr)\sum_{j=1}^{l}\bigl(\overline{e_{nj}^{R}}p_j^R+\overline{e_{nj}^{I}}q_j^R\bigr)\bigl(v_j^t\bigr)^2+\bigl(1+|c_n|\bigr)\sum_{j=1}^{l}\bigl(\overline{e_{nj}^{R}}p_j^I+\overline{e_{nj}^{I}}q_j^I\bigr)\bigl(u_j^t\bigr)^2\Biggr\}\\
&+\lambda N, \qquad (5)
\end{aligned}
$$

where N is a positive constant.

Since

$$
\begin{aligned}
\frac{dV_{2n}(t)}{dt} &= \lambda\Bigg\{\frac{1+|c_n|}{1-\sigma}\sum_{j=1}^{l}\big\{E_{nj}u_j^2(t)-E_{nj}x_j^2\big(t-\tau_{ij}(t)\big)\big(1-\tau_{nj}'(t)\big) \\
&\quad + F_{nj}v_j^2(t)-F_{nj}v_j^2\big(t-\tau_{nj}(t)\big)\big(1-\tau_{nj}'(t)\big)\big\} \\
&\quad + |c_n|\sum_{j=1}^{l}\big(B_{nj\delta}+\delta^2\overline{P_n^R}\big)u_n^2(t)-|c_n|\sum_{j=1}^{l}\big(B_{nj\delta}+\delta^2\overline{P_n^R}\big)u_n^2(t-\tau) \\
&\quad + |c_n|\sum_{j=1}^{l}\big(B_{nj\delta}^*+\delta^2\overline{P_n^I}\big)v_n^2(t)-|c_n|\sum_{j=1}^{l}\big(B_{nj\delta}^*+\delta^2\overline{P_n^I}\big)v_n^2(t-\tau)\Bigg\} \\
&\le \lambda\sum_{j=1}^{l}\Bigg\{\frac{E_{nj}(1+|c_n|)}{1-\sigma}u_j^2(t)-E_{nj}\big(1+|c_n|\big)u_j^2\big(t-\tau_{nj}(t)\big) \\
&\quad + \frac{F_{nj}(1+|c_n|)}{1-\sigma}v_j^2(t)-\big(1+|c_n|\big)F_{nj}v_j^2\big(t-\tau_{nj}(t)\big)\Bigg\} \\
&\quad + |c_n|\sum_{j=1}^{l}\big(B_{nj\delta}+\delta^2\overline{P_n^R}\big)u_n^2(t)-|c_n|\sum_{j=1}^{l}\big(B_{nj\delta}+\delta^2\overline{P_n^R}\big)u_n^2(t-\tau) \\
&\quad + |c_n|\sum_{j=1}^{l}\big(B_{nj\delta}^*+\delta^2\overline{P_n^I}\big)v_n^2(t)-|c_n|\sum_{j=1}^{l}\big(B_{nj\delta}^*+\delta^2\overline{P_n^I}\big)v_n^2(t-\tau)\},
\end{aligned}
\tag{6}
$$

from (5) and (6), we have

$$
\begin{aligned}
\frac{dV_n(t)}{dt} &\le \lambda\Bigg\{\Bigg[-2\underline{d_n}+|c_n|\overline{d_n}+\sum_{j=1}^{l}A_{nj\delta}+|c_n|\Bigg(\sum_{j=1}^{l}B_{nj\delta}+\delta^2\overline{P_n^R}\Bigg)\Bigg]u_n^2(t) \\
&\quad + \big(1+|c_n|\big)\sum_{j=1}^{l}\bigg(U_{nj}+\frac{E_{nj}}{1-\sigma}\bigg)u_j^2(t)+\big(1+|c_n|\big)\sum_{j=1}^{l}\bigg(V_{nj}+\frac{F_{nj}}{1-\sigma}\bigg)v_j^2(t) \\
&\quad + \Bigg[-2\underline{d_n}+|c_n|\overline{d_n}+\sum_{j=1}^{l}A_{nj\delta}^*+|c_n|\Bigg(\sum_{j=1}^{l}B_{nj\delta}^*+\delta^2\overline{P_n^I}\Bigg)\Bigg]v_n^2(t)\Bigg\}+\lambda N \\
&= \lambda\sum_{j=1}^{l}\Bigg\{\bigg[-2\frac{\underline{d_n}}{l}+|c_n|\frac{\overline{d_n}}{l}+A_{nj\delta}+|c_n|\big(B_{nj\delta}+l\delta^2\overline{P_n^R}\big)+\delta\bigg]u_n^2(t) \\
&\quad + |c_n|)\sum_{j=1}^{l}(V_{nj}+\big(1+|c_n|\big)\bigg(U_{nj}+\frac{E_{nj}}{1-\sigma}\bigg)u_j^2(t) \\
&\quad + \big(1+|c_n|\big)\bigg(V_{nj}+\frac{F_{nj}}{1-\sigma}\bigg)v_j^2(t) \\
&\quad + \bigg[-2\frac{\underline{d_n}}{l}+|c_n|\frac{\overline{d_n}}{l}+A_{nj\delta}^*+|c_n|\big(B_{nj\delta}^*+l\delta^2\overline{P_n^I}\big)+\delta\bigg]v_n^2(t) \\
&\quad - \delta\big[u_n^2(t)+v_n^2(t)\big]+N\Bigg\}.
\end{aligned}
\tag{7}
$$

By using (h_5) and (h_6), from (7), it follows that

$$\begin{aligned}\frac{dV_n(t)}{dt} \le{}& \lambda \sum_{j=1}^{l} \Bigg\{ \bigg[2\frac{\underline{d_j}}{l} - |c_j|\frac{\overline{d_j}}{l} - A_{jn\delta} - |c_j|(B_{jn\delta} + l\delta^2\overline{P_j^R} - \delta \bigg] u_j^2(t)) \\ & - \bigg[2\frac{\underline{d_n}}{l} - |c_n|\frac{\overline{d_n}}{l} - A_{nj\delta} - |c_n|\big(B_{nj\delta} + l\delta^2\overline{P_n^R}\big) - \delta \bigg] u_n^2(t) \Bigg\} \\ & + \lambda \sum_{j=1}^{l} \Bigg\{ \bigg[2\frac{\underline{d_j}}{l} - |c_j|\frac{\overline{d_j}}{l} - A^*_{jn\delta} - |c_j|\big(B^*_{jn\delta} + l\delta^2\overline{P_j^I}\big) - \delta \bigg] v_j^2(t) \\ & - \bigg[2\frac{\underline{d_n}}{l} - |c_n|\frac{\overline{d_n}}{l} - A^*_{nj\delta} - |c_n|\big(B^*_{nj\delta} + l\delta^2\overline{P_n^I}\big) - \delta \bigg] v_n^2(t) \Bigg\} \\ & - \delta\big[u_n^2(t) + v_n^2(t)\big] + N\}. \end{aligned} \tag{8}$$

Letting $b_{nj} = 1$ $(n \neq j)$, $b_{nj} = 0$, $n = j$, $G_{nj}(u_n^2(t), u_j^2(t)) = [2\frac{\underline{d_j}}{l} - |c_j|\frac{\overline{d_j}}{l} - A_{jn\delta} - |c_j|(B_{jn\delta} + l\delta^2\overline{P_j^R}) - \delta]u_j^2(t) - [2\frac{\underline{d_n}}{l} - |c_n|\frac{\overline{d_n}}{l} - A_{nj\delta} - |c_n|(B_{nj\delta} + l\delta^2\overline{P_n^R}) - \delta]u_n^2(t)$ and $p_n(u_n^2(t)) = [2\frac{\underline{d_n}}{l} - |c_n|\frac{\overline{d_n}}{l} - A_{nj\delta} - |c_n|(B_{nj\delta} + l\delta^2\overline{P_n^R}) - \delta]u_n^2(t)$; $b^*_{nj} = 1$ $(n \neq j)$, $b^*_{nj} = 0$, $n = j$, $G^*_{nj}(v_n^2(t), v_j^2(t)) = [2\frac{\underline{d_j}}{l} - |c_j|\frac{\overline{d_j}}{l} - A^*_{jn\delta} - |c_j|(B^*_{jn\delta} + l\delta^2\overline{P_j^I}) - \delta]v_j^2(t) - [2\frac{\underline{d_n}}{l} - |c_n|\frac{\overline{d_n}}{l} - A^*_{nj\delta} - |c_n|(B^*_{nj\delta} + l\delta^2\overline{P_n^I}) - \delta]v_n^2(t)$ and $p^*_n(v_n^2(t)) = [2\frac{\underline{d_n}}{l} - |c_n|\frac{\overline{d_n}}{l} - A^*_{nj\delta} - |c_n|(B^*_{nj\delta} + l\delta^2\overline{P_n^I}) - \delta]v_n^2(t)$, then we have, from (8),

$$\begin{aligned}\frac{dV_n(t)}{dt} \le{}& \lambda \Bigg\{ \sum_{j=1}^{l} b_{nj} G_{nj}\big(u_n^2(t), u_j^2(t)\big) \\ & + \sum_{j=1}^{l} b^*_{nj} G^*_{nj}\big(v_n^2(t), v_j^2(t)\big) - \sum_{j=1}^{l} \delta\big[u_n^2(t) + v_n^2(t)\big] + \frac{N}{l} \Bigg\}, \end{aligned} \tag{9}$$

$$G_{nj}\big(u_n^2(t), u_j^2(t)\big) = p_j\big(u_j^2(t)\big) - p_n\big(u_n^2(t)\big), \tag{10}$$

and

$$G^*_{nj}\big(v_n^2(t), v_j^2(t)\big) = p^*_j\big(v_j^2(t)\big) - p^*_n\big(v_n^2(t)\big). \tag{11}$$

We construct the following Lyapunov function for system (3):

$$V(t) = \sum_{n=1}^{l} c^*_n V_n(t),$$

where $c^*_n > 0$ is the cofactor of the nth diagonal element of the Laplacian matrix of (g, B). From (9), we have

$$\begin{aligned}\frac{dV(t)}{dt} &= \sum_{n=1}^{l} c^*_n \frac{dV_n(t)}{dt} \\ &= \lambda \sum_{n=1}^{l} c^*_n \sum_{j=1}^{l} \big\{ b_{nj} G_{nj}\big(u_n^2(t), u_j^2(t)\big) + b^*_{nj} G^*_{nj}\big(v_j^2(t), v_j^2(t)\big) \\ &\quad - \delta\big[u_n^2(t) + v_n^2(t)\big] + N\big\}. \end{aligned} \tag{12}$$

From Lemma 2.2, it follows that

$$\sum_{n=1}^{l}\sum_{j=1}^{l} c_n^* b_{nj} G_{nj}\big(u_n^2(t), u_j^2(t)\big) = \sum_{Q\in\Omega} W(Q) \sum_{(n,j)\in K(C_\Omega)} G_{nj}\big(u_n^2(t), u_j^2(t)\big), \tag{13}$$

$$\sum_{n=1}^{l}\sum_{j=1}^{l} c_n^* b_{nj}^* G_{nj}^*\big(v_n^2(t), v_j^2(t)\big) = \sum_{Q\in\Omega} W(Q) \sum_{(n,j)\in K(C_\Omega)} G_{nj}^*\big(v_n^2(t), v_j^2(t)\big). \tag{14}$$

By substituting (10) into (13) and substituting (11) into (14), it follows that, from the fact $W(Q) > 0$,

$$\begin{aligned}&\sum_{n=1}^{l}\sum_{j=1}^{l} c_n^* b_{nj} G_{nj}\big(u_n^2(t), u_j^2(t)\big)\\ &\quad= \sum_{Q\in\Omega} W(Q) \sum_{(n,j)\in K(C_\Omega)} \big[p_j\big(u_j^2(t)\big) - p_n\big(u_n^2(t)\big)\big] \le 0\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\sum_{n=1}^{l}\sum_{j=1}^{l} c_n^* b_{nj}^* G_{nj}^*\big(v_n^2(t), v_j^2(t)\big)\\ &\quad= \sum_{Q\in\Omega} W(Q) \sum_{(n,j)\in K(C_\Omega)} \big[p_j^*\big(v_j^2(t)\big) - p_n^*\big(v_n^2(t)\big)\big] \le 0.\end{aligned} \tag{16}$$

Substituting (15) and (16) into (12) gives

$$\frac{dV(t)}{dt} \le \lambda \sum_{n=1}^{l} c_n^* \sum_{j=1}^{l}\big(-\delta\big[u_n^2(t) + v_n^2(t)\big] + N\big). \tag{17}$$

Integrating (17) from 0 to ω gives

$$\int_0^{\omega} \sum_{n=1}^{l} c_n^* \delta\big[u_n^2(s) + v_n^2(s)\big]\,ds \le \omega N \sum_{n=1}^{l} c_n^*. \tag{18}$$

Integrating (17) from 0 to t gives

$$\begin{aligned}V(t) &\le V(0) + \int_0^{t} \sum_{n=1}^{l} c_n^* l\big(\delta\big[u_n^2(s) + v_n^2(s)\big] + N\big)\,ds\\ &\le V(0) + \int_0^{\omega} \sum_{n=1}^{l} c_n^* l\big(\delta\big[u_n^2(s) + v_n^2(s)\big] + N\big)\,ds.\end{aligned} \tag{19}$$

Substituting (18) into (19) gives

$$V(t) \le V(0) + 2l\omega N \sum_{n=1}^{l} c_n^*. \tag{20}$$

From (20) and the definitions of $V_n(t)$ and $V_{1n}(t)$, we have

$$\begin{aligned}&\sum_{n=1}^{l} c_n^*\{[u_n(t)-c_nu_n(t-\tau_1)]^2+[v_n(t)-c_nv_n(t-\tau_1)]^2\}\\&\quad\le V(0)+2l\omega N\sum_{n=1}^{l}c_n^*.\end{aligned}\tag{21}$$

Letting $|u_n(\xi)|=\max_{t\in[0,\omega]}|u_n(t)|$, $|v_n(\eta)|=\max_{t\in[0,\omega]}|v_n(t)|$, then from (21), it follows that

$$\sum_{n=1}^{l} c_n^*(1-|c_n|)^2[u_n^2(\xi)+v_n^2(\eta)]\le V(0)+2l\omega N\sum_{n=1}^{l}c_n^*.$$

Hence there exists a positive constant M such that $\|(u(t),v(t))^T\|\le M$. This completes the proof of Lemma 3.1. □

Theorem 3.1 *Assume that* (h_1)–(h_4) *hold and* $|c_n|<1$. *Then system* (2) *has at least one* ω *periodic solution.*

Proof We will prove the existence of periodic solutions of system (2) by means of using Lemma 2.1. We are concerned with the Banach spaces: $X^*=Z^*=\{(u(t),v(t))^T\in C(R,R^{2l}): u(t+\omega)=u(t), v(t+\omega)=v(t)\}$ with the norm $\|(u(t),v(t))^T\|=\sum_{n=1}^{l}\max_{t\in[0,\omega]}(|u_n(t)|+|v_n(t)|)$. Set $L^*:\operatorname{Dom}L^*\subset X^*\to X^*$, $L^*(u(t),v(t))=(\frac{d[K_1u_1(t)]}{dt},\frac{d[K_2u_2(t)]}{dt},\dots,\frac{d[K_lu_l(t)]}{dt},\frac{d[K_1v_1(t)]}{dt},\frac{d[K_2v_2(t)]}{dt},\dots,\frac{d[K_lv_1(t)]}{dt})^T$ and

$$N^*(u(t),v(t))=(f_1(t),f_2(t),\dots,f_l(t),f_1^*(t),f_2^*(t),\dots,f_l^*(t)),$$

where, for $n\in\mathbf{L}$,

$$\begin{aligned}f_n(t)={}&-d_n(t)u_n(t)+\sum_{j=1}^{l}b_{nj}^R(t)F_j^R(u_j(t),v_j(t))-\sum_{j=1}^{l}b_{nj}^I(t)F_j^I(u_j(t),v_j(t))\\&+\sum_{j=1}^{l}e_{nj}^R(t)G_j^R(u_j^t,v_j^t)-\sum_{j=1}^{l}e_{nj}^I(t)G_j^I(u_j^t,v_j^t)+P_n^R(t),\\f_n^*(t)={}&-d_n(t)v_n(t)+\sum_{j=1}^{l}b_{nj}^R(t)F_j^I(u_j(t),v_j(t))+\sum_{j=1}^{l}b_{nj}^I(t)F_j^R(u_j(t),v_j(t))\\&+\sum_{j=1}^{l}e_{nj}^R(t)G_j^I(u_j^t,v_j^t)+\sum_{j=1}^{l}e_{nj}^I(t)G_j^R(u_j^t,v_j^t)+P_n^I(t).\end{aligned}$$

Thus $\operatorname{Ker}L^*=\{u=(u(t),v(t))^T\in X^*:u\in R^{2l}\}$, $\operatorname{Im}L^*=\{w\in Z^*:\int_0^\omega w(t)\,dt=0\}$ is closed in Z^* and $\operatorname{Dim}\operatorname{Ker}L^*=2l=\operatorname{Codim}\operatorname{Im}L^*$. Hence, the operator L^* is a Fredholm mapping of index 0. We construct the projectors $P^*:X^*\cap\operatorname{Dom}L^*\to\operatorname{Ker}L^*$ and $Q^*:Z^*\to Z^*$ as

$$\begin{aligned}&P^*u=\frac{1}{\omega}\int_0^\omega u(t)\,dt,\quad u\in X^*;\\&Q^*w=\frac{1}{\omega}\int_0^\omega w(t)\,dt,\quad w\in Z^*.\end{aligned}$$

Therefore, $\mathrm{Lm}\,P^* = \mathrm{Ker}\,L^*$, $\mathrm{Lm}\,L^* = \mathrm{Ker}\,Q^* = \mathrm{Im}(I - Q^*)$. Moreover, the generalized inverse K_p of L^* is given as $K_p = (L^*)^{-1}(\int_0^t w(s)\,ds)$. Since $|c_n| < 1$, from Lemma 2.3, it is not difficult to show that N^* is L^*-compact on $\overline{\Omega}$. The concrete form of the operator equation $L^*(u,v) = \lambda N^*(u,v)$, $(u,v)^T \in X^*$, $\lambda \in (0,1)$ is system (2). From Lemma 3.1, for every periodic solution $(u(t),v(t))^T = (u_1(t),u_2(t),\dots,u_l(t),v_1(t),v_2(t),\dots,v_l(t))^T$ of system (2), there exists a positive constant M such that $\|(u(t),v(t))^T\| < M$. We set $\Omega = \{(u(t),v(t))^T \in X^* : \|(u(t),v(t))^T\| < M\}$, $M > \sqrt{\frac{2Nl\sum_{n=1}^{l} c_n^*}{\min_{1\le n\le l}\{c_n^*\}}}$. Then, for each $(u(t),v(t))^T \in \partial\Omega \cap \mathrm{Dom}\,L^*$, $L^*(u(t),v(t)) \neq \lambda N^*(u(t),v(t))$, $\lambda \in (0,1)$. Hence, condition (1) in Lemma 2.2 is satisfied. Secondly, we will show that when $(u(t),v(t))^T \in \partial\Omega \cap \mathrm{Ker}\,L^*$, $Q^*N^*(u(t),v(t)) \neq 0$. Since $(u,v)^T \in \partial\Omega \cap \mathrm{Ker}\,L^*$, $(u,v)^T$ is a constant vector with $\|(u,v)^T\| = M$, then when $(u,v)^T \in \partial\Omega \cap \mathrm{Ker}\,L^*$, $Q^*N^*(u,v) = (f_1(\xi_1),f_2(\xi_2),\dots,f_l(\xi_l),f_1^*(\xi_1),f_2^*(\xi_2),\dots,f_l^*(\xi_l))$, where ξ_i $(i = 1,2,\dots,l) \in [0,\omega]$. When $(u,v)^T \in \partial\Omega \cap \mathrm{Ker}\,L^*$, we have

$$
\begin{aligned}
&[u_n - c_n u_n, v_n - c_n v_n]\big[Q^*N^*(u,v)_n\big]^T \\
&\quad= [u_n - c_n u_n, v_n - c_n v_n]\big(f_n(\xi_n), f_n^*(\xi_n)\big)^T \\
&\quad= (u_n - c_n u_n)\Bigg[-d_n(\xi_n)u_n + \sum_{j=1}^{l} b_{nj}^R(\xi_n)F_j^R(u_j,v_j) \\
&\qquad - \sum_{j=1}^{l} b_{nj}^I(\xi_n)F_j^I(u_j,v_j) + \sum_{j=1}^{l} e_{nj}^R(\xi_n)G_j^R(u_j,v_j) - \sum_{j=1}^{l} e_{nj}^I(\xi_n)G_j^I(u_j,v_j) + P_n^R(\xi_n)\Bigg] \\
&\qquad + (v_n - c_n v_n)\Bigg[-d_n(\xi_n)v_n + \sum_{j=1}^{l} b_{nj}^R(\xi_n)F_j^I(u_j,v_j) \\
&\qquad - \sum_{j=1}^{l} b_{nj}^I(\xi_n)F_j^R(u_j,v_j) + \sum_{j=1}^{l} e_{nj}^R(\xi_n)G_j^I(u_j,v_j) - \sum_{j=1}^{l} e_{nj}^I(\xi_n)G_j^R(u_j,v_j) + P_n^I(\xi_n)\Bigg] \\
&\qquad + 0.
\end{aligned}
\tag{22}
$$

It is obvious that

$$
\begin{aligned}
0 &= |c_n|\left(\sum_{j=1}^{l} B_{nj}\delta + \delta^2\overline{P_n^R}\right)\big(u_n^2 - u_n^2\big) + \frac{1+|c_n|}{1-\sigma}\sum_{j=1}^{l} E_{nj}\big(u_j^2 - u_j^2\big) \\
&\quad + |c_n|\left(\sum_{j=1}^{l} B_{nj\delta}^* + \delta^2\overline{P_n^I}\right)\big(v_n^2 - v_n^2\big) + \frac{1+|c_n|}{1-\sigma}\sum_{j=1}^{l} f_{nj}\big(v_j^2 - v_j^2\big).
\end{aligned}
\tag{23}
$$

Substituting (23) into (22) gives

$$
\begin{aligned}
&[u_n - c_n u_n, v_n - c_n v_n]\big[QN(u,v)_n\big]^T \\
&\quad= [u_n - c_n u_n, v_n - c_n v_n]\big(f_n(\xi_n), f_n^*(\xi_n)\big)^T \\
&\quad\le (u_n - c_n u_n)\Bigg[-d_n(\xi_n)u_n + \sum_{j=1}^{l} b_{nj}^R(\xi_n)F_j^R(u_j,v_j) - \sum_{j=1}^{l} b_{nj}^I(\xi_n)F_j^I(u_j,v_j)
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=1}^{l} e_{nj}^{R}(\xi_n) G_j^{R}(x_j, y_j) - \sum_{j=1}^{l} e_{nj}^{I}(\xi_n) G_j^{I}(u_j, v_j) + P_n^{R}(\xi_n)\Bigg] \\
&+ (v_n - c_n v_n)\Bigg[-d_n(\xi_n) v_n + \sum_{j=1}^{l} b_{nj}^{R}(\xi_n) F_j^{I}(u_j, v_j) + \sum_{j=1}^{l} b_{nj}^{I}(\xi_n) F_j^{R}(u_j, v_j) \\
&+\sum_{j=1}^{l} e_{nj}^{R}(\xi_n) G_j^{I}(u_j, v_j) + \sum_{j=1}^{l} e_{nj}^{I}(\xi_n) G_j^{R}(u_j, v_j) + P_n^{I}(\xi_n)\Bigg] \\
&+ |c_n|\left(\sum_{j=1}^{l} B_{nj}\delta + \delta^2 \overline{P_n^{R}}\right)\left(u_n^2 - u_n^2\right) + \frac{1 + |c_n|}{1 - \sigma} \sum_{j=1}^{l} E_{nj}\left(u_j^2 - u_j^2\right) \\
&+ |c_n|\Big(\sum_{j=1}^{l} B_{nj\delta}^{*} + \delta^2 \overline{P_n^{I}}\left(v_n^2 - v_n^2\right) + \frac{1 + |c_n|}{1 - \sigma} \sum_{j=1}^{l} f_{nj}\left(v_j^2 - v_j^2\right).
\end{aligned}
\tag{24}
$$

From (24), the same proofs as those of (7)–(17) give

$$
\begin{aligned}
&\sum_{n=1}^{l} c_n^{*}[u_n - c_n y_n, v_n - c_n v_n]\left[Q^{*} N^{*}(u, v)_n\right]^{T} \\
&\quad = \sum_{n=1}^{l} c_n^{*}[u_n - c_n u_n, v_n - c_n v_n]\left(f_n(\xi_n), f_n^{*}(\xi_m)\right)^{T} \\
&\quad \le \sum_{n=1}^{l} c_n^{*} \sum_{j=1}^{l} \Bigg\{\left[2\frac{\underline{d_j}}{l} - |c_j|\frac{\overline{d_j}}{l} - A_{jn\delta} - |c_j|\left(B_{jn\delta} + l\delta^2 \overline{P_j^{R}}\right) - \delta\right] u_j^2 \\
&\qquad - \left[2\frac{\underline{d_n}}{l} - |c_n|\frac{\overline{d_n}}{l} - A_{nj\delta} - |c_n|\left(B_{nj\delta} + l\delta^2 \overline{P_n^{R}}\right) - \delta\right] u_n^2\Bigg\} \\
&\qquad + \sum_{j=1}^{l} \Bigg\{\left[2\frac{\underline{d_j}}{l} - |c_j|\frac{\overline{d_j}}{l} - A_{jn\delta}^{*} - |c_j|\left(B_{jn\delta}^{*} + l\delta^2 \overline{P_j^{I}}\right) - \delta\right] v_j^2 \\
&\qquad - \left[2\frac{\underline{d_n}}{l} - |c_n|\frac{\overline{d_n}}{l} - A_{nj\delta}^{*} - |c_n|\left(B_{nj\delta}^{*} + l\delta^2 \overline{P_n^{I}}\right) - \delta\right] v_n^2(t)\Bigg\} \\
&\qquad - \delta\left(u_n^2 + v_n^2\right) + N\big\} \\
&\quad \le \sum_{n=1}^{l} \sum_{j=1}^{l} c_n^{*}\left[-\delta\left(u_n^2 + v_n^2\right) + N\right].
\end{aligned}
\tag{25}
$$

Since $\sum_{n=1}^{l}(|u_n| + |v_n|) = M$, then

$$
M^2 \le l \sum_{n=1}^{l}\left(u_n^2 + v_n^2 + 2|u_n||v_n|\right) \le 2l \sum_{n=1}^{l}\left(u_n^2 + v_n^2\right).
$$

Namely,

$$
\sum_{n=1}^{l}\left(u_n^2 + v_n^2\right) \ge \frac{M^2}{2l}.
\tag{26}
$$

Substituting (26) into (25) gives

$$\begin{aligned}
&\sum_{n=1}^{l} c_n^*[u_n - c_n y_n, v_n - c_n v_n]\bigl[Q^* N^*(u,v)_n\bigr]^T \\
&\quad \le -\min_{1\le n\le l}\{c_n^*\}\frac{M^2}{2} + Nl\sum_{n=l}^{l}\{c_n^*\} < 0.
\end{aligned} \tag{27}$$

Thus when $(u,v) \in \partial\Omega \cap \operatorname{Ker} L^*$, $Q^*N^*(u,v) \neq 0$. Thus, condition (b) in Lemma 2.1 is satisfied.

Thirdly, we show that when $(u,v)^T \in \partial\Omega \cap \operatorname{Ker} L^*$, $\deg\{J^*Q^*N^*, \Omega \cap \operatorname{Ker} L^*, 0\} \neq 0$. We construct a mapping $H(u,v,\mu^*)$ by setting

$$\begin{aligned}
H\bigl(u,v,\mu^*\bigr) &= -\mu^*(\underline{d_1}u_1, \underline{d_2}u_2, \dots, \underline{d_l}v_l, \underline{d_1}v_1, \underline{d_2}v_2, \dots, \underline{d_l}v_l) \\
&\quad + \bigl(1-\mu^*\bigr)\bigl(f_1(\xi_1), f_2(\xi_2), \dots, f_l(\xi_l), f_1^*(\xi_1), f_2^*(\xi_2), \dots, f_l^*(\xi_l)\bigr),
\end{aligned}$$

where $\forall(u,v,\mu^*) \in \partial\Omega \cap \operatorname{Ker} L^* \times [0,1]$. If when $(u,v,\mu^*) \in \partial\Omega \cap \operatorname{Ker} L^* = R^{2l} \cap \operatorname{Ker} L^*$, $H(u,v,\mu^*) = 0$, then for $n \in \mathbf{L}$,

$$0 = -\mu^*\underline{d_n}u_n + \bigl(1-\mu^*\bigr)f_n(\xi_n) \tag{28}$$

and

$$0 = -\mu^*\underline{d_n}u_n + \bigl(1-\mu^*\bigr)f_n^*(\xi_n). \tag{29}$$

From (28) and (29), we have

$$\begin{aligned}
0 &= (u_n - c_n u_n)\bigl[-\mu^*\underline{d_n}u_n + \bigl(1-\mu^*\bigr)f_n(\xi_n)\bigr] \\
&\quad + (v_n - c_n v_n)\bigl[-\mu^*\underline{d_n}v_n + \bigl(1-\mu^*\bigr)f_n^*(\xi_n)\bigr] \\
&\le -(1-c_n)\mu^*\underline{d_n}u_n^2 - (1-c_n)\bigl(1-\mu^*\bigr)\underline{d_n}u_n^2 + u_n\Biggl\{\sum_{j=1}^{l} b_{nj}^R(\xi_n)F_j^R(u_j,v_j) \\
&\quad - \sum_{j=1}^{l} b_{nj}^I(\xi_n)F_j^I(u_j,v_j) + \sum_{j=1}^{l} e_{nj}^R(\xi_n)G_j^R(u_j,v_j) - \sum_{j=1}^{l} e_{nj}^I(\xi_n)G_j^I(u_j,v_j) + P_n^R(\xi_n)]\Biggr\} \\
&\quad - (1-c_n)\mu^*\underline{d_n}v_n^2 - (1-c_n)\bigl(1-\mu^*\bigr)\overline{d_n}v_n^2 + v_n\Biggl\{\sum_{j=1}^{l} b_{nj}^R(\xi_n)F_j^I(u_j,v_j) \\
&\quad + \sum_{j=1}^{l} b_{nj}^I(\xi_n)F_j^R(u_j,v_j) + \sum_{j=1}^{l} e_{nj}^R(\xi_n)G_j^I(u_j,v_j) + \sum_{j=1}^{l} e_{nj}^I(\xi_n)G_j^R(u_j,v_j) + P_n^I(\xi_n)\Biggr\} \\
&\le -(1-c_n)\underline{d_n}u_n^2 + (1-c_n)\bigl(1-\mu^*\bigr)|u_n|\Biggl\{\sum_{j=1}^{l}\overline{b_{nj}^R}\bigl|F_j^R(u_j,v_j)\bigr| \\
&\quad + \sum_{j=1}^{l}\overline{b_{nj}^I}\bigl|F_j^I(u_j,v_j)\bigr| + \sum_{j=1}^{l}\overline{e_{nj}^R}\bigl|G_j^R(u_j,v_j)\bigr| + \sum_{j=1}^{l}\overline{e_{nj}^I}\bigl|G_j^I(u_j,v_j)\bigr| + \overline{P_n^R}\Biggr\}
\end{aligned}$$

$$
\begin{aligned}
&- (1-c_n)\underline{d_n}v_n^2 + (1-c_n)\bigl(1-\mu^*\bigr)|v_n|\Biggl\{\sum_{j=1}^{l}\overline{b_{nj}^R}\bigl|G_j^I(u_j,v_j)\bigr| \\
&+ \sum_{j=1}^{l}\overline{b_{nj}^I}\bigl|F_j^R(u_j,v_j)\bigr| + \sum_{j=1}^{l}\overline{e_{nj}^R}\bigl|G_j^I(u_j,v_j)\bigr| + \sum_{j=1}^{l}\overline{e_{nj}^I}\bigl|G_j^R(u_j,v_j)\bigr| + \overline{P_n^I}\Biggr\}.
\end{aligned} \tag{30}
$$

By the same proofs as those in (7)–(17), from (30), we obtain

$$
\begin{aligned}
&-(1-c_n)\underline{d_n}u_n^2 + (1-c_n)\bigl(1-\mu^*\bigr)|u_n|\Biggl\{\sum_{j=1}^{l}\overline{b_{nj}^R}\bigl|F_j^R(u_j,v_j)\bigr| \\
&\quad + \sum_{j=1}^{l}\overline{b_{nj}^I}\bigl|F_j^I(u_j,v_j)\bigr| + \sum_{j=1}^{l}\overline{e_{nj}^R}\bigl|G_j^R(u_j,v_j)\bigr| + \sum_{j=1}^{l}\overline{e_{nj}^I}\bigl|G_j^I(u_j,v_j)\bigr| + \overline{P_n^R}\Biggr\} \\
&\quad - (1-c_n)\underline{d_n}v_n^2 + (1-c_n)\bigl(1-\mu^*\bigr)|v_n|\{\sum_{j=1}^{l}\overline{b_{nj}^R}\bigl|F_j^I(u_j,v_j)\bigr| \\
&\quad + \sum_{j=1}^{l}\overline{b_{nj}^I}|F_j^R(u_j,v_j) + \sum_{j=1}^{l}\overline{e_{nj}^R}\bigl|G_j^I(u_j,v_j)\bigr| + \sum_{j=1}^{l}\overline{e_{nj}^I}\bigl|G_j^R(u_j,v_j)\bigr| \\
&< 0.
\end{aligned} \tag{31}
$$

Equation (31) contradicts (30), hence $H(u,v,\mu^*) \neq 0$ when $(u,v,\mu^*) \in \partial\Omega \cap R^{2l} \cap \operatorname{Ker} L^*$. Hence, $L^*(u,v,\mu^*)$ is a homotopic mapping. Thus, we have

$$
\begin{aligned}
&\deg\bigl(J^*Q^*N^*\bigl(u,v,\mu^*\bigr), \partial\Omega \cap \operatorname{Ker} L^*, (0,0,\ldots,0)\bigr) \\
&\quad = \deg\bigl(H(u,v,0), \partial\Omega \cap \operatorname{Ker} L^*, (0,0,\ldots,0)\bigr) \\
&\quad = \deg\bigl(H(u,v,1), \partial\Omega \cap \operatorname{Ker} L^*, (0,0,\ldots,0)\bigr) \\
&\quad \neq 0.
\end{aligned}
$$

Thus condition (c) in Lemma 2.1 is satisfied. Hence the proof of Theorem 3.1 is complete. □

4 Exponential stability

Theorem 4.1 *Let the conditions in Theorem* 3.1 *be satisfied. Then the unique ω-periodic solution of system* (2) *is globally exponentially stable.*

Proof According to Theorem 3.1, system (2) has an ω-periodic solution. Let

$$
\bigl(u^*(t), v^*(t)\bigr)^T = \bigl(u_1^*(t), u_2^*(t), \ldots, u_l^*(t), v_1^*(t), v_2^*(t), \ldots, v_l^*(t)\bigr)^T
$$

be an ω-periodic solution. From (h_3) and (h_4), it follows that there exist two positive numbers δ^* and α such that

(h_7) $(1+|c_n|)(U_{nj} + \frac{E_{nj}e^{\alpha\sigma^*}}{1-\sigma}) < 2\frac{d_j}{l} - \frac{|c_j|}{l}\overline{d_j} - A_{jn} - |c_j|e^{\alpha\tau}B_{jn} - \delta^* - \frac{\alpha}{l}[e^{\alpha\tau}(1+c_n^2) + 1 + |c_n|]$.

(h_8) $(1+|c_n|)(V_{nj} + \frac{F_{nj}e^{\alpha\sigma^*}}{1-\sigma}) < \frac{2d_j}{l} - \frac{|c_j|}{l}\overline{d_j} - A_{jn}^* - |c_j|e^{\alpha\tau}B_{jn}^* - \frac{\alpha}{l}[e^{\alpha\tau}(1+c_n^2) + 1 + |c_n|] - \delta^*$.

Let $(u(t),v(t))^T=(u_1(t),u_2(t),\dots,u_l(t),v_1(t),v_2(t),\dots,v_l(t))^T$ be an arbitrary solution of system (2). We define the following Lyapunov functional: $V_n(t)=V_{1n}(t)+V_{2n}(t)$, $n\in\mathbf{L}$,

$$\begin{aligned}
V_{1n}(t) &= e^{\alpha t}\bigl(K_nX_n(t)\bigr)^2+e^{\alpha t}\bigl(K_nY_n(t)\bigr)^2,\\
V_{2n}(t) &= |c_n|\int_{t-\tau}^{t}e^{\alpha(s+\tau)}\sum_{j=1}^{l}B_{nj}X_n^2(s)\,ds\\
&\quad+\frac{(1+|c_n|)}{1-\sigma}\sum_{j=1}^{l}E_{nj}\int_{t-\tau_{nj}(t)}^{t}e^{\alpha(s+\sigma^*)}X_j^2(s)\,ds\\
&\quad+\frac{(1+|c_n|)}{1-\sigma}\sum_{j=1}^{l}F_{nj}\int_{t-\tau_{nj}(t)}^{t}e^{\alpha(s+\sigma^*)}Y_j^2(s)\,ds\\
&\quad+|c_n|\int_{t-\tau}^{t}e^{\alpha(s+\tau)}\sum_{j=1}^{l}B^*_{nj}Y_n^2(s)\,ds\\
&\quad+\alpha\int_{t-\tau}^{t}\bigl(1+c_n^2\bigr)e^{\alpha(s+\tau)}X_n^2(s)\,ds\\
&\quad+\alpha\int_{t-\tau}^{t}\bigl(1+c_n^2\bigr)e^{\alpha(s+\tau)}Y_n^2(s)\,ds,
\end{aligned}$$

where $X_n(t)=u_n(t)-u_n^*(t)$, $Y_n(t)=v_n(t)-v_n^*(t)$. Then we can get, along with the solutions of system (2),

$$\begin{aligned}
\frac{dV_{1n}(t)}{dt} &= e^{\alpha t}\Biggl\{\bigl[X_m(t)-c_nX_n(t-\tau)\bigr]\Biggl(-d_n(t)X_n(t)\\
&\quad+\sum_{j=1}^{l}b_{nj}^R(t)\bigl[F_j^R\bigl(u_j(t),v_j(t)\bigr)-F_j^R\bigl(u_j^*(t),v_j^*(t)\bigr)\bigr]\\
&\quad-\sum_{j=1}^{l}b_{nj}^I(t)\bigl[F_j^I\bigl(u_j(t),v_j(t)\bigr)-F_j^I\bigl(u_j^*(t),v_j^*(t)\bigr)\bigr]\\
&\quad+\sum_{j=1}^{l}e_{nj}^R(t)\bigl[G_j^R\bigl(u_j^t,v_j^t\bigr)-G_j^R\bigl(u_j^{*t},v_j^{*t}\bigr)\bigr]\\
&\quad-\sum_{j=1}^{l}e_{nj}^I(t)\bigl[G_j^I\bigl(u_j^t,v_j^t\bigr)-G_j^I\bigl(u_j^{*t},v_j^{*t}\bigr)\bigr]\Biggr)\\
&\quad+\bigl[Y_n(t)-c_nv_n(t-\tau)\bigr]\Biggl(-d_n(t)Y_n(t)\\
&\quad+\sum_{j=1}^{l}b_{nj}^R(t)\bigl[F_j^I\bigl(u_j(t),v_j(t)\bigr)-F_j^I\bigl(u_j^*(t),v_j^*(t)\bigr)\bigr]\\
&\quad+\sum_{j=1}^{l}b_{nj}^I(t)\bigl[F_j^R\bigl(u_j(t),v_j(t)\bigr)-F_j^R\bigl(u_j^*(t),v_j^*(t)\bigr)\bigr]
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=1}^{l} e_{nj}^{R}(t)\bigl[G_j^I\bigl(u_j^t, v_j^t\bigr) - G_j^I\bigl(u_j^{*t}, v_j^{*t}\bigr)\bigr] \\
&+ \sum_{j=1}^{l} e_{nj}^{I}(t)\bigl[G_j^R\bigl(u_j^t, v_j^t\bigr) - G_j^R\bigl(u_j^{*t}, v_j^{*t}\bigr)\bigr]\biggr)\biggr\} \\
&+ \alpha e^{\alpha t}\bigl\{\bigl[X_n - c_n X_n(t-\tau)\bigr]^2 + \bigl[Y_n - c_n Y_n(t-\tau)\bigr]^2\bigr\}
\end{aligned}
\tag{32}
$$

and

$$
\begin{aligned}
\frac{dV_{2n}(t)}{dt} = e^{\alpha t}\biggl\{ &\frac{1+|c_n|}{1-\sigma} \sum_{j=1}^{l} \bigl(E_{nj} e^{\alpha\sigma^*} X_j^2(t) - E_{nj} X_j^2\bigl(t-\tau_{nj}(t)\bigr)\bigl(1-\tau_{nj}'(t)\bigr) \\
&+ F_{nj} e^{\alpha\sigma^*} Y_j^2(t) - F_{nj} Y_j^2\bigl(t-\tau_{nj}(t)\bigr)\bigl(1-\tau_{nj}'(t)\bigr)\bigr) \\
&+ |c_n| \sum_{j=1}^{l} B_{nj\delta} e^{\alpha\tau} X_n^2(t) - |c_n| \sum_{j=1}^{l} B_{nj} X_n^2(t-\tau) \\
&+ |c_n| \sum_{j=1}^{l} B_{nj\delta}^{*} e^{\alpha\tau} Y_n^2(t) - |c_n| \sum_{j=1}^{l} B_{nj}^{*} Y_n^2(t-\tau) \\
&+ \alpha\bigl(1+c_n^2\bigr)\bigl[e^{\alpha\tau} X_n^2(t) - X_n^2(t-\tau)\bigr] \\
&+ \alpha\bigl(1+c_n^2\bigr)\bigl[e^{\alpha\tau} Y_n^2(t) - Y_n^2(t-\tau)\bigr]\biggr\}.
\end{aligned}
\tag{33}
$$

From (32) and (33), by using arguments similar to (7)–(17), we have

$$
\begin{aligned}
\frac{dV_n(t)}{dt} \le e^{\alpha t}\Biggl\{&\Biggl[-2\underline{d_n} + |c_n|\overline{d_n} + \sum_{j=1}^{l} A_{nj} + |c_n| e^{\alpha\tau} \sum_{j=1}^{l} B_{nj} \\
&+ \alpha\bigl[e^{\alpha\tau}\bigl(1+c_n^2\bigr) + 1 + |c_n|\bigr]\Biggr] X_n^2(t) \\
&+ \bigl(1+|c_n|\bigr) \sum_{j=1}^{l} \biggl(U_{nj} + \frac{E_{nj} e^{\alpha\sigma^*}}{1-\sigma}\biggr) X_j^2(t) + \bigl(1+|c_n|\bigr) \sum_{j=1}^{n} \biggl(V_{nj} + \frac{F_{nj} e^{\alpha\sigma^*}}{1-\sigma}\biggr) Y_j^2(t) \\
&+ \Biggl[-2\underline{d_n} + |c_n|\overline{d_n} + \sum_{j=1}^{l} A_{nj}^{*} + |c_n| e^{\alpha\tau} \sum_{j=1}^{l} B_{nj}^{*} \\
&+ \alpha\bigl[e^{\alpha\tau}\bigl(1+c_n^2\bigr) + 1 + |c_n|\bigr]\Biggr] Y_n^2(t)\Biggr\} \\
= e^{\alpha t} \sum_{j=1}^{l} \Biggl\{&\Biggl[-2\frac{\underline{d_n}}{l} + \frac{|c_n|\overline{d_n}}{l} + A_{nj} + |c_n| e^{\alpha\tau} B_{nj} \\
&+ \frac{\alpha}{l}\bigl[e^{\alpha\tau}\bigl(1+c_n^2\bigr) + 1 + |c_n|\bigr] + \delta^*\Biggr] X_n^2(t) \\
&+ \bigl(1+|c_n|\bigr)\biggl(U_{nj} + \frac{E_{nj} e^{\alpha\sigma^*}}{1-\sigma}\biggr) X_j^2(t) + \bigl(1+|c_n|\bigr)\biggl(V_{nj} + \frac{F_{nj} e^{\alpha\sigma^*}}{1-\sigma}\biggr) Y_j^2(t)
\end{aligned}
$$

$$+\left[-2\frac{d_n}{l}+\frac{|c_n|}{l}\overline{d_n}+A^*_{nj}+|c_n|e^{\alpha\tau}B^*_{nj}\right.$$
$$\left.+\frac{\alpha}{l}\left[e^{\alpha\tau}\left(1+c_n^2\right)+1+|c_n|\right]+\delta^*\right]Y_n^2(t)-\delta^*\left[X_n^2(t)+Y_n^2(t)\right]\Bigg\}. \tag{34}$$

By using (h_7) and (h_8), from (34), we obtain

$$\frac{dV_n(t)}{dt}\le e^{\alpha t}\sum_{j=1}^{l}\Bigg\{\Bigg[2\frac{d_j}{l}-\frac{|c_j|}{l}\overline{d_j}-A_{jn}-|c_j|e^{\alpha\tau}B_{jn}$$
$$-\frac{\alpha}{l}\left[e^{\alpha\tau}\left(1+c_n^2\right)+1+|c_n|\right]-\delta^*\Bigg]X_j^2(t)$$
$$-\Bigg[2\frac{d_n}{l}-\frac{|c_n|}{l}\overline{d_n}-A_{nj}-|c_n|e^{\alpha\tau}B_{nj}$$
$$-\frac{\alpha}{l}\left[e^{\alpha\tau}\left(1+c_n^2\right)+1+|c_n|\right]-\delta^*\Bigg]X_n^2(t)\Bigg\}$$
$$+\sum_{j=1}^{l}\Bigg\{\Bigg[2\frac{d_j}{l}-\frac{|c_j|}{l}\overline{d_j}-A^*_{jn}-|c_j|e^{\alpha\tau}B^*_{jn}$$
$$-\frac{\alpha}{l}\left[e^{\alpha\tau}\left(1+c_n^2\right)+1+|c_n|\right]-\delta^*\Bigg]Y_j^2(t)$$
$$-\Bigg[2\frac{d_n}{l}-\frac{|c_n|}{l}\overline{d_n}-A^*_{nj}-|c_n|e^{\alpha\tau}B^*_{nj}$$
$$-\frac{\alpha}{l}\left[e^{\alpha\tau}\left(1+c_n^2\right)+1+|c_n|\right]-\delta^*\Bigg]Y_n^2(t)\Bigg\}-\delta^*\left[X_n^2(t)+Y_n^2(t)\right]\}. \tag{35}$$

Letting $e_{nj}=1$ $(n\neq j)$, $e_{nj}=0$, $n=j$, $G_{nj}(X_n^2(t),X_j^2(t))=[2\frac{d_j}{l}-\frac{|c_j|}{l}\overline{d_j}-A_{jn}-|c_j|e^{\alpha\tau}B_{jn}-\frac{\alpha}{l}[e^{\alpha\tau}(1+c_n^2)+1+|c_n|]-\delta^*]X_j^2(t)-[2\frac{d_n}{l}-\frac{|c_n|}{l}\overline{d_n}-A_{nj}-|c_n|e^{\alpha\tau}B_{nj}-\frac{\alpha}{l}[e^{\alpha\tau}(1+c_n^2)+1+|c_n|]-\delta^*]X_n^2(t)$ and $p_n(X_n^2(t))=[2\frac{d_n}{l}-\frac{|c_n|}{l}\overline{d_n}-A_{nj}-|c_n|e^{\alpha\tau}B_{nj}-\frac{\alpha}{l}[e^{\alpha\tau}(1+c_n^2)+1+|c_n|]-\delta^*]X_n^2(t)$; $b^*_{nj}=1$ $(n\neq j)$, $b^*_{nj}=0$, $n=j$, $G^*_{nj}(Y_n^2(t),Y_j^2(t))=[2\frac{d_j}{l}-\frac{|c_j|}{l}\overline{d_j}-A^*_{jn\delta}-|c_j|e^{\alpha\tau}B^*_{jn}-\frac{\alpha}{l}[e^{\alpha\tau}(1+c_n^2)+1+|c_n|]-\delta^*]Y_j^2(t)-[\frac{d_n}{l}-\frac{|c_n|}{l}\overline{d_n}-A^*_{nj}-|c_n|e^{\alpha\tau}B^*_{jn}-\frac{\alpha}{l}[e^{\alpha\tau}(1+c_n^2)+1+|c_n|]-\delta^*]Y_n^2(t)$ and $p^*_n(Y_n^2(t))=[\frac{d_n}{l}-\frac{|c_n|}{l}\overline{d_n}-A^*_{nj}-|c_n|e^{\alpha\tau}B^*_{nj}-\frac{\alpha}{l}[e^{\alpha\tau}(1+c_n^2)+1+|c_n|]-\delta^*]Y_n^2(t)$, then we have from (35)

$$\frac{dV_n(t)}{dt}\le e^{\alpha t}\Bigg\{\sum_{j=1}^{l}e_{nj}G_{nj}\left(X_n^2(t),X_j^2(t)\right)$$
$$+\sum_{j=1}^{l}b^*_{nj}G^*_{nj}\left(Y_n^2(t),Y_j^2(t)\right)-\sum_{j=1}^{l}\delta^*\left[X_n^2(t)+Y_n^2(t)\right]\Bigg\}, \tag{36}$$
$$G_{nj}\left(X_n^2,X_j^2(t)\right)=p_j\left(X_j^2(t)\right)-p_n\left(X_n^2(t)\right), \tag{37}$$

and

$$G^*_{nj}\left(Y_n^2,Y_j^2(t)\right)=p^*_j\left(Y_j^2(t)\right)-p^*_n\left(Y_n^2(t)\right). \tag{38}$$

From (36)–(38), using the same proofs as those of (7)–(17), we have

$$\frac{dV(t)}{dt} \leq e^{\alpha t} \sum_{n=1}^{l} \sum_{j=1}^{l} \bigl(-c_n^* \delta^* \bigl[u_n^2(t) + v_n^2(t)\bigr]\bigr) < 0.$$

The rest of the proof is similar to that of the corresponding part in global exponential stability in [1] and it is omitted.

When $c_n = 0$, system (1) and system (2) reduce, respectively, to the following complex-valued neural networks with time delays and real-valued neural networks with time delays:

$$\begin{aligned} z_n'(t) &= -d_n(t) z_n(t) + \sum_{j=1}^{l} b_{nj}(t) F_j\bigl(z_j(t)\bigr) \\ &\quad + \sum_{j=1}^{l} e_{nj}(t) G_j(z_j\bigl(t - \tau_{nj}(t)\bigr) + P_n(t) \end{aligned} \tag{39}$$

and

$$\begin{aligned} \frac{d[u_n(t)]}{dt} &= -d_n(t) u_n(t) + \sum_{j=1}^{l} b_{nj}^R F_j^R\bigl(u_j(t), v_j(t)\bigr) - \sum_{j=1}^{l} b_{nj}^I F_j^I\bigl(u_j(t), v_j(t)\bigr) \\ &\quad + \sum_{j=1}^{l} e_{nj}^R(t) G_j^R\bigl(u_j^t, v_j^t\bigr) - \sum_{j=1}^{l} e_{nj}^I(t) G_j^I\bigl(u_j^t, v_j^t\bigr) + P_n^R(t), \\ \frac{d[v_n(t)]}{dt} &= -d_n(t) v_n(t) + \sum_{j=1}^{l} b_{nj}^R F_j^I\bigl(u_j(t), v_j(t)\bigr) + \sum_{j=1}^{l} b_{nj}^I F_j^R\bigl(u_j(t), v_j(t)\bigr) \\ &\quad + \sum_{j=1}^{l} e_{nj}^R(t) G_j^I\bigl(u_j^t, v_j^t\bigr) + \sum_{j=1}^{l} e_{nj}^I(t) G_j^R\bigl(u_j^t, v_j^t\bigr) + P_n^I(t). \end{aligned} \tag{40}$$

From Theorem 4.1, we can obtain the following corollary. □

Corollary 1 *Assume that* (h_1) *and* (h_2) *hold. Further assume that*

(h_9)

$$U_{nj} + \frac{E_{nj}}{1-\sigma} < \frac{2\underline{d_j}}{l} - A_{jn}.$$

(h_{10})

$$V_{nj} + \frac{F_{nj}}{1-\sigma} < \frac{2\underline{d_j}}{l} - A_{jn}^*.$$

Then system (39) *or system* (40) *has an* ω*-periodic solution which is globally exponentially stable.*

Remark 2 In [2, 14, 15], and [16], the existence and global exponential/asymptotic stability of periodic solutions for complex-valued neural networks have been obtained by using coincidence degree theory, LMI method, and Lypunov functional method. In our paper, by combining graph theory with coincidence degree theory to study periodic solutions, we establish new sufficient conditions to guarantee the existence and global exponential stability of periodic solutions for complex-valued neural networks. Hence, a new study method of periodic solutions for neural networks is introduced in our paper.

Remark 3 In [1], the activation functions were assumed to be bounded, but in our paper, the activation functions are not bounded; hence, our results of exponential stability of periodic solutions for complex-valued neural networks of neutral type are less conservative than those obtained in [1].

Remark 4 Up to now, the results of exponential stability of periodic solutions for neural networks with time delays have not been published by means of graph theory. Hence, our work to study periodic solutions of neural networks by applying graph theory are novel in comparison to those obtained by using only coincidence degree theory or fixed point theorems.

Remark 5 So far, coincidence degree theory has been widely applied to investigate the existence of periodic solutions for neural networks. In recent years, combination of graph theory with coincidence degree theory has been applied to studying the existence of periodic solutions for coupled networks [31–35]. Recently, we have established some sufficient conditions for the existence and global stability of periodic solutions for neural networks by combining coincidence degree theory with Lyapunov functional method [14, 15, 37]. However, the results on the existence and global stability of periodic solutions for neural networks have not been obtained by combining coincidence degree theory with graph theory as well as Lyapunov functional method. Hence, our results on the existence and global exponential stability of periodic solutions for neural networks by using graph theory are novel and complementary to the existing papers.

Remark 6 So far, the existence result of periodic solutions has been different from that of global exponential/asymptotic stability for dynamical systems and differential equations by using coincidence degree theory or fixed point theorems in the existing papers. In our paper, by combining coincidence degree theory with graph theory as well as Lyapunov functional method, by constructing the same Lyapunov functionals in the proofs of the existence of periodic solutions and global exponential stability of periodic solutions, we can obtain novel identical sufficient conditions for the existence of periodic solutions and global exponential stability of periodic solutions. Hence, our study method of periodic solutions is new and our result of global exponential stability for neural networks is concise and easy to verify.

5 Numerical examples

In this section, we give an example for showing our results.

Example 1 We consider the neutral-type system (2) with the following parameters: $n = 1, 2$, $c_n = -0.1$, $l = 2$, $d_n(t) = 10 + 0.5\sin 5t$, $b^R_{nj}(t) = b^I_{nj}(t) = e^R_{nj}(t) = e^I_{nj}(t) = P^R_n(t) = P^I_n(t) = 0.01 + 0.09\sin 5t$, $\tau_{nj}(t) = 0.1(2 + \sin 5t)$, $F^R_j(u_j(t), v_j(t)) = F^I_j(u_j(t), v_j(t)) = G^R_j(u_j(t), v_j(t)) = 0.01|u_j(t)| + 0.01|v_j(t)|$, $G^I_j(u_j(t), v_j(t)) = 0.01|u_j(t)| + 0.01|v_j(t)|$. Then, in Theorem 4.1,

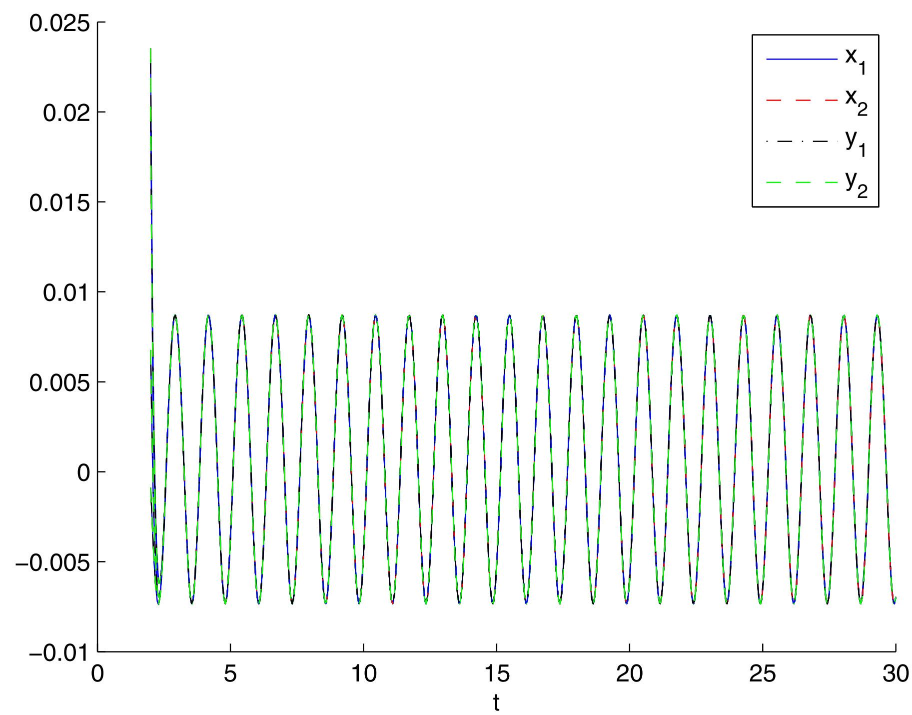

Figure 1 Global exponential stability of periodic solutions in Example 1

$c_n = -0.1$, $n = 1,2$, $l = 2$, $\overline{d_j} = 10.5$, $\underline{d_j} = 9.5$, $\overline{b^R_{jn}} = \overline{b^I_{jn}} = \overline{e^R_{nj}} = \overline{e^I_{nj}} = 0.1$, $l^R_n = l^I_n = k^R_n = k^I_n = q^R_n = q^I_n = p^R_n = p^I_n = 0.01$, $\sigma = 0.1$, $\sigma^* = 0.1$.

Since the activation functions in [1] are bounded, while the activation functions in Example 1 are not bounded, hence the global exponential stability of periodic solutions for Example 1 cannot be verified by the result in [1].

It is easy to verify that

$$\begin{aligned}
&(1+|c_n|)\left(U_{nj} + \frac{E_{nj}}{1-\sigma}\right) = 1.1(0.004 + 0.0044) = 0.00924,\\
&\frac{2\underline{d_j}}{l} - \frac{|c_j|}{l}\overline{d_j} - A_{jn} - |c_j|B_{jn} = 9.5 - 0.525 - 0.008 - 0.0008 = 8.9662,\\
&(1+|c_n|)\left(V_{nj} + \frac{F_{nj}}{1-\sigma}\right) = 1.1(0.004 + 0.0044) = 0.00924,\\
&\frac{2\underline{d_j}}{l} - \frac{|c_j|}{l}\overline{d_j} - A^*_{jn} - |c_j|B^*_{jn} = 9.5 - 0.525 - 0.008 - 0.0008 = 8.9662.
\end{aligned}$$

Hence

$$\begin{aligned}
&(1+|c_n|)\left(U_{nj} + \frac{E_{nj}}{1-\sigma}\right) < \frac{2\underline{d_j}}{l} - \frac{|c_j|}{l}\overline{d_j} - A_{jn} - |c_j|B_{jn},\\
&(1+|c_n|)\left(V_{nj} + \frac{F_{nj}}{1-\sigma}\right) < \frac{2\underline{d_j}}{l} - \frac{|c_j|}{l}\overline{d_j} - A^*_{jn} - |c_j|B^*_{jn}.
\end{aligned}$$

Namely (h_3) and (h_4) in Theorem 4.1 are satisfied. Thus all the conditions in Theorem 4.1 are satisfied; therefore, by Theorem 4.1, system (2) in Example 1 has a unique $\frac{2\pi}{5}$ periodic solution which is globally exponentially stable.

The global exponential stability of periodic solutions of the neutral-type complex-valued neural networks in Example 1 is shown in Fig. 1.

6 Conclusion

By combining graph theory with coincidence degree theory as well as Lyapunov functional method, by constructing the same Lyapunov functionals in the proofs of the existence of periodic solutions and global exponential stability of periodic solutions, novel identical sufficient conditions on the existence of periodic solutions and global exponential stability of periodic solutions for neutral-type complex-valued neural networks are established. In our results, the assumption on the boundedness for the activation functions in [1] is removed and the inequality conditions in [1] are replaced with new inequalities. Hence, our results are less conservative than those obtained in [1] and easy to verify. In near future, we will study nonlinear control of delayed systems [19, 20].

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Competing interests
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Authors' contributions
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