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Dynamical behavior of a generalized eco-epidemiological system with prey refuge

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Abstract

A generalized eco-epidemiological system with prey refuge is proposed in this paper. The saturation incidence kinetics and a generalized functional response are used to describe the contact process and the predation process, respectively. Based on mathematical issue, the local and global stability properties, Hopf bifurcation, and permanence of the dynamical system are investigated. Based on the ecological aspects, the impact of prey refuge on the dynamical consequences of the eco-epidemiological system and the mechanism of prey refuge are discussed. The results reveal that the stabilizing and destabilizing effects occur under some certain conditions. Based on epidemiological issue, the controlling strategies of the infectious disease are proposed. The results show that the prey refuge can control the spread of disease by the relative level of prey refuge. This study has resolved some basic and interesting issues for an eco-epidemiological system with a generalized response function and the effect of prey refuge.

Keywords: Eco-epidemiological system; Generalized function response; Prey refuge; Dynamical behavior; Stabilizing/Destabilizing effect; Method of disease control

1 Introduction

Eco-epidemiological systems, which are applied to describe predator and prey interactions with diseases in one population or both populations, have become important tools in analyzing the spread and control of infectious diseases, and hence have received much attention since the Kermac–Mckendric SIR model was proposed [1–10]. In eco-epidemiology, researchers study an ecological system with disease either in prey or in predator or in both populations [10–16]. Anderson and May [1] proposed an animal (predator) and plant (prey) model with infectious diseases and investigated the invasion, persistence, and spread of diseases. Chattopadhyay et al. investigated a predator–prey system with disease in the prey [4], and then applied their research to study the pelicans at risk in the Salton sea [5]. Saifuddin et al. [13] explored an eco-epidemiological system with disease in the prey and weak Allee in predator, and considered the complex dynamics including stability properties and bifurcations. Bairagi et al. [10] noticed the fact that the functional response plays an important role in determining the dynamical consequences of the population interactions, and hence conducted a comparative research on the stability aspects of a predator–prey system with a class of functional responses. Besides the published works mentioned above, more and more researchers have focused on the population interactions with diseases in prey or predator or both populations [1–16].

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However, most of published research on eco-epidemiological systems incorporated a certain response function (e.g., Holling type functional response, Beddington–DeAngelis functional response) and a certain incidence rate (e.g., bilinear incidence, standard incidence, and saturation incidence), and investigated the dynamical behaviors of the considered systems [10, 11, 14–16]. As far as we know, few works have focused on a model with a generalized functional response and obtained a generalized conclusion. Motivated by these, in this paper we will propose a class of eco-epidemiological systems. That is to say, a generalized functional response is incorporated into an eco-epidemiological model with saturation incidence. Hence, the existing eco-epidemiological models become some special cases of ours.

In fact, there exist many ecological effects which influence the dynamical consequences of the species interactions, such as the effect of Allee effect, habitat complexity, harvesting, and prey refuge [17–19]. For the effect of prey refuge, the theoretical research and the field observations give a general conclusion that prey refuge can stabilize or destabilize the considered predation systems, and can prevent the prey extinction [16–36]. Here, the stabilizing effect means that the interior equilibrium point changes from an unstable state to a stable state due to increase in the degree of prey refuge [16, 21, 26]. Otherwise, the destabilizing effect is observed [19, 35]. For examples, Gonzalez–Olivares and Ramos–Jiliberto [21], and Ruxton [16] proposed two continuous-time predator–prey systems with the assumption that a constant proportion of prey could move to refuges. Their studies found a stabilizing effect on the dynamical consequences of the considered systems. The stabilizing effect was also observed in a generalized predator–prey system under some certain conditions [18, 19]. Most interestingly and importantly, Ma et al. [18, 19] proposed two generalized predator–prey systems incorporating prey refuge and observed a destabilizing effect. The above cited references reveal that the functional response of predator to prey population plays an important role in determining dynamical consequences of the interacting populations. However, few studies incorporate the effect of prey refuge into eco-epidemiological systems. Hence, this paper incorporates prey refuge into a species interaction with disease in prey.

Motivated by above analyses, in this paper we present a generalized eco-epidemiological system with the effect of prey refuge and the saturation incidence, and focus on the dynamical consequences of the proposed system and the explanations of the realistic meanings.

2　Model formulation

The basic model comprises two population subclasses: (i) prey population with density $N(t)$ and (ii) predators with density $Y(t)$, and is based on the following generalized predation model:

$$
\begin{aligned}
\dot{N}(t) &= rN\left(1 - \frac{N}{K}\right) - cY\varphi\big((1-\gamma)N\big), \\
\dot{Y}(t) &= ecY\varphi\big((1-\gamma)N\big) - d_2 Y,
\end{aligned}
\tag{2.1}
$$

where all parameters are positive and have the following ecological meanings: r is the intrinsic growth rate of prey population, K is the environmental carrying capacity of prey population, c is the predation coefficients of predators to prey population, d_2 is the natural death rate of predators, e is the conversion efficiency, γ ($0 < \gamma < 1$) denotes that a constant

proportion of prey use refuges. The function $\varphi(N)$ denotes the functional response of predators and satisfies the following assumptions:

$$\varphi(0) = 0, \qquad \varphi'(N) > 0 \quad (N > 0).$$

Furthermore, it is assumed that the prey population ($N(t)$) is divided into two subclasses: the susceptible prey ($S(t)$) and the infected prey ($I(t)$) due to infectious disease. Beside, this paper gives the following assumptions:

(1) The susceptible prey is capable of reproducing only and the infected prey is removed by death at a natural rate d_1.

(2) The disease is spread only among the prey population and the disease is not genetically inherited. The infected prey does not become immune.

(3) Susceptible prey becomes infected with the saturation incidence kinetics $\frac{\beta SI}{\alpha + I}$, where β measures the force of infection and α is the inhibition effect.

(4) Predators consume susceptible and infected prey with predation coefficients c_1 and c_2, respectively. The consumed prey is converted into predator with efficiency e.

Combining the generalized predation model (2.1) and the above assumptions, a generalized eco-epidemiological system with prey refuge and disease in prey is proposed by the following equations:

$$
\begin{aligned}
\dot{S}(t) &= rS\left(1 - \frac{S+I}{K}\right) - \frac{\beta SI}{\alpha + I} - c_1 Y\varphi\big((1-\gamma)S\big), \\
\dot{I}(t) &= \frac{\beta SI}{\alpha + I} - d_1 I - c_2 Y\varphi\big((1-\gamma)I\big), \\
\dot{Y}(t) &= ec_1 Y\varphi\big((1-\gamma)S\big) + ec_2 Y\varphi\big((1-\gamma)I\big) - d_2 Y,
\end{aligned}
\tag{2.2}
$$

with the initial conditions

$$S(0) = S_0 > 0; \qquad I(0) = I_0 > 0; \qquad Y(0) = Y_0 > 0. \tag{2.3}$$

Using the following change of variables

$$\Phi : \big(R_0^+\big)^3 \to \big(R_0^+\big)^3, \qquad \Phi(S, I, Y) = \left(\frac{\bar{S}}{(1-\gamma)}, \frac{\bar{I}}{(1-\gamma)}, \frac{\bar{Y}}{(1-\gamma)}\right)$$

and rewriting system (2.1) with (S, I, Y), the following system can be obtained:

$$
\begin{aligned}
\dot{S}(t) &= rS\left(1 - \frac{S+I}{(1-\gamma)K}\right) - \frac{\beta SI}{\alpha(1-\gamma) + I} - c_1 Y\varphi(S), \\
\dot{I}(t) &= \frac{\beta SI}{\alpha(1-\gamma) + I} - d_1 I - c_2 Y\varphi(I), \\
\dot{Y}(t) &= ec_1 Y\varphi(S) + ec_2 Y\varphi(I) - d_2 Y.
\end{aligned}
\tag{2.4}
$$

3 Equilibria

All equilibrium points of system (2.4) can be obtained by solving the following equations:

$$
\begin{cases}
rS(1 - \frac{S+I}{(1-\gamma)K}) - \frac{\beta SI}{\alpha(1-\gamma)+I} - c_1 Y \varphi(S) = 0, \\
\frac{\beta SI}{\alpha(1-\gamma)+I} - d_1 I - c_2 Y \varphi(I) = 0, \\
ec_1 Y \varphi(S) + ec_2 Y \varphi(I) - d_2 Y = 0.
\end{cases}
\tag{3.1}
$$

(1) The trivial equilibrium point $E_0(0,0,0)$.

(2) The equilibrium point in the absence of the infected prey and predators
$E_1((1-\gamma)K, 0, 0)$.

(3) The disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$, where

$$
\tilde{S} = \varphi^{-1}\left(\frac{d_2}{ec_1}\right), \qquad \tilde{Y} = \frac{er\tilde{S}}{d_2}\left(1 - \frac{\tilde{S}}{(1-\gamma)K}\right).
$$

If $\gamma < 1 - \frac{1}{K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ has its ecological meanings.

(4) The coexisting equilibrium point $E_3(S^*, I^*, Y^*)$, where

$$
S^* = \frac{(d_1 I^* + c_2 Y^* \varphi(I^*) + I^*)((1-\gamma)\alpha + I^*)}{\beta I^*},
$$

$$
I^* = \varphi^{-1}\left(\frac{d_2 - ec_1 \varphi(S^*)}{ec_2}\right),
$$

$$
Y^* = \frac{rS^*(1 - \frac{S^*+I^*}{(1-\gamma)K})S^* - I^* - d_1 I^*}{c_1 \varphi(S^*) + c_2 \varphi(I^*)}.
$$

If $0 < \gamma < 1 - \frac{rS^*(S^*+I^*)}{K(rS^*-d_1 I^*)}$, then the equilibrium point $E_3(S^*, I^*, Y^*)$ is a positive equilibrium point. Otherwise, it has no ecological meanings anymore.

4 Stability property

4.1 Stability of the disease-free equilibrium

4.1.1 Local stability of the disease-free equilibrium

In this section, we study the local stability properties of the equilibrium points of system (2.4). Especially, the local stability analysis for the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is investigated in this section.

Firstly, it is easy to show that the trivial equilibrium point $E_0(0,0,0)$ is a saddle point and the equilibrium point $E_1((1-\gamma)K, 0, 0)$ is stable if $\beta < \frac{\alpha d_1}{K}$ and $\gamma > 1 - \frac{1}{K}\varphi^{-1}(\frac{d_2}{ec_1})$.

Again, the roots of the characteristic equation of the community matrix corresponding to $E_2(\tilde{S}, 0, \tilde{Y})$ are $\frac{\beta\tilde{S}}{\alpha} - d_1 - c_2(1-\gamma)\varphi'(0)\tilde{Y}$ and the roots of the following equation:

$$
\lambda^2 - A\lambda + B = 0,
\tag{4.1}
$$

where

$$
A = r\left(1 - \frac{2\tilde{S}}{(1-\gamma)K}\right) - c_1(1-\gamma)\varphi'\big((1-\gamma)\tilde{S}\big)\tilde{Y}
$$

and

$$B = c_1^2 e(1 - \gamma)\varphi'\big((1 - \gamma)\tilde{S}\big)\varphi\big((1 - \gamma)\tilde{S}\big)\tilde{Y} > 0.$$

Hence, the roots of Eq. (4.1) will have negative real parts if $A < 0$, which implies that

$$\left[d_2 - ec_1\varphi'\left(\varphi^{-1}\left(\frac{d_2}{ec_1}\right)\right)\varphi^{-1}\left(\frac{d_2}{ec_1}\right) \right]$$
$$- \frac{\tilde{S}}{(1 - \gamma)K}\left[2d_2 - ec_1\varphi'\left(\varphi^{-1}\left(\frac{d_2}{ec_1}\right)\right)\varphi^{-1}\left(\frac{d_2}{ec_1}\right) \right] < 0.$$

Therefore, the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ is locally asymptotically stable iff

$$\frac{\beta\tilde{S}}{\alpha} - d_1 - c_2(1 - \gamma)\varphi'(0)\tilde{Y} < 0 \tag{4.2}$$

and

$$\left[d_2 - ec_1\varphi'\left(\varphi^{-1}\left(\frac{d_2}{ec_1}\right)\right)\varphi^{-1}\left(\frac{d_2}{ec_1}\right) \right]$$
$$- \frac{\tilde{S}}{(1 - \gamma)K}\left[2d_2 - ec_1\varphi'\left(\varphi^{-1}\left(\frac{d_2}{ec_1}\right)\right)\varphi^{-1}\left(\frac{d_2}{ec_1}\right) \right] < 0. \tag{4.3}$$

Inequality (4.2) is equivalent to the following term:

$$\gamma < 1 - \frac{\varphi^{-1}(\frac{d_2}{ec_1})}{\alpha K}\left[\frac{\beta d_2 K + \alpha c_2 er\varphi'(0)\varphi^{-1}(\frac{d_2}{ec_1})}{d_1 d_2 + c_2 er\varphi^{-1}(\frac{d_2}{ec_1})} \right]. \tag{4.4}$$

Furthermore, inequality (4.2) is equivalent to the following cases:

(1) If $\dfrac{d_2}{c_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})} < e < \dfrac{2d_2}{c_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})}$, then inequality (4.3) always holds.

(2) If $e < \dfrac{d_2}{c_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})}$, then inequality (4.3) holds when

$$\gamma > 1 - \frac{\varphi^{-1}(\frac{d_2}{ec_1})}{K}\left[\frac{2d_2 - ec_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})}{d_2 - ec_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})} \right].$$

(3) If $e > \dfrac{2d_2}{c_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})}$, then inequality (4.3) holds when

$$\gamma < 1 - \frac{\varphi^{-1}(\frac{d_2}{ec_1})}{K}\left[\frac{2d_2 - ec_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})}{d_2 - ec_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})} \right].$$

Next, define

$$\underline{E} = \frac{d_2}{c_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})},$$

$$\overline{E} = \frac{2d_2}{c_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})},$$

$$K_1 = \frac{2d_2 - ec_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})}{d_2 - ec_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})},$$

$$K_2 = \frac{\alpha e c_2 r \varphi'(0)\varphi^{-1}(\frac{d_2}{ec_1})}{d_2 - ec_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})},$$

$$K_3 = \frac{\beta d_2 K + \alpha c_2 e r \varphi'(0)\varphi^{-1}(\frac{d_2}{ec_1})}{d_1 d_2 + c_2 e r \varphi^{-1}(\frac{d_2}{ec_1})},$$

$$\underline{B} = \frac{\alpha d_1}{K}, \qquad \overline{B} = \frac{\alpha d_1 K_1}{K} + K_2.$$

According to the above analyses, the following conclusions are obtained.

Theorem 4.1 *Supposing that $\underline{B} < \beta < \overline{B}$ and $e < \underline{E}$, we have*

(1) *If $0 < \gamma < 1 - \dfrac{K_1}{K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is unstable;*

(2) *If $1 - \dfrac{K_1}{K}\varphi^{-1}(\frac{d_2}{ec_1}) < \gamma < 1 - \frac{K_3}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is locally asymptotically stable;*

(3) *If $1 - \frac{K_3}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1}) < \gamma < 1 - \frac{1}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is unstable.*

Theorem 4.2 *Assuming that $\underline{B} < \beta < \overline{B}$ and $e > \overline{E}$, we obtain*

(1) *If $0 < \gamma < 1 - \frac{K_3}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is locally asymptotically stable;*

(2) *If $1 - \frac{K_3}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1}) < \gamma < 1 - \frac{1}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is unstable.*

Theorem 4.3 *Assuming that $\beta < \underline{B}$ and $e < \underline{E}$, we obtain*

(1) *If $0 < \gamma < 1 - \dfrac{K_1}{K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is unstable;*

(2) *If $1 - \dfrac{K_1}{K}\varphi^{-1}(\frac{d_2}{ec_1}) < \gamma < 1 - \frac{1}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is locally asymptotically stable.*

Theorem 4.4 *Supposing that $e > \overline{E}$ and $\beta > \overline{B}$ or $\underline{E} < e < \overline{E}$ and $\beta > \underline{B}$, we have*

(1) *If $0 < \gamma < 1 - \frac{K_3}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is locally asymptotically stable;*

(2) *If $1 - \frac{K_3}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1}) < \gamma < 1 - \frac{1}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is unstable.*

Theorem 4.5 *Supposing that $\beta < \underline{B}$ and $e > \overline{E}$, we have*

(1) *If $0 < \gamma < 1 - \dfrac{K_1}{K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is locally asymptotically stable;*

(2) *If $1 - \dfrac{K_1}{K}\varphi^{-1}(\frac{d_2}{ec_1}) < \gamma < 1 - \frac{1}{\alpha K}\varphi^{-1}(\frac{d_2}{ec_1})$, then the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is unstable.*

Theorem 4.6 *Assuming that $\beta < \underline{B}$ and $\underline{E} < e < \overline{E}$, we have that the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is always locally asymptotically stable.*

Next, three tables are given to list the above results. Define the following notations:

$$M_1 = 1 - \frac{K_1}{K}\varphi^{-1}\left(\frac{d_2}{ec_1}\right), \qquad M_2 = 1 - \frac{K_3}{\alpha K}\varphi^{-1}\left(\frac{d_2}{ec_1}\right), \qquad M_3 = 1 - \frac{1}{\alpha K}\varphi^{-1}\left(\frac{d_2}{ec_1}\right),$$

Table 1 The stability properties of the disease-free equilibrium point when $e < \underline{E}$

Stability	γ		
β	$(0, M_1)$	(M_1, M_2)	(M_2, M_3)
$(0, \underline{B})$	U. S.	S.	S.
$(\underline{B}, \overline{B})$	U. S.	S.	U. S.
$(\overline{B}, 1)$	U. D.	U. D.	U. D.

Table 2 The stability properties of the disease-free equilibrium point when $\underline{E} < e < \overline{E}$

Stability	γ		
β	$(0, M_1)$	(M_1, M_2)	(M_2, M_3)
$(0, \underline{B})$	A. S.	A. S.	A. S.
$(\underline{B}, \overline{B})$	S.	S.	U. S.
$(\overline{B}, 1)$	S.	S.	U. S.

Table 3 The stability properties of the disease-free equilibrium point when $e > \overline{E}$

Stability	γ		
β	$(0, M_1)$	(M_1, M_2)	(M_2, M_3)
$(0, \underline{B})$	S.	U. S.	U. S.
$(\underline{B}, \overline{B})$	S.	S.	U. S.
$(\overline{B}, 1)$	S.	S.	U. S.

the conditions and conclusions of Theorem 4.1–*Theorem* 4.6 *are listed in Table* 1, *Table* 2, *and Table* 3 *in which*

(1) *S. means the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ is stable*;

(2) *U.S. means the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ is unstable*;

(3) *A.S. means the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ is always stable*;

(4) *U.D. means the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ is undefined*.

Table 1 shows the stability properties of the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ when $e < \underline{E}$.

Table 2 shows the stability properties of the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ when $\underline{E} < e < \overline{E}$.

Table 3 shows the stability properties of the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ when $e > \overline{E}$.

4.1.2 Global stability of the disease-free equilibrium

In this section, we consider the global stability of the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$.

Theorem 4.7 *If $\beta > \frac{\bar{\alpha}(d_1 + \varphi'(0)c_2)\tilde{Y}}{\tilde{S}}$ in which $\bar{\alpha} = \alpha(1 - \gamma)$, then system (2.4) with initial conditions (2.3) is to be globally asymptotically stable around the disease-free equilibrium point $E_2(\tilde{S}, 0, \tilde{Y})$ in the region $\Sigma_1 = \{(S, I, Y) | S \geq \tilde{S}, I \geq 0, Y \geq \tilde{Y}\}$.*

Proof We rewrite system (2.4) with initial conditions (2.3) as the following form:

$$\dot{S}(t) = SF_1(S, I, Y),$$
$$\dot{I}(t) = IF_2(S, I, Y), \tag{4.5}$$
$$\dot{Y}(t) = YF_3(S, I, Y),$$

where

$$F_1(S, I, Y) = r\left(1 - \frac{S + I}{(1 - \gamma)K}\right) - \frac{\beta I}{\alpha(1 - \gamma) + I} - c_1 Y \varphi(S)/S,$$

$$F_2(S, I, Y) = \frac{\beta S}{\alpha(1 - \gamma) + I} - d_1 - c_2 Y \varphi(I)/I,$$

$$F_3(S, I, Y) = ec_1\varphi(S) + ec_2\varphi(I) - d_2.$$

Let us define

$$G_1(I) = -F_1(\bar{S}, I, \bar{Y}), \qquad G_2(S) = F_2(S, 0, \bar{Y}), \qquad G_3(Y) = F_1(\bar{S}, 0, Y) + F_2(\bar{S}, 0, Y).$$

Next, we consider a Lyapunov function defined as follows:

$$V(S(t), I(t), Y(t)) = \int_{\tilde{S}}^{S} \frac{G_2(u)}{u}\, du + \int_0^I \frac{G_1(v)}{v}\, dv + \int_{\tilde{Y}}^{Y} \frac{G_3(w)}{w}\, dw.$$

Now, by simple computation, we obtain that

$$\frac{dV}{dt} = G_1 F_2(S, I, Y) + G_2 F_1(S, I, Y) + G_3 F_3(S, I, Y)$$

$$= G_1\big(F_2(S, I, Y) - F_2(S, 0, \bar{Y})\big) + G_2\big(F_1(S, I, Y) - F_1(\bar{S}, I, \bar{Y})\big) + G_3\big(F_3(S, I, Y)$$

$$\quad + G_2 F_1(\bar{S}, I, \bar{Y}) + G_1 F_2(S, 0, \bar{Y})\big) + G_3 F_1(\bar{S}, 0, Y) - F_1(\bar{S}, 0, Y)$$

$$= G_2\left[(S - S_1)\frac{\partial F_1}{\partial S}(\bar{S}, I, Y) + (Y - Y_1)\frac{\partial F_1}{\partial Y}(S, I, \bar{Y})\right]$$

$$\quad + G_1\left[I\frac{\partial F_2}{\partial I}(S, \bar{\bar{I}}, Y) + (Y - Y_1)\frac{\partial F_2}{\partial Y}(S, I, \bar{\bar{Y}})\right]$$

$$\quad + G_3\left[(S - S_1)\frac{\partial F_3}{\partial S}(\bar{\bar{S}}, I, Y) + I\frac{\partial F_3}{\partial I}(S, \bar{\bar{I}}, Y)\right].$$

Again, we have

$$\frac{\partial F_1}{\partial S} = -\frac{r}{K} - \frac{c_1 Y \varphi'(S)S - \varphi(S)}{S^2} < 0, \qquad \frac{\partial F_1}{\partial Y} = -c_1\varphi(S) < 0,$$

$$\frac{\partial F_2}{\partial I} = -\frac{\bar{\alpha}\beta S}{(\bar{\alpha} + I)^2} - \frac{c_2 Y \varphi'(I)I - \varphi(I)}{I^2} < 0, \qquad \frac{\partial F_2}{\partial Y} = -c_2\varphi(I) < 0,$$

$$\frac{\partial F_3}{\partial S} = ec_1\varphi'(S) > 0, \qquad \frac{\partial F_2}{\partial I} = ec_2\varphi'(I) > 0.$$

Furthermore, we obtain that

$$G_1(I) = -\left[r\left(1 - \frac{\tilde{S} + I}{K}\right) - \frac{\beta I}{\tilde{\alpha} + I} - \frac{c_1 \tilde{Y} \varphi(\tilde{S})}{\tilde{S}} \right] > 0$$

and

$$G_2(S) = \frac{\beta S}{\tilde{\alpha}} - d_1 - ac_2 \tilde{Y} \geq \frac{\beta \tilde{S}}{\tilde{\alpha}} - d_1 - ac_2 \tilde{Y},$$

in which $a = \lim_{x \to 0} \frac{\varphi(x)}{x} = \varphi'(0)$.

Thus, it is obtained that $G_2 > 0$ while $\beta > \frac{\tilde{\alpha}(d_1 + \varphi'(0)c_2)\tilde{Y}}{\tilde{S}}$.

Therefore, $\frac{dV}{dt} < 0$ in the region $\Sigma = \{(S, I, Y) | S \geq \tilde{S}, I \geq 0, Y \geq \tilde{Y}\}$. Hence the theorem is proved. □

4.2 Stability of the positive equilibrium

4.2.1 Local stability of the positive equilibrium

In this section, we consider the local stability of the positive equilibrium point $E_3(S^*, I^*, Y^*)$.

Theorem 4.8 *If $A_1 < 0$ and $A_1 A_2 < A_3 < 0$, then the positive equilibrium point $E_3(S^*, I^*, Y^*)$ is locally asymptotically stable.*

Proof The characteristic equation of the community matrix corresponding to the positive equilibrium point $E_3(S^*, I^*, Y^*)$ is as follows:

$$\lambda^3 - (a_{11} + a_{22})\lambda^2 + (a_{11}a_{22} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32})\lambda$$
$$- (a_{11}a_{23}a_{32} + a_{13}a_{31}a_{22} - a_{21}a_{13}a_{32} - a_{12}a_{23}a_{31}) = 0,$$

where

$$a_{11} = -\frac{rS^*}{K} - c_1 Y^* (\varphi(S^*) - \varphi'(S^*)),$$

$$a_{12} = -\frac{rS^*}{K} - \frac{\alpha\beta(1 - \gamma)}{((1 - \gamma)\alpha + I^*)^2} c_1 Y^* (\varphi(S^*) - \varphi'(S^*)), \qquad a_{13} = -c_1 \varphi(S^*),$$

$$a_{21} = \frac{\beta I^*}{(1 - \gamma)\alpha + I^*},$$

$$a_{22} = \frac{\alpha\beta(1 - \gamma)}{((1 - \gamma)\alpha + I^*)^2} - \frac{\beta S^*}{(1 - \gamma)\alpha + I^*} - c_2 Y^* (\varphi(I^*) - \varphi'(I^*)),$$

$$a_{23} = -c_2 \varphi(I^*),$$

$$a_{31} = ec_1 Y^* \varphi'(S^*), \qquad a_{32} = ec_2 Y^* \varphi'(I^*).$$

Now, define

$$A_1 = a_{11} + a_{22}, \qquad A_2 = a_{11}a_{22} - a_{12}a_{21} - a_{13}a_{31} - a_{23}a_{32},$$

$$A_3 = a_{11}a_{23}a_{32} + a_{13}a_{31}a_{22} - a_{21}a_{13}a_{32} - a_{12}a_{23}a_{31}.$$

Therefore, the characteristic equation of the positive equilibrium point $E_3(S^*, I^*, Y^*)$ can be rewritten as the following form:

$$\lambda^3 - A_1\lambda^2 + A_2\lambda - A_3 = 0.$$

According to Routh–Hurwitz's criteria, the necessary and sufficient conditions for local stability of positive point are $A_1 < 0$ and $A_1A_2 < A_3 < 0$. Hence the theorem is proved. □

4.2.2 Global stability of the positive equilibrium

In this section, we consider the global stability of the positive equilibrium point $E_3(S^*, I^*, Y^*)$.

Theorem 4.9 *System* (2.4) *with initial conditions* (2.3) *is to be globally asymptotically stable around the positive equilibrium point $E_3(S^*, I^*, Y^*)$ in the region $\Sigma_2 = \{(S, I, Y)|Y > Y^*, 0 < S < S^*, 0 < I < I^* or Y < Y^*, S > S^*, I > I^*\}$.*

Proof We first choose a Lyapunov function which is defined as follows:

$$W\big(S(t), I(t), Y(t)\big) = \int_{S^*}^{S} \frac{u - S^*}{u}\,du + \frac{r}{\beta K(\alpha(1-\gamma) + I^*)} \int_{I^*}^{I} \frac{v - I^*}{v}\,dv + \int_{Y^*}^{Y} \frac{w - Y^*}{w}\,dw.$$

Now, by simple computation, we obtain that

$$\begin{aligned}
\frac{dW}{dt} &= \frac{S - S^*}{S}\frac{dS}{dt} + \frac{r}{\beta K(\alpha(1-\gamma) + I^*)}\frac{I - I^*}{I}\frac{dI}{dt} + \frac{Y - Y^*}{Y}\frac{dY}{dt} \\
&= (S - S^*)\left[r\left(1 - \frac{S+I}{(1-\gamma)K}\right) - \frac{\beta I}{\alpha(1-\gamma) + I} - c_1 Y\varphi(S)/S\right] \\
&\quad + \frac{r}{\beta K(\alpha(1-\gamma) + I^*)}(I - I^*)\left[\frac{\beta S}{\alpha(1-\gamma) + I} - d_1 - c_2 Y\varphi(I)/I\right] \\
&\quad + (Y - Y^*)\left[ec_1\varphi(S) + ec_2\varphi(I) - d_2\right] \\
&= (S - S^*)\left[r\left(1 - \frac{S+I}{(1-\gamma)K}\right) - r\left(1 - \frac{S^* + I^*}{(1-\gamma)K}\right)\right] \\
&\quad + \frac{r}{\beta K(\alpha(1-\gamma) + I^*)}(I - I^*)\left[\frac{\beta S}{\alpha(1-\gamma) + I} - \frac{\beta S^*}{\alpha(1-\gamma) + I^*}\right] \\
&\quad + (Y - Y^*)\left[ec_1\varphi(S) + ec_2\varphi(I) - ec_1\varphi(S^*) + ec_2\varphi(I^*)\right] \\
&= -\frac{r}{K}(S - S^*)^2 - \frac{rS^*}{K(\alpha(1-\gamma) + I^*)}(I - I^*)^2 \\
&\quad + ec_1\varphi'(S^*)(S - S^*)(Y - Y^*) + ec_2\varphi'(I^*)(I - I^*)(Y - Y^*).
\end{aligned}$$

It is clear that $\frac{dW}{dt} < 0$ in Σ_2. Hence the theorem is proved. □

5 Hopf bifurcation

In this section, we consider the Hopf bifurcation near the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ of system (2.4).

The characteristic equation of system (2.4) at the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ is given by the following form:

$$\lambda^3 - \big(A(\gamma) + B(\gamma)\big)\lambda^2 + \big(A(\gamma)B(\gamma) + C(\gamma)\big)\lambda - A(\gamma)C(\gamma) = 0, \tag{5.1}$$

where

$$A(\gamma) = \frac{\beta\tilde{S}}{\alpha} - d_1 - c_2(1 - \gamma)\varphi'(0)\tilde{Y},$$

$$B(\gamma) = r\left(1 - \frac{2\tilde{S}}{(1 - \gamma)K}\right) - c_1(1 - \gamma)\varphi'\big((1 - \gamma)\tilde{S}\big)\tilde{Y},$$

$$C(\gamma) = c_1^2 e(1 - \gamma)\varphi'\big((1 - \gamma)\tilde{S}\big)\varphi\big((1 - \gamma)\tilde{S}\big)\tilde{Y} > 0.$$

It is noted that the expressions $A(\gamma)$, $B(\gamma)$, and $C(\gamma)$ are smooth functions of γ.

In order to determine the instability of system (2.4), let us consider γ (the effect of prey refuge) as a bifurcation parameter. For this purpose, let us firstly give the following lemma.

Lemma 5.1 (Hopf bifurcation theorem [16]) *If $A(\gamma)$, $B(\gamma)$, and $C(\gamma)$ are smooth functions of γ in an open interval about $\gamma \in R$, such that the characteristic equation (5.1) has*

(1) *a pair of complex eigenvalues $\lambda = p(\gamma) \pm iq(\gamma)$ with $p(\gamma)$ and $q(\gamma) \in R$, so that they become purely imaginary at $\gamma = \gamma_0$ and $\frac{dp(\gamma)}{d\gamma}|(\gamma = \gamma_0) \neq 0$,*

(2) *the other eigenvalue is negative at $\gamma = \gamma_0$,*

then a Hopf bifurcation occurs around an equilibrium point of the considered system at $\gamma = \gamma_0$ (i.e., a stability change of an equilibrium point of the considered system accompanied by the creation of a limit cycle at $\gamma = \gamma_0$).

Based on Lemma 5.1, the following conclusion can be obtained.

Theorem 5.2 *If $A < 0$ and $B < 0$, then system (2.4) possesses a Hopf bifurcation around the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$.*

Proof Suppose the value γ_0 is equal to

$$1 - \frac{\varphi^{-1}(\frac{d_2}{ec_1})}{\alpha K}\left[\frac{\beta d_2 K + \alpha c_2 er\varphi'(0)\varphi^{-1}(\frac{d_2}{ec_1})}{d_1 d_2 + c_2 er\varphi^{-1}(\frac{d_2}{ec_1})}\right],$$

or

$$1 - \frac{\varphi^{-1}(\frac{d_2}{ec_1})}{K}\left[\frac{2d_2 - ec_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})}{d_2 - ec_1\varphi'(\varphi^{-1}(\frac{d_2}{ec_1}))\varphi^{-1}(\frac{d_2}{ec_1})}\right],$$

which are the roots of $A = 0$ or $B = 0$, respectively.

Firstly, for $\gamma = \gamma_0$, the characteristic equation of system (2.4) at the disease-free equilibrium $E_2(\tilde{S}, 0, \tilde{Y})$ becomes the following form:

$$(\lambda - A)(\lambda^2 + C) = 0.$$

It is clear that the roots of the above equation are $\lambda_1 = A < 0$, $\lambda_2 = i\sqrt{C}$, and $\lambda_1 = -i\sqrt{C}$.

Hence, there exist a pair of purely imaginary eigenvalues and a strictly negative real eigenvalue.

Secondly, for γ in a neighborhood of γ_0, the roots have the form $\lambda_1 = A < 0$, $\lambda_2 = p_1(\gamma) + ip_2(\gamma)$ and $\lambda_3 = p_1(\gamma) - ip_2(\gamma)$ in which $p_1(\gamma)$ and $p_2(\gamma)$ are real.

Next, we shall verify the transversality condition:

$$\frac{d}{d\gamma}\big(\operatorname{Re}\lambda_i(\gamma)\big)\big|(\gamma = \gamma_0) \neq 0, \quad i = 1, 2.$$

Substituting $\lambda_2 = p_1(\gamma) + ip_2(\gamma)$ into the characteristic Eq. (5.1), we get

$$\big(p_1(\gamma) + ip_2(\gamma)\big)^3 - \big(A(\gamma) + B(\gamma)\big)\big(p_1(\gamma) + ip_2(\gamma)\big)^2 + \big(A(\gamma)B(\gamma)$$
$$+ C(\gamma)\big)\big(p_1(\gamma) + ip_2(\gamma)\big) - A(\gamma)C(\gamma) = 0. \tag{5.2}$$

Taking derivatives of both sides of (5.2) with respect to γ, we have

$$3\big(p_1(\gamma) + ip_2(\gamma)\big)^2\big(p_1'(\gamma) + ip_2'(\gamma)\big) - \big(A'(\gamma) + B'(\gamma)\big)\big(p_1(\gamma) + ip_2(\gamma)\big)^2$$
$$- 2\big(A(\gamma) + B(\gamma)\big)\big(p_1(\gamma) + ip_2(\gamma)\big)\big(p_1'(\gamma) + ip_2'(\gamma)\big)$$
$$+ \big(A'(\gamma)B(\gamma) + A(\gamma)B'(\gamma) + C'(\gamma)\big)\big(p_1(\gamma) + ip_2(\gamma)\big)$$
$$+ \big(A(\gamma)B(\gamma) + C(\gamma)\big)\big(p_1'(\gamma) + ip_2'(\gamma)\big) - A'(\gamma)C(\gamma) - A(\gamma)C'(\gamma) = 0. \tag{5.3}$$

Comparing real and imaginary parts from both sides of Eq. (5.3), we obtain

$$\big(3p_1^2 - 3p_2^2 - 2p_1D_1 + D_2\big)p_1' - \big(6p_1p_2 - 2p_2D_1\big)p_2'$$
$$+ \big(p_1p_2 + P_1D_2' - P_1^2D_1' - D_3'\big) = 0, \tag{5.4}$$
$$\big(6p_1p_2 - 2p_2D_1\big)p_1' - \big(3p_1^2 - 3p_2^2 - 2p_1D_1 + D_2\big)p_2' + \big(P_2D_2' - 2P_1P_1D_1'\big) = 0,$$

where

$$D_1 = A(\gamma) + B(\gamma), \qquad D_2 = A(\gamma)B(\gamma) + C(\gamma), \qquad D_3 = A(\gamma)B(\gamma).$$

Define $E_1 = (3p_1^2 - 3p_2^2 - 2p_1D_1 + D_2)$, $E_2 = (6p_1p_2 - 2p_2D_1)$, $E_3 = (p_1p_2 + P_1D_2' - P_1^2D_1' - D_3')$, and $E_4 = (P_2D_2' - 2P_1P_1D_1')$, then Eqs. (5.4) become of the following form:

$$E_1p_1' - E_2p_2' + E_3 = 0,$$
$$E_2p_1' + E_1p_2' + E_4 = 0. \tag{5.5}$$

The value of p_1' can be obtained by solving Eqs. (5.5)

$$p_1' = \frac{E_1E_3 + E_2E_4}{E_1^2 + E_1^2}.$$

For $p_1 = 0$ and any other possible value of p_2 at $\gamma = \gamma_0$, the $E_1E_3 + E_2E_4$ is always unequal to zero since $\frac{D_2D_3}{3D_3' + 2D_1D_2'} \neq \frac{A^2 - 4B}{C^2}$.

Therefore, $\lambda_1 = A < 0$ and

$$\frac{d}{d\gamma}\left(\operatorname{Re}\lambda_i(\gamma)\right)|(\gamma = \gamma_0) = \frac{E_1E_3 + E_2E_4}{E_1^2 + E_1^2} \neq 0, \quad i = 1,2.$$

Hence, according to Lemma 5.1, the theorem is proved. □

6 Permanence

In this section, we prove the permanence of system (2.4) with initial conditions (2.3) under the condition $X\varphi'(X) < \varphi(X)$ $(X > 0)$.

Definition 6.1 ([6]) If there exist positive constants m_S, M_S, m_I, M_I, m_Y, and M_Y such that each solution $(S(t), I(t), Y(t))$ of system (2.4) satisfies

$$0 < m_S \leq \liminf_{t\to+\infty} S(t) \leq \limsup_{t\to+\infty} S(t) \leq M_S,$$

$$0 < m_I \leq \liminf_{t\to+\infty} I(t) \leq \limsup_{t\to+\infty} I(t) \leq M_I,$$

$$0 < m_Y \leq \liminf_{t\to+\infty} Y(t) \leq \limsup_{t\to+\infty} Y(t) \leq M_Y,$$

then system (2.4) is permanent. Otherwise, it is non-permanent.

In order to consider the permanence of system (2.4), we consider the following auxiliary system:

$$\dot{S}(t) = rS\left(1 - \frac{S+I}{K}\right) - \frac{\beta SI}{\alpha + A} - c_1 BS,$$

$$\dot{I}(t) = \frac{\beta SI}{\alpha + C} - dI - c_2 DI,$$

$$(6.1)$$

in which A, C, and D are non-negatively constant numbers, B is positive and bounded by $\frac{r}{c_1}$.

For system (6.1), we can obtain the following result.

Lemma 6.2 *The positive equilibrium point of system* (6.1) *is globally asymptotically stable when it exists.*

Proof The positive equilibrium point of system (6.1) is $\bar{P}(\bar{S}, \bar{I})$, where

$$\bar{S} = \frac{(\alpha + C)(d + c_2 D)}{\beta}, \qquad \bar{I} = \frac{K(r - c_1 B) - r\bar{S}}{r(\alpha + A) + \beta K}.$$

The equilibrium point $\bar{P}(\bar{S}, \bar{I})$ is positive if $\beta > \frac{r(\alpha+C)(d+c_2D)}{K(r-c_1B)}$.
The Jacobian matrix of system (6.1) at $\bar{P}(\bar{S}, \bar{I})$ is

$$J = \begin{pmatrix} -\frac{r\bar{S}}{K} & -\frac{(r(\alpha+A)+\beta K)\bar{S}}{K(\alpha+A)} \\ \frac{\beta \bar{I}}{\alpha+C} & 0 \end{pmatrix}.$$

Clearly, $\operatorname{Tr} J = -\frac{r\bar{S}}{K} < 0$ and $\operatorname{Det} J = \frac{\beta(r(\alpha+A)+\beta K)\bar{S}\bar{I}}{K(\alpha+A)(\alpha+C)} > 0$.

Therefore, the positive equilibrium point $\bar{P}(\bar{S}, \bar{I})$ is locally asymptotically stable.

Next, we show its global asymptotic stability.

We consider the following function:

$$V = \left(S - \bar{S} - \bar{S} \ln \frac{S}{\bar{S}} \right) + E\left(I - \bar{I} - \bar{I} \ln \frac{I}{\bar{I}} \right).$$

From the construction of V, it is easily seen that V is positive definite in the region $\Sigma = \{(S, I) | S \geq 0, Y \geq 0\}$ and $V(\bar{S}, \bar{I}) = 0$.

By simple computation, we obtain that

$$\frac{dV}{dt} = \frac{S - \bar{S}}{S} \dot{S}(t) + \frac{I - \bar{I}}{I} \dot{S}(I)$$

$$= -\frac{r(S - \bar{S})^2}{K} + \left[\frac{\beta E}{\alpha + C} - \frac{r}{K} - \frac{\beta}{\alpha + A} \right] (S - \bar{S})(I - \bar{I}).$$

Let $E = \frac{(\alpha + C)(r(\alpha + A) + \beta K)}{\beta K(\alpha + A)} > 0$, then $\frac{dV}{dt} \leq 0$ in the region $\Sigma = \{(S, I) | S \geq 0, Y \geq 0\}$.

Hence the theorem is proved. $\qquad\qquad\square$

Next, we consider the permanence of system (2.4) with initial conditions (2.3).

Theorem 6.3 *If $\beta > \frac{r(\alpha + (1-\gamma)K)(d_1 + c_2\varphi'(0)M_Y)}{(1-\gamma)K(r - c_1\varphi'(0)M_Y)} > 0$ and $0 < M_Y < \frac{r}{c_1\varphi'(0)}$, then system (2.4) with initial conditions (2.3) is permanent. Otherwise, it is non-permanent.*

Proof From the first and second equations of system (2.4), we obtain that

$$\dot{S}(t) \leq rS\left(1 - \frac{S + I}{(1-\gamma)K} \right) - \frac{\beta SI}{(1-\gamma)\alpha + K}, \qquad \dot{I}(t) \leq \frac{\beta SI}{(1-\gamma)\alpha} - d_1 I.$$

By Lemma 6.2 and a standard comparison theorem, we have

$$\limsup_{t \to +\infty} S(t) \leq \hat{S} \doteq M_S, \qquad \limsup_{t \to +\infty} I(t) \leq \hat{I} \doteq M_I,$$

where $\hat{S} = \frac{\alpha d_1(1-\gamma)}{\beta} > 0, \hat{I} = \frac{((1-\gamma)\alpha + K)((1-\gamma)rK - \hat{S})}{(1-\gamma)(r\alpha + \beta K) + K} > 0$ if $\beta > \frac{\alpha d_1}{rK}$.

Thus, for any given $\varepsilon > 0$, there exists $T_1 > 0$ such that, for any $t > T_1 > 0$, we get

$$S(t) \leq M_S + \varepsilon, \qquad I(t) \leq M_I + \varepsilon.$$

From the third equation, we obtain that

$$\dot{Y}(t) \leq \left[ec_1\varphi\big((M_S + \varepsilon)\big) + ec_2\varphi\big((M_I + \varepsilon)\big) - d_2 \right] Y.$$

It is easy to show that there is $M_Y > 0$ such that

$$\limsup_{t \to +\infty} Y(t) \leq M_Y.$$

Hence, for any given $\varepsilon > 0$, there exists $T_2 > T_1 > 0$ such that, for any $t > T_2 > 0$, we have

$$Y(t) \leq M_Y + \varepsilon.$$

Again, from the first and second equations, we get

$$\dot{S}(t) \geq rS\left(1 - \frac{S+I}{K}\right) - \frac{\beta SI}{\alpha} - c_1 M(M_Y + \varepsilon)S,$$

$$\dot{I}(t) \geq \frac{\beta SI}{\alpha + K} - d_1 I - c_2 M(M_Y + \varepsilon)I,$$

where M is the maximum value of the function $F(X)$, where

$$F(X) = \begin{cases} \varphi(X)/X & (0 < X \leq K), \\ \varphi'(0) & (X = 0). \end{cases} \tag{6.2}$$

Notice that the function $F(X)$ has the maximum and minimum values since it is continuous on the closed interval $[0, K]$.

By Lemma 6.2 and a standard comparison theorem, we have

$$\liminf_{t \to +\infty} S(t) \geq S^* \doteq m_S, \qquad \liminf_{t \to +\infty} I(t) \geq I^* \doteq m_I,$$

where

$$S^* = \frac{(\alpha + (1-\gamma)K)(d_1 + c_2\varphi'(0)M_Y)}{\beta} > 0, \qquad I^* = \frac{(1-\gamma)K(r - c_1\varphi'(0)M_Y) - rS^*}{\alpha r + (1-\gamma)\beta K} > 0,$$

when $\beta > \frac{r(\alpha+(1-\gamma)K)(d_1+c_2\varphi'(0)M_Y)}{(1-\gamma)K(r-c_1\varphi'(0)M_Y)} > 0$ and $0 < M_Y < \frac{r}{c_1\varphi'(0)}$.

Thus, for any given $\varepsilon > 0$, there exists $T_3 > T_2 > T_1 > 0$ such that, for any $t > T_3 > 0$, we obtain

$$S(t) \geq m_S - \varepsilon, \qquad I(t) \geq m_I - \varepsilon.$$

Again, from the third equation, we get

$$\dot{Y}(t) \geq \left[ec_1\varphi\big((1-\gamma)(m_S - \varepsilon)\big) + ec_2\varphi\big((1-\gamma)(m_I - \varepsilon)\big) - d_2 \right]Y.$$

It is easy to show that there is $m_Y > 0$ such that

$$\liminf_{t \to +\infty} Y(t) \geq m_Y > 0.$$

According to Definition 6.1, system (2.4) with initial conditions (2.3) is permanent under some strict conditions. Hence the theorem is proved. □

7 Examples

Example 1 If $\varphi(X) = \frac{X}{a+X}$, then system (2.2) becomes the following system:

$$\dot{S}(t) = rS\left(1 - \frac{S+I}{K}\right) - \frac{\beta SI}{\alpha+I} - \frac{c_1(1-\gamma)XY}{a+(1-\gamma)X},$$

$$\dot{I}(t) = \frac{\beta SI}{\alpha+I} - d_1 I - \frac{c_2(1-\gamma)IY}{a+(1-\gamma)I}, \tag{7.1}$$

$$\dot{Y}(t) = \frac{ec_1(1-\gamma)XY}{a+(1-\gamma)X} + \frac{ec_2(1-\gamma)IY}{a+(1-\gamma)I} - d_2 Y.$$

It is easy to obtain the disease-free equilibrium point $E_1(S_1, 0, Y_1)$ of system (7.1), where

$$S_1 = \frac{ad_2}{(1-\gamma)(c_1 e - d_2)}, \qquad Y_1 = \frac{erS_1}{d_2}\left(1 - \frac{S_1}{K}\right).$$

By simple computation, the equilibrium point $E_1(S_1, 0, Y_1)$ has its ecological meanings if $\gamma < 1 - \frac{ad_2}{K(c_1 e - d_2)}$.

According to the theorems in Sect. 4, we can obtain the following propositions:

(1) Assuming that $\frac{\alpha d_1}{K} < \beta < \frac{\alpha d_1}{K} + \frac{\alpha c_1 e(c_2 er + d_1(c_1 e - d_2))}{d_2 K(c_1 e - d_2)}$, then

   (a) If $0 < \gamma < 1 - \frac{a(c_1 e + d_2)}{K(c_1 e - d_2)}$, then the disease-free equilibrium point of system (7.1) is unstable;

   (b) If $1 - \frac{a(c_1 e + d_2)}{K(c_1 e - d_2)} < \gamma < 1 - \frac{ad_2(\alpha c_2 er + \beta K(c_1 e - d_2))}{\alpha K(c_1 e - d_2)(c_2 er + d_1(c_1 e - d_2))}$, then the disease-free equilibrium point of system (7.1) is locally asymptotically stable;

   (c) If $1 - \frac{ad_2(\alpha c_2 er + \beta K(c_1 e - d_2))}{\alpha K(c_1 e - d_2)(c_2 er + d_1(c_1 e - d_2))} < \gamma < 1 - \frac{ad_2}{K(c_1 e - d_2)}$, then the disease-free equilibrium point of system (7.1) is unstable.

(2) Assuming that $\beta < \frac{\alpha d_1}{K}$, then the disease-free equilibrium point of system (7.1) is always locally asymptotically stable.

Example 2 If $\varphi(X) = \frac{X^2}{a + X^2}$, then system (2.2) becomes the following system:

$$\dot{S}(t) = rS\left(1 - \frac{S + I}{K}\right) - \frac{\beta SI}{\alpha + I} - \frac{c_1(1-\gamma)^2 X^2 Y}{a + (1-\gamma)^2 X^2},$$

$$\dot{I}(t) = \frac{\beta SI}{\alpha + I} - d_1 I - \frac{c_2(1-\gamma)^2 I^2 Y}{a + (1-\gamma)^2 I^2}, \tag{7.2}$$

$$\dot{Y}(t) = \frac{ec_1(1-\gamma)^2 X^2 Y}{a + (1-\gamma)^2 X^2} + \frac{ec_2(1-\gamma)^2 I^2 Y}{a + (1-\gamma)^2 I^2} - d_2 Y.$$

It is easy to obtain the disease-free equilibrium point $E_2(S_2, 0, Y_2)$ of system (7.2), where

$$S_2 = \frac{1}{1-\gamma}\sqrt{\frac{ad_2}{c_1 e - d_2}}, \qquad Y_2 = \frac{erS_2}{d_2}\left(1 - \frac{S_2}{K}\right).$$

Clearly, if $\gamma < 1 - \frac{1}{K}\sqrt{\frac{ad_2}{c_1 e - d_2}}$, then the equilibrium point $E_2(S_1, 0, Y_1)$ has its ecological meanings.

According to the theorems in Sect. 4, we can obtain the following propositions:

(1) Assuming that $\frac{\alpha d_1}{K} < \beta < \frac{2\alpha d_1 d_2}{K(2d_2 - c_1 e)}$, then we have

   (a) If $0 < \gamma < 1 - \frac{2d_2}{K(2d_2 - c_1 e)}\sqrt{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.2) is unstable;

   (b) If $1 - \frac{2d_2}{K(2d_2 - c_1 e)}\sqrt{\frac{ad_2}{c_1 e - d_2}} < \gamma < 1 - \frac{\beta}{\alpha d_1}\sqrt{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.2) is locally asymptotically stable;

   (c) If $1 - \frac{\beta}{\alpha d_1}\sqrt{\frac{ad_2}{c_1 e - d_2}} < \gamma < 1 - \frac{1}{K}\sqrt{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.2) is unstable.

(2) Assuming that $\beta < \frac{\alpha d_1}{K}$, then we have

   (a) If $0 < \gamma < 1 - \frac{2d_2}{K(2d_2 - c_1 e)}\sqrt{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.2) is unstable;

   (b) If $1 - \frac{2d_2}{K(2d_2 - c_1 e)}\sqrt{\frac{ad_2}{c_1 e - d_2}} < \gamma < 1 - \frac{1}{K}\sqrt{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.2) is locally asymptotically stable.

Example 3 If $\varphi(X) = \frac{X^p}{a+X^p}$, then system (2.2) becomes the following system:

$$
\dot{S}(t) = rS\left(1 - \frac{S+I}{K}\right) - \frac{\beta SI}{\alpha+I} - \frac{c_1(1-\gamma)^p X^p Y}{a+(1-\gamma)^p X^p},
$$

$$
\dot{I}(t) = \frac{\beta SI}{\alpha+I} - d_1 I - \frac{c_2(1-\gamma)^p I^p Y}{a+(1-\gamma)^p I^p}, \tag{7.3}
$$

$$
\dot{Y}(t) = \frac{ec_1(1-\gamma)^p X^p Y}{a+(1-\gamma)^p X^p} + \frac{ec_2(1-\gamma)^p I^p Y}{a+(1-\gamma)^p I^p} - d_2 Y.
$$

It is easy to obtain the disease-free equilibrium point $E_3(S_3, 0, Y_3)$ of system (7.3), where

$$
S_3 = \frac{1}{1-\gamma}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}, \qquad Y_3 = \frac{erS_3}{d_2}\left(1 - \frac{S_3}{K}\right).
$$

Clearly, if $\gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}$, then the equilibrium point $E_2(S_1, 0, Y_1)$ has its ecological meanings.

According to the theorems in Sect. 4, we can obtain the following propositions:

(I) Supposing that $\frac{\alpha d_1}{K} < \beta < \frac{\alpha d_1(2c_1 e - p(c_1 e - d_2))}{K(c_1 e - p(c_1 e - d_2))}$, then we obtain that

    (1) Assuming that $p < \frac{c_1 e}{c_1 e - d_2}$, then

        (a) If $0 < \gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}[\frac{2c_1 e - p(c_1 e - d_2)}{c_1 e - p(c_1 e - d_2)}]$, then the disease-free equilibrium point of system (7.3) is unstable;

        (b) If $1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}[\frac{2c_1 e - p(c_1 e - d_2)}{c_1 e - p(c_1 e - d_2)}] < \gamma < 1 - \frac{\beta}{\alpha d_1}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.3) is locally asymptotically stable;

        (c) If $1 - \frac{\beta}{\alpha d_1}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}} < \gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.3) is unstable.

    (2) Assuming that $p > \frac{2c_1 e}{c_1 e - d_2}$, then

        (a) If $0 < \gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}[\frac{2c_1 e - p(c_1 e - d_2)}{c_1 e - p(c_1 e - d_2)}]$, then the disease-free equilibrium point of system (7.3) is locally asymptotically stable;

        (b) If $1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}[\frac{2c_1 e - p(c_1 e - d_2)}{c_1 e - p(c_1 e - d_2)}] < \gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.3) is unstable.

    (3) Assuming that $\frac{c_1 e}{c_1 e - d_2} < p < \frac{2c_1 e}{c_1 e - d_2}$, then

        (a) If $0 < \gamma < 1 - \frac{\beta}{\alpha d_1}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.3) is locally asymptotically stable;

        (b) If $1 - \frac{\beta}{\alpha d_1}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}} < \gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (7.3) is unstable.

(II) Supposing that $\beta < \frac{\alpha d_1}{K}$, then we obtain

    (1) Assuming that $p < \frac{c_1 e}{c_1 e - d_2}$, then

        (a) If $0 < \gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}[\frac{2c_1 e - p(c_1 e - d_2)}{c_1 e - p(c_1 e - d_2)}]$, then the disease-free equilibrium point of system (20) is unstable;

        (b) If $1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}[\frac{2c_1 e - p(c_1 e - d_2)}{c_1 e - p(c_1 e - d_2)}] < \gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}$, then the disease-free equilibrium point of system (20) is locally asymptotically stable.

    (2) Assuming that $p > \frac{2c_1 e}{c_1 e - d_2}$, then

        (a) If $0 < \gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1 e - d_2}}[\frac{2c_1 e - p(c_1 e - d_2)}{c_1 e - p(c_1 e - d_2)}]$, then the disease-free equilibrium point of system (20) is locally asymptotically stable;

    (b) If $1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1e-d_2}}\left[\frac{2c_1e-p(c_1e-d_2)}{c_1e-p(c_1e-d_2)}\right] < \gamma < 1 - \frac{1}{K}\sqrt[p]{\frac{ad_2}{c_1e-d_2}}$, then the disease-free equilibrium point of system (20) is unstable.

(3) Assuming that $\frac{c_1e}{c_1e-d_2} < p < \frac{2c_1e}{c_1e-d_2}$, then the disease-free equilibrium point of system (20) is always locally asymptotically stable.

8 Discussion

In this paper, a generalized system describing predator–prey interaction with prey refuges and disease in prey is proposed. Based on mathematical issues, the dynamical properties (stability, Hopf bifurcation, and permanence) are investigated, and the sufficient conditions which guarantee these properties are obtained (see Sects. 4, 5, 6). However, based on ecological and epidemiological issues, our analyses reveal that the effect of prey refuge, the force of infection, and the converting efficiency of predators play an important role in the dynamical properties of the proposed system. The ecological and epidemiological meanings will be discussed according to the following aspects:

(1) If the infectious rate of prey is smaller than the threshold $\underline{B}$, then the effect of prey refuge has a stabilizing impact when the converting ratio is relatively small. The contrary effect happens under the larger converting ratio for predators. However, the effect of prey refuge has no influence under the middle converting ratio and the disease will vanish. These results show that the infectious disease can be prevented by controlling the degree of the effect of prey refuge under the certain converting ratio.

(2) If the infectious rate of prey is larger than the threshold $\overline{B}$, the dynamical consequences of the considered system will be relatively simple. In this case, in order to control the infectious disease which will break out, the effect of prey refuge must be relatively small.

(3) If the infectious rate is smaller than $\underline{B}$ and larger than $\overline{B}$, the stabilizing effect and destabilizing effect will occur under the small converting ratio, and the destabilizing effect happens while the converting ratio is larger than the threshold $\overline{B}$. Therefore, for the small converting ratio, the middle level of prey refuge can be applied to control the infectious disease. But, for the larger converting ratio, the smaller level of prey refuge is required to control the disease.

According to the above discussions, we find that the effect of prey refuge has a destabilizing impact on the considered eco-epidemiological system in most cases. The reasons for this phenomenon may be the disease in prey population, the relatively large converting efficiency of predators, and the complex functional response to prey population. However, according to the published research works, the stabilizing effect is often observed in a simple predator–prey system with no disease in prey and/or predators. The stabilizing effect of prey refuge is observed when the infectious rate of prey and the converting ratio of predators are relatively small. Furthermore, the most complex dynamics is observed when the converting ratio of predators is relatively small and the infectious rate of prey is moderate. At this moment, three kinds of equilibria can be reached: the unstable point with stable orbits, the stable coexistence of prey and predators, and then the unstable point again. That is to say, the increase of the infectious rate can lead to the lost of stability. Therefore, the stabilizing effect and/or destabilizing effects mainly are determined by the essential properties of predator, such as the converting coefficient, the type of functional response of predators to prey population. Hence, it is rational to guess that the predation

strategy and the evolutionary strategy of predator population are the main factors which induce the partially hiding behavior (prey refuge) of prey population.

On the other hand, our results show that the effect of prey refuge can be applied to control the spread of the infectious disease. The level of prey refuge needed to control the disease spread is mainly determined by the infectious rate and the converting ratio of predators.

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