Xing and Wen *Advances in Difference Equations* (2018) 2018:238
https://doi.org/10.1186/s13662-018-1689-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

A conservative difference scheme for the Riesz space-fractional sine-Gordon equation

Zhiyong Xing[1,2*] and Liping Wen[1]

*Correspondence:
201690110064@smail.xtu.edu.cn
[1]School of Mathematics and Computational Science, Xiangtan University, Xiangtan, P.R. China
[2]Department of Mathematics, Shaoyang University, Shaoyang, P.R. China

Abstract

In this paper, we study a conservative difference scheme for the sine-Gordon equation (SGE) with the Riesz space fractional derivative. We rigorously establish the conservation property and solvability of the difference scheme. We discuss the stability and convergence of the difference scheme in the L_∞ norm. To reduce the computational complexity, we introduce a revised Newton method for implementing the difference scheme. Finally, we provide several numerical experiments to support the theoretical results.

Keywords: Space-fractional sine-Gordon equation; Riesz fractional derivative; Conservation law; Newton method; Convergence and stability

1 Introduction

The nonlinear SGE

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \sin u = 0, \qquad \mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n, \tag{1.1}$$

arises in many different areas, such as stability of fluid motions, differential geometry, Josephson junctions, models of particle physics [1], the propagation of fluxon [2], the motion of a rigid pendulum attached to a stretched wire [3], the phenomenon of supra-transmission in nonlinear media [4], and so on. Therefore the investigation for the SGE has attracted attention of some researchers, and many significant achievements have been made.

A remarkable property of (1.1) is the energy conservation:

$$\frac{1}{2}\int_{\mathbb{R}} \left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2 + 2P(u) \right] dx = \text{const},$$

where $P(u) = 1 - \cos u$; it was studied by many researchers [5–9]. The conserved quantity is good for the analysis of the nonlinear stability of the numerical schemes proposed, although it is difficult to apply them [10, 11].

Since the fractional calculus is frequently better than the integer calculus in the description of many physical laws, various classical partial differential equations have been extended to the corresponding fractional-order differential equations [12–19]. However, there are a few works on fractional-order SGEs. Ray [20] combined the modified decomposition method and Fourier transform to approximate the solution of a fractional SGE. In [21], a family of breather-like solutions for the fractional SGE is found numerically by using the approach that is called sometimes the rotating wave approximation. Macías-Díaz [4] employed an explicit finite difference scheme to simulate a space-fractional SGE. His result supported the fact that nonlinear supratransmission is present in the Riesz space-fractional model. He also pointed out that numerical simulations for fractional SGE require enormous amount of computer time.

In this paper, we consider the space-fractional SGE

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^\alpha u}{\partial |x|^\alpha} + \sin u = 0, \quad -\infty < x < +\infty, 0 < t \leq T, \tag{1.2}$$

with the boundary and initial conditions

$$u(x,0) = \varphi(x), \qquad \frac{\partial u(x,0)}{\partial t} = \psi(x), \quad -\infty < x < +\infty, \tag{1.3}$$

$$\lim_{|x|\to\infty} u(x,t) = 0, \quad 0 < t < T, \tag{1.4}$$

where $1 < \alpha \leq 2$.

The Riesz fractional derivative of order α is defined by [22]

$$\frac{\partial^\alpha u}{\partial |x|^\alpha}(x,t) = -(-\Delta)^{\frac{\alpha}{2}} u(x,t) = -\frac{1}{2\cos(\pi\alpha/2)}\bigl({}_{-\infty}D_x^\alpha u(x,t) + {}_x D_{+\infty}^\alpha u(x,t)\bigr),$$

where ${}_{-\infty}D_x^\alpha u(x,t)$ and ${}_x D_{+\infty}^\alpha u(x,t)$ are the left- and right-side Riemann–Liouville fractional derivatives, respectively.

Multiplying (1.2) by $\frac{\partial u}{\partial t}$ and then integrating with respect to x, we obtain

$$\frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\biggl[\biggl(\frac{\partial u}{\partial t}\biggr)^2 + \bigl((-\Delta)^{\frac{\alpha}{4}} u\bigr)^2 + 2P(u)\biggr]\,dx = 0,$$

that is, the conservation of energy

$$\frac{1}{2}\biggl\|\frac{\partial u}{\partial t}\biggr\|_{L^2}^2 + \frac{1}{2}\bigl\|(-\Delta)^{\frac{\alpha}{4}} u\bigr\|_{L^2}^2 + \int_{\mathbb{R}} P(u)\,dx = \text{const}, \tag{1.5}$$

where $P(u)$ is consistent with the aforementioned.

To the best of our knowledge, there are very few works developing conservative numerical methods for fractional SGEs. The main objective of this paper is to propose a conservative numerical method for space-fractional SGEs and try to reduce the computational complexity.

This paper is arranged as follows. In the next section, we propose a conservative difference scheme for space-fractional SGEs. Subsequently, we prove that the difference scheme preserves the energy conservation law. The boundedness, solvability, and convergence of the difference scheme are rigorously established. In Sect. 4, we introduce a revised Newton method for implementation of the difference scheme. In Sect. 5, we present some numerical results to demonstrate the effectiveness of the difference scheme. Finally, we give a simple conclusion.

2 A conservative difference scheme for fractional SGEs

2.1 Notation

To develop a finite difference scheme for problem (1.2)–(1.4), we assume that its solution is negligibly small outside of the interval $\Omega = (x_R, x_L)$, that is, $u|_{x\in\mathbb{R}\setminus\Omega} = 0$. we choose the time step $\tau = \frac{T}{N}$ and mesh size $h = \frac{x_R - x_L}{M}$ with two positive integers N and M. Denote

$$\Omega_h = \{x_i | x_i = x_L + ih, 1 \le i \le M-1\}, \qquad \Omega_\tau = \{t_n | t_n = n\tau, 0 \le n \le N\}$$

and

$$u_i^n \approx u(x_i, t_n), \qquad U_i^n = u(x_i, t_n).$$

Let $v_h = \{w | w = (w_1, w_2, \dots, w_{M-1})^T\}$ be the space of grid functions. For a given grid function $w = \{w_i^n | (x_i, t_n) \in \Omega_h \times \Omega_\tau\}$, we define the finite difference operators

$$\begin{aligned} &(w_i^n)_t = \frac{w_i^{n+1} - w_i^n}{\tau}, \qquad (w_i^n)_{\bar{t}} = \frac{w_i^n - w_i^{n-1}}{\tau}, \\ &(w_i^n)_{\hat{t}} = \frac{w_i^{n+1} - w_i^{n-1}}{2\tau}, \qquad \bar{w}_i^n = \frac{w_i^{n+1} + w_i^{n-1}}{2}. \end{aligned}$$

For any two grid functions u^n and v^n, we define

$$(u^n, v^n) = h\sum_{i=1}^{M-1} u_i^n v_i^n, \qquad \|u^n\|^2 = (u^n, u^n), \qquad \|u^n\|_{l_h^\infty} = \sup_{i\in\mathbb{Z}} |u_i|. \tag{2.1}$$

Let $0 \le \sigma \le 1$ be given. For any $u \in l_h^2$, the fractional Sobolev norm $\|u\|_{H^\sigma}$ and seminorm $|u|_{H^\sigma}$ can be defined as

$$\|u\|_{H^\sigma}^2 = h\int_{-\pi}^{\pi} \left(1 + h^{-2\sigma}|\mathbf{k}|^{2\sigma}\right)\left|\widehat{u}(\mathbf{k})\right|^2 d\mathbf{k}, \qquad |u|_{H^\sigma}^2 = h\int_{-\pi}^{\pi} h^{-2\sigma}|\mathbf{k}|^{2\sigma}\left|\widehat{u}(\mathbf{k})\right|^2 d\mathbf{k}.$$

2.2 A conservative implicit difference scheme

Lemma 2.1 ([23]) *Suppose that* $u \in \mathcal{L}^{2+\alpha}(\mathcal{R})$. *Then*

$$-h^{-\alpha}\triangle_h^\alpha f(x) = -(-\triangle)^{\frac{\alpha}{2}} f(x) + O(h^2), \tag{2.2}$$

where

$$\triangle_h^\alpha f(x) = \sum_{k=-\infty}^{\infty} g_k^{(\alpha)} f(x - kh), \qquad g_k^{(\alpha)} = \frac{(-1)^k\Gamma(\alpha+1)}{\Gamma(\alpha/2 - k + 1)\Gamma(\alpha/2 + k + 1)}.$$

If we define

$$u^*(x,t) = \begin{cases} u(x,t) & \text{if } x \in [x_L, x_R], \\ 0 & \text{if } x \in (-\infty, x_L) \cup (x_R, \infty), \end{cases}$$

then, for $1 < \alpha \le 2$, we can get

$$-(-\triangle)^{\frac{\alpha}{2}} u_i^n = -h^{-\alpha} \sum_{l=1}^{M-1} g_{i-l}^{(\alpha)} u_l^n + O(h^2). \tag{2.3}$$

Denote

$$\delta_h^\alpha u_i^n = h^{-\alpha} \sum_{l=1}^{M-1} g_{i-l}^{(\alpha)} u_l^n.$$

Considering equation (1.2) at points (x_i, t_n), we derive

$$\frac{\partial^2 u}{\partial t^2}(x_i, t_n) - \frac{\partial^\alpha u}{\partial |x|^\alpha}(x_i, t_n) = -\sin u(x_i, t_n), \quad 1 \le i \le M-1, 1 \le n \le N-1. \tag{2.4}$$

Combining the Taylor expansion and equation (2.3), we get

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(x_i, t_n) = \left(U_i^n\right)_{t\bar{t}} + O\left(\tau^2\right), \\ &\frac{\partial^\alpha u}{\partial |x|^\alpha}(x_i, t_n) = \delta_h^\alpha \bar{U}_i^n + O\left(h^2 + \tau^2\right), \\ &-\sin u(x_i, t_n) = \zeta\left(\bar{U}_i^n\right) + O\left(\tau^2\right), \end{aligned} \tag{2.5}$$

where $\zeta(\bar{U}_i^n) = \frac{\cos(U_i^{n+1}) - \cos(U_i^{n-1})}{U_i^{n+1} - U_i^{n-1}}$.

Substituting (2.5) into (2.4), we get

$$\left(U_i^n\right)_{t\bar{t}} + \delta_h^\alpha \bar{U}_i^n = \zeta\left(\bar{U}_i^n\right) + r_i^n, \tag{2.6}$$

where $r_i^n = O(h^2 + \tau^2)$.

In addition, from conditions (1.3) and (1.4) we have

$$U_0^n = 0, \qquad U_M^n = 0, \quad 0 \le n \le N, \tag{2.7}$$

$$U_i^0 = \varphi(x_i), \qquad \frac{\partial U}{\partial t}(x_i, 0) = \psi(x_i), \quad 1 \le i \le M-1. \tag{2.8}$$

Omitting the small term r_i^n in (2.6) and using the numerical solution u_i^n to replace U_i^n, we obtain the following difference scheme for solving problem (1.2)–(1.4):

$$\left(u_i^n\right)_{t\bar{t}} + \delta_h^\alpha \bar{u}_i^n = \zeta\left(\bar{u}_i^n\right), \quad 1 \le i \le M-1, 1 \le n \le N-1, \tag{2.9}$$

$$u_0^n = 0, \qquad u_M^n = 0, \quad 0 \le n \le N, \tag{2.10}$$

$$u_i^0 = \varphi(x_i), \qquad u_i^1 = u_i^0 + \tau\psi(x_i) - \frac{\tau^2}{2}\left[\delta_h^\alpha u_i^0 + \sin\left(u_i^0\right)\right], \quad 1 \le i \le M-1. \tag{2.11}$$

3 Numerical analysis

3.1 Some useful lemmas

Lemma 3.1 ([13]) *If u, v are any two grid functions in v_h, then there exists a linear operator $\Lambda_h^\alpha = h^{-\frac{\alpha}{2}} G^{\frac{1}{2}}$ such that*

$$(\delta_h^\alpha u, v) = (h^{-\alpha} G u, v) = (\Lambda_h^\alpha u, \Lambda_h^\alpha v),$$

where

$$G = \begin{pmatrix} g_0^{(\alpha)} & g_{-1}^{(\alpha)} & \cdots g_{-M+2}^{(\alpha)} \\ g_1^{(\alpha)} & g_0^{(\alpha)} & \cdots g_{-M+3}^{(\alpha)} \\ \vdots & \vdots & \ddots \vdots \\ g_{M-2}^{(\alpha)} & g_{M-3}^{(\alpha)} & \cdots g_0^{(\alpha)} \end{pmatrix};$$

$G^{\frac{1}{2}}$ is the unique positive definite square root of G, that is, $(G^{\frac{1}{2}})^2 = G$. Moreover, we have

$$(\delta_h^\alpha u, u) = (\Lambda_h^\alpha u, \Lambda_h^\alpha u) = \|\Lambda_h^\alpha u\|^2, \tag{3.1}$$

where $\delta_h^\alpha u = (\delta_h^\alpha u_1, \delta_h^\alpha u_2, \ldots, \delta_h^\alpha u_{M-1})^T$.

Lemma 3.2 *If $u^n \in v_h$, then we have*

$$(u_{t\bar{t}}^n, 2u_{\hat{t}}^n) = (\|u_t^n\|^2)_{\bar{t}}, \tag{3.2}$$

$$(\delta_h^\alpha \bar{u}^n, 2u_{\hat{t}}^n) = \|\Lambda_h^\alpha u^n\|_{\hat{t}}^2. \tag{3.3}$$

Proof Equality (3.2) can be found in [8], so we omit it here. Equality (3.3) can be proved as follows:

$$\begin{aligned} (\delta_h^\alpha \bar{u}^n, 2u_{\hat{t}}^n) &= \frac{1}{2\tau}(\delta_h^\alpha u^{n+1} + \delta_h^\alpha u^{n-1}, u^{n+1} - u^{n-1}) \\ &= \frac{1}{2\tau}\left[(\delta_h^\alpha u^{n+1}, u^{n+1}) - (\delta_h^\alpha u^{n-1}, u^{n-1}) + (\delta_h^\alpha u^{n-1}, u^{n+1}) - (\delta_h^\alpha u^{n+1}, u^{n-1})\right] \\ &= \frac{1}{2\tau}\left[(\delta_h^\alpha u^{n+1}, u^{n+1}) - (\delta_h^\alpha u^{n-1}, u^{n-1})\right] \\ &= \|\Lambda_h^\alpha u^n\|_{\hat{t}}^2. \end{aligned}$$

The proof ends. □

Lemma 3.3 ([8]) *Let $\omega(k)$ and $\rho(k)$ be nonnegative mesh functions. If $C > 0$, $\rho(k)$ is non-decreasing, and*

$$\omega(k) \le \rho(k) + C\tau \sum_{l=0}^{k-1} \omega(l)$$

for all k, then

$$\omega(k) \le \rho(k) e^{Ck\tau}$$

for all k.

Lemma 3.4 (Discrete Sobolev inequality [24]) *For every* $\frac{1}{2} < \sigma \le 1$, *there exists a constant* $C = C(\sigma) > 0$ *independent of* $h > 0$ *such that*

$$\|u\|_{l_h^\infty} \le C\|u\|_{H^\sigma}$$

for all $u \in l_h^2$.

Lemma 3.5 (Uniform norm equivalence [25]) *For any grid function* $u = \{u_j\}$ *and every* $1 < \alpha \le 2$, *we have*

$$\left(\frac{2}{\pi}\right)^\alpha |u|^2_{H^{\alpha/2}} \le \left(h^{-\alpha}\Delta_h^\alpha u, u\right) \le |u|^2_{H^{\alpha/2}}.$$

Lemma 3.6 ([26], Lemma 1.4, Ch. 2) *Let* X *be a finite-dimensional Hilbert space with scalar product* $[\cdot,\cdot]$ *and norm* $[\cdot]$, *and let* $\mathcal{F}$ *be a continuous mapping from* X *into itself such that*

$$\left[\mathcal{F}(\xi), \xi\right] > 0 \quad \textit{for any } [\xi] = k > 0.$$

Then there exists $\xi \in X$ *with* $[\xi] \le k$ *such that*

$$\mathcal{F}(\xi) = 0.$$

3.2 Conservation

Theorem 3.1 *The scheme* (2.9)–(2.11) *is conservative in the sense that*

$$\varepsilon^n = \varepsilon^{n-1} = \cdots = \varepsilon^0,$$

where

$$\varepsilon^n = \frac{1}{2}\left\|u_t^n\right\|^2 + \frac{1}{4}\left\|\Lambda_h^\alpha u^{n+1}\right\|^2 + \frac{1}{4}\left\|\Lambda_h^\alpha u^n\right\|^2 + \frac{h}{2}\sum_{i=0}^{M-1}\left[\left(1-\cos u_i^{n+1}\right) + \left(1-\cos u_i^n\right)\right]$$

is the energy in the discrete sense.

Proof Making the discrete inner product of (2.9) with $2u_{\hat{t}}^n$, we get that

$$\left(\left(u^n\right)_{t\bar{t}}, 2u_{\hat{t}}^n\right) + \left(\delta_h^\alpha \bar{u}^n, 2u_{\hat{t}}^n\right) = \left(\zeta\left(\bar{u}^n\right), 2u_{\hat{t}}^n\right). \tag{3.4}$$

Directly computing, we have

$$\begin{aligned}\left(\zeta\left(\bar{u}^n\right), 2u_{\hat{t}}^n\right) &= \frac{h}{\tau}\sum_{i=0}^{M-1}\left(\cos u_i^{n+1} - \cos u_i^{n-1}\right) \\ &= -\frac{h}{\tau}\sum_{i=0}^{M-1}\Big\{\left[\left(1-\cos u_i^{n+1}\right) + \left(1-\cos u_i^n\right)\right] \\ &\quad - \left[\left(1-\cos u_i^n\right) + \left(1-\cos u_i^{n-1}\right)\right]\Big\}.\end{aligned} \tag{3.5}$$

Notice that other inner products in (3.4) can be calculated by Lemma 3.2. Substituting (3.2), (3.3), and (3.5) into (3.4), we obtain

$$\begin{aligned}
&\left(\|u_t^n\|^2\right)_{\bar{t}} + \frac{1}{2\tau}\left[\left(\|\Lambda_h^\alpha u^{n+1}\|^2 + \|\Lambda_h^\alpha u^n\|^2\right) - \left(\|\Lambda_h^\alpha u^n\|^2 + \|\Lambda_h^\alpha u^{n-1}\|^2\right)\right] \\
&\quad + \frac{h}{\tau}\sum_{i=0}^{M-1}\left\{\left[\left(1-\cos u_i^{n+1}\right) + \left(1-\cos u_i^n\right)\right] - \left[\left(1-\cos u_i^n\right) + \left(1-\cos u_i^{n-1}\right)\right]\right\} \\
&= 0,
\end{aligned}$$

and hence

$$\begin{aligned}
\varepsilon^n &= \frac{1}{2}\|u_t^n\|^2 + \frac{1}{4}\left[\left(\|\Lambda_h^\alpha u^{n+1}\|^2 + \|\Lambda_h^\alpha u^n\|^2\right)\right] + \frac{h}{2}\sum_{i=0}^{M-1}\left[\left(1-\cos u_i^{n+1}\right) + \left(1-\cos u_i^n\right)\right] \\
&= \frac{1}{2}\|u_t^{n-1}\|^2 + \frac{1}{4}\left(\|\Lambda_h^\alpha u^n\|^2 + \|\Lambda_h^\alpha u^{n-1}\|^2 + \frac{h}{2}\sum_{i=0}^{M-1}\left[\left(1-\cos u_i^n\right) + \left(1-\cos u_i^{n-1}\right)\right]\right.,
\end{aligned}$$

that is,

$$\varepsilon^n = \varepsilon^{n-1} = \cdots = \varepsilon^0. \qquad \square$$

3.3 The boundedness, solvability, and convergence

Theorem 3.2 *Assume that* $\{u_i^n | 0 \le i \le M, 0 \le n \le N\}$ *is a solution of the difference scheme* (2.9)–(2.11). *Then*

$$\|u_t^n\| \le C, \qquad \|\Lambda_h^\alpha u^n\| \le C, \qquad \|u^n\| \le C, \qquad \|u^n\|_{l_h^\infty} \le C.$$

Here and later, C denotes a generic positive constant; in different places, it may represent different constants.

Proof By Theorem 3.1 we get

$$2\varepsilon^n = \|u_t^n\|^2 + \frac{1}{2}\left[\left(\|\Lambda_h^\alpha u^{n+1}\|^2 + \|\Lambda_h^\alpha u^n\|^2\right)\right] + h\sum_{i=0}^{M-1}\left[\left(1-\cos u_i^{n+1}\right) + \left(1-\cos u_i^n\right)\right] = C.$$

Notice that

$$\left(1-\cos u_i^{n+1}\right) + \left(1-\cos u_i^n\right) \ge 0.$$

Then

$$\|u_t^n\| \le C, \qquad \|\Lambda_h^\alpha u^n\| \le C.$$

Moreover,

$$\frac{\|u^{k+1}\| - \|u^k\|}{\tau} \le \|u_t^k\| \le C.$$

By summing this equality for $k = 0, 1, \dots, n-1$ ($n\tau \le T$), we get

$$\|u^n\| \le n\tau C + \|u^0\| \le CT + \|u^0\| \le C.$$

Combining Lemma 3.1 and Lemma 3.5, we derive

$$|u^n|^2_{H^{\alpha/2}} \le \left(\frac{\pi}{2}\right)^\alpha \left(h^{-\alpha}\Delta_h^\alpha u^n, u^n\right) = \left(\frac{\pi}{2}\right)^\alpha \left(h^{-\alpha}Gu^n, u^n\right) = \left(\frac{\pi}{2}\right)^\alpha \|\Lambda_h^\alpha u^n\|^2 \le C.$$

Hence

$$\|u^n\|^2_{H^{\alpha/2}} = \|u^n\|^2 + |u^n|^2_{H^{\alpha/2}} \le C.$$

According to Lemma (3.4),

$$\|u^n\|_{l_h^\infty} \le C\|u^n\|_{H^{\alpha/2}} \le C.$$

Thus we get the desired results. □

For the existence and uniqueness of a solution for the difference scheme (2.9)–(2.11), we have the following theorems.

Theorem 3.3 *The difference scheme* (2.9)–(2.11) *has at least one solution.*

Proof We carry out the proof by mathematical induction. Assume that $u^0, u^1, \dots, u^n$ are given solutions. We aim to prove that there exists yet one u^{n+1} satisfying the difference scheme. Let $X = v_h$ with the scalar product $(\cdot,\cdot)$ defined in (2.1). We define the mapping $\mathcal{F} : X \to X$ such that

$$\begin{aligned}\left(\mathcal{F}(\omega), v\right) = {}& \left(\frac{\omega}{\tau^2}, v\right) - \left(\frac{2}{\tau}u_t^{n-1}, v\right) + \left(\frac{1}{2}\delta_h^\alpha \omega, v\right) + \left(\delta_h^\alpha u^{n-1}, v\right) \\ & + \left(\frac{\cos u^{n-1} - \cos(\omega + u^{n-1})}{\omega}, v\right), \quad \forall \omega, v \in X,\end{aligned}$$

where

$$\begin{aligned}&\frac{\cos u^{n-1} - \cos(\omega + u^{n-1})}{\omega} = (\chi_1, \chi_2, \dots, \chi_{M-1})^T, \\ &\chi_i = \frac{\cos u_i^{n-1} - \cos(\omega_i + u_i^{n-1})}{\omega_i}, \quad 1 \le i \le M-1.\end{aligned}$$

Obviously, the mapping $\mathcal{F}$ is continuous. Moreover, we have

$$\begin{aligned}\left(\mathcal{F}(\omega), \omega\right) = {}& \frac{\|\omega\|^2}{\tau^2} - \frac{2}{\tau}\left(u_t^{n-1}, \omega\right) + \frac{1}{2}\left(\delta_h^\alpha \omega, \omega\right) + \left(\delta_h^\alpha u^{n-1}, \omega\right) \\ & + \left(\frac{\cos u^{n-1} - \cos(\omega + u^{n-1})}{\omega}, \omega\right).\end{aligned}$$

Applying Young's inequality and Theorem 3.2, we have

$$\begin{aligned}\frac{2}{\tau}\left(u_t^{n-1},\omega\right) &= \left(2\sqrt{2}u_t^{n-1},\frac{1}{\sqrt{2}\tau}\omega\right)\\ &\le 4\left\|u_t^{n-1}\right\|^2+\frac{1}{4}\frac{\|\omega\|^2}{\tau^2}\\ &\le 8\varepsilon^0+\frac{1}{4}\frac{\|\omega\|^2}{\tau^2}\end{aligned}$$

and

$$\begin{aligned}\left(\delta_h^\alpha u^{n-1},\omega\right) &\le \frac{\tau^2}{2}\left\|\delta_h^\alpha u^{n-1}\right\|^2+\frac{1}{2}\frac{\|\omega\|^2}{\tau^2}\\ &\le 4\tau^2h^{-\alpha}g_0^{(\alpha)}\varepsilon^0+\frac{1}{2}\frac{\|\omega\|^2}{\tau^2},\end{aligned}$$

where we used the fact that

$$\begin{aligned}\left\|\delta_h^\alpha u^{n-1}\right\|^2 &= \left(h^{-\alpha}Gu^{n-1},h^{-\alpha}Gu^{n-1}\right)\\ &\le h^{-\alpha}\lambda_{\max}(G)\left(h^{-\alpha}Gu^{n-1},u^{n-1}\right)\\ &\le 2h^{-\alpha}g_0^{(\alpha)}\left\|\Lambda_h^\alpha u^{n-1}\right\|^2\\ &\le 8h^{-\alpha}g_0^{(\alpha)}\varepsilon^0.\end{aligned}$$

Noticing that

$$\left(\delta_h^\alpha\omega,\omega\right)=\left(\Lambda_h^\alpha\omega,\Lambda_h^\alpha\omega\right)\ge 0$$

and

$$\left(\frac{\cos u^{n-1}-\cos(\omega+u^{n-1})}{\omega},\omega\right)=h\sum_{i=1}^{M-1}\left(\cos u_i^{n-1}-\cos\left(\omega_i+u_i^{n-1}\right)\right)\le 2L,$$

where $L=X_R-X_L$, we have

$$\left(\mathcal{F}(\omega),\omega\right)\ge\frac{1}{4}\frac{\|\omega\|^2}{\tau^2}-\left[\left(8+4\tau^2h^{-\alpha}g_0^{(\alpha)}\right)\varepsilon^0+2L\right].$$

Let

$$K=2\tau\sqrt{\left(8+4\tau^2h^{-\alpha}g_0^{(\alpha)}\right)\varepsilon^0+2L}.$$

Then, for any $\|\omega\|>K$, we have $(\mathcal{F}(\omega),\omega)>0$. By Lemma 3.6 there exists $\omega\in X$ such that $\mathcal{F}(\omega)=0$. Setting $\omega=u^{n+1}-u^{n-1}$, we get the desired result. □

Theorem 3.4 *The solution of the difference scheme* (2.9)–(2.11) *is unique*.

Proof Suppose that two sequences $u^0, u^1, \dots, u^n, v$ and $u^0, u^1, \dots, u^n, v'$ both satisfy the difference scheme (2.9)–(2.11). Let $v^* = v - v'$. Then we obtain that

$$\frac{1}{\tau^2} v_i^* + \delta_h^\alpha v_i^* - p_i = 0, \quad 1 \le i \le M-1, \tag{3.6}$$

where $\{p_i\}$ are given by (3.8).

Making the inner product of (3.6) with v^*, we obtain that

$$\left(\frac{1}{\tau^2} v^*, v^*\right) + \left(\delta_h^\alpha v^*, v^*\right) - \left(p, v^*\right) = 0. \tag{3.7}$$

Noticing that

$$\begin{aligned} p_i &= \frac{\cos v_i - \cos u_i^{n-1}}{v_i - u_i^{n-1}} - \frac{\cos v_i' - \cos u_i^{n-1}}{v_i' - u_i^{n-1}} \\ &= \int_0^1 \sin\left[\lambda v_i' + (1-\lambda) u_i^{n-1}\right] d\lambda - \int_0^1 \sin\left[\lambda v_i + (1-\lambda) u_i^{n-1}\right] d\lambda \\ &= -2 \int_0^1 \cos\left[\lambda \frac{v_i + v_i'}{2} + (1-\lambda) u_i^{n-1}\right] \sin\left(\lambda \frac{v_i^*}{2}\right) d\lambda, \end{aligned} \tag{3.8}$$

we have

$$\left|p_i v_i^*\right| \le \frac{1}{2}\left(v_i^*\right)^2,$$

and therefore

$$\left|\left(p, v^*\right)\right| \le \frac{1}{2}\left\|v^*\right\|^2.$$

Thus for the right-hand side of (3.7), we get

$$\left(\frac{1}{\tau^2} v^*, v^*\right) + \left(\delta_h^\alpha v^*, v^*\right) - \left(p, v^*\right) \ge \left(\frac{1}{\tau^2} - \frac{1}{2}\right)\left\|v^*\right\|^2. \tag{3.9}$$

Then supposing that $\tau < \sqrt{2}$ and combining (3.7) and (3.9), we derive

$$\left\|v^*\right\| = 0, \quad \text{i.e., } v = v'.$$

The proof ends. □

Denote

$$e_i^n = U_i^n - u_i^n, \quad 0 \le i \le M, 0 \le n \le N.$$

Theorem 3.5 *Assume that problem (1.2)–(1.4) has a smooth solution and $\{u_i^n | 0 \le i \le M, 0 \le n \le N\}$ is the solution of the finite difference scheme (2.9)–(2.11). If $\frac{\tau^2}{h^\alpha} \le S$ $(0 < S < +\infty)$, then there exists a positive constant C such that*

$$\left\|e^n\right\|_{l_h^\infty} \le C\left(h^2 + \tau^2\right).$$

Proof Subtracting (2.9)–(2.11) from (2.6)–(2.8), respectively, we get the following error equations:

$$\big(e_i^n\big)_{t\bar{t}} + \delta_h^\alpha \bar{e}_i^n = V\big(\bar{e}_i^n\big) + r_i^n, \quad 1 \le i \le M-1, 1 \le n \le N, \tag{3.10}$$

$$e_0^n = 0, \qquad e_M^n = 0, \quad 0 \le n \le N, \tag{3.11}$$

$$e_i^0 = 0, \quad 1 \le i \le M-1, \tag{3.12}$$

where

$$\begin{aligned} V\big(\bar{e}_i^n\big) &= \zeta\big(\bar{U}_i^n\big) - \zeta\big(\bar{u}_i^n\big) \\ &= -2\int_0^1 \cos\left[\lambda \frac{U_i^{n+1} + u_i^{n+1}}{2} + (1-\lambda)\frac{U_i^{n-1} + u_i^{n-1}}{2}\right] \\ &\quad \times \sin\left[\lambda \frac{U_i^{n+1} - u_i^{n+1}}{2} + (1-\lambda)\frac{U_i^{n-1} - u_i^{n-1}}{2}\right] d\lambda. \end{aligned} \tag{3.13}$$

Similarly to the proof of Theorem 3.1, making the discrete inner product of (3.10) with $2e_{\hat{t}}^n$, we get that

$$\begin{aligned} &\big(\|e_t^n\|^2\big)_{\bar{t}} + \frac{1}{2\tau}\big[\big(\|\Lambda_h^\alpha e^{n+1}\|^2 + \|\Lambda_h^\alpha e^n\|^2\big) - \big(\|\Lambda_h^\alpha e^n\|^2 + \|\Lambda_h^\alpha e^{n-1}\|^2\big)\big] \\ &\quad = \big(V\big(\bar{e}^n\big) + r^n, 2e_{\hat{t}}^n\big). \end{aligned} \tag{3.14}$$

By summing this equality for $n = 1, 2, \ldots, k$ ($k\tau \le T$), we derive

$$\begin{aligned} &\|e_t^k\|^2 + \frac{1}{2}\|\Lambda_h^\alpha e^{k+1}\|^2 + \frac{1}{2}\|\Lambda_h^\alpha e^k\|^2 \\ &\quad = \|e_t^0\|^2 + \frac{1}{2}\|\Lambda_h^\alpha e^1\|^2 + \frac{1}{2}\|\Lambda_h^\alpha e^0\|^2 + \tau\sum_{n=1}^k \big(V\big(\bar{e}^n\big) + r^n, 2e_{\hat{t}}^n\big). \end{aligned} \tag{3.15}$$

From (3.13) we get

$$\begin{aligned} \big|V\big(\bar{e}_i^n\big)\big| &\le \left|2\int_0^1 \sin\left[\lambda\frac{e_i^{n+1}}{2} + (1-\lambda)\frac{e_i^{n-1}}{2}\right] d\lambda\right| \\ &\le \int_0^1 \big|\lambda e_i^{n+1} + (1-\lambda)e_i^{n-1}\big|\, d\lambda \\ &\le \int_0^1 \big|\tau\lambda\big(e_i^n\big)_t + \lambda e_i^n + (1-\lambda)e_i^{n-1}\big|\, d\lambda \\ &= \frac{1}{2}\tau\big|\big(e_i^n\big)_t\big| + \frac{1}{2}\big|e_i^n\big| + \frac{1}{2}\big|e_i^{n-1}\big|, \end{aligned}$$

and hence

$$\big\|V\big(\bar{e}^n\big)\big\|^2 \le \frac{3}{4}\big(\tau^2\|e_t^n\|^2 + \|e^n\|^2 + \|e^{n-1}\|^2\big). \tag{3.16}$$

Noticing that $e_i^n = e_i^0 + \tau \sum_{j=0}^{n-1} (e_i^j)_t$ and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\|e^n\|^2 &\le 2\|e^0\|^2 + 2\tau^2 \left\| \sum_{j=0}^{n-1} e_t^j \right\|^2 \\ &\le 2\|e^0\|^2 + 2T\tau \sum_{j=0}^{n-1} \|e_t^j\|^2. \end{aligned} \tag{3.17}$$

According to (3.16)–(3.17) and noticing that $e_{\hat{t}}^n = \frac{1}{2}e_t^n + \frac{1}{2}e_t^{n-1}$, we get

$$\begin{aligned}\left|\tau \sum_{n=1}^{k} \bigl(V(\bar{e}^n), 2e_{\hat{t}}^n\bigr)\right| &\le \tau \sum_{n=1}^{k} \left(\|V(\bar{e}^n)\|^2 + \frac{1}{2}\|e_t^n\|^2 + \frac{1}{2}\|e_t^{n-1}\|^2 \right) \\ &\le 3T\|e^0\|^2 + \frac{3\tau^3 + 2\tau}{4}\|e_t^k\|^2 \\ &\quad + \left(\frac{3}{4}\tau^3 + 3T^2\tau + \tau \right) \sum_{n=0}^{k-1} \|e_t^n\|^2. \end{aligned} \tag{3.18}$$

On the other hand,

$$\begin{aligned}\left|\tau \sum_{n=1}^{k} \bigl(r^n, 2e_{\hat{t}}^n\bigr)\right| &\le \tau \sum_{n=1}^{k} \bigl(\|r^n\|^2 + \|e_{\hat{t}}^n\|^2\bigr) \\ &\le \tau \sum_{n=1}^{k} \|r^n\|^2 + \frac{1}{2}\tau \sum_{n=0}^{k} \|e_t^n\|^2 + \frac{1}{2}\tau \sum_{n=0}^{k-1} \|e_t^n\|^2. \end{aligned} \tag{3.19}$$

Substituting (3.18) and (3.19) into (3.15) and noticing that $e^0 = 0$, we get

$$\begin{aligned}&\left(1 - \left(\frac{3\tau^3}{4} + \tau\right)\right)\|e_t^k\|^2 + \frac{1}{2}\|\Lambda_h^\alpha e^{k+1}\|^2 + \frac{1}{2}\|\Lambda_h^\alpha e^k\|^2 \\ &\quad \le \|e_t^0\|^2 + \frac{1}{2}\|\Lambda_h^\alpha e^1\|^2 + \tau \sum_{n=1}^{k} \|r^n\|^2 \\ &\qquad + \left(\frac{3}{4}\tau^3 + 3T^2\tau + 2\tau\right) \sum_{n=0}^{k-1} \|e_t^n\|^2. \end{aligned} \tag{3.20}$$

In addition,

$$\begin{aligned}&\left(1 - \left(\frac{3\tau^3}{4} + \tau\right)\right)\|e_t^k\|^2 + \frac{1}{2}\|\Lambda_h^\alpha e^{k+1}\|^2 + \frac{1}{2}\|\Lambda_h^\alpha e^k\|^2 \\ &\quad \ge \left(1 - \left(\frac{3\tau^3}{4} + \tau\right)\right)\|e_t^k\|^2 + \frac{1}{2}\tau^2\|\Lambda_h^\alpha e_t^k\|^2 \\ &\quad \ge \left(1 - \left(\frac{3\tau^3}{4} + \tau\right) + \frac{1}{2}\tau^2 h^{-\alpha}\lambda_{\min}(G)\right)\|e_t^k\|^2. \end{aligned}$$

Supposing that $\sigma = 1 - (\frac{3\tau^3}{4} + \tau) + \frac{1}{2}\tau^2 h^{-\alpha}\lambda_{\min}(G) > 0$ (it is easily to verify that when $\tau \leq 0.72$, the condition is satisfied), then inequality (3.20) can be rewritten as

$$\begin{aligned} \|e_t^k\|^2 + \frac{1}{2\sigma}\|\Lambda_h^\alpha e^k\|^2 \leq{}& \frac{1}{\sigma}\|e_t^0\|^2 + \frac{1}{2\sigma}\|\Lambda_h^\alpha e^1\|^2 + \frac{\tau}{\sigma}\sum_{n=1}^{k}\|r^n\|^2 \\ &+ \frac{3\tau^2 + 12T^2 + 8}{4\sigma}\tau\sum_{n=0}^{k-1}\left(\|e_t^n\|^2 + \frac{1}{2\sigma}\|\Lambda_h^\alpha e^n\|^2\right). \end{aligned} \tag{3.21}$$

Let

$$\begin{aligned} \rho(k) &= \frac{1}{\sigma}\|e_t^0\|^2 + \frac{1}{2\sigma}\|\Lambda_h^\alpha e^1\|^2 + \frac{\tau}{\sigma}\sum_{n=1}^{k}\|r^n\|^2, \\ H(k) &= \|e_t^k\|^2 + \frac{1}{2\sigma}\|\Lambda_h^\alpha e^k\|^2. \end{aligned} \tag{3.22}$$

Substituting (3.22) into (3.21), we derive

$$H(k) \leq \rho(k) + \frac{3\tau^2 + 12T^2 + 8}{4\sigma}\tau\sum_{n=0}^{k-1}H(n).$$

Applying Lemma 3.3 to $\rho(k)$ and $H(k)$, we arrive at

$$H(k) \leq \rho(k)e^{\frac{3\tau^2+12T^2+8}{4\sigma}k\tau} \leq \rho(k)e^{\frac{(3\tau^2+12T^2+8)T}{4\sigma}}. \tag{3.23}$$

Noticing that if τ and h are sufficiently small, then we have

$$e_i^0 = 0, \qquad r_i^n = O(h^2 + \tau^2), \quad 1 \leq i \leq M-1, 1 \leq n \leq N-1.$$

Applying the Taylor expansion, we get

$$u_i^1 = u_i^0 + \tau\psi(x_i) + \frac{\tau^2}{2}\left[-(-\Delta)^{\frac{\alpha}{2}}u_i^0 - \sin(u_i^0)\right] + O(\tau^3), \quad 1 \leq i \leq M-1.$$

According to Lemma 2.1, we obtain

$$-(-\Delta)^{\frac{\alpha}{2}}u_i^0 = -\delta_h^\alpha u_i^0 + O(h^2).$$

Therefore, we have the following equality:

$$e_i^1 = O(\tau^3 + h^2\tau^2), \quad 1 \leq i \leq M-1. \tag{3.24}$$

Due to $e_i^0 = 0$, $\frac{\tau^2}{h^\alpha} < S$, and (3.24), we derive

$$\begin{aligned} \|e_t^0\|^2 &\leq L\max_{1\leq i\leq M-1}\left(\frac{e_i^1}{\tau}\right)^2 \leq C(\tau^2 + h^2\tau)^2 \leq C(\tau^2 + h^2)^2 = O(\tau^2 + h^2)^2, \\ \|\Lambda_h^\alpha e^1\|^2 &\leq h^{-\alpha}\lambda_{\max}(G)\|e^1\|^2 \end{aligned} \tag{3.25}$$

$$
\begin{aligned}
&\le 2h^{-\alpha} g_0^{(\alpha)} L \max_{1\le i\le M-1} \left(e_i^1\right)^2 \\
&\le C\frac{\tau^2}{h^\alpha}\left(\tau^2 + h^2\tau\right)^2 \\
&\le C \cdot S\left(\tau^2 + h^2\right)^2 \\
&= O\left(\tau^2 + h^2\right)^2. \qquad (3.26)
\end{aligned}
$$

On the other hand,

$$
\tau \sum_{n=1}^{k} \left\|r^n\right\|^2 \le T \max_{1\le n\le k} \left\|r^n\right\|^2 \le T\cdot L \max_{1\le n\le k, 1\le i\le M-1} \left(r_i^n\right)^2 = O\left(\tau^2 + h^2\right)^2. \qquad (3.27)
$$

Combining (3.25), (3.26), and (3.27), we derive

$$
\rho(k) = O\left(h^2 + \tau^2\right)^2. \qquad (3.28)
$$

Furthermore, from (3.23) and (3.28) we immediately obtain the following results:

$$
\left\|e_t^k\right\| \le C\left(h^2 + \tau^2\right), \qquad \left\|\Lambda_h^\alpha e^k\right\| \le C\left(h^2 + \tau^2\right).
$$

Hence

$$
\frac{\|e^{k+1}\| - \|e^k\|}{\tau} \le \left\|e_t^k\right\| \le C\left(h^2 + \tau^2\right).
$$

By summing this equality for $k = 0, 1, \ldots, n-1$ ($n\tau \le T$), we get

$$
\left\|e^n\right\| \le n\tau C\left(h^2 + \tau^2\right) + \left\|e^0\right\| = O\left(h^2 + \tau^2\right).
$$

It follows from Lemma 3.1 and Lemma 3.5 that

$$
\left|e^n\right|_{H^{\alpha/2}}^2 \le \left(\frac{\pi}{2}\right)^\alpha \left\|\Lambda_h^\alpha e^n\right\|^2 = O\left(h^2 + \tau^2\right)^2,
$$

and hence

$$
\left\|e^n\right\|_{H^{\alpha/2}}^2 = \left\|e^n\right\|^2 + \left|e^n\right|_{H^{\alpha/2}}^2 = O\left(h^2 + \tau^2\right)^2.
$$

According to Lemma 3.4,

$$
\left\|e^n\right\|_{l_h^\infty} \le C\left\|e^n\right\|_{H^{\alpha/2}} = O\left(h^2 + \tau^2\right).
$$

Thus we get the desired results. □

4 The implementation issue and feasibility analysis of the numerical scheme

4.1 A revised Newton method for the proposed difference scheme

In this subsection, we mainly introduce how to implement the difference scheme (2.9)–(2.11). First, we rewrite the difference scheme as follows:

$$
\left(\frac{1}{\tau^2} I + \frac{1}{2h^\alpha} G\right) u^{n+1} - Q\left(u^{n+1}, u^{n-1}\right) + g = 0 \quad (n = 1, 2, \ldots, N-1), \qquad (4.1)
$$

where

$$Q(u^{n+1},u^{n-1}) = \left(\zeta(\bar{u}_1^n),\zeta(\bar{u}_2^n),\ldots,\zeta(\bar{u}_{M-1}^n)\right)^T,$$
$$u^n = \left(u_1^n,u_2^n,\ldots,u_{M-1}^n\right)^T,$$
$$g = \frac{1}{2h^\alpha}Gu^{n-1} - \frac{1}{\tau^2}\left(2u^n - u^{n-1}\right).$$

Note that, at each time level $k = n+1$, u^n, u^{n-1}, and g are constants. If we define

$$F(x) = \left(\frac{1}{\tau^2}I + \frac{1}{2h^\alpha}G\right)x - Q\left(x,u^{n-1}\right) + g \quad \left(x \in R^{M-1}\right),$$

then u^{n+1} turns out to be a solution of the nonlinear equation

$$F(x) = 0. \tag{4.2}$$

In addition, letting $u^{n+1} = x$, any solution of the nonlinear equation (4.2) satisfies the difference equations (2.9). Hence the solution of (4.2) is unique.

Applying the Newton method to solve system (4.2), we get

$$J\left(x^s\right)\left(x^{s+1} - x^s\right) = -F\left(x^s\right), \quad s = 0,1,2,\ldots, \tag{4.3}$$

that is,

$$J\left(u^{n+1(s)}\right)\left(u^{n+1(s+1)} - u^{n+1(s)}\right) = -F\left(u^{n+1(s)}\right),$$

where

$$J\left(u^{n+1(s)}\right) = \frac{1}{\tau^2}I + \frac{1}{2h^\alpha}G - \operatorname{diag}\left(q_1^{n+1},q_2^{n+1},\ldots,q_{M-1}^{n+1}\right)$$

and

$$q_i^{n+1} = \frac{d}{dx_i}Q\left(x_i,u_i^{n-1}\right)\Big|_{x_i=u_i^{n+1(s)}} = \begin{cases} -\cos(u_i^{n+1(s)}) & \text{if } u_i^{n+1(s)} = u_i^{n-1}, \\ \Theta(u_i^{n+1(s)},u_i^{n-1}) & \text{if } u_i^{n+1(s)} \neq u_i^{n-1}, \end{cases}$$

with

$$\begin{aligned} \left|\Theta\left(u_i^{n+1(s)},u_i^{n-1}\right)\right| &= \left|\left[\frac{d}{dx_i}\int_0^1 \sin\left(\lambda x_i + (1-\lambda)u_i^{n-1}\right)d\lambda\right]_{x_i=u_i^{n+1(s)}}\right| \\ &= \left|\int_0^1 \lambda\cos\left(\lambda u_i^{n+1(s)} + (1-\lambda)u_i^{n-1}\right)d\lambda\right| \\ &\le \int_0^1 \lambda\, d\lambda \\ &= \frac{1}{2}. \end{aligned}$$

Therefore $|q_i^{n+1}| \le 1$ $(i = 1, 2, \ldots, M-1)$. Hence, as $\tau \to 0$ or $h \to 0$, we have

$$\frac{|q_i^{n+1}|}{\frac{1}{\tau^2} + \frac{1}{2h^\alpha} g_0^{(\alpha)}} \to 0, \quad i = 1, 2, \ldots, M-1. \tag{4.4}$$

If we apply the Newton method to solve system (4.2), then we have to renew the Jacobian matrix of F at each stage, which will be very expensive. However, from (4.4) we observe that the contribution of the term $Q(x, u^{n-1})$ to the Jacobian matrix can be omitted when τ and h are sufficiently small. So we revise the Jacobian matrix as

$$J = \frac{1}{\tau^2} I + \frac{1}{2h^\alpha} G.$$

Then the revised Newton method for solving equations (4.2) is given as follows:

$$J\bigl(x^{s+1} - x^s\bigr) = -F\bigl(x^s\bigr), \quad s = 0, 1, 2, \ldots.$$

4.2 Feasibility analysis

Now we consider the feasibility of the revised Newton method for solving equations (4.2).

Denote

$$A = J\bigl(x^s\bigr), \qquad y = \bigl(x^{s+1} - x^s\bigr), \qquad b = -F\bigl(x^s\bigr).$$

Then equations (4.3) change into $Ay = b$. Using the revised Newton method for equations (4.2), we can rewrite the corresponding perturbed equations as

$$(A + \delta A)(y + \delta y) = b,$$

where

$$A + \delta A = J, \qquad \delta A = -\operatorname{diag}\bigl(q_1^{n+1}, q_2^{n+1}, \ldots, q_{M-1}^{n+1}\bigr).$$

Applying the classic perturbation analysis theory, we get

$$\frac{\|\delta y\|}{\|y\|} \le \frac{\operatorname{cond}(A)\frac{\|\delta A\|}{\|A\|}}{1 - \operatorname{cond}(A)\frac{\|\delta A\|}{\|A\|}},$$

where

$$\|y\|^2 = y^T y, \qquad \|A\| = \max\bigl\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^T A\bigr\},$$
$$\operatorname{cond}(A) = \bigl\|A^{-1}\bigr\| \|A\| = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}.$$

According to (4.4), as $\tau, h \to 0$, we have $\frac{\|\delta A\|}{\|A\|} \to 0$ and

$$\operatorname{cond}(A) \le \frac{\frac{1}{\tau^2} + \frac{1}{h^\alpha} g_0^{(\alpha)} + \max\{q_i^{n+1} : i = 1, 2, \ldots, M-1\}}{\frac{1}{\tau^2} + \min\{q_i^{n+1} : i = 1, 2, \ldots, M-1\}} \approx 1 + \frac{\tau^2}{h^\alpha} g_0^{(\alpha)}.$$

Table 1 The convergence order in spatial direction with $\alpha = 1.75, 1.95, \tau = 1/256$ at time $t = 1$

h	$E(h)$ ($\alpha = 1.75$)	order	$E(h)$ ($\alpha = 1.95$)	order
1	0.071175803333281	–	0.083949535804123	
1/2	0.018280687400889	1.9611	0.020521577513857	2.0324
1/4	0.004463589467647	2.0340	0.004906614930621	2.0643
1/8	0.001108210787441	2.0100	0.001213495898485	2.0156

Hence, if $\tau, h \to 0$ and $\frac{\tau^2}{h^\alpha} < \infty$, we have

$$\frac{\|\delta y\|}{\|y\|} \to 0.$$

In summary, when h, τ are sufficiently small, the revised Newton method for solving equations (4.2) is feasible.

5 Numerical results

In this section, we present a few numerical results to verify the effectiveness of the conservative difference scheme (2.9)–(2.11) and the efficiency of the revised Newton method developed in Sect. 4 for implementation of the difference scheme. All numerical computations were carried out using MATLAB on a DELL OptiPlex 3046 computer Intel(R) Core(TM) I5-6500, 3.2 GHz and 4 GB RAM.

Example 1 We take the following initial conditions:

$$\begin{cases} u(x,0) = 0, & x \in [x_L, x_R], \\ u_t(x,0) = 4\,\mathrm{sech}\, x, & x \in [x_L, x_R]. \end{cases} \tag{5.1}$$

Firstly, we test the convergence order of the difference scheme (2.9)–(2.11). Denote

$$\begin{aligned} E(h) &= \max_{0\le i\le M} \left| u_i^n(\tau,h) - u_{2i}^n(\tau,h/2)\right|, \\ E(\tau) &= \max_{0\le i\le M} \left| u_i^n(\tau,h) - u_i^{2n}(\tau/2,h)\right|, \\ p &= \log_2\left[E(h)/E(h/2)\right], \\ q &= \log_2\left[E(\tau)/E(\tau/2)\right], \end{aligned}$$

where p, q denote the space convergence order and time convergence order, respectively. We apply the difference scheme (2.9)–(2.11) with fixed time step $\tau = 1/256$ and different space steps to solve the problem with different values of fractional order α with $-x_L = x_R = 40$ and $T = 1$. The computation results are presented in Table 1. Also with a fixed space step $h = 1/128$, we use different time steps to solve these problems. The computation results are listed in Table 2. According to Tables 1 and 2, the convergence orders in space and time are both consistent with theoretical results. Moreover, we depict the numerical solution of the problem with $\alpha = 1.75$, $h = 1/16$, and $\tau = 1/200$ in Fig. 1, which shows that the results we get are qualitatively similar to those obtained in [4, 20].

Secondly, we pay attention to investigate the relationship between the fractional order α and the shape of the soliton for the problem with different fractional order α. We choose

Table 2 The convergence order in temporal direction with $\alpha = 1.75, 1.95, h = 1/128$ at time $t = 1$

τ	$E(\tau)$ ($\alpha = 1.75$)	order	$E(\tau)$ ($\alpha = 1.95$)	order
1/4	0.076360499008544	–	0.077602598777544	–
1/8	0.020013445313199	1.9319	0.020188901056871	1.9425
1/16	0.005060174214254	1.9837	0.005092254511141	1.9872
1/32	0.001268571271052	1.9960	0.001275824597592	1.9969

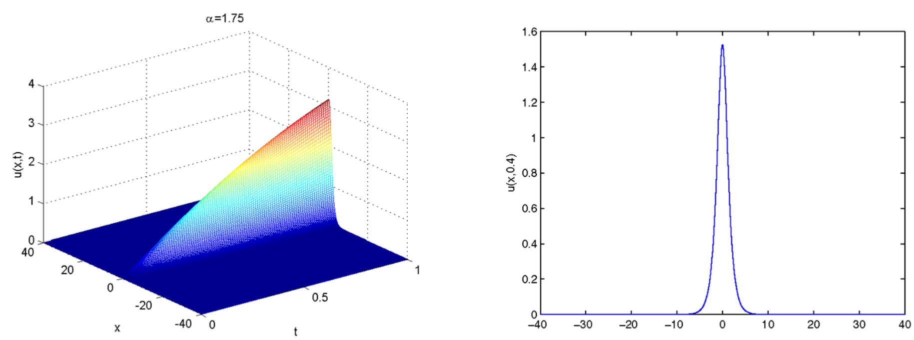

Figure 1 The numerical solution of the initial boundary value problem in Example 1 with $\alpha = 1.75$, $h = 1/16, \tau = 1/200$ and corresponding solution for $u(x,t)$ when $t = 0.4$

$-x_L = x_R = 100$ and $T = 150$ for the problem. The numerical results are presented in Fig. 2, from which we derive that the shape of the soliton is changes dramatically when value of the order α is changed from 2.0 to 1.99. When $1 < \alpha < 2$, along with α becoming smaller, the period of the soliton turns out to be smaller.

Finally, we test the discrete energy conservation law of the difference scheme (2.9)–(2.11). The values of the discrete energy at different moments for $\alpha = 1.1, 1.75, 1.99, 2.0$ are listed in Table 3, where the numerical results are derived with $h = \tau = \frac{1}{10}$. From Table 3 we obtain that the difference scheme preserves the energy conservation very well and the energy varies with different α.

Example 2 We take the following initial conditions:

$$\begin{cases} u(x,0) = 3.2\,\mathrm{sech}\,x, & x \in [x_L, x_R], \\ u_t(x,0) = 0, & x \in [x_L, x_R]. \end{cases}$$

Firstly, the convergence order of the difference scheme (2.9)–(2.11) in space and time are tested for the problem with different values of fractional order α with $-x_L = x_R = 40$ and $T = 1$. The computation results are presented in Tables 4 and 5. According to Tables 4 and 5, the convergence orders in space and time are both consistent with theoretical results.

Secondly, we test the discrete energy conservation law of the difference scheme (2.9)–(2.11) for the problem with different values of fractional order α with $-x_L = x_R = 100$ and $T = 100$. The values of the discrete energy at different moments for $\alpha = 1.1, 1.5, 1.95, 2.0$ are listed in Table 6, where the numerical results are derived with $h = \frac{1}{10}$ and $\tau = \frac{1}{20}$. From Table 6 we can observe that the difference scheme can preserve the energy conservation very well in a long time interval and the energy depends on the order α. Moreover, the numerical results are presented in Fig. 3. We find that

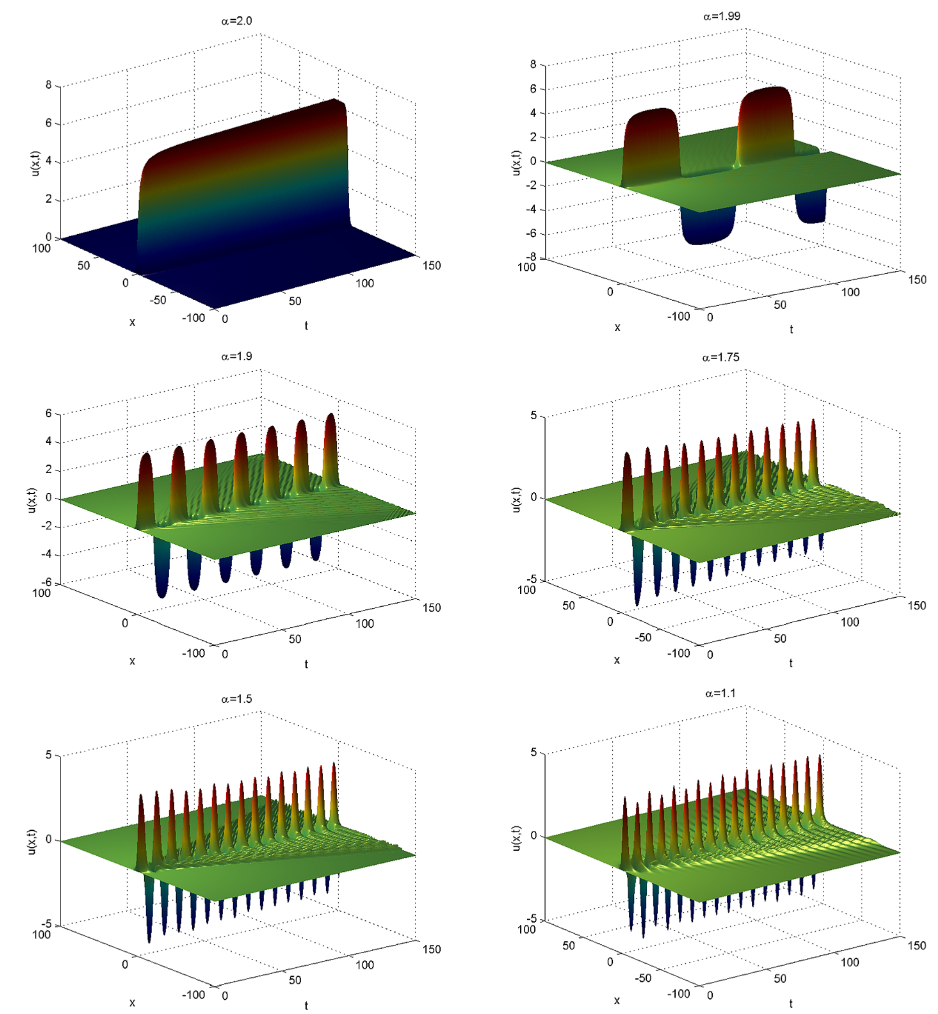

Figure 2 Approximate solution of the initial boundary value problem in Example 1 with various orders α with $h = 1/10$ and $\tau = 1/20$

the fractional order α affects the shape of the wave of the initial boundary problem.

Finally, we test the efficiency of the revised Newton method developed in Sect. 4. In Table 7, we display the consumed CPU time to implement the difference scheme by the regular Newton method and the revised Newton method for different space steps for the problem in Example 2 with $\alpha = 1.75, -x_L = x_R = 40, T = 1$, and $\tau = 1/200$. We conclude that the revised Newton method significantly reduces the computational cost of the difference scheme. It reduces the CPU time from more than 2 hours 12 minutes consumed by the regular Newton method to less than 8 minutes for the model with space step $h = \frac{1}{128}$.

6 Conclusions

In this paper, we propose a conservative implicit difference scheme for SGEs with the Riesz space fractional derivative. We give a rigorous theoretical analysis of its conservation property, boundedness, and convergence. We introduce a revised Newton iterative

Table 3 ε^n at $t = t_n$ for different values of α with $h = \frac{1}{10}, \tau = \frac{1}{20}$

t	$\alpha = 1.1$	$\alpha = 1.75$	$\alpha = 1.99$	$\alpha = 2.0$
0	16.028375166427221	16.026853085312613	16.026621262922639	16.026614500015587
10	16.028375167453508	16.026853083065745	16.026621263595434	16.026614500665112
20	16.028375167026866	16.026853083341813	16.026621263623838	16.026614500693057
30	16.028375169429935	16.026853082109128	16.026621263611123	16.026614500697043
40	16.028375169276917	16.026853082184989	16.026621262942438	16.026614500697779
50	16.028375174037464	16.026853082023688	16.026621263929581	16.026614500697725
60	16.028375172694766	16.026853081374647	16.026621263991700	16.026614500697306
70	16.028375173733984	16.026853080731076	16.026621263988194	16.026614500697072
80	16.028375174760189	16.026853081190453	16.026621263850092	16.026614500696731
90	16.028375175956295	16.026853082788968	16.026621264363811	16.026614500696208
100	16.028375178909357	16.026853082989092	16.026621264523360	16.026614500696372
110	16.028375178244517	16.026853083445701	16.026621264528124	16.026614500695700
120	16.028375179698422	16.026853085066683	16.026621264471139	16.026614500695260
130	16.028375180814812	16.026853085383230	16.026621264061959	16.026614500694873
140	16.028375181940007	16.026853087126149	16.026621264895454	16.026614500694507
150-τ	16.028375183321515	16.026853087643513	16.026621264907444	16.026614500694265

Table 4 The convergence order in spatial direction with $\alpha = 1.5, 1.95, \tau = 1/400$ at time $t = 1$

h	$E(h)$ ($\alpha = 1.5$)	order	$E(h)$ ($\alpha = 1.95$)	order
1	0.175728570098761	–	0.130474954167071	–
1/2	0.030927558960760	2.0768	0.033480284841195	2.3920
1/4	0.007465839195053	2.0505	0.007679396860666	2.1242
1/8	0.002086595144613	1.8392	0.001888044160022	2.0241

Table 5 The convergence order in temporal direction with $\alpha = 1.5, 1.95, h = 1/300$ at time $t = 1$

τ	$E(\tau)$ ($\alpha = 1.5$)	order	$E(\tau)$ ($\alpha = 1.95$)	order
1/20	8.0262e–004	–	6.1903e–004	–
1/40	2.0469e–004	1.9713	1.6227e–004	1.9317
1/80	5.1690e–005	1.9855	4.1427e–005	1.9697
1/160	1.2989e–005	1.9926	1.0459e–005	1.9858

Table 6 ε^n at $t = t_n$ for different values of α with $h = \frac{1}{10}$ and $\tau = \frac{1}{20}$

t	$\alpha = 1.1$	$\alpha = 1.5$	$\alpha = 1.95$	$\alpha = 2.0$
0	10.087375775801739	9.515393309522924	9.204583910317352	9.186084289597257
10	10.087375779384356	9.515393313821262	9.204583911659553	9.186084292149856
20	10.087375780023113	9.515393315532492	9.204583913085234	9.186084295906156
30	10.087375780648971	9.515393316223834	9.204583914395832	9.186084297443383
40	10.087375781657538	9.515393316456663	9.204583914934780	9.186084298335000
50	10.087375783376320	9.515393317054322	9.204583915129753	9.186084299068263
60	10.087375785301962	9.515393318461506	9.204583915590725	9.186084300186360
70	10.087375786753782	9.515393320281280	9.204583917024371	9.186084300778976
80	10.087375787805106	9.515393321510466	9.204583919287133	9.186084299627677
90	10.087375788916159	9.515393321706426	9.204583919632452	9.186084299527717
100-τ	10.087375790133651	9.515393321480675	9.204583918662443	9.186084299808456

method for implementation of the proposed difference scheme. Both theoretical analysis and numerical experiments show that the difference scheme is efficient for solving space-fractional SGE.

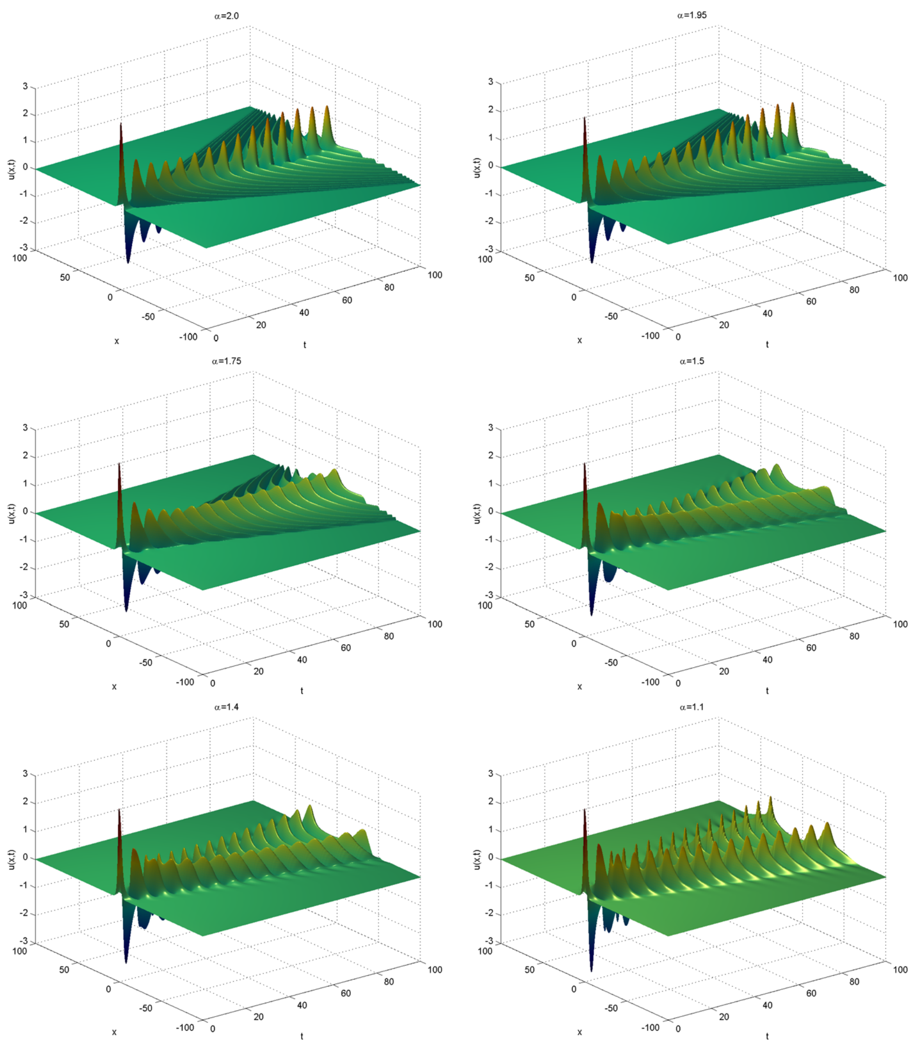

Figure 3 Approximate solution of the initial boundary value problem in Example 2 with various orders α with $h = 1/10$ and $\tau = 1/20$

Table 7 The consumed CPU time of the regular Newton method and the revised Newton method

h	CPU time of the regular Newton method
1/8	18.7031 s
1/16	72.8281 s
1/32	302.9375s
1/64	1.1886e+03 s
1/128	7.9377e+03 s≈ 2 h 12 m 18 s

h	CPU time of the revised Newton method
1/8	4.9688 s
1/16	14.0938 s
1/32	42.7500 s
1/64	142.3594 s
1/128	467.7500 s ≈ 7 m 48 s

Acknowledgements
The authors would like to express the sincere thanks to the referees for their valuable comments and suggestions, which helped to improve the original paper.

Funding
This work is supported by NSF of China (No. 11371302) and Postgraduate Innovation project of Hunan Province, China (No. CX2017B270).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors contributed equally to writing of this paper. Both authors read and approved the final manuscript.

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 15 December 2017 Accepted: 11 June 2018 Published online: 11 July 2018

References

1. Ablowitz, M., Clarkson, P.: Solitons, Nonlinear Evolution Equations and Inverse Scattering. Cambridge University Press, Cambridge (1991)
2. Perring, J.K., Skyrme, T.H.: A model unified field equation. Nucl. Phys. **31**, 550–555 (1962)
3. Mohebbi, A., Dehghan, M.: High-order solution of one-dimensional sine-Gordon equation using compact finite difference and DIRKN methods. Math. Comput. Model. **51**, 537–549 (2010)
4. Macías-Díaz, J.: Numerical study of the process of nonlinear supratransmission in Riesz space-fractional sine-Gordon equations. Commun. Nonlinear Sci. Numer. Simul. **46**, 89–102 (2017)
5. Shi, L., Hao, C., Li, C., Huang, N.: Conservation laws in sine-Gordon equation. Chin. Phys. Lett. **20**, 1003–1005 (2003)
6. Zhang, F., Vázquez, L.: Two energy conserving numerical schemes for the sine-Gordon equation. Appl. Math. Comput. **45**, 17–30 (1991)
7. Zhang, L., Chang, Q.: A conserving nine points finite difference scheme for the sine-Gordon equation. Math. Appl. **12**, 30–35 (1999)
8. Guo, B., Pascual, P., Rodriguez, M., Vázquez, L.: Numerical solution of the sine-Gordon equation. Appl. Math. Comput. **18**, 1–14 (1986)
9. Jiang, C.L., Sun, J.Q., Li, H.C., Wang, Y.F.: A fourth-order AVF method for the numerical integration of sine-Gordon equation. Appl. Math. Comput. **313**, 144–158 (2017)
10. Lu, X., Schmid, R.: Symplectic integration of sine-Gordon type systems. Math. Comput. Simul. **50**, 255–263 (1999)
11. Voss, D.A., Khaliq, A.Q.M.: Parallel LOD methods for second order time dependent PDEs. Comput. Math. Appl. **10**, 25–35 (1995)
12. Ran, M., Zhang, C.: A conservative difference scheme for solving the strongly coupled nonlinear fractional Schrödinger equations. Commun. Nonlinear Sci. Numer. Simul. **41**, 64–83 (2016)
13. Wang, P., Huang, C.: An energy conservative difference scheme for the nonlinear fractional Schrödinger equations. J. Comput. Phys. **293**, 238–251 (2015)
14. Guo, B., Han, Y., Xin, J.: Existence of the global smooth solution to the period boundary value problem of fractional nonlinear Schrödinger equation. Appl. Math. Comput. **204**, 468–477 (2008)
15. Herzallah, M., Gepreel, K.: Approximate solution to the time-space fractional cubic nonlinear Schrödinger equation. Appl. Math. Model. **36**, 5678–5685 (2012)
16. Li, M., Huang, C., Wang, P.: Galerkin finite element method for nonlinear fractional Schrödinger equations. Numer. Algorithms **74**, 499–525 (2017)
17. Hu, Y., Li, C., Li, H.: The finite difference method for Caputo-type parabolic equation with fractional Laplacian: one-dimensional case. Chaos Solitons Fractals **102**, 361–371 (2017)
18. Bu, W., Tang, Y., Yang, J.: Galerkin finite element method for two-dimensional Riesz space fractional diffusion equations. J. Comput. Phys. **276**, 26–38 (2014)
19. Jiang, Y., Ma, J.: High-order finite element methods for time-fractional partial differential equations. J. Comput. Appl. Math. **235**, 3285–3290 (2011)
20. Ray, S.: A new analytical modelling for nonlocal generalized Riesz fractional sine-Gordon equation. J. King Saud Univ., Sci. **28**, 48–54 (2016)
21. Alfimov, G., Pierantozzi, T., Vázquez, L.: Numerical study of a fractional sine-Gordon equation. Fract. Differ. Appl. FDA. **4**, 644–649 (2004)
22. Yang, Q., Liu, F., Turner, I.: Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. Appl. Math. Model. **34**, 200–218 (2010)
23. Sun, Z., Gao, G.: A Finite Difference Method for Fractional Differential Equations. Science Press, China (2015)
24. Kirkpatrick, K., Lenzmann, E., Staffilani, G.: On the continuum limit for discrete NLS with long-range lattice interactions. Commun. Math. Phys. **317**, 563–591 (2013)
25. Wang, D., Xiao, A., Yang, W.: Maximum-norm error analysis of a difference scheme for the space fractional CNLS. Appl. Math. Comput. **257**, 241–251 (2015)
26. Temam, R.: Navier-Stokes Equations: Theory and Numerical Analysis. North-Holland, Amsterdam (1977)