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Existence and uniqueness of solutions for a class of nonlinear integro-differential equations on unbounded domains in Banach spaces

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Abstract

In this paper, the existence and uniqueness of solutions for a class of nonlinear integro-differential equations on unbounded domains in Banach spaces are established under more general conditions by constructing a special Banach space and using cone theory and the Banach contraction mapping principle. The results obtained herein improve and generalize some well-known results.

MSC: 47H07; 47H10; 47G20; 34G20

Keywords: Fixed points of operator; Banach contraction mapping principle; Integro-differential equation

1 Introduction

Nonlinear integro-differential equations in abstract spaces arise in different fields of physical sciences, engineering, biology, and applied mathematics. The theory of nonlinear integro-differential equations in abstract spaces is a fast growing field with important applications to a number of areas of analysis as well as other branches of science [1]. In recent years, there has been a significant development in nonlinear integro-differential equations (see [1–20] and the references therein).

Using the upper and lower solutions method and monotone iterative technique, Guo, Liu and Zheng et al. [2–4] studied the existence and uniqueness of solutions for the first order integro-differential equations

$$\begin{cases} u'(t) = f(t, u(t), (Tu)(t), (Su)(t)), \quad t \in I = [t_0, t_0 + a], \\ u(t_0) = u_0, \end{cases}$$

in Banach space E, where $u_0 \in E, f : I \times E \times E \times E \to E$, for any $u \in C[I, E], g(t) = f(t, u(t), (Tu)(t), (Su)(t)) : I \to E$ is continuous, and T is a Volterra integral operator defined by

$$(Tu)(t) = \int_{t_0}^{t} k(t, s)u(s)\, ds,$$

S is a Hammerstein integral operator defined by

$$(Su)(t) = \int_{t_0}^{t_0+a} h(t,s)u(s)\,ds,$$

where $k \in C[D_1, R], h \in C[D_2, R], D_1 = \{(t,s) \in I \times I : t \geq s\}, D_2 = \{(t,s) \in I \times I\}$.

In [5], the authors studied the second order integro-differential equations

$$\begin{cases} u''(t) = f(t, u(t), (Tu)(t)), & t \in I = [0,a], \\ u(0) = u_0, & u'(0) = u_1, \end{cases} \tag{1.1}$$

in Banach space E, where $u_0, u_1 \in E, f : I \times E \times E \to E$, for any $u \in C[I,E], g(t) = f(t, u(t), (Tu)(t)) : I \to E$ is continuous, and T is a Volterra integral operator defined by

$$(Tu)(t) = \int_0^t k(t,s)u(s)\,ds,$$

where $k \in C[D, R], D = \{(t,s) \in I \times I : t \geq s\}$. By using an iteration method, the existence and uniqueness results for second order integro-differential equations (1.1) are obtained without demanding the existence of upper and lower solutions and monotonicity conditions.

In [1], Guo, Lakshmikantham and Liu studied the IVP for first order integro-differential equations of Volterra type on an infinite interval in Banach space E:

$$\begin{cases} u'(t) = f(t, u(t), (Tu(t))), & t \in J = [0,\infty), \\ u(0) = u_0, \end{cases} \tag{1.2}$$

where $u_0 \in E, f : J \times E \times E \to E$, for any $u \in C[J,E], g(t) = f(t, u(t), (Tu)(t)) : J \to E$ is continuous, and T is a Volterra integral operator defined by

$$(Tu)(t) = \int_0^t k(t,s)u(s)\,ds,$$

where $k \in C[D, R], D = \{(t,s) \in J \times J : t \geq s\}$. By using the Banach contraction mapping principle, the authors obtained the following results.

Theorem 1.1 *If the following conditions are satisfied*:

(H_1) $\|f(t,u,v) - f(t,\overline{u},\overline{v})\| \leq \beta(t)(a\|u-\overline{u}\| + b\|v-\overline{v}\|), \forall t \in J, u, v, \overline{u}, \overline{v} \in E$, *where constants* $a \geq 0, b \geq 0, \beta \in C[J, R^+]$;

(H_2)

$$k^* = \sup_{t\in J} \int_0^t \left|k(t,s)\right| ds < \infty,$$

$$\beta^* = \int_0^\infty \beta(t)\,dt < \infty,$$

$$a^* = \int_0^\infty \left\|f(t,\theta,\theta)\right\| dt < \infty,$$

here θ *denotes the zero element of* E;

(H_3) $c_0 = (a + bk^*)\beta^* < 1$, *then IVP* (1.2) *has a unique solution* $x^*(t)$ *in* $C^1[J,E] \cap BC[J,E]$.

Remark 1.1 Clearly, the assumptions imposed in Theorem 1.1 are too strict. As is well known, there is no research focused on the improvement of the assumptions in Theorem 1.1. By constructing a special Banach space and using cone theory and the Banach contraction mapping principle, this paper improves Theorem 1.1 without assumptions (H_2) and (H_3).

2 Preliminaries and lemmas

Let $(E, \|\cdot\|)$ be a real Banach space and P be a cone in E which defines a partial ordering in E by $x \leq y$ if and only if $y - x \in P$. θ denotes the zero element in E. A cone P is said to be normal if there exists a constant $N > 0$ such that, for any $x, y \in E$, $\theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$. A cone P is said to be generating if $E = P - P$, i.e., every element $x \in E$ can be represented in the form $x = y - z$, where $y, z \in P$. A cone is called solid if it contains interior points, i.e., $\mathring{P} \neq \emptyset$. An operator $T : E \to E$ is said to be a positive operator if $x \geq \theta$ implies $Tx \geq \theta$. The detailed contents of the cone and partial ordering may be found in [6–8].

Let $\beta(t)$ be a nonnegative continuous function on J, $k(t,s)$ be continuous on $D = \{(t,s) | 0 \leq s \leq t < \infty\}$. Set

$$\lambda(t) = \max\{t, 1\}, \qquad \phi(t) = \max\{a, b, 1\} \cdot \sup_{s\in[0,t]} \{\beta(s) + 1\},$$

$$\varphi(t,s) = \max\{|k(t,s)|, 1\}, \qquad K(t) = \sup_{0\leq s\leq t} \{\varphi(t,s)\},$$

$$\Phi_1(t) = \int_0^t \left[\lambda(s)\phi(s)K(s) + \|f(s,\theta,\theta)\|\right] ds, \qquad \Phi_2(t) = \int_0^t \lambda(s)\phi(s)K(s)\, ds,$$

$$(T_1u)(t) = \int_0^t \varphi(t,s)u(s)\, ds,$$

$$\|u\|_{\beta 1} = \sup_{t\in J}\{e^{-4\Phi_1(t)}\|u(t)\|\}, \qquad \|u\|_{\beta 2} = \sup_{t\in J}\{[\lambda(t)]^{-1}e^{-4\Phi_2(t)}\|u(t)\|\},$$

$$C_{\beta 1} = \{\|u\|_{\beta 1} < \infty | u : J \to E \text{ is continuous}\},$$

$$C_{\beta 2} = \{\|u\|_{\beta 2} < \infty | u : J \to E \text{ is continuous}\}.$$

Then $C_{\beta 1}$ is a Banach space with norm $\|\cdot\|_{\beta 1}$, and $C_{\beta 2}$ is a Banach space with norm $\|\cdot\|_{\beta 2}$.

Let P be a cone of E, $P_{\beta 2} = \{u \in C_{\beta 2} | u \geq \theta, t \in J\}$. Clearly, $P_{\beta 2}$ is a cone of $C_{\beta 2}$.

Lemma 2.1 *Let*

$$(\sigma u)(t) = \int_0^t \beta(s)\big(a\|u(s)\| + b\|(Tu)(s)\|\big)\, ds, \quad u \in C_{\beta 1}, t \in J,$$

then

$$(\sigma u)(t) \leq \frac{1}{2}\|u\|_{\beta 1}e^{4\Phi_1(t)}.$$

Proof Direct calculations shows that, for $u \in C_{\beta 1}$ and $t \in J$,

$$(\sigma u)(t) = \int_0^t \beta(s)\big(a\|u(s)\| + b\|(Tu)(s)\|\big)\, ds$$

$$
\begin{aligned}
&\leq \int_0^t \phi(s)\left(\|u(s)\| + \int_0^s K(s)\|u(\tau)\|d\tau\right) ds \\
&\leq \int_0^t \phi(s)\left(\|u\|_{\beta 1}e^{4\Phi_1(s)} + \int_0^s K(s)\|u\|_{\beta 1}e^{4\Phi_1(\tau)}\,d\tau\right) ds \\
&\leq \|u\|_{\beta 1}\int_0^t \phi(s)\left(e^{4\Phi_1(s)} + K(s)\int_0^s \big(\lambda(\tau)\phi(\tau)K(\tau) + \|f(\tau,\theta,\theta)\|\big)e^{4\Phi_1(\tau)}\,d\tau\right) ds \\
&\leq \|u\|_{\beta 1}\int_0^t \phi(s)\left(e^{4\Phi_1(s)} + \frac{1}{4}K(s)e^{4\Phi_1(s)}\right) ds \\
&\leq 2\|u\|_{\beta 1}\int_0^t \lambda(s)\phi(s)K(s)e^{4\Phi_1(s)}\,ds \\
&\leq \frac{1}{2}\|u\|_{\beta 1}e^{4\Phi_1(t)}.
\end{aligned}
$$
□

Lemma 2.2 *Let*

$$(\delta u)(t) = \int_0^t (t-s)\beta(s)\big(a\|u(s)\| + b\|(T_1u)(s)\|\big)\,ds, \quad u \in C_{\beta 2}, t \in J,$$

then

$$(\delta u)(t) \leq \frac{1}{2}\|u\|_{\beta 2}\lambda(t)e^{4\Phi_2(t)}.$$

Proof Direct calculations show that, for $u \in C_{\beta 2}$ and $t \in J$,

$$
\begin{aligned}
(\delta u)(t) &= \int_0^t (t-s)\beta(s)\big(a\|u(s)\| + b\|(T_1u)(s)\|\big)\,ds \\
&\leq \int_0^t \lambda(t)\phi(s)\left(\|u(s)\| + \int_0^s K(s)\|u(\tau)\|d\tau\right) ds \\
&\leq \lambda(t)\int_0^t \phi(s)\left(\|u\|_{\beta 2}\lambda(s)e^{4\Phi_2(s)} + \int_0^s K(s)\|u\|_{\beta 2}\lambda(\tau)e^{4\Phi_2(\tau)}\,d\tau\right) ds \\
&\leq \|u\|_{\beta 2}\lambda(t)\int_0^t \phi(s)\left(\lambda(s)e^{4\Phi_2(s)} + K(s)\int_0^s \lambda(\tau)\phi(\tau)K(\tau)e^{4\Phi_2(\tau)}\,d\tau\right) ds \\
&\leq \|u\|_{\beta 2}\lambda(t)\int_0^t \phi(s)\big(\lambda(s)e^{4\Phi_2(s)} + K(s)e^{4\Phi_2(s)}\big)\,ds \\
&\leq 2\|u\|_{\beta 2}\lambda(t)\int_0^t \lambda(s)\phi(s)K(s)e^{4\Phi_2(s)}\,ds \\
&\leq \frac{1}{2}\|u\|_{\beta 2}\lambda(t)e^{4\Phi_2(t)}.
\end{aligned}
$$
□

Lemma 2.3 ([6]) *Let P be a generating normal cone and B a positive linear operator whose spectral radius satisfies* $r(B) < 1$. *If the operator A satisfies the estimate*

$$-B(x-y) \leq Ax - Ay \leq B(x-y) \quad (x, y \in E; x \geq y),$$

then A has a unique fixed point in E, and for each initial approximation $x_0 \in E$, *let* $x_n = Ax_{n-1} (n = 1, 2, \ldots)$, *then we have* $x_n \to x^*$ $(n \to \infty)$.

3 Main results

Theorem 3.1 *Suppose that* (H_1) *holds. Then IVP* (1.2) *has a unique solution* $u \in C_{\beta 1}$.

Proof It is well known that u is a solution of IVP (1.2) if and only if

$$u(t) = u_0 + \int_0^t f\bigl(s, u(s), (Tu)(s)\bigr)\,ds, \quad t \in J.$$

Define the operator $A : C_{\beta 1} \to C_{\beta 1}$ by

$$(Au)(t) = u_0 + \int_0^t f\bigl(s, u(s), (Tu)(s)\bigr)\,ds, \quad t \in J. \tag{3.1}$$

It follows from (H_1) that

$$\bigl\|f(t,u,v)\bigr\| \le \bigl\|f(t,\theta,\theta)\bigr\| + \beta(t)\bigl(a\|u\| + b\|v\|\bigr), \quad \forall t \in J, u, v \in E. \tag{3.2}$$

For any $u \in C_{\beta 1}$, by (H_1), (3.1) and (3.2),

$$\begin{aligned}\bigl\|(Au)(t)\bigr\| &\le \|u_0\| + \int_0^t \bigl[\bigl\|f(s,\theta,\theta)\bigr\| + \beta(s)\bigl(a\bigl\|u(s)\bigr\| + b\bigl\|(Tu)(s)\bigr\|\bigr)\bigr]\,ds \\ &\le \|u_0\| + \Phi_1(t) + \frac{1}{2}\|u\|_{\beta 1} e^{4\Phi_1(t)}, \quad \forall t \in J,\end{aligned}$$

then $Au \in C_{\beta 1}$, so $A : C_{\beta 1} \to C_{\beta 1}$.

On the other hand, for any $u, v \in C_{\beta 1}$, by (H_1) and Lemma 2.1,

$$\begin{aligned}\bigl\|Au(t) - Av(t)\bigr\| &\le \int_0^t \beta(s)\bigl(a\bigl\|u(s) - v(s)\bigr\| + b\bigl\|(Tu)(s) - (Tv)(s)\bigr\|\bigr)\,ds \\ &\le \frac{1}{2}\|u - v\|_{\beta 1} e^{4\Phi_1(t)},\end{aligned}$$

then $\|Au - Av\|_{\beta 1} \le \frac{1}{2}\|u - v\|_{\beta 1}$. Thus the Banach contraction mapping principle implies that A has a unique fixed point in $C_{\beta 1}$. □

In the following, we consider the second order nonlinear integro-differential equations of Volterra type on an infinite interval,

$$\begin{cases} u'' = f(t, u(t), (Tu)(t)), & t \in J = [0, \infty), \\ u(0) = u_0, & u'(0) = u_1. \end{cases} \tag{3.3}$$

Suppose $f : J \times E \times E \to E$, for any $u \in C[J, E]$, $g(t) = f(t, u(t), (Tu)(t)) : J \to E$ is continuous, $u_0, u_1 \in E$.

Theorem 3.2 *Let* P *be a normal solid cone of* E. *Assume that there exists* $\beta \in C[J, R^+]$ *such that, for any* $y_1, y_2, \overline{y}_1, \overline{y}_2, \in E$, $y_1 \ge \overline{y}_1$, $y_2 \ge \overline{y}_2$, *we have*

$$\begin{aligned}-\beta(t)\bigl[a(y_1 - \overline{y}_1) + b(y_2 - \overline{y}_2)\bigr] &\le f(t, y_1, y_2) - f(t, \overline{y}_1, \overline{y}_2) \\ &\le \beta(t)\bigl[a(y_1 - \overline{y}_1) + b(y_2 - \overline{y}_2)\bigr].\end{aligned} \tag{3.4}$$

Then IVP (3.3) *has a unique solution in* $C_{\beta 2}$.

Proof It is clear that u is a solution of IVP (3.3) if and only if u is a solution of the following integral equation:

$$u(t) = u_0 + tu_1 + \int_0^t (t-s)f\bigl(s, u(s), (Tu)(s)\bigr)\,ds, \quad t \in J.$$

Define operators A and B by

$$(Au)(t) = u_0 + tu_1 + \int_0^t (t-s)f\bigl(s, u(s), (Tu)(s)\bigr)\,ds, \quad t \in J.$$

$$(Bu)(t) = \int_0^t (t-s)\beta(s)\bigl(au(s) + b(T_1u)(s)\bigr)\,ds, \quad t \in J.$$

By Lemma 2.2, it is easy to see that B is a positive linear bounded operator on $C_{\beta 2}$, and $\|B\| < \frac{1}{2}$, then $r(B) < \frac{1}{2}$. By (3.4), for any $u, v \in C_{\beta 2}, u \geq v$,

$$-B\bigl(u(t) - v(t)\bigr) \leq (Au)(t) - (Av)(t) \leq B\bigl(u(t) - v(t)\bigr), \quad \forall t \in J.$$

Since P is a normal cone of E, it is easy to show that $P_{\beta 2}$ is normal in $C_{\beta 2}$. Since P is a solid cone of E, by Lemma 2.1.2 in [1] we see that $P_{\beta 2}$ is also a solid cone in $C_{\beta 2}$, and so, from Lemma 1.4.1 in [7], we know that $P_{\beta 2}$ is a generating cone in $C_{\beta 2}$. Hence all the conditions of Lemma 2.3 are satisfied, and the conclusion of Theorem 3.2 holds. □

Remark 3.1 In most of the early work, for example [1], the conditions (H_2) and (H_3) play an important role in the proof of the main results. Undoubtedly, it is interesting and important to remove these conditions, which is very helpful for the applications of IVPs (1.2) and (3.3). In this paper, the existence and uniqueness of solutions for a class of nonlinear integro-differential equations of mixed type on unbounded domains in Banach spaces are established under more general conditions. The restrictive conditions (H_2) and (H_3) are removed; this implies that our results in essence improve and generalize the corresponding conclusions of [1–20].

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