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Optimal dynamic mean-variance asset-liability management under the Heston model

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Abstract

This paper studies a continuous-time mean-variance asset-liability management problem under the Heston model. Specifically, an asset-liability manager is allowed to invest in a risk-free asset and a risky asset whose price process is governed by the Heston model. By applying the Lagrange duality theorem and stochastic control theory, we derive the closed-form expressions of the efficient investment strategy and the efficient frontier. Moreover, we provide numerical experiments to analyze the sensitivity of the efficient frontier with respect to the relevant parameters in the Heston model.

Keywords: Continuous-time mean-variance; Asset-liability management; Heston model; Efficient investment strategy; Efficient frontier

1 Introduction

Asset-liability management (ALM) is essential for financial security systems such as banks, life insurance companies, property insurance companies and pension funds. In recent years, dynamic allocation strategies for mean–variance (M–V) ALM problems have been studied widely. These studies consider optimization problems of selecting optimal portfolios that can yield sufficient returns in compensating the corresponding liability. Sharpe and Tint [1] first consider an ALM problem under the static M–V framework. Keel and Müller [2] study a portfolio choice with liabilities and show that liabilities affect the efficient frontier. Based on the multi-period M–V framework, Leippold et al. [3] investigate a multi-period ALM problem and derive explicit expressions for the efficient investment strategy and the efficient frontier. By using the stochastic linear-quadratic control theory, Chiu and Li [4] study a continuous-time ALM problem where the risky assets' prices and the liability value are both governed by geometric Brownian motions. Xie et al. [5] also study a continuous-time ALM problem while the liability process is governed by a Brownian motion with drift. Chen et al. [6] and Chen and Yang [7] extend the work of Chiu and Li [4] and Leippold et al. [3] to the cases with a Markovian regime switching market. Chiu and Wong [8, 9] apply the backward stochastic differential equation (BSDE) method to study the M–V ALM problems with cointegrated risky assets. Yao et al. [10, 11], respectively, consider a continuous-time M–V ALM problem and a multi-period M–V ALM problem with uncertain time horizon. Chiu and Wong [12] investigate a M–V ALM problem with asset correlation risks, which are modeled by a multivariate Wishart process. Pan and Xiao [13] study an optimal M–V ALM problem with stochastic interest rates and inflation risks.

In most of the existing literature, it is standard to assume that the price of the risky asset (stock) follows a geometric Brownian motion, which implies that the volatility of risky asset price is a constant or a deterministic function. However, many phenomena, such as the volatility clustering, the volatility smile, the heavy-tailed nature of return distributions, etc., cannot be explained within the framework of deterministic volatility models. As natural extensions, stochastic volatility (SV) models have been proposed by many scholars such as the constant elasticity of variance (CEV) model (see Cox and Ross [14]), Stein–Stein model (see Stein and Stein [15]) and Heston model (see Heston [16]), where the CEV model and the Heston model have widely used in portfolio selection problems. Zhang and Chen [17] investigate a M–V ALM problem under the CEV model and derive the corresponding efficient investment strategy and efficient frontier by using the method of BSDE. Li et al. [18] consider a derivative-based optimal investment strategy for a M–V ALM problem under the Heston model and apply the BSDE method to obtain the explicit expressions of the efficient investment strategies and efficient frontiers for three special cases. In this paper, we also consider a continuous-time M–V ALM problem under the Heston model. Different from the work of Li et al. [18], we use a generalized Brownian motion to characterize the liability process, in which the expected growth rate of liabilities is cointegrated with the volatility of risky assets. Moreover, we apply the Hamilton–Jacobi–Bellman (HJB) equation method to derive the closed-form expressions of the efficient investment strategy and the efficient frontier. To the best of our knowledge, this problem has not yet received a complete treatment in the existing literature. The reason is that when both the Heston model and the more general liability model are introduced, this ALM problem becomes more complicated for adding the two new state variables. The dimension enlargement in the state space drastically increases the difficulty level in solving the associated HJB equation which comes from the dynamic programming approach. Here we need to point out that to reduce the dimension and obtain the closed-form expression, many scholars directly absorb the liability process into the dynamic process of assets instead of the traditional surplus process (the difference of liability from the asset). To solve this problem, we first apply the Lagrange multiplier method to transform the original problem into a standard stochastic optimal control problem and establish the corresponding extended HJB equation. Then we obtain the closed-form expression of the optimal investment strategy by solving the extended HJB equation. Furthermore, by the Lagrange duality theorem, we derive the efficient investment strategy and efficient frontier of this M–V ALM problem explicitly, which is our main contribution. We also discuss the effects on the efficient frontier of the stochastic volatility model parameters.

The remainder of this paper is organized as follows. Section 2 formulates the dynamic M–V ALM problem under the Heston model. Section 3 gives the efficient investment strategy and efficient frontier of the M–V ALM problem by applying the stochastic control theory and partial differential equation (PDE) method. Section 4 provides numerical examples to analyze the sensitivity of the efficient frontier based on the Heston model. Section 5 concludes this paper.

2 Problem formulation

In this section, we introduce the financial market and establish the optimal dynamic M–V ALM problem under the Heston model.

2.1 The financial market

Let $(\Omega, F, \{F_s\}_{t\leq s\leq T}, \mathrm{P})$ be a filtered complete probability space satisfying the usual conditions, where $T > 0$ is a finite constant representing the investment time horizon; $\{F_s\}_{t\leq s\leq T}$ is generated by two standard one-dimensional Brownian motions $W_S(t)$ and $W_m(t)$, P is a real-world probability measure, and the expectation with respect to P is denoted $\mathrm{E}[\cdot]$.

In this probability space, we consider the manager who can invest in a risk-free asset (cash) and a risky asset (stock). The price process of the risk-free asset $B(t)$ is governed by

$$\frac{dB(t)}{B(t)} = r\,dt, \quad B(0) = B_0 > 0, \tag{1}$$

where the constant r represents the risk-free interest rate. The price process of the risky asset $S(t)$ follows the Heston model,

$$\begin{cases} \frac{dS(t)}{S(t)} = [r + \lambda_1 m(t)]\,dt + \sqrt{m(t)}\,dW_S(t), & S(0) = S_0 > 0, \\ dm(t) = k[\theta - m(t)]\,dt + \sigma_m\sqrt{m(t)}\,dW_m(t), & m(0) = m_0 > 0, \end{cases} \tag{2}$$

where $\lambda_1 > 0$ is a constant coefficient capturing the market price of the risk factor $W_S(t)$; k, θ and σ_m are all positive constants and denote the mean-reversion rate, the long-run mean and the volatility coefficient of the instantaneous variance process $m(t)$, respectively; $W_S(t)$ and $W_m(t)$ are correlated with a constant correlation coefficient $\rho_{Sm} \in [-1, 1]$, namely, $\mathrm{Cov}(dW_S(t), dW_m(t)) = \rho_{Sm}\,dt$.

Assumption 1 To ensure that $m(t) > 0$, there exist parameters k, θ and σ_m such that $2k\theta \geq \sigma_m^2$.

In what follows, we consider the situation where the manager endowed with an initial wealth $X(0) = X_0 > 0$ at time 0 plans to invest in the financial market dynamically in the horizon $[0, T]$. Let $\pi(t)$ be the proportion of money invested in the stock at time t, and suppose that there are no transaction costs as well as other restrictions in the market. According to Eqs. (1) and (2), the dynamic of the total asset process $X(t)$ is

$$\begin{aligned} \frac{dX(t)}{X(t)} &= \pi(t)\frac{dS(t)}{S(t)} + \left[1 - \pi(t)\right]\frac{dB(t)}{B(t)} \\ &= \left[r + \lambda_1\pi(t)m(t)\right]dt + \pi(t)\sqrt{m(t)}\,dW_S(t). \end{aligned} \tag{3}$$

Here $\pi(t)$ is called an admissible portfolio strategy, i.e., if $\pi(t)$ is F-adapted, $\mathrm{E}[\int_0^T \pi^2(s)\,ds] < +\infty$ and the stochastic differential equation in Eq. (3) together with X_0 has a unique strong solution $X(\cdot)$ corresponding to $\pi(\cdot)$. The set of all admissible portfolio strategies is denoted by Π.

On the other hand, in the course of investment practice, the investor may encounter uncontrollable liabilities and assume that the accumulative liability process $L(t)$ is described by

$$dL(t) = \big[\alpha(t) + \lambda_2 m(t)\big]\,dt + \sigma_L\sqrt{m(t)}\,dW_S(t), \quad L(0) = L_0 > 0, \tag{4}$$

where $\sigma_L\sqrt{m(t)}$ is the volatility of $L(t)$ and σ_L is a non-negative constant. Moreover, $\alpha(t) + \lambda_2 m(t)$ denotes the appreciation rate of $L(t)$, where $\alpha(t)$ is assumed to be a deterministic function of time t and λ_2 is a constant.

Remark 1 To derive an explicit solution, we assume that the random term of liabilities is $\sqrt{m(t)}\,dW_S(t)$ in the interval $(t, t + dt)$.

2.2 The mean-variance asset-liability management optimization problem

In ALM models, one main concern is the surplus which is the difference of asset value and liability value. Thus ALM is also known as the surplus management. Let $Y(t) = X(t) - L(t)$ be the surplus. Then, by Eqs. (3) and (4), $Y(t)$ satisfies the following stochastic differential equation:

$$\begin{aligned} dY(t) &= \big[\big(Y(t) + L(t)\big)\big(r + \lambda_1\pi(t)m(t)\big) - \alpha(t) - \lambda_2 m(t)\big]\,dt \\ &\quad + \big[\big(Y(t) + L(t)\big)\pi(t) - \sigma_L\big]\sqrt{m(t)}\,dW_S(t), \end{aligned} \tag{5}$$

where the initial value $Y(0) = X_0 - L_0 > 0$.

With the initial surplus $Y(0)$, the manager's objective is to find a strategy $\pi(t) \in \Pi$ to minimize the variance of the terminal surplus for a given level of the expected terminal surplus. More specifically, we consider the following optimization problem:

$$\begin{cases} \min_{\pi(t)\in\Pi} \operatorname{Var}[Y(T)] = \mathrm{E}[Y(T) - C]^2, \\ \text{subject to} \begin{cases} \mathrm{E}[Y(T)] = C, \\ Y(t) \text{ satisfies Eq. (5).} \end{cases} \end{cases} \tag{6}$$

The solution $\pi^*(t)$ to Problem (6) is called an efficient investment strategy for $C \geq C^*$, where C^* is the expected terminal surplus corresponding to the global minimum variance of the terminal surplus over all feasible strategies. The point $(\operatorname{Var}[Y(T)], \mathrm{E}[Y(T)])$ corresponds to an efficient investment strategy is called an efficient point. The set of all efficient points forms the efficient frontier in the variance-mean plane.

3 Solution of the optimization problem

In this section, we apply the Lagrange multiplier technique and stochastic control method to obtain a closed-form solution of Problem (6).

3.1 Transformation of the original problem

As widely adopted in the literature, we apply the Lagrange multiplier technique to deal with the constraint $\mathrm{E}[Y(T)] = C$. Define

$$\begin{aligned} J\big(Y(t), t; \pi(t), \gamma\big) &= \mathrm{E}\big[Y(T) - C\big]^2 + 2\gamma\mathrm{E}\big[Y(T) - C\big] \\ &= \mathrm{E}\big[Y(T) + \gamma - C\big]^2 - \gamma^2, \end{aligned} \tag{7}$$

where $\gamma \in \mathrm{R}$ is the Lagrange multiplier. Then by the Lagrange duality theorem (see Luenberger [19]), the original M–V portfolio selection problem (6) is equivalent to the following max–min stochastic control problem:

$$\begin{cases} \max_{\gamma\in\mathrm{R}} \min_{\pi(t)\in\Pi} J(Y(t),t;\pi(t),\gamma) = \mathrm{E}[Y(T)+\gamma-C]^2-\gamma^2, \\ \text{subject to } Y(t) \text{ satisfies Eq. (5).} \end{cases} \tag{8}$$

Clearly, to solve the above max–min stochastic control problem (8), we first need to consider the following quadratic loss minimization problem:

$$\begin{cases} \min_{\pi(t)\in\Pi} J_0(Y(t),t;\pi(t),\gamma) = \mathrm{E}[Y(T)-u]^2, \\ \text{subject to } Y(t) \text{ satisfies Eq. (5),} \end{cases} \tag{9}$$

where $u = C - \gamma$.

Problem (9) can be solved by using the stochastic control method. We now consider a truncated form of Problem (9) beginning at time t and define the corresponding value function as

$$H(t,m,y,l) = \inf_{\pi(t)\in\Pi} \mathrm{E}\bigl\{\bigl[Y(T)-u\bigr]^2 \mid m(t)=m, Y(t)=y, L(t)=l\bigr\} \tag{10}$$

with the boundary condition $H(T,m,y,l) = (y-u)^2$.

If $H(t,m,y,l) \in \mathrm{C}^{1,2,2,2}([0,T]\times\mathrm{R}^+\times\mathrm{R}\times\mathrm{R})$, then by the principle of dynamic programming, $H(t,m,y,l)$ satisfies the following HJB equation:

$$\begin{aligned} &H_t + k(\theta-m)H_m + \frac{\sigma_m^2}{2}mH_{mm} + \bigl[\alpha(t)+\lambda_2 m\bigr]H_l + \frac{\sigma_L^2}{2}mH_{ll} \\ &\quad + \rho_{Sm}\sigma_m\sigma_L mH_{ml} + \bigl[(y+l)r-\alpha(t)-\lambda_2 m\bigr]H_y - m\sigma_L[\sigma_L H_{yl} \\ &\quad + \sigma_m\rho_{Sm}H_{ym}] + \inf_{\pi(t)\in\Pi}\biggl\{(y+l)\lambda_1\pi(t)mH_y + (y+l)m\pi(t)[\sigma_L H_{yl} \\ &\quad + \sigma_m\rho_{Sm}H_{ym}] + \frac{[(y+l)\pi(t)-\sigma_L]^2 m}{2}H_{yy}\biggr\} = 0, \end{aligned} \tag{11}$$

where H is short for $H(t,m,y,l)$, H_t, H_m, H_y, H_l, H_{mm}, H_{yy}, H_{yl}, H_{lm}, H_{ym} and H_{ll} denote the partial derivatives of first and second orders with respect to t, m, y and l, respectively.

We first assume that $H_{yy} > 0$, which will be verified later. The first-order condition for the optimization problem in the HJB Eq. (11) yields the optimal control as

$$\pi^*(t) = \frac{\sigma_L H_{yy} - \sigma_L H_{yl} - \sigma_m\rho_{Sm}H_{ym} - \lambda_1 H_y}{(y+l)H_{yy}}. \tag{12}$$

Substituting (12) into (11), after simplification, we have

$$\begin{aligned} &H_t + k(\theta-m)H_m + \frac{\sigma_m^2}{2}mH_{mm} + \bigl[\alpha(t)+\lambda_2 m\bigr]H_l + \frac{\sigma_L^2}{2}mH_{ll} \\ &\quad + \rho_{Sm}\sigma_m\sigma_L mH_{ml} + \bigl[(y+l)r-\alpha(t)+m\lambda_1\sigma_L-\lambda_2 m\bigr]H_y \end{aligned}$$

$$
\begin{aligned}
&-\frac{m}{2H_{yy}}\big[\sigma_L^2 H_{yl}^2+\lambda_1^2 H_y^2+\sigma_m^2\rho_{Sm}^2 H_{ym}^2+2\lambda_1\sigma_L H_y H_{yl}\\
&+2\sigma_m\sigma_L\rho_{Sm}H_{yl}H_{ym}+2\lambda_1\sigma_m\rho_{Sm}H_yH_{ym}\big]=0.
\end{aligned}
\tag{13}
$$

Therefore, Problem (10) can be transformed into the following non-linear PDE problem:

$$
\begin{cases}
H_t+k(\theta-m)H_m+\frac{\sigma_m^2}{2}mH_{mm}+[\alpha(t)+\lambda_2 m]H_l+\frac{\sigma_L^2}{2}mH_{ll}\\
\quad+\rho_{Sm}\sigma_m\sigma_L mH_{ml}+[(y+l)r-\alpha(t)+m\lambda_1\sigma_L-\lambda_2 m]H_y\\
\quad-\frac{m}{2H_{yy}}[\sigma_L^2H_{yl}^2+\lambda_1^2H_y^2+\sigma_m^2\rho_{Sm}^2H_{ym}^2+2\lambda_1\sigma_LH_yH_{yl}\\
\quad+2\sigma_m\sigma_L\rho_{Sm}H_{yl}H_{ym}+2\lambda_1\sigma_m\rho_{Sm}H_yH_{ym}]=0,\\
H(T,m,y,l)=(y-u)^2.
\end{cases}
$$

Here we need to point out that it is very difficult to obtain an explicit solution of the complicated non-linear PDE problem for the market is not self-financing (i.e., the manager has a continuous payment $dL(t)$ in the interval $(t,t+dt)$). However, in a particular case, this problem can be solved.

3.2 Solution of the auxiliary problem (10)

In this subsection, we shall apply the variable transform techniques and PDE method to a special solution of the auxiliary problem (10). By the terminal condition $H(T,m,y,l)=(y-u)^2$, we may look for a candidate solution in the form

$$
H(t,m,y,l)=f(t,m)\big[y+l-g(t,m,l)\big]^2,
\tag{14}
$$

where $f(t,m)$ and $g(t,m,l)$ are two undetermined functions with the boundary conditions $f(T,m)=1$ and $g(T,m,l)=l+u$.

After substituting (14) into (13) and by some tedious calculations, we have

$$
\begin{aligned}
&[y+l-g]^2\bigg\{f_t+\big[k(\theta-m)-2\lambda_1\sigma_m\rho_{Sm}m\big]f_m+\frac{\sigma_m^2}{2}mf_{mm}\\
&-m\sigma_m^2\rho_{Sm}^2\frac{f_m^2}{f}+\big[2r-m\lambda_1^2\big]f\bigg\}-2f[y+l-g]\bigg\{g_t+\bigg[k(\theta-m)\\
&-m\sigma_m^2\big(\rho_{Sm}^2-1\big)\frac{f_m}{f}-m\lambda_1\sigma_m\rho_{Sm}\bigg]g_m+\frac{\sigma_m^2}{2}mg_{mm}+\frac{\sigma_L^2}{2}mg_{ll}+\big[\alpha(t)\\
&+\lambda_2m-m\sigma_L\lambda_1\big]g_l+m\sigma_L\sigma_m\rho_{Sm}g_{lm}-rg\bigg\}+mfg_m^2\sigma_m^2\big(1-\rho_{Sm}^2\big)=0,
\end{aligned}
\tag{15}
$$

where f and g are short for $f(t,m)$ and $g(t,m,l)$, respectively.

Note that Eq. (15) can be taken as a polynomial of variable $y+l-g$. Thus, by the boundary conditions $f(T,m)=1$ and $g(T,m,l)=l+u$, we have

$$
\begin{cases}
f_t+[k(\theta-m)-2\lambda_1\sigma_m\rho_{Sm}m]f_m+\frac{\sigma_m^2}{2}mf_{mm}\\
\quad-m\sigma_m^2\rho_{Sm}^2\frac{f_m^2}{f}+[2r-m\lambda_1^2]f=0,\\
f(T,m)=1,
\end{cases}
\tag{16}
$$

$$\begin{cases} g_t + [k(\theta - m) - m\sigma_m^2(\rho_{Sm}^2 - 1)\frac{f_m}{f} - m\lambda_1\sigma_m\rho_{Sm}]g_m + \frac{\sigma_m^2}{2}mg_{mm} \\ \quad + \frac{\sigma_L^2}{2}mg_{ll} + [\alpha(t) - m\sigma_L\lambda_1 + m\lambda_2]g_l + m\sigma_L\sigma_m\rho_{Sm}g_{lm} - rg = 0, \\ g(T,m,l) = l + u, \end{cases} \tag{17}$$

$$mfg_m^2\sigma_m^2(1 - \rho_{Sm}^2) = 0. \tag{18}$$

In what follows, we aim to solve the two terminal value problems (16) and (17) based on Eq. (18). We first solve the problem (16) and postulate that $f(t,m)$ has the following exponential affine form:

$$f(t,m) = \mathrm{e}^{A_1(t)m + A_2(t)}, \tag{19}$$

where $A_1(t)$ and $A_2(t)$ are two undetermined functions with the boundary conditions $A_1(T) = 0$ and $A_2(T) = 0$.

Note that $\sigma_m > 0$, $m > 0$ and $f > 0$. Then Eq. (18) is equivalent to $g_m = 0$ (false) or $\rho_{Sm}^2 = 1$. In such a case, substituting (19) into (16) and by some simplifications, Problem (16) can be decomposed into the following ordinary differential equation (ODE) problems:

$$\begin{cases} \frac{dA_1(t)}{dt} = \frac{\sigma_m^2}{2}A_1^2(t) + (k + 2\lambda_1\sigma_m\rho_{Sm})A_1(t) + \lambda_1^2, \\ A_1(T) = 0, \end{cases} \tag{20}$$

$$\begin{cases} \frac{dA_2(t)}{dt} + k\theta A_1(t) + 2r = 0, \\ A_2(T) = 0. \end{cases} \tag{21}$$

The following proposition presents an explicit solution for Problem (20).

Proposition 1 *The solution to Problem* (20) *is given by*

$$A_1(t) = \begin{cases} \frac{n_1 n_2(1 - \mathrm{e}^{\sqrt{\Delta}(T-t)})}{n_1 - n_2\mathrm{e}^{\sqrt{\Delta}(T-t)}}, & \rho_{Sm}^2 = 1 \text{ and } \Delta > 0, \\ \frac{\sigma_m^2(T-t)n^2}{\sigma_m^2(T-t)n - 2}, & \rho_{Sm}^2 = 1 \text{ and } \Delta = 0, \\ \sqrt{\frac{-\Delta}{\sigma_m^4}}\tan(\arctan(\frac{k + 2\lambda_1\sigma_m\rho_{Sm}}{\sqrt{-\Delta}}) - \frac{\sqrt{-\Delta}(T-t)}{2}) + n, & \rho_{Sm}^2 = 1 \text{ and } \Delta < 0, \end{cases} \tag{22}$$

where

$$\Delta = (k + 2\lambda_1\sigma_m\rho_{Sm})^2 - 2\lambda_1^2\sigma_m^2,$$

$$n_1 = \frac{-(k + 2\lambda_1\sigma_m\rho_{Sm}) + \sqrt{\Delta}}{\sigma_m^2},$$

$$n_2 = \frac{-(k + 2\lambda_1\sigma_m\rho_{Sm}) - \sqrt{\Delta}}{\sigma_m^2},$$

$$n = \frac{-(k + 2\lambda_1\sigma_m\rho_{Sm})}{\sigma_m^2}.$$

Proof See the Appendix. □

According to the result of Proposition 1, the solution to Problem (21) can be expressed in terms of $A_1(t)$, that is,

$$A_2(t) = \int_t^T k\theta A_1(s)\,ds + 2r(T-t). \tag{23}$$

Furthermore, we have the following proposition.

Proposition 2 *The solution to Problem* (16) *is given by*

$$f(t,m) = \mathrm{e}^{A_1(t)m + A_2(t)},$$

where $A_1(t)$ *and* $A_2(t)$ *are given by Eqs.* (22) *and* (23), *respectively.*

We now solve Problem (17). Let

$$g(t,m,l) = A_3(t)l + A_4(t)m + A_5(t), \tag{24}$$

where $A_3(t)$, $A_4(t)$ and $A_5(t)$ are three undetermined functions with the boundary conditions $A_3(T) = 1$, $A_4(T) = 0$ and $A_5(T) = u$.

Substituting (24) into (17) and by some simplifications, Problem (17) can be decomposed into the following ODE problems:

$$\begin{cases} \frac{dA_3(t)}{dt} - rA_3(t) = 0, \\ A_3(T) = 1, \end{cases} \tag{25}$$

$$\begin{cases} \frac{dA_4(t)}{dt} - (k + \lambda_1\sigma_m\rho_{Sm} + r)A_4(t) - (\lambda_1\sigma_L - \lambda_2)A_3(t) = 0, \\ A_4(T) = 0, \end{cases} \tag{26}$$

$$\begin{cases} \frac{dA_5(t)}{dt} - rA_5(t) + k\theta A_4(t) + \alpha(t)A_3(t) = 0, \\ A_5(T) = u. \end{cases} \tag{27}$$

By a simple calculation, we have

$$A_3(t) = \mathrm{e}^{r(t-T)}. \tag{28}$$

Then the solutions of Problems (26)–(27) are given by

$$A_4(t) = \begin{cases} -(\lambda_1\sigma_L - \lambda_2)\mathrm{e}^{r(t-T)}(T-t), & k + \lambda_1\sigma_m\rho_{Sm} = 0, \\ \frac{(\lambda_1\sigma_L - \lambda_2)\mathrm{e}^{r(t-T)}[\mathrm{e}^{(k+\lambda_1\sigma_m\rho_{Sm})(t-T)} - 1]}{k + \lambda_1\sigma_m\rho_{Sm}}, & k + \lambda_1\sigma_m\rho_{Sm} \neq 0, \end{cases} \tag{29}$$

and

$$A_5(t) = \left\{ u + \int_t^T \left[k\theta A_4(s) + \alpha(s)A_3(s)\right]\mathrm{e}^{r(T-s)}\,ds \right\}\mathrm{e}^{r(t-T)}, \tag{30}$$

respectively. Furthermore, we have the following conclusion.

Proposition 3 *The solution to Problem* (17) *is given by*

$$g(t,m,l) = A_3(t)l + A_4(t)m + A_5(t),$$

where $A_3(t)$, $A_4(t)$ *and* $A_5(t)$ *are given by Eqs.* (28), (29) *and* (30), *respectively.*

Based on the results of Propositions 2–3 and the expression for $H(t,m,y,l)$ in Eq. (14), we have

$$H(t,m,y,l) = \mathrm{e}^{A_1(t)m+A_2(t)}\bigl[y - A_4(t)m + \bigl(1 - A_3(t)\bigr)l - A_5(t)\bigr]^2. \tag{31}$$

Obviously, $H_{yy} = 2\mathrm{e}^{A_1(t)m+A_2(t)} > 0$, which means that Problem (10) does have the optimal solution. Substituting (31) into (12) further reveals the optimal investment strategy of the optimal control problem (10) as

$$\pi^*(t) = \frac{\sigma_L A_3(t) + \sigma_m \rho_{Sm} A_4(t) - \lambda_1[X(t) - A_3(t)l - A_4(t)m - A_5(t)]}{X(t)}. \tag{32}$$

To sum up, we have the following theorem.

Theorem 1 *For any* $t \in [0,T]$, *the value function of the optimization problem* (10) *under the condition of* $\rho_{Sm}^2 = 1$ *is given by Eq.* (31), *while the corresponding optimal solution is given by Eq.* (32).

3.3 Efficient investment strategy and efficient frontier

In this subsection, we shall apply the Lagrange duality theorem (see Luenberger [19]) to derive the efficient investment strategy of Problem (6). Since the optimal control problem (9) is the same as the optimal control problem (10) when $t = 0$, the value function of Problem (9) is

$$H(0,m_0,y_0,l_0) = \mathrm{e}^{A_1(0)m_0+A_2(0)}\bigl[y_0 - A_4(0)m_0 + \bigl(1 - A_3(0)\bigr)l_0 - A_5(0)\bigr]^2, \tag{33}$$

where $y_0 = Y(0)$ and $l_0 = L(0)$. By the analysis of Sect. 3.1, to solve the original M–V portfolio selection problem, we only need to maximize the following function:

$$\begin{aligned} J\bigl(Y(0),0;\pi(t),\gamma\bigr) = {}& \mathrm{e}^{A_1(0)m_0+A_2(0)}\bigl[y_0 - A_4(0)m_0 \\ & + \bigl(1 - A_3(0)\bigr)l_0 - A_5(0)\bigr]^2 - \gamma^2 \end{aligned} \tag{34}$$

over γ. By the expression of $A_5(t)$ and $u = C - \gamma$, Eq. (34) can be reduced to

$$\begin{aligned} J\bigl(Y(0),0;\pi(t),\gamma\bigr) = {}& \gamma^2\bigl[\mathrm{e}^{A_1(0)m_0+A_2(0)-2rT} - 1\bigr] - 2\gamma\mathrm{e}^{A_1(0)m_0+A_2(0)-2rT}\biggl\{C \\ & - \mathrm{e}^{rT}\biggl[y_0 - A_4(0)m_0 + \bigl(1 - A_3(0)\bigr)l_0 \\ & - \int_0^T \bigl(k\theta A_4(s) + \alpha(s)A_3(s)\bigr)\mathrm{e}^{-rs}\,ds\biggr]\biggr\} \end{aligned}$$

$$\begin{aligned} &+ \mathrm{e}^{A_1(0)m_0+A_2(0)-2rT}C^2 \\ &+ \mathrm{e}^{A_1(0)m_0+A_2(0)}\biggl[y_0 - A_4(0)m_0 + \bigl(1-A_3(0)\bigr)l_0 \\ &- \int_0^T \bigl(k\theta A_4(s) + \alpha(s)A_3(s)\bigr)\mathrm{e}^{-rs}\,ds\biggr]^2 \\ &- 2C\mathrm{e}^{A_1(0)m_0+A_2(0)-rT}\biggl[y_0 - A_4(0)m_0 + \bigl(1-A_3(0)\bigr)l_0 \\ &- \int_0^T \bigl(k\theta A_4(s) + \alpha(s)A_3(s)\bigr)\mathrm{e}^{-rs}\,ds\biggr]. \end{aligned} \tag{35}$$

Note that Eq. (35) is a quadratic function with respect to γ, which implies that for (35) there may exist a finite maximum value while the existence of finite maximum value depends on the coefficient of γ^2. For this purpose, in this paper, we need to add one more assumption as follows.

Assumption 2 There exists a constant m_0 such that $\mathrm{e}^{A_1(0)m_0+A_2(0)-2rT} - 1 < 0$.

Under Assumption 2, the optimal value of $J(Y(0),0;\pi(t),\gamma)$ can be achieved when γ is given by

$$\gamma^* = \frac{C - [y_0 - A_4(0)m_0 + (1-A_3(0))l_0 - \int_0^T (k\theta A_4(s) + \alpha(s)A_3(s))\mathrm{e}^{-rs}\,ds]\mathrm{e}^{rT}}{1 - \mathrm{e}^{-[A_1(0)m_0+A_2(0)-2rT]}}. \tag{36}$$

Substituting (36) into (35), the optimal investment strategy and the minimum variance of the M–V ALM problem (9) are given by

$$\begin{aligned} \pi^*(t) = \frac{1}{X(t)}\biggl\{&\sigma_L A_3(t) + \sigma_m\rho_{Sm}A_4(t) - \lambda_1\biggl[X(t) - A_3(t)l - A_4(t)m \\ &- \biggl\{C - \gamma^* + \int_t^T \bigl(k\theta A_4(s) + \alpha(s)A_3(s)\bigr)\mathrm{e}^{r(T-s)}\,ds\biggr\}\mathrm{e}^{r(t-T)}\biggr]\biggr\} \end{aligned} \tag{37}$$

and

$$\begin{aligned} \mathrm{Var}^*\bigl[Y(T)\bigr] = \frac{1}{\mathrm{e}^{-[A_1(0)m_0+A_2(0)]} - \mathrm{e}^{-2rT}}\biggl\{&C\mathrm{e}^{-rT} - [y_0 - A_4(0)m_0 \\ &+ \bigl(1-A_3(0)\bigr)l_0 - \int_0^T \bigl(k\theta A_4(s) + \alpha(s)A_3(s)\bigr)\mathrm{e}^{-rs}\,ds\biggr\}^2, \end{aligned} \tag{38}$$

respectively. Furthermore, setting

$$C^* = \mathrm{e}^{rT}\biggl[y_0 - A_4(0)m_0 + \bigl(1-A_3(0)\bigr)l_0 - \int_0^T \bigl(k\theta A_4(s) + \alpha(s)A_3(s)\bigr)\mathrm{e}^{-rs}\,ds\biggr],$$

then we can get the global minimum variance $\mathrm{Var}^*_{\min}[Y(T)] = 0$. Moreover, according to the definition of efficient investment strategy, the rational investors should not select the expected terminal wealth less than C^*.

In conclusion, we summarize the above results in the following theorem.

Theorem 2 *Under Assumptions 1–2, the efficient investment strategy and the efficient frontier of this M–V ALM problem under the condition of* $\rho_{Sm}^2 = 1$ *for* $C \geq C^*$ *are given by Eqs.* (37) *and* (38), *respectively.*

Remark 2 Based on the result of Theorem 2, we can derive the efficient investment strategy and efficient frontier for the corresponding asset allocation problem (i.e., the case of no liability).

4 Numerical analysis

In this section, we give some numerical examples to analyze the sensitivity of the efficient frontier with respect to the parameters derived from the Heston model. The basic parameters in the model are given by

$$\begin{aligned} &r = 0.05, \quad \lambda_1 = -4, \quad k = 5, \quad \theta = 0.0169, \quad \sigma_m = 0.25, \quad m_0 = 0.0225, \\ &\alpha(t) = 0.05, \quad \lambda_2 = 0.5, \quad \sigma_L = 2, \quad X_0 = 1, \quad L_0 = 0.1, \quad \rho_{Sm} = 1, \\ &t = 0, \quad T = 5. \end{aligned}$$

Here the above parameters are the same as those in Li et al. [18] and satisfy Assumptions 1–2. We now analyze how the main parameters in the model affect the efficient frontier.

From Fig. 1, we can see clearly that, as the parameter λ_1 increases from –5 to 5, the efficient frontier rapidly moves upwards. This means that the expected terminal wealth with the same terminal variance $\mathrm{Var}[Y(T)]$ becomes higher. The main reason is that by Eq. (2), the parameter λ_1 determines the additional appreciation rate of the stock. In the case of other parameters unchanged, the bigger λ_1 leads to the higher additional appreciation rate of the stock, which makes the manager get the higher expected returns by investing in the stock.

Figures 2–4 depict the impacts of the parameters k, σ_m, m_0 in the Heston model on the efficient frontiers. From Fig. 2, we find that, as the mean-reversion rate k increases, the efficient frontier moves downwards, i.e., the terminal variance $\mathrm{Var}[Y(T)]$ with the same expected terminal wealth $\mathrm{E}[Y(T)]$ becomes larger. This is mainly because in the case of other parameters having remained unchanged, a bigger k means a higher risk from stochastic volatility, which makes the terminal variance larger. In Fig. 3, we find that when σ_m increases from 0.0225 to 0.0625, the efficient frontier moves up and the expected terminal wealth increases rapidly. In this case, the manger adopts a more aggressive investment

Figure 1 Impact of λ_1 on the efficient frontier

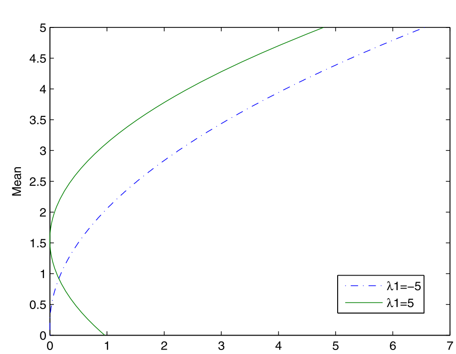


Figure 2 Impact of k on the efficient frontier

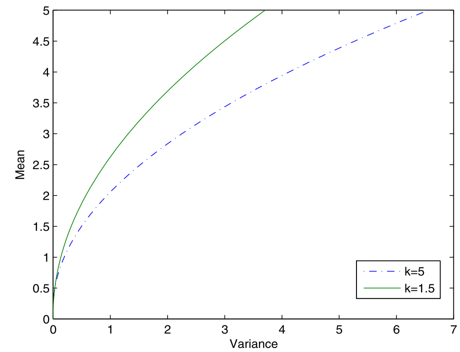


Figure 3 Impact of σ_m on the efficient frontier

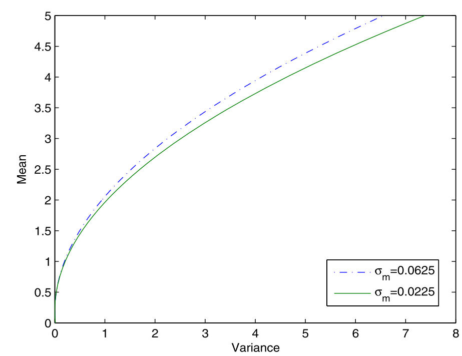
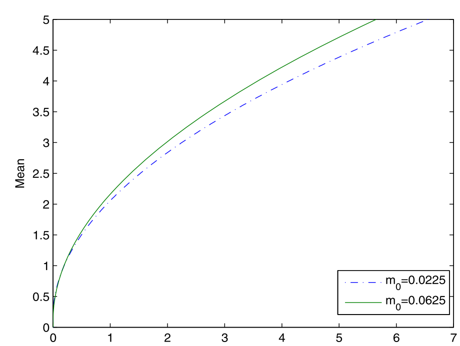


Figure 4 Impact of m_0 on the efficient frontier



strategy (i.e., more money is allocated to the stock). Similarly, we can analyze the effect of m_0 on the efficient frontier.

By Eq. (2), we know that the uncertainty of the stock's price is related to the value of ρ_{Sm}, i.e., if $\rho_{Sm} > 0$, the uncertainty of the stock's price and its volatility $m(t)$ changes in the same way, while $\rho_{Sm} < 0$, the uncertainty of $m(t)$ and $S(t)$ changes in the different way. In Fig. 5, we find that when ρ_{Sm} varies from -1 to 1, the efficient frontier moves upward and the expectation improvement increases remarkably. In the current market environment, the manager expects to earn more money from investing in the stock.

In Fig. 6, we can see that the efficient frontier moves down with the increase of λ_2, but at a slow pace. This can be attributed to the fact that in the case of other parameters unchanged, as the parameter λ_2 increases, the appreciation rate of liability becomes higher

Figure 5 Impact of ρ_{Sm} on the efficient frontier

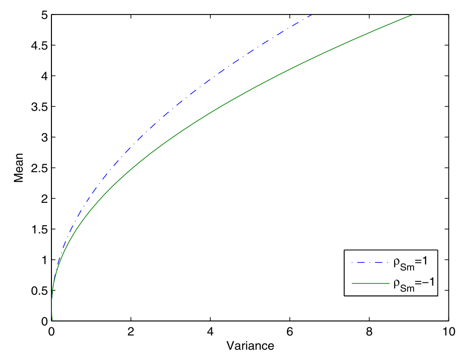
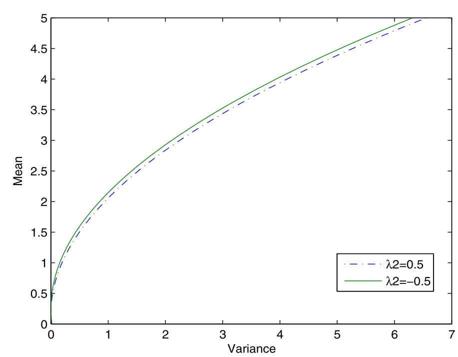


Figure 6 Impact of λ_2 on the efficient frontier



and so is the liability value. To hedge against the risk from the liability, the manager invests more money in the stock, which makes the terminal variance larger.

5 Conclusions

In this paper, we study the optimal dynamic M–V ALM problem under the Heston model. Due to some technical difficulties, we only derive the closed-form expressions of the efficient investment strategy and the efficient frontier for the special case: $dW_S(t)$ and $dW_m(t)$ are perfectly correlated or anti-correlated ($\rho_{Sm} = \pm 1$). Nevertheless, we are convinced that the results can provides some effective methods for managers in characterizing their optimal portfolio strategies under stochastic volatility environment. Moreover, we provide numerical examples to illustrate how the main parameters of the model affect the efficient frontier. From numerical examples, we find that the effective frontier is obviously affected by the parameters of the stochastic volatility model, which also shows the effectiveness of our model.

Appendix

In this Appendix, we give the proof of Proposition 1.

Let $\Delta = k^2 + 4k\lambda_1\sigma_m\rho_{Sm} + 2\lambda_1^2\sigma_m^2$ be the discriminant of the following equation:

$$\frac{1}{2}\sigma_m^2 A_1^2(t) + (k + 2\lambda_1\sigma_m\rho_{Sm})A_1(t) + \lambda_1^2 = 0. \tag{39}$$

If $\Delta > 0$, then (39) has two different real roots, namely,

$$n_1 = \frac{-(k + 2\lambda_1\sigma_m\rho_{Sm}) + \sqrt{\Delta}}{\sigma_m^2}, \qquad n_2 = \frac{-(k + 2\lambda_1\sigma_m\rho_{Sm}) - \sqrt{\Delta}}{\sigma_m^2}.$$

In this case, Problem (20) is equivalent to

$$\begin{cases} \frac{dA_1(t)}{dt} = \frac{\sigma_m^2}{2}[A_1(t) - n_1][A_1(t) - n_2], \\ A_1(T) = 0. \end{cases} \tag{40}$$

By applying the separation variable method to the ODE in Problem (40), we have

$$\frac{dA_1(t)}{[A_1(t) - n_1][A_1(t) - n_2]} = \frac{\sigma_m^2}{2}\,dt. \tag{41}$$

Furthermore, (41) can be rewritten as

$$\frac{dA_1(t)}{A_1(t) - n_1} - \frac{dA_1(t)}{A_1(t) - n_2} = \sqrt{\Delta}\,dt. \tag{42}$$

Integrating (42) on both sides with respect to t, we obtain

$$\int_t^T \frac{dA_1(s)}{A_1(s) - n_1} - \int_t^T \frac{dA_1(s)}{A_1(s) - n_2} = \sqrt{\Delta}(T - t).$$

Combined with the boundary condition $A_1(T) = 0$, the solution to Problem (40) is

$$A_1(t) = \frac{n_1 n_2 (1 - \mathrm{e}^{\sqrt{\Delta}(T-t)})}{n_1 - n_2 \mathrm{e}^{\sqrt{\Delta}(T-t)}}.$$

If $\Delta = 0$, then (39) has only one real root, namely,

$$n = \frac{-(k + 2\lambda_1\sigma_m\rho_{Sm})}{\sigma_m^2}.$$

Similar to the former case, $A_1(t)$ can be derived by solving the following ODE problem:

$$\begin{cases} \frac{dA_1(t)}{[A_1(t)-n]^2} = \frac{\sigma_m^2}{2}\,dt, \\ A_1(T) = 0. \end{cases} \tag{43}$$

Integrating the ODE in Problem (43) on both sides with respect to t, we have

$$\int_t^T \frac{dA_1(s)}{[A_1(s) - n]^2} = \frac{\sigma_m^2}{2}(T - t).$$

Furthermore, by the boundary condition $A_1(T) = 0$, we obtain

$$A_1(t) = \frac{n^2\sigma_m^2(T - t)}{n\sigma_m^2(T - t) - 2}.$$

If $\Delta < 0$, (39) has two imaginary roots. However, we consider the optimization problem in real spaces instead of complex spaces. It is gratifying that in this case $A_1(t)$ can be derived by solving the following ODE problem:

$$\begin{cases} \frac{dA_1(t)}{[A_1(t)+\frac{k+2\lambda_1\sigma_m\rho_{Sm}}{\sigma_m^2}]^2+\frac{-\Delta}{\sigma_m^4}} = \frac{\sigma_m^2}{2}\,dt, \\ A_1(T) = 0. \end{cases} \tag{44}$$

Applying the separation variable method to Problem (44) and after the more complex integral calculation, we have

$$A_1(t) = \sqrt{\frac{-\Delta}{\sigma_m^4}}\tan\left(\arctan\left(\frac{k+2\lambda_1\sigma_m\rho_{Sm}}{\sqrt{-\Delta}}\right) - \frac{\sqrt{-\Delta}(T-t)}{2}\right) + n.$$

Thus, the proof is completed.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study is realized in collaboration with the same responsibility. All authors are read and approved the manuscript.

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