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Existence of positive periodic solutions for the impulsive Lotka–Volterra cooperative population model with time-delay and harvesting control on time scales

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Abstract

In this paper, we consider a class of impulsive Lotka–Volterra cooperative population models with time delay and harvesting control on time scales. Using the fixed point theorem of strict-set-contraction, we analyze the existence conditions of positive periodic solutions for this model. As applications, we analyze the existence conditions of positive periodic solutions for some common Lotka–Volterra systems on time scales.

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1 Introduction

It is well known that the application of theories of functional differential equations in mathematical ecology or biology has developed rapidly and effectively. The Lotka–Volterra ecological population model proposed by Lotka [1] and Volterra [2] is one of the most famous and important population dynamics models described by functional differential equations. Owing to their theoretical and practical significance, Lotka–Volterra systems have been studied extensively [3–19]. To protect the balanced development or sustainable economic benefits of various species in the ecosystem, a number of artificial controls have been applied to the ecosystem. This needs to consider the influence of artificial control in the ecological mathematical model. For example, we add the control harvest rate $h_i(t)$ ($i = 1, 2, \dots$) to our model (1.1). However, dynamics in each equally spaced time interval may vary continuously. So it may be more realistic to assume that the population dynamics involves the hybrid discrete-continuous processes. For example, Gamarra and Solé [20] pointed out that such hybrid processes appear in the population dynamics of certain species that feature nonoverlapping generations: the change in population from one generation to the next is discrete and so is modeled by a difference equation, whereas within-generation dynamics varies continuously (due to mortality rates, resource consumption, predation, interaction, etc.) and thus is described by a differential equation.

However, it is often difficult to study discrete and continuous differential systems in a unified way. Fortunately, the theory of calculus on time scales (see [21, 22] and references therein) proposed by Hilger in his Ph.D. thesis [23] can unify continuous and discrete analysis, and it has become an effective approach to the study of mathematical models involving hybrid discrete-continuous processes. There are many achievements in the study of hybrid discrete-continuous mathematical models (see [24–32]).

Motivated by the preceding, in this paper, we deal with the impulsive Lotka–Volterra cooperative population model with time delay and harvesting control on time scales of the form

$$\begin{cases} u_i^\Delta(t) = u_i(t)[r_i(t) + F_i(t, u(t)) + G_i(t, u(t)) - h_i(t)], & t \neq t_k, t \in \mathbb{T}, \\ u_i(t_k^+) = u_i(t_k^-) + I_{ik}(u(t_k)), & k = 1, 2, \dots, \end{cases} \quad (1.1)$$

where $i = 1, 2, \dots, n$, $u = (u_1, u_2, \dots, u_n)$, \mathbb{T} is an ω -periodic time scale satisfying $s \pm t \in \mathbb{T}$ for all $s, t \in \mathbb{T}$, $\omega > 0$ is a constant,

$$F_i(t, u(t)) = F_i(t, u_1(t - \tau_{i1}(t)), u_2(t - \tau_{i2}(t)), \dots, u_n(t - \tau_{in}(t)))$$

and

$$G_i(t, u(t)) = G_i\left(t, \int_{-\infty}^0 K_{i1}(s)u_1(t+s)\Delta s, \dots, \int_{-\infty}^0 K_{in}(s)u_n(t+s)\Delta s\right).$$

For each interval I of \mathbb{R} , we denote by $I_{\mathbb{T}} = I \cap \mathbb{T}$; $u_i(t_k^+)$ and $u_i(t_k^-)$ represent the right and left limits at t_k in the sense of time scales, respectively; in addition, if t_k is right-scattered, then $u_i(t_k^+) = u_i(t_k)$, whereas if t_k is left-scattered, then $u_i(t_k^-) = u_i(t_k)$. We assume that: $h_i(t) > r_i(t)$, $r_i, h_i \in C_{\text{rd}}(\mathbb{T}, (0, \infty))$ and $\tau_{ij} \in C_{\text{rd}}(\mathbb{T}, (0, \infty)_{\mathbb{T}})$ ($j = 1, 2, \dots, n$) are ω -periodic functions; $F_i, G_i \in C_{\text{rd}}(\mathbb{T} \times \mathbb{R}^n, \mathbb{R})$ ($i = 1, 2, \dots, n$) are ω -periodic with respect to their first arguments, respectively; $K_{ij} \in C_{\text{rd}}((-\infty, 0]_{\mathbb{T}}, (0, \infty))$ with $\int_{-\infty}^0 K_{ij}(s)\Delta s = 1$; $I_{ik} \in C([0, \infty)^n, [0, \infty))$ and there exists a positive integer p such that $t_{k+p} = t_k + \omega$ and $I_{i,k+p} = I_{ik}$ for $k \in \mathbb{Z}$. Without loss of generality, we also assume that $[0, \omega)_{\mathbb{T}} \cap \{t_k : k \in \mathbb{Z}\} = \{t_1, t_2, \dots, t_p\}$.

To maintain the vitality and economic benefits of the ecosystem, we always assume that the rate of controlled harvesting for each species is greater than the birth rate of the species, namely, $h_i(t) > r_i(t)$ ($i = 1, 2, \dots, n$) in the whole paper. Therefore, throughout this paper, we assume that

(H₁) the ω -periodic time scale \mathbb{T} satisfies $s \pm t \in \mathbb{T}$ for all $s, t \in \mathbb{T}$;

(H₂) $r_i(t) < h_i(t)$ for all $t \in \mathbb{T}$, $i = 1, 2, \dots, n$;

(H₃) $\sup_{t \in [0, \omega]_{\mathbb{T}}} \{\mu(t)[h_i(t) - r_i(t)]\} < 1$, $i = 1, 2, \dots, n$.

System (1.1) contain some mathematical population models of differential equations and difference equations. For example, if the time scale $\mathbb{T} = \mathbb{R}$, then system (1.1) is changed into the following continuous system:

$$\begin{cases} u_i'(t) = u_i(t)[r_i(t) + F_i(t, u(t)) + G_i(t, u(t)) - h_i(t)], & t \neq t_k, \\ u_i(t_k^+) = u_i(t_k^-) + I_{ik}(u(t_k)), & k = 1, 2, \dots, \end{cases} \quad (1.2)$$

where $i = 1, 2, \dots, n$, $u = (u_1, u_2, \dots, u_n)$,

$$F_i(t, u(t)) = F_i(t, u_1(t - \tau_{i1}(t)), u_2(t - \tau_{i2}(t)), \dots, u_n(t - \tau_{in}(t))),$$

$$G_i(t, u(t)) = G_i\left(t, \int_{-\infty}^0 K_{i1}(s)u_1(t+s)ds, \dots, \int_{-\infty}^0 K_{in}(s)u_n(t+s)ds\right),$$

$u_i(t_k^+)$ and $u_i(t_k^-)$ are the right and left limits at t_k , respectively, $u_i(t_k^-) = u_i(t_k)$, $h_i(t) > r_i(t)$, $r_i, h_i \in C(\mathbb{R}, (0, \infty))$ and $\tau_{ij} \in C(\mathbb{R}, \mathbb{R})$ ($j = 1, 2, \dots, n$) are ω -periodic functions; $F_i, G_i \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ ($i = 1, 2, \dots, n$) are ω -periodic with respect to their first arguments, respectively, $K_{ij} \in C((-\infty, 0], (0, \infty))$ with $\int_{-\infty}^0 K_{ij}(s)ds = 1$, $I_{ik} \in C([0, \infty)^n, [0, \infty))$, there exists a positive integer p such that $t_{k+p} = t_k + \omega$ and $I_{i,k+p} = I_{ik}$ for all $k \in \mathbb{Z}$. Without loss of generality, we also assume that $[0, \omega) \cap \{t_k : k \in \mathbb{Z}\} = \{t_1, t_2, \dots, t_p\}$.

If the time scale $\mathbb{T} = \mathbb{Z}$, system (1.1) is changed into the following difference system:

$$\begin{cases} \Delta u_i(t) = u_i(t)[r_i(t) + F_i(t, u(t)) + G_i(t, u(t)) - h_i(t)], & t \neq t_k, \\ u_i(t_k^+) = u_i(t_k^-) + I_{ik}(u(t_k)), & k = 1, 2, \dots, \end{cases} \quad (1.3)$$

where $t \in \mathbb{Z}$, $\Delta u_i(t) = u_i(t+1) - u_i(t)$, $i = 1, 2, \dots, n$, $u = (u_1, u_2, \dots, u_n)$,

$$\begin{aligned} F_i(t, u(t)) &= F_i(t, u_1(t - \tau_{i1}(t)), u_2(t - \tau_{i2}(t)), \dots, u_n(t - \tau_{in}(t))), \\ G_i(t, u(t)) &= G_i\left(t, \sum_{s=-\infty}^0 K_{i1}(s)u_1(t+s), \dots, \sum_{s=-\infty}^0 K_{in}(s)u_n(t+s)\right), \end{aligned}$$

$u_i(t_k^+)$ and $u_i(t_k^-)$ are the right and left limits at t_k , respectively, $u_i(t_k^-) = u_i(t_k)$, $h_i(t) > r_i(t)$, $r_i, h_i \in C(\mathbb{Z}, (0, \infty))$ and $\tau_{ij} \in C(\mathbb{Z}, \mathbb{Z})$ ($j = 1, 2, \dots, n$) are ω -periodic functions, $F_i, G_i \in C(\mathbb{Z} \times \mathbb{R}^n, \mathbb{R})$ ($i = 1, 2, \dots, n$) are ω -periodic with respect to their first arguments, respectively, $K_{ij} \in C((-\infty, 0] \cap \mathbb{Z}, (0, \infty))$ with $\sum_{s=-\infty}^0 K_{ij}(s) = 1$, and $I_{ik} \in C([0, \infty)^n, [0, \infty))$, and there exists a positive integer p such that $t_{k+p} = t_k + \omega$ and $I_{i,k+p} = I_{ik}$ for all $k \in \mathbb{Z}$. Without loss of generality, we also assume that $[0, \omega) \cap \{t_k : k \in \mathbb{Z}\} = \{t_1, t_2, \dots, t_p\}$.

To the best of our knowledge, few papers have been published on the existence of positive periodic solutions of system (1.1). Our main purpose of this paper is by using a fixed point theorem of strict-set-contraction to establish some sufficient conditions to guarantee the existence of positive periodic solutions of system (1.1) on time scales.

2 Preliminaries on time scales

In this section, we briefly recall some basic definitions and lemmas on time scales. For more detail, we refer to [21–23].

Let \mathbb{T} be a nonempty closed subset (time scale) of \mathbb{R} . The forward and backward jump operators $\sigma, \rho : \mathbb{T} \rightarrow \mathbb{T}$ and the graininess $\mu : \mathbb{T} \rightarrow \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If \mathbb{T} has a left-scattered maximum m , then $\mathbb{T}^k = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$. If \mathbb{T} has a right-scattered minimum m , then $\mathbb{T}_k = \mathbb{T} \setminus \{m\}$; otherwise, $\mathbb{T}^k = \mathbb{T}$.

Let $\omega > 0$. Throughout this paper, we assume that the time scale \mathbb{T} is ω -periodic, that is, $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$ and $\mu(t + \omega) = \mu(t)$. In particular, the time scale \mathbb{T} under consideration is unbounded above and below.

Definition 2.1 A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called *regulated* if its right-side limits exist (finite) at all right-side points in \mathbb{T} and its left-side limits exist (finite) at all left-side points in \mathbb{T} .

Definition 2.2 A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called rd-continuous if it is continuous at right-dense points in \mathbb{T} and its left-side limits exist (finite) at left-dense points in \mathbb{T} . The set of rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ will be denoted by $C_{\text{rd}} = C_{\text{rd}}(\mathbb{T}) = C_{\text{rd}}(\mathbb{T}, \mathbb{R})$.

Definition 2.3 Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$. Then we define $f^\Delta(t)$ to be the number (if it exists) such that, for all $\varepsilon > 0$, there exists a neighborhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$| [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] | < \varepsilon |\sigma(t) - s|$$

for all $s \in U$. We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t . The set of differentiable functions $f : \mathbb{T} \rightarrow \mathbb{R}$ with rd-continuous derivatives is denoted by $C_{\text{rd}}^1 = C_{\text{rd}}^1(\mathbb{T}) = C_{\text{rd}}^1(\mathbb{T}, \mathbb{R})$.

If f is continuous, then f is rd-continuous. If f is rd-continuous, the f is regulated. If f is delta differentiable at t , then f is continuous at t .

Lemma 2.1 Let f be regulated. Then there exists a function F which is delta differentiable with region of differentiation D such that

$$F^\Delta(t) = f(t) \quad \text{for all } t \in D.$$

Definition 2.4 Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a regulated function. Any function F as in Lemma 2.1 is called a Δ -antiderivative of f . We define the indefinite integral of a regulated function f by

$$\int f(t) \Delta t = F(t) + C,$$

where C is an arbitrary constant, and F is a Δ -antiderivative of f . We define the Cauchy integral by

$$\int_a^b f(s) \Delta s = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ if

$$F^\Delta(t) = f(t) \quad \text{for all } t \in \mathbb{T}^k.$$

Lemma 2.2 If $a, b \in \mathbb{T}$, $\alpha, \beta \in \mathbb{R}$, and $f, g \in C(\mathbb{T}, \mathbb{R})$, then we have:

- (i) $\int_a^b [\alpha f(t) + \beta g(t)] \Delta t = \alpha \int_a^b f(t) \Delta t + \beta \int_a^b g(t) \Delta t$;
- (ii) if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t) \Delta t \geq 0$;
- (iii) if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then $|\int_a^b f(t) \Delta t| \leq \int_a^b g(t) \Delta t$.

Definition 2.5 ([33]) A time scale \mathbb{T} is called periodic if there exists $p > 0$ such that if $t \in \mathbb{T}$, then $t \pm p \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive p is called the period of the time scale \mathbb{T} .

Definition 2.6 ([33]) Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period $p > 0$. The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is called periodic with period ω if there exists a natural number n such that $\omega = np$, $f(t + \omega) = f(t)$ for all $t \in \mathbb{T}$, and ω is the smallest number such that $f(t + \omega) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, then we say that f is periodic with period $\omega > 0$ if ω is the smallest positive number such that $f(t + \omega) = f(t)$ for all $t \in \mathbb{R}$.

A function $p : \mathbb{T} \rightarrow \mathbb{R}$ is called regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive rd-continuous functions $f : \mathbb{T} \rightarrow \mathbb{R}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set \mathcal{R}^+ of all positively regressive elements of \mathcal{R} by $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. If p is a regressive function, then the generalized exponential function e_p is defined by $e_p(t, s) = \exp\{\int_s^t \xi_{\mu(\tau)}(p(\tau)) \Delta \tau\}$ for $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\text{Log}(1+hz)}{h} & \text{if } h \neq 0, \\ z & \text{if } h = 0. \end{cases}$$

For two regressive functions $p, q : \mathbb{T} \rightarrow \mathbb{R}$, we define

$$p \oplus q = p + q + \mu p q, \quad \ominus p = -\frac{p}{1 + \mu p}, \quad p \ominus q = p \oplus p(\ominus q).$$

The generalized exponential function has the following properties.

Lemma 2.3 ([21]) Let $p, q : \mathbb{T} \rightarrow \mathbb{R}$ be two regressive functions. Then

- (1) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- (2) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- (3) $\frac{1}{e_p(t, s)} = e_{\ominus p}(t, s)$;
- (4) $e_p(t, s) = \frac{1}{e_p(s, t)} = e_{\ominus p}(s, t)$;
- (5) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- (6) $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$;
- (7) $\frac{e_p(t, s)}{e_q(t, s)} = e_{p \ominus q}(t, s)$;
- (8) $[e_p(t, s)]^\Delta = p(t)e_p(t, s)$;
- (9) $[e_p(c, \cdot)]^\Delta = -p[e_p(c, \cdot)]^\sigma$ for $c \in \mathbb{T}$;
- (10) $\frac{d}{dz}[e_z(t, s)] = (\int_s^t \frac{1}{1 + \mu(\tau)z} \Delta \tau)e_z(t, s)$.

For convenience, we now introduce some notation:

$$B(0, R) = \{(x_1, x_2, \dots, x_n)^T \in \mathbb{R}^n : \|(x_1, x_2, \dots, x_n)\| \leq R\},$$

$$f^M = \max_{t \in [0, \omega]_{\mathbb{T}}} \{f(t)\},$$

$$F_i^0 = \limsup_{\sum_{i=1}^n x_i \rightarrow 0} \max_{t \in [0, \omega]_{\mathbb{T}}} \frac{F_i(t, x_1, x_2, \dots, x_n)}{\sum_{i=1}^n x_i},$$

$$G_i^0 = \limsup_{\sum_{i=1}^n x_i \rightarrow 0} \max_{t \in [0, \omega]_{\mathbb{T}}} \frac{G_i(t, x_1, x_2, \dots, x_n)}{\sum_{i=1}^n x_i},$$

$$F_i^\infty = \liminf_{\sum_{i=1}^n x_i \rightarrow \infty} \min_{t \in [0, \omega]_{\mathbb{T}}} \frac{F_i(t, x_1, x_2, \dots, x_n)}{\sum_{i=1}^n x_i},$$

$$G_i^\infty = \liminf_{\sum_{i=1}^n x_i \rightarrow \infty} \min_{t \in [0, \omega]_{\mathbb{T}}} \frac{G_i(t, x_1, x_2, \dots, x_n)}{\sum_{i=1}^n x_i},$$

$$\gamma_i = \limsup_{\sum_{i=1}^n x_i \rightarrow 0} \frac{\sum_{k=1}^p I_{ik}(x_1, x_2, \dots, x_n)}{\sum_{i=1}^n x_i},$$

where $i = 1, 2, \dots, n$, and f is an rd-continuous ω -periodic function.

Lemma 2.4 ([21]) *Let $r : \mathbb{T} \rightarrow \mathbb{R}$ be right-dense continuous and regressive. For $a \in \mathbb{T}$ and $y_a \in \mathbb{R}$, the unique solution of the initial value problem*

$$y^\Delta(t) = r(t)y(t) + h(t), \quad y(a) = y_a,$$

is given by

$$y(t) = e_r(t, a)y_a + \int_a^t e_r(t, \sigma(s))h(s)\Delta s.$$

The existence of periodic solutions of system (1.1) is equivalent to the existence of periodic solutions of the corresponding integral system. So the following lemma is important in our discussion.

Lemma 2.5 *A function $u(t) = (u_1(t), u_2(t), \dots, u_n(t))^T$ is an ω -periodic solution of (1.1) if and only if $u(t)$ is an ω -periodic solution of the integral system*

$$u_i(t) = \int_t^{t+\omega} H_i(t, s)u_i(s)[F_i(s, u(s)) + G_i(s, u(s))]\Delta s$$

$$+ \sum_{t_k \in [t, t+\omega)_{\mathbb{T}}} H_i(t, t_k)e_{(r_i-h_i)}(\sigma(t_k), t_k)I_{ik}(u(t_k)), \quad i = 1, 2, \dots, n, \quad (2.1)$$

where

$$H_i(t, s) = \frac{e_{(r_i-h_i)}(t, \sigma(s))}{e_{(r_i-h_i)}(0, \omega) - 1}, \quad s \in [t, t + \omega]_{\mathbb{T}}, i = 1, 2, \dots, n. \quad (2.2)$$

Proof If $u(t)$ is an ω -periodic solution of (1.1), then for all $t \in \mathbb{T}$, there exists $k \in \mathbb{N}^+$ such that t_k is the first impulsive point after t . Applying Lemma 2.4 and equation (1.1), for $s \in [t, t_k]_{\mathbb{T}}$, we have

$$u_i(s) = e_{(r_i-h_i)}(s, t)u_i(t) + \int_t^s e_{(r_i-h_i)}(s, \sigma(\tau))u_i(\tau)[F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))]\Delta \tau,$$

and thus

$$u_i(t_k) = e_{(r_i-h_i)}(t_k, t)u_i(t) + \int_t^{t_k} e_{(r_i-h_i)}(t_k, \sigma(\tau))u_i(\tau)[F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))]\Delta \tau.$$

Again using Lemma 2.4, for $s \in (t_k, t_{k+1}]_{\mathbb{T}}$, we have

$$u_i(s) = e_{(r_i-h_i)}(s, t_k)u_i(t_k^+) + \int_{t_k}^s e_{(r_i-h_i)}(s, \sigma(\tau))u_i(\tau)[F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))]\Delta \tau$$

$$\begin{aligned}
&= e_{(r_i-h_i)}(s, t_k) u_i(t_k) + \int_{t_k}^s e_{(r_i-h_i)}(s, \sigma(\tau)) u_i(\tau) [F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))] \Delta \tau \\
&\quad + e_{(r_i-h_i)}(s, t_k) I_{ik}(u(t_k)).
\end{aligned}$$

Thus, for $s \in [t, t_{k+1}]_{\mathbb{T}}$, we get

$$\begin{aligned}
u_i(s) &= e_{(r_i-h_i)}(s, t) u_i(t) + \int_t^s e_{(r_i-h_i)}(s, \sigma(\tau)) u_i(\tau) [F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))] \Delta \tau \\
&\quad + e_{(r_i-h_i)}(s, t_k) I_{ik}(u(t_k)).
\end{aligned}$$

Repeating this process for $s \in [t, t + \omega]_{\mathbb{T}}$, we obtain

$$\begin{aligned}
u_i(s) &= e_{(r_i-h_i)}(s, t) u_i(t) + \int_t^s e_{(r_i-h_i)}(s, \sigma(\tau)) u_i(\tau) [F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))] \Delta \tau \\
&\quad + \sum_{t_k \in [t, t+\omega]_{\mathbb{T}}} e_{(r_i-h_i)}(s, t_k) I_{ik}(u(t_k)).
\end{aligned}$$

Let $s = t + \omega$ in this equality and notice that $u_i(t) = u_i(t + \omega)$, $e_{(r_i-h_i)}(t, t + \omega) = e_{(r_i-h_i)}(0, \omega)$, $e_{(r_i-h_i)}(t + \omega, \sigma(\tau)) = e_{(r_i-h_i)}(t, \sigma(\tau)) e_{(r_i-h_i)}(t + \omega, t)$, $e_{(r_i-h_i)}(t, t_k) = e_{(r_i-h_i)}(t, \sigma(t_k)) e_{(r_i-h_i)}(\sigma(t_k), t_k)$, and $e_{(r_i-h_i)}(t, t + \omega) e_{(r_i-h_i)}(t + \omega, t) = 1$, we have

$$\begin{aligned}
u_i(t) &= u_i(t + \omega) \\
&= e_{(r_i-h_i)}(t + \omega, t) u_i(t) + \int_t^{t+\omega} e_{(r_i-h_i)}(t + \omega, \sigma(\tau)) u_i(\tau) \\
&\quad \times [F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))] \Delta \tau \\
&\quad + \sum_{t_k \in [t, t+\omega]_{\mathbb{T}}} e_{(r_i-h_i)}(t + \omega, t_k) I_{ik}(u(t_k)) \\
&= e_{(r_i-h_i)}(\omega, 0) u_i(t) + \int_t^{t+\omega} e_{(r_i-h_i)}(t, \sigma(\tau)) e_{(r_i-h_i)}(\omega, 0) u_i(\tau) \\
&\quad \times [F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))] \Delta \tau \\
&\quad + \sum_{t_k \in [t, t+\omega]_{\mathbb{T}}} e_{(r_i-h_i)}(t, \sigma(t_k)) e_{(r_i-h_i)}(\omega, 0) e_{(r_i-h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)),
\end{aligned}$$

which implies that

$$\begin{aligned}
u_i(t) &= \int_t^{t+\omega} H_i(t, \tau) u_i(\tau) [F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))] \Delta \tau \\
&\quad + \sum_{t_k \in [t, t+\omega]_{\mathbb{T}}} H_i(t, t_k) e_{(r_i-h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)).
\end{aligned}$$

Thus, we conclude that $u(t)$ satisfies (2.1).

Let $u(t)$ be an ω -periodic solution of (2.1). Noting that the above reduction is completely reversible, we see that $x(t)$ is also an ω -periodic solution of (1.1). This completes the proof of Lemma 2.5. \square

Lemma 2.6 *If conditions (H_1) – (H_3) hold, then $H_i(t, s)$ ($i = 1, 2, \dots, n$) defined by (2.2) satisfy the following:*

- (1) $\frac{1}{\sigma_i - 1} \leq H_i(t, s) \leq \frac{\sigma_i}{\sigma_i - 1}$, $\forall s \in [t, t + \omega]_{\mathbb{T}}$, where $\sigma_i = e_{(r_i - h_i)}(0, \omega)$, $i = 1, 2, \dots, n$;
- (2) $H_i(t + \omega, s + \omega) = H_i(t, s)$, $i = 1, 2, \dots, n$.

Proof According to conditions (H_1) – (H_3) , since $\mu(t) = \sigma(t) - t \geq 0$ and $r_i(t) - h_i(t) < 0$, we have $0 < 1 + \mu(t)[r_i(t) - h_i(t)] < 1$. In addition, by the definition of the generalized exponential function we get $\sigma_i = e_{(r_i - h_i)}(0, \omega) > 1$, $i = 1, 2, \dots, n$. Noticing that $t \leq s \leq \sigma(s) \leq t + \omega$, we have

$$\frac{1}{\sigma_i - 1} = \frac{e_{(r_i - h_i)}(t, t)}{\sigma_i - 1} \leq H_i(t, s) \leq \frac{e_{(r_i - h_i)}(t, t + \omega)}{\sigma_i - 1} = \frac{\sigma_i}{\sigma_i - 1}.$$

Thus, assertion (1) holds. Now we show that the assertion (2) also holds. In fact, since $\sigma(t + \omega) = \sigma(t) + \omega$, by integration by substitution we have

$$\begin{aligned} H_i(t + \omega, s + \omega) &= \frac{e_{(r_i - h_i)}(t + \omega, \sigma(s + \omega))}{\sigma_i - 1} = \frac{e_{(r_i - h_i)}(t + \omega, \sigma(s) + \omega)}{\sigma_i - 1} \\ &= \frac{e_{(r_i - h_i)}(t, \sigma(s))}{\sigma_i - 1} = H_i(t, s), \quad i = 1, 2, \dots, n. \end{aligned}$$

The proof of Lemma 2.6 is complete. \square

To obtain the existence of a periodic solution of system (2.1), we need the some preparations. Let X be a real Banach space, and let K be a closed nonempty subset of X . Then K is a cone if

- (i) $k\alpha + l\beta \in K$ for all $\alpha, \beta \in K$ and $k, l \geq 0$;
- (ii) $\alpha, -\alpha \in K$ imply $\alpha = \theta$, where θ is the zero element of X .

Let E be a Banach space, and let K be a cone in E . The semiorder induced by the cone K is denoted by \leq , that is, $x \leq y$ if and only if $y - x \in K$. In addition, for a bounded subset $A \subset E$, let $\alpha_E(A)$ denote the (Kuratowski) measure of noncompactness, namely

$$\alpha_E(A) = \inf \{ \delta > 0 : A \text{ admits a finite cover by subsets of } A_i \subset A \text{ such that } \text{diam}(A_i) \leq \delta \},$$

where $\text{diam}(A_i)$ denotes the diameter of a set A_i .

Let E, F be two Banach spaces and $D \subset E$. A continuous bounded map $\Phi : \overline{D} \rightarrow F$ is called k -set-contractive if for any bounded set $S \subset D$, we have

$$\alpha_F(\Phi(S)) \leq k\alpha_E(\Phi(S)).$$

The map Φ is called strict-set-contractive if it is k -set-contractive for some $0 \leq k < 1$. Particularly, completely continuous operators are 0-set-contractive.

The following lemma is useful for the proof of our main results.

Lemma 2.7 ([34, 35]) *Let K be a cone in the real Banach space X , and let $K_{r,R} = \{u \in K : r \leq \|u\| \leq R\}$ with $R > r > 0$. Suppose that $\Phi : K_{r,R} \rightarrow K$ is strict-set-contractive such that one of the following two conditions is satisfied:*

- (i) $\Phi u \not\leq u, \forall u \in K, \|u\| = r$ and $\Phi u \not\leq u, \forall u \in K, \|u\| = R$.
(ii) $\Phi u \not\leq u, \forall u \in K, \|u\| = r$ and $\Phi u \not\leq u, \forall u \in K, \|u\| = R$.

Then Φ has at least one fixed point in $K_{r,R}$.

Define

$$PC(\mathbb{T}) = \left\{ u = (u_1, u_2, \dots, u_n) : \mathbb{T} \rightarrow \mathbb{R}^n, u|_{(t_k, t_{k+1})} \in C_{rd}((t_k, t_{k+1}), \mathbb{R}^n), \right. \\ \left. \exists u(t_k^-) = u(t_k), u(t_k^+), k \in \mathbb{N}^+ \right\}.$$

Set

$$X = \left\{ u : u \in PC(\mathbb{T}), u(t + \omega) = u(t), t \in \mathbb{T} \right\}$$

be equipped with the norm $\|u\| = \sum_{i=1}^n |u_i|_0$, where $|u_i|_0 = \sup_{t \in [0, \omega]_{\mathbb{T}}} |u_i(t)|$, $i = 1, 2, \dots, n$. Then X is a Banach space. In view of Lemma 2.6, we define the cone K in X as

$$K = \left\{ u = (u_1, \dots, u_n) \in X : u_i(t) \geq \frac{1}{\sigma_i} |u_i|_0, \forall t \in [0, \omega]_{\mathbb{T}}, i = 1, 2, \dots, n \right\}.$$

Let the map Φ be defined by

$$(\Phi u)(t) = ((\Phi_1 u)(t), (\Phi_2 u)(t), \dots, (\Phi_n u)(t))^T, \quad (2.3)$$

where $x \in K, t \in \mathbb{T}$,

$$(\Phi_i u)(t) = \int_t^{t+\omega} H_i(t, s) u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s \\ + \sum_{t_k \in [t, t+\omega)_{\mathbb{T}}} H_i(t, t_k) e_{(r_i - h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)), \quad i = 1, 2, \dots, n,$$

and $H_i(t, s)$ ($i = 1, 2, \dots, n$) are defined by (2.2).

Lemma 2.8 Assume that $(H_1)-(H_3)$ hold. Then $\Phi : K \rightarrow K$ defined by (2.3) is well defined, that is, $\Phi(K) \subset K$.

Proof It is clear that $\Phi u \in PC(\mathbb{T})$ for all $u \in K$. In view of Lemma 2.6 and (2.3), we obtain

$$(\Phi_i u)(t + \omega) = \int_{t+\omega}^{t+2\omega} H_i(t + \omega, s) u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s \\ + \sum_{t_k \in [t+\omega, t+2\omega)_{\mathbb{T}}} H_i(t + \omega, t_k) e_{(r_i - h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)) \\ = \int_t^{t+\omega} H_i(t + \omega, \tau + \omega) u_i(\tau + \omega) [F_i(\tau + \omega, u(\tau + \omega)) \\ + G_i(\tau + \omega, u(\tau + \omega))] \Delta \tau \\ + \sum_{t_l \in [t, t+\omega)_{\mathbb{T}}} H_i(t + \omega, t_l + \omega) e_{(r_i - h_i)}(\sigma(t_l + \omega), t_l + \omega) I_{il}(u(t_l + \omega))$$

$$\begin{aligned}
&= \int_t^{t+\omega} H_i(t, \tau) u_i(\tau) [F_i(\tau, u(\tau)) + G_i(\tau, u(\tau))] \Delta \tau \\
&\quad + \sum_{t_l \in [t, t+\omega)_{\mathbb{T}}} H_i(t, t_l) e_{(r_i-h_i)}(\sigma(t_l), t_l) I_{il}(u(t_l)) = (\Phi_i u)(t),
\end{aligned}$$

that is, $(\Phi_i u)(t + \omega) = (\Phi_i u)(t)$, $\forall t \in \mathbb{T}$, $i = 1, 2, \dots, n$. So $\Phi u \in X$. For any $u \in K$, we have

$$\begin{aligned}
|\Phi_i u|_0 &\leq \frac{\sigma_i}{\sigma_i - 1} \left[\int_0^\omega u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s \right. \\
&\quad \left. + \sum_{k=1}^p e_{(r_i-h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)) \right], \quad i = 1, 2, \dots, n
\end{aligned}$$

and

$$\begin{aligned}
&(\Phi_i u)(t) \\
&\geq \frac{1}{\sigma_i - 1} \left[\int_t^{t+\omega} u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s + \sum_{k=1}^p e_{(r_i-h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)) \right] \\
&= \frac{1}{\sigma_i - 1} \left[\int_0^\omega u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s + \sum_{k=1}^p e_{(r_i-h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)) \right] \\
&\geq \frac{1}{\sigma_i} |\Phi_i u|_0, \quad i = 1, 2, \dots, n.
\end{aligned}$$

So $\Phi u \in K$. This completes the proof of Lemma 2.8. \square

Lemma 2.9 Assume that $(H_1) - (H_2)$ hold. Then $\Phi : K \rightarrow K$ defined by (2.3) is completely continuous.

Proof It is easy to see that Φ is continuous and bounded. Now we let us show that Φ maps bounded sets into relatively compact sets. Let $\Omega \subset K$ be an arbitrary open bounded set in K . Then there exists $R > 0$ such that $\|u\| < R$ for any $u = (u_1, u_2, \dots, u_n)^T \in \Omega$. We prove that $\overline{\Phi(\Omega)}$ is compact. In fact, for any $u \in \Omega$ and $t \in [0, \omega]_{\mathbb{T}}$, we have

$$\begin{aligned}
&|(\Phi_i u)(t)| \\
&= \int_t^{t+\omega} H_i(t, s) u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s \\
&\quad + \sum_{t_k \in [t, t+\omega)_{\mathbb{T}}} H_i(t, t_k) e_{(r_i-h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)) \\
&\leq \frac{\sigma_i}{\sigma_i - 1} \left[\int_0^\omega u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s + \sum_{k=1}^p e_{(r_i-h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)) \right] \\
&\leq \frac{\sigma_i}{\sigma_i - 1} \left[R \omega \left(\max_{s \in [0, \omega]_{\mathbb{T}}, u \in B(0, R)} \{F_i(s, u)\} + \max_{s \in [0, \omega]_{\mathbb{T}}, u \in B(0, R)} \{G_i(s, u)\} \right) \right. \\
&\quad \left. + \sigma_i \sum_{k=1}^p \max_{u \in B(0, R)} \{I_{ik}(u)\} \right] \triangleq A_i, \quad i = 1, 2, \dots, n
\end{aligned}$$

and

$$\begin{aligned} |(\Phi_i u)^\Delta(t)| &= |[r_i(t) - h_i(t)](\Phi_i x)(t) + u_i(t)[F_i(t, u(t)) + G_i(t, u(t))]| \\ &\leq (r_i^M + h_i^M)A_i + R \left(\max_{t \in [0, \omega]_{\mathbb{T}}, u \in B(0, R)} \{F_i(t, u)\} + \max_{t \in [0, \omega]_{\mathbb{T}}, u \in B(0, R)} \{G_i(t, u)\} \right) \\ &\triangleq B_i, \quad i = 1, 2, \dots, n. \end{aligned}$$

Hence $\|(\Phi u)\| \leq \sum_{i=1}^n A_i$ and $\|(\Phi u)^\Delta\| \leq \sum_{i=1}^n B_i$. It follows from Lemma 2.4 in [36] that $\Phi(\bar{\Omega})$ is relatively compact in X . The proof of Lemma 2.9 is complete. \square

3 Main results

In this section, we give our main results.

Theorem 3.1 *Assume that*

$$(H_4) \quad \max_{1 \leq i \leq n} \left\{ \frac{\gamma_i \sigma_i^2}{\sigma_i - 1} \right\} < 1 \text{ and}$$

$$(H_5) \quad F_i^0 < \infty, G_i^0 < \infty, F_i^\infty > 0, G_i^\infty > 0, i = 1, 2, \dots, n.$$

If (H_1) – (H_5) hold, then system (1.1) has at least one ω -periodic solution.

Proof By assumptions (H_4) and (H_5) there exists $\delta > 0$ such that $\max_{1 \leq i \leq n} \left\{ \frac{\gamma_i \sigma_i^2}{\sigma_i - 1} \right\} + \delta < 1$ and for any $0 < \epsilon < \frac{1}{2} \min\{1, \min_{1 \leq i \leq n} \{F_i^\infty + G_i^\infty\}\}$, there exist two positive numbers $r_0 < R_0$ such that, for $i = 1, 2, \dots, n, j = 1, 2, \dots, m$,

$$\begin{aligned} I_{ik}(u_1, u_2, \dots, u_n) &< (\gamma_i + \epsilon) \sum_{l=1}^n u_l, \quad \text{for } 0 < \sum_{l=1}^n u_l < r_0, \\ F_i(t, u_1, u_2, \dots, u_n) &< (F_i^0 + \epsilon) \sum_{l=1}^n u_l, \quad \text{for } 0 < \sum_{l=1}^n u_l < r_0, \\ G_i(t, u_1, u_2, \dots, u_n) &< (G_i^0 + \epsilon) \sum_{l=1}^n u_l, \quad \text{for } 0 < \sum_{l=1}^n u_l < r_0, \\ F_i(t, u_1, u_2, \dots, u_n) &> (F_i^\infty - \epsilon) \sum_{l=1}^n u_l, \quad \text{for } \sum_{l=1}^n u_l > R_0, \\ G_i(t, u_1, u_2, \dots, u_n) &< (G_i^\infty - \epsilon) \sum_{l=1}^n u_l, \quad \text{for } \sum_{l=1}^n u_l > R_0. \end{aligned}$$

Take

$$0 < r < \min \left\{ \min_{1 \leq i \leq n} \left\{ \frac{\delta(\sigma_i - 1)}{\omega \sigma_i (F_i^0 + G_i^0 + 1)} \right\}, r_0 \right\}$$

and

$$R = \max \left\{ \min_{1 \leq i \leq n} \left\{ \frac{1}{\sigma_i} \right\} R_0, \omega \min_{1 \leq i \leq n} \left\{ \frac{1}{\sigma_i} \right\} \left[\min_{1 \leq i \leq n} \left\{ \frac{F_i^\infty + G_i^\infty - 2\epsilon}{\sigma_i(\sigma_i - 1)} \right\} \right]^{-1} \right\}.$$

Then we have $0 < r < R$. It follows from Lemmas 2.8 and 2.9 that Φ is strict-set-contractive on $K_{r,R}$. By Lemma 2.5 it is easy to see that if there exists $u^* \in K$ such that $\Phi u^* = u^*$, then

u^* is a positive ω -periodic solution of system (1.1). Now, we will prove that condition (ii) of Lemma 2.7 holds.

First, we prove that $\Phi u \not\leq u, \forall u \in K, \|u\| = r$. Otherwise, there exists $u \in K, \|u\| = r$, such that $\Phi u \leq u$. So, $\|u\| > 0$ and $\Phi u - u \in K$, which implies that

$$(\Phi_i u)(t) - u_i(t) \geq \frac{1}{\sigma_i} |\Phi_i u - u_i|_0 \geq 0, \quad \forall t \in [0, \omega]_{\mathbb{T}}, i = 1, 2, \dots, n. \quad (3.1)$$

Moreover, for $t \in [0, \omega]_{\mathbb{T}}$, we have

$$\begin{aligned} & (\Phi_i u)(t) \\ &= \int_t^{t+\omega} H_i(t, s) u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s \\ & \quad + \sum_{t_k \in [t, t+\omega]_{\mathbb{T}}} H_i(t, t_k) e_{(r_i - h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)) \\ &\leq \frac{\sigma_i}{\sigma_i - 1} \left[\int_0^\omega u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s + \sum_{k=1}^p e_{(r_i - h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)) \right] \\ &\leq \frac{\omega \sigma_i |u_i|_0}{\sigma_i - 1} \left[(F_i^0 + \epsilon) \sum_{i=1}^n |u_i|_0 + (G_i^0 + \epsilon) \sum_{i=1}^n |u_i|_0 \right] + \frac{\sigma_i^2}{\sigma_i - 1} (\gamma_i + \epsilon) \sum_{i=1}^n |u_i|_0 \\ &\leq \frac{\omega \sigma_i (F_i^0 + G_i^0 + 2\epsilon)}{\sigma_i - 1} r^2 + \frac{\sigma_i^2 (\gamma_i + \epsilon)}{\sigma_i - 1} r \\ &< \left[\frac{\omega \sigma_i (F_i^0 + G_i^0 + 2\epsilon)}{\sigma_i - 1} \times \frac{\delta(\sigma_i - 1)}{\omega \sigma_i (F_i^0 + G_i^0 + 1)} + \frac{\sigma_i^2 (\gamma_i + \epsilon)}{\sigma_i - 1} \right] r \\ &= \left[\frac{\delta(F_i^0 + G_i^0 + 2\epsilon)}{F_i^0 + G_i^0 + 1} + \frac{\sigma_i^2 (\gamma_i + \epsilon)}{\sigma_i - 1} \right] r, \quad i = 1, 2, \dots, n. \end{aligned} \quad (3.2)$$

From (3.1)–(3.2) and the arbitrariness of ϵ we get

$$\|u\| \leq \|\Phi u\| \leq \left(\max_{1 \leq i \leq n} \left\{ \frac{\gamma_i \sigma_i^2}{\sigma_i - 1} \right\} + \delta \right) r < r = \|u\|,$$

which is a contradiction. Next, we prove that $\Phi u \not\leq u, \forall u \in K, \|u\| = R$. Indeed, we only need to prove that $\Phi u \not\leq u, \forall u \in K, \|u\| = R$. For contradiction, suppose that there exists $u \in K$ with $\|u\| = R$ such that $\Phi u \leq u$. Thus $u - \Phi u \in K \setminus \{\theta = (0, 0, \dots, 0)^T\}$. Furthermore, for any $t \in [0, \omega]_{\mathbb{T}}$, we have

$$u_i(t) - (\Phi u)(t) \geq \frac{1}{\sigma_i} |u_i - \Phi_i u|_0 \geq 0, \quad i = 1, 2, \dots, n. \quad (3.3)$$

Since $u \in K$ and $\|u\| = R$, for all $s \in [0, \omega]_{\mathbb{T}}$, we find

$$\begin{aligned} \sum_{l=1}^n u_l(s - \tau_{il}(s)) &\geq \sum_{l=1}^n \frac{1}{\sigma_l} |u_l|_0 \geq \min_{1 \leq l \leq n} \left\{ \frac{1}{\sigma_l} \right\} \sum_{l=1}^n |u_l|_0 \\ &= \min_{1 \leq l \leq n} \left\{ \frac{1}{\sigma_l} \right\} R \geq R_0, \quad i = 1, 2, \dots, n \end{aligned}$$

and

$$\begin{aligned} & \sum_{l=1}^n \int_{-\infty}^0 K_{il}(\tau) u_l(s+\tau) \Delta \tau \\ & \geq \sum_{l=1}^n \frac{1}{\sigma_l} |u_l|_0 \int_{-\infty}^0 K_{il}(\tau) \Delta \tau \\ & \geq \min_{1 \leq l \leq n} \left\{ \frac{1}{\sigma_l} \right\} \sum_{l=1}^n |u_l|_0 = \min_{1 \leq l \leq n} \left\{ \frac{1}{\sigma_l} \right\} R \geq R_0, \quad i = 1, 2, \dots, n. \end{aligned}$$

In addition, for any $t \in [0, \omega]_{\mathbb{T}}$, we have

$$\begin{aligned} & (\Phi_i u)(t) \\ & = \int_t^{t+\omega} H_i(t, s) u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s \\ & \quad + \sum_{t_k \in [t, t+\omega]_{\mathbb{T}}} H_i(t, t_k) e_{(r_i-h_i)}(\sigma(t_k), t_k) I_{ik}(u(t_k)) \\ & \geq \frac{|u_i|_0}{\sigma_i(\sigma_i - 1)} \int_0^\omega [F_i(s, u(s)) + G_i(s, u(s))] \Delta s \\ & \geq \frac{\omega |u_i|_0}{\sigma_i(\sigma_i - 1)} \left[(F_i^\infty - \epsilon) \sum_{i=1}^n \frac{1}{\sigma_i} |u_i|_0 + (G_i^\infty - \epsilon) \sum_{l=1}^n \int_{-\infty}^0 K_{il}(\tau) \frac{1}{\sigma_l} |u_l|_0 \Delta \tau \right] \\ & \geq \frac{\omega |u_i|_0}{\sigma_i(\sigma_i - 1)} (F_i^\infty + G_i^\infty - 2\epsilon) \min_{1 \leq i \leq n} \left\{ \frac{1}{\sigma_i} \right\} R, \quad i = 1, 2, \dots, n. \end{aligned} \quad (3.4)$$

It follows from (3.3) and (3.4) that

$$\begin{aligned} \|\Phi u\| & = \sum_{i=1}^n |\Phi_i u|_0 \geq \sum_{i=1}^n \frac{\omega |u_i|_0}{\sigma_i(\sigma_i - 1)} (F_i^\infty + G_i^\infty - 2\epsilon) \min_{1 \leq i \leq n} \left\{ \frac{1}{\sigma_i} \right\} R \\ & \geq \min_{1 \leq i \leq n} \left\{ \frac{F_i^\infty + G_i^\infty - 2\epsilon}{\sigma_i(\sigma_i - 1)} \right\} \min_{1 \leq i \leq n} \left\{ \frac{1}{\sigma_i} \right\} R^2 \omega \geq R. \end{aligned} \quad (3.5)$$

From (3.3)–(3.5) we obtain $\|u\| > \|\Phi u\| \geq R$, a contradiction. Therefore, condition (ii) of Lemma 2.7 holds. By Lemma 2.7 we see that Φ has at least one nonzero fixed point in $K_{r,R}$. Therefore, system (1.1) has at least one positive ω -periodic solution. The proof of Theorem 3.1 is complete. \square

Similarly to the proof of Theorem 3.1, we can show that the existence of positive ω -periodic solutions for the impulsive system without infinite distributed time delays or with pure infinite distributed delays on time scales.

Theorem 3.2 *In system (1.1), assume that $G_i(t, \cdot, \cdot) \equiv 0$, $F_i^0 < \infty$, $F_i^\infty > 0$ ($i = 1, \dots, n$), and (H_1) – (H_4) hold. Then system (1.1) has at least one positive ω -periodic solution.*

Theorem 3.3 *In system (1.1), assume that $F_i(t, \cdot, \cdot) \equiv 0$, $G_i^0 < \infty$, $G_i^\infty > 0$ ($i = 1, \dots, n$), and (H_1) – (H_4) hold. Then system (1.1) has at least one positive ω -periodic solution.*

In system (1.1), if $I_{ik} \equiv 0$ ($i = 1, 2, \dots, n$), then system (1.1) changes into the following non-impulsive system:

$$u_i^\Delta(t) = u_i(t)[r_i(t) + F_i(t, u(t)) + G_i(t, u(t)) - h_i(t)], \quad t \neq t_k, t \in \mathbb{T}, \quad (3.6)$$

where \mathbb{T} , r_i , h_i , F_i , G_i ($i = 1, 2, \dots, n$) are the same as in system (1.1). We have the following:

Theorem 3.4 *Assume that $F_i^0 < \infty$, $G_i^\infty > 0$ ($i = 1, \dots, n$), and (H_1) – (H_3) hold. Then system (3.6) has at least one positive ω -periodic solution.*

Proof Let $X = \{u : u \in C_{\text{rd}}(\mathbb{T}, \mathbb{R}^n), u(t + \omega) = u(t), t \in \mathbb{T}\}$ with the norm $\|u\| = \sum_{i=1}^n |u_i|_0$, where $|u_i|_0 = \sup_{t \in [0, \omega]_{\mathbb{T}}} |u_i(t)|$, $i = 1, 2, \dots, n$. Then X is a Banach space. Define the cone K in X by

$$K = \left\{ u : u = (u_1, u_2, \dots, u_n)^T \in X, u_i(t) \geq \frac{1}{\sigma_i} |u_i|_0, t \in [0, \omega]_{\mathbb{T}}, i = 1, \dots, n \right\}$$

and the map Φ by

$$(\Phi u)(t) = ((\Phi_1 u)(t), (\Phi_2 u)(t), \dots, (\Phi_n u)(t))^T,$$

where $u \in K, t \in \mathbb{T}$,

$$(\Phi_i u)(t) = \int_t^{t+\omega} H_i(t, s) u_i(s) [F_i(s, u(s)) + G_i(s, u(s))] \Delta s, \quad i = 1, 2, \dots, n,$$

and $H_i(t, s)$ ($i = 1, 2, \dots, n$) are defined by (2.2). The remainder of the proof is similar to the proof of Theorem 3.1 and is omitted here. The proof of Theorem 3.4 is complete. \square

Similarly, we can prove that

Theorem 3.5 *In system (3.6), assume that $G_i(t, \cdot, \cdot) \equiv 0$, $F_i^0 < \infty$, $F_i^\infty > 0$ ($i = 1, \dots, n$), and (H_1) – (H_3) hold. Then system (3.6) has at least one positive ω -periodic solution.*

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Remark 3.1 Similarly to the previous arguments, we can easily get the corresponding results for systems (1.2) and (1.3).

4 Applications

In this section, as applications of our main results, we give some existence results of positive periodic solutions for Lotka–Volterra systems with or without impulses.

Consider the following Lotka–Volterra system with impulses and time delays on time scales:

$$\begin{cases} u_i^\Delta(t) = u_i(t) \left[r_i(t) + \sum_{j=1}^n a_{ij}(t) u_j(t - \tau_{ij}(t)) \right. \\ \quad \left. + \sum_{j=1}^n b_{ij}(t) \int_{-\infty}^0 K_{ij}(s) u_j(t+s) \Delta s - h_i(t) \right], & t \neq t_k, t \in \mathbb{T}, \\ u_i(t_k^+) = u_i(t_k^-) + I_{ik}(u_i(t_k)), & k = 1, 2, \dots, \end{cases} \quad (4.1)$$

where $i = 1, 2, \dots, n$, \mathbb{T} is an ω -periodic time scale, $\omega > 0$ is a constant, $r_i, h_i \in C_{\text{rd}}(\mathbb{T}, (0, \infty))$, $a_{ij}, b_{ij} \in C_{\text{rd}}(\mathbb{T}, (0, \infty))$, $\tau_{ij} \in C_{\text{rd}}(\mathbb{T}, (0, \infty)_{\mathbb{T}})$ ($i, j = 1, 2, \dots, n$) are rd-continuous ω -periodic functions. $K_{ij} \in C_{\text{rd}}((-\infty, 0)_{\mathbb{T}}, (0, \infty))$ with $\int_{-\infty}^0 K_{ij}(s) \Delta s = 1$ ($i, j = 1, 2, \dots, n$), $I_{ik} \in C((0, \infty), (0, \infty))$, there exists a positive integer p such that $t_{i,k+p} = t_k + \omega$ and $I_{i,k+p} = I_{ik}$ for $k \in \mathbb{Z}$, and $[0, \omega)_{\mathbb{T}} \cap \{t_k : k \in \mathbb{Z}\} = \{t_1, t_2, \dots, t_p\}$.

Theorem 4.1 Assume that $r_i(t) < h_i(t)$, $\sup_{t \in [0, \omega]_{\mathbb{T}}} \{\mu(t)[h_i(t) - r_i(t)]\} < 1$, $\max_{1 \leq i \leq n} \{\frac{\gamma_i \sigma_i^2}{\sigma_i - 1}\} < 1$, and $a_{ij}(t), b_{ij}(t) > 0$ ($i, j = 1, 2, \dots, n$). Then system (4.1) has at least one positive ω -periodic solution.

Proof In this case,

$$\begin{aligned} F_i(t, u_1, u_2, \dots, u_n) &= \sum_{j=1}^n a_{ij}(t) u_j, G_i(t, u_1, u_2, \dots, u_n) = \sum_{j=1}^n b_{ij}(t) u_j, \quad i = 1, 2, \dots, n, \\ F_i^0 &= \limsup_{\sum_{i=1}^n u_i \rightarrow 0} \max_{t \in [0, \omega]_{\mathbb{T}}} \frac{F_i(t, u_1, \dots, u_n)}{\sum_{i=1}^n u_i} \leq \max_{1 \leq j \leq n} \left\{ \max_{t \in [0, \omega]_{\mathbb{T}}} \{a_{ij}(t)\} \right\} < \infty, \quad i = 1, 2, \dots, n, \\ G_i^0 &= \limsup_{\sum_{i=1}^n u_i \rightarrow 0} \max_{t \in [0, \omega]_{\mathbb{T}}} \frac{G_i(t, u_1, \dots, u_n)}{\sum_{i=1}^n u_i} \leq \max_{1 \leq j \leq n} \left\{ \max_{t \in [0, \omega]_{\mathbb{T}}} \{b_{ij}(t)\} \right\} < \infty, \quad i = 1, 2, \dots, n, \\ F_i^\infty &= \liminf_{\sum_{i=1}^n u_i \rightarrow 0} \min_{t \in [0, \omega]_{\mathbb{T}}} \frac{F_i(t, u_1, \dots, u_n)}{\sum_{i=1}^n u_i} \geq \min_{1 \leq j \leq n} \left\{ \min_{t \in [0, \omega]_{\mathbb{T}}} \{a_{ij}(t)\} \right\} > 0, \quad i = 1, 2, \dots, n, \\ G_i^\infty &= \liminf_{\sum_{i=1}^n u_i \rightarrow 0} \min_{t \in [0, \omega]_{\mathbb{T}}} \frac{G_i(t, u_1, \dots, u_n)}{\sum_{i=1}^n u_i} \geq \min_{1 \leq j \leq n} \left\{ \min_{t \in [0, \omega]_{\mathbb{T}}} \{b_{ij}(t)\} \right\} > 0, \quad i = 1, 2, \dots, n. \end{aligned}$$

It follows from Theorem 3.1 that system (4.1) has at least one positive ω -periodic solution. The proof of Theorem 4.1 is complete. \square

Considering the existence of positive periodic solutions for system (4.1) without impulses, we conclude by the following assertion, which follows from Theorem 3.4.

Theorem 4.2 In system (4.1), assume that $I_{ik} \equiv 0$, $r_i(t) < h_i(t)$, $\sup_{t \in [0, \omega]_{\mathbb{T}}} \{\mu(t)[h_i(t) - r_i(t)]\} < 1$, $a_{ij}(t), b_{ij}(t) > 0$ ($i, j = 1, 2, \dots, n$). Then system (4.1) has at least one positive ω -periodic solution.

5 Conclusions

In this paper, we study a class of impulsive Lotka–Volterra cooperative population models with time-delay and harvesting control on time scales \mathbb{T} . This model unifies the continuous and difference cases of impulsive Lotka–Volterra cooperative population model with time delay. Applying the fixed point theorem of strict-set-contraction, we have established some new existence conditions of positive periodic solutions for this model. As applications, the existence conditions of positive periodic solutions are analyzed for some common Lotka–Volterra systems on time scales.

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