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Monotone iterative technique and positive solutions to a third-order differential equation with advanced arguments and Stieltjes integral boundary conditions

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Abstract

We treat the existence of monotonic iteration positive solutions to a third-order boundary value problem with advanced arguments and Stieltjes integral boundary conditions. In our work, the main tool is a monotone iterative technique. Meanwhile, at the end of this paper, an example is presented to show that this method can be well used to get the main results.

MSC: 34K10; 34B15

Keywords: Stieltjes integral boundary conditions; Monotone Iterative; Completely continuous

1 Introduction

Third-order equations play a significant role in various aspects of applied mathematics and physics. Take draining or coating fluid-flow problems for example, surface tension forces are important in these problems and third-order ordinary differential equations serve well to describe them. Besides, third-order equations also work out well in the deflection of a curved beam having a constant or varying cross-section, a three-layer beam, electromagnetic waves, or gravity driven flows, etc. To learn more about the applications of the third-order boundary value problems, readers can refer to [1] and related themes.

Recently, many authors have widely studied the existence of multiple solutions to some boundary value problems; to see the details, we refer the readers to [1–5] and the references therein.

Since science and technology are developing at an unprecedented speed, a lot of boundary value problems with integral boundary conditions are applied in different industries and fields, for instance, thermal conduction, chemical engineering, semiconductor, underground water flow, hydrodynamic, thermo-elasticity, etc.; these can be found in [6–8] and related topics. The point is that boundary value problems with integral boundary conditions are made up of a very interesting and significant class of problems, since they include two, three, multi-point, and nonlocal boundary value problems as special cases.

Although the existence of multiple solutions to some boundary value problems with integral boundary conditions has been studied widely by many authors nowadays, we find that most authors study the second-order and fourth-order differential equations involving integral boundary conditions; to give example, the readers can see [9–13] and the references therein. As far as we are concerned, there are few papers dealing with third-order differential equations with Stieltjes integral boundary conditions in the existing literature.

We noted that Graef and Webb in [14] dealt with the existence of multiple positive solutions to the boundary value problem

$$\begin{cases} u'''(t) = g(t)f(t,u(t)), \quad 0 < t < 1, \\ u(0) = 0, \qquad u'(p) = 0, \qquad u''(1) = \lambda[u''], \end{cases} \tag{1.1}$$

where $p > \frac{1}{2}$, and $\lambda[v] = \int_0^1 v(t)\,d\Lambda(t)$ is a typical linear function on $C[0,1]$ along with a Riemann–Stieltjes integral and Λ is a function of bounded variation which suits well. Given that λ can cover sums and integrals as well, it is a more ordinary setup.

We also noted that in [15] Jankowski proved the existence of at least three non-negative solutions to the following nonlocal boundary value problem by using a fixed point theorem:

$$\begin{cases} x'''(t) + g(t)f(t,x(\alpha(t))) = 0, \quad 0 < t < 1, \\ x(0) = x''(0) = 0, \qquad x(1) = \beta x(\eta) + \lambda[x], \quad \beta > 0, 0 < \eta < 1, \end{cases} \tag{1.2}$$

where λ is a linear function under the circumstance that $\lambda[x] = \int_0^1 x(t)\,d\Lambda(t)$, which bears Stieltjes integral with a befitting function Λ of bounded variation.

We cannot help pointing out that only $x(1)$ is related to a Stieltjes integral in the above boundary condition in (1.2), and this is the specific source where our thoughts came from. That is, whether there will be some interesting findings when $x(0)$ is also related to a Stieltjes integral.

Under the motivation of the above works we talked about, we finally decided to research the successive iteration and positive solutions to the following third-order boundary value problem with advanced arguments and Stieltjes integral boundary conditions in this paper:

$$\begin{cases} u'''(t) + q(t)f(t,u(\alpha(t))) = 0, \quad 0 < t < 1, \\ u(0) = \beta u(\eta) + \lambda[u], \qquad u''(0) = 0, \qquad u(1) = \gamma u(\eta) + \lambda[u], \end{cases} \tag{1.3}$$

where $0 < \eta < 1, 0 \leq \gamma^2 \leq \beta < \gamma < 1, \alpha(t) \in C([0,1] \to [0,1])$, and $\alpha(t) \geq t$ for $t \in [0,1]$, Λ is an appropriate bounded variation function and $\lambda[u] = \int_0^1 u(t)\,d\Lambda(t)$ bearing a Riemann–Stieltjes integral function. It is worthy to mention that $\lambda[u]$ is not always positive for all positive u here.

When it comes to our work here, we will show that not only can we prove the existence of positive solutions to problem (1.3), but also a few of successive iterative schemes with either a known constant function starting point or a simple linear function one will be set up to approach the solutions. Last but not least, a perfect example is shown at the end of our paper to represent the applicability of the above mentioned methods and results. We must point out that acquiring the knowledge of how to find the solutions is perhaps the most significant skill when we turn to numerical analysis and application.

2 Preliminaries

At the beginning, we will show some important and necessary definitions here by using the theories of cones in Banach spaces.

Definition 2.1 Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided the following hypotheses are satisfied:

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$, and

(ii) $u, -u \in P$ implies $u = 0$.

Definition 2.2 The map α is identified as being concave on $[0,1]$ if

$$\alpha\bigl(tu + (1-t)v\bigr) \geq t\alpha(u) + (1-t)\alpha(v)$$

for all $u, v \in [0,1]$ and $t \in [0,1]$.

Definition 2.3 An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

We are concerned with the Banach space $E = C[0,1]$ facilitated with the norm $\|u\| = \max_{0 \leq t \leq 1} |u(t)|$. And we denote the cone $P \subset E$ by

$$P = \Bigl\{u \in \bigm| u(t) \geq 0, u \text{ is concave on } [0,1], \text{and } \min_{\eta \leq t \leq 1} u(t) \geq \delta\|u\|, \lambda[u] \geq 0\Bigr\},$$

where

$$\delta = \min\left\{\frac{\gamma(1-\eta)}{1-\gamma\eta}, \frac{\gamma\eta}{1-\beta(1-\eta)}\right\}.$$

Lemma 2.1 *For $y \in L[0,1]$, then the boundary value problem*

$$\begin{cases} u'''(t) = -y(t), \quad 0 < t < 1, \\ u(0) = \beta u(\eta) + \lambda[u], \qquad u''(0) = 0, \qquad u(1) = \gamma u(\eta) + \lambda[u] \end{cases} \tag{2.1}$$

has a unique solution

$$\begin{aligned} u(t) = {} & \frac{1 - (\gamma-\beta)\eta + (\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\lambda[u] + \frac{\beta + (\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\int_0^1 F(\eta,s)y(s)\,ds \\ & + \int_0^1 F(t,s)y(s)\,ds, \end{aligned}$$

where

$$F(t,s) = \begin{cases} \frac{1}{2}t(1-s)^2, & 0 \leq t \leq s \leq 1, \\ \frac{1}{2}(1-t)(t-s^2), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.2}$$

Proof Facilitated by (2.1), we have

$$u(t) = u(0) + u'(0)t - \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds, \quad 0 \leq t \leq 1.$$

And then we can obtain

$$u'(0) = u(1) - u(0) + \frac{1}{2}\int_0^1 (1-s)^2 y(s)\,ds.$$

Under the boundary condition in (2.1), we can get that

$$u(t) = \bigl(\beta + (\gamma-\beta)t\bigr)u(\eta) + \lambda[u] + \int_0^1 F(t,s)y(s)\,ds, \quad 0 \le t \le 1.$$

Thus

$$u(\eta) = \frac{1}{1-\beta-(\gamma-\beta)\eta}\lambda[u] + \frac{1}{1-\beta-(\gamma-\beta)\eta}\int_0^1 F(\eta,s)y(s)\,ds, \quad 0 \le t \le 1.$$

Therefore, we get the solution to the problem as follows:

$$u(t) = \frac{1-(\gamma-\beta)\eta+(\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\lambda[u] + \frac{\beta+(\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\int_0^1 F(\eta,s)y(s)\,ds + \int_0^1 F(t,s)y(s)\,ds.$$

So, the proof is completed. □

Now we define an operator $T: P \to E$ by

$$\begin{aligned}(Tu)(t) &= u(t) \\ &= \frac{1-(\gamma-\beta)\eta+(\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\lambda[u] + \frac{\beta+(\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\int_0^1 F(\eta,s)q(s)f\bigl(s,u(\alpha(s))\bigr)\,ds \\ &\quad + \int_0^1 F(t,s)q(s)f\bigl(s,u(\alpha(s))\bigr)\,ds. \end{aligned} \tag{2.3}$$

According to Lemma 2.1, boundary value problem (1.3) has a solution $u = u(t)$ if and only if u is a fixed point of T.

During the following proof, we will assume that (H_1)–(H_4) are satisfied.

(H_1): $f(t,x) \in C([0,1] \times [0,+\infty) \to [0,+\infty))$.

(H_2): $q(t)$ is a nonnegative continuous function on $[0,1]$, $q(t) \not\equiv 0$ on any subinterval of $(0,1)$.

(H_3): $\int_0^1 d\Lambda(t) \ge 0$, $\int_0^1 t\,d\Lambda(t) \ge 0$, $\int_0^1 F(t,s)\,d\Lambda(t) \ge 0$, $0 < s < 1$.

(H_4): $1-\beta-(\gamma-\beta)\eta > 0$.

Lemma 2.2 *Since* (H_1)–(H_4) *hold, then* $T: P \to P$ *defined by* (2.3) *is completely continuous.*

Proof Through (2.3), we have

$$(Tu)''(t) = -\int_0^t q(s)f\bigl(s,u(\alpha(s))\bigr)\,ds \le 0.$$

It is easy to notice that $(Tu)(t)$ is concave on $[0,1]$.

Besides, by (2.3)

$$(Tu)(0) = \frac{1-(\gamma-\beta)\eta}{1-\beta-(\gamma-\beta)\eta}\lambda[u] + \frac{\beta}{1-\beta-(\gamma-\beta)\eta}\int_0^1 F(\eta,s)q(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds \ge 0,$$

and

$$\begin{aligned}(Tu)(1) &= \frac{1-(\gamma-\beta)\eta+(\gamma-\beta)}{1-\beta-(\gamma-\beta)\eta}\lambda[u] \\ &\quad + \frac{\gamma}{1-\beta-(\gamma-\beta)\eta}\int_0^1 F(\eta,s)q(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds \ge 0.\end{aligned}$$

Then it follows that Tu is nonnegative on $[0,1]$.

Then, according to (H_3), we can get

$$\begin{aligned}\Lambda[Tu] &= \int_0^1 \biggl(\frac{1-(\gamma-\beta)\eta+(\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\lambda[u] \\ &\quad + \frac{\beta+(\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\int_0^1 F(\eta,s)q(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds \\ &\quad + \int_0^1 F(t,s)q(s)f\bigl(s,u\bigl(\alpha(s)\bigr)\bigr)\,ds\biggr)\,d\Lambda(t) \ge 0.\end{aligned}$$

On the other hand, we must show that $\min_{\eta\le t\le 1}(Tu)(t) \ge \delta\|Tu\|$.

Because of the concavity of Tu, we can obtain that there exists $\sigma \in [0,1]$ such that $\|Tu\| = (Tu)(\sigma)$.

When $\eta \ge \sigma$, we have $\min_{\eta\le t\le 1}(Tu)(t) = (Tu)(1)$ and

$$\frac{(Tu)(\sigma)-(Tu)(1)}{1-\sigma} \le \frac{(Tu)(\eta)-(Tu)(1)}{1-\eta},$$

then

$$\|Tu\| \le \frac{1}{1-\eta}(Tu)(\eta) - \frac{\eta}{1-\eta}(Tu)(1).$$

Uniting the boundary condition in (1.3), we can have that

$$\min_{\eta\le t\le 1}(Tu)(t) \ge \frac{\gamma(1-\eta)}{1-\gamma\eta}\|Tu\|. \tag{2.4}$$

When $\eta < \sigma$, we have $\min_{\eta\le t\le 1}(Tu)(t) = \min\{(Tu)(\eta),(Tu)(1)\}$ and

$$\frac{(Tu)(\sigma)-(Tu)(0)}{\sigma} \le \frac{(Tu)(\eta)-(Tu)(0)}{\eta},$$

then

$$\|Tu\| \le \frac{1}{\eta}(Tu)(\eta) - \frac{1-\eta}{\eta}(Tu)(0).$$

Combining the boundary condition in (1.3), we can obtain that

$$(Tu)(\eta) \geq \frac{\eta}{1-\beta(1-\eta)}\|Tu\|,$$

and

$$(Tu)(1) \geq \frac{\gamma\eta}{1-\beta(1-\eta)}\|Tu\|.$$

Hence, we get

$$\min_{\eta\leq t\leq 1}(Tu)(t) \geq \frac{\gamma\eta}{1-\beta(1-\eta)}\|Tu\|. \tag{2.5}$$

Given (2.4) and (2.5), we can get that $\min_{\eta\leq t\leq 1}(Tu)(t) \geq \delta\|Tu\|$.

It is easy to see that T is continuous. Then, let $\Omega \subset P$ be a bounded set, the proof that $T\Omega$ is bounded and equicontinuous is easy and obvious. Then the Arzela–Ascoli theorem makes sure that $T\Omega$ is relatively compact, which means T is compact. Then we obtain that T is completely continuous.

So, based on what has been discussed above, we can arrive at the conclusion that $T: P \to P$ is completely continuous. □

3 Main results

For the convenience of next work, we denote

$$A = \frac{\frac{\gamma}{1-\beta-(\gamma-\beta)\eta}\int_0^1 F(\eta,s)q(s)\,ds + \int_0^1 \frac{1}{8}(1+s)^2(1-s)^2 q(s)\,ds}{1-\frac{1+(\gamma-\beta)(1-\eta)}{1-\beta-(\gamma-\beta)\eta}\int_0^1 d\Lambda(t)},$$

$$B = \frac{\gamma}{1-\beta-(\gamma-\beta)\eta}\int_\eta^1 F(\eta,s)q(s)\,ds.$$

Theorem 3.1 *Assume that* (H_1)–(H_4) *hold and there exists* $0 < b < a$ *such that*

(H_5): $f(t,x_1) \leq f(t,x_2)$ *for any* $0 \leq t \leq 1$, $0 \leq x_1 \leq x_2 \leq a$;

(H_6): $\sup_{0\leq t\leq 1} f(t,a) \leq \frac{a}{A}$, $\inf_{\eta\leq t\leq 1} f(t,\delta b) \geq \frac{b}{B}$;

(H_7): $f(t,0) \not\equiv 0$ *for* $0 \leq t \leq 1$.

Thus we can say that the boundary value problem (1.3) *has at least two positive concave solutions* w^* *and* v^* *such that*

$$b \leq \|w^*\| \leq a, \qquad \min_{\eta\leq t\leq 1} w^*(t) \geq \delta\|w^*\|, \quad \textit{and}$$

$$w^* = \lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0, \quad \textit{where } w_0(t) = a, 0 \leq t \leq 1,$$

$$b \leq \|v^*\| \leq a, \qquad \min_{\eta\leq t\leq 1} v^*(t) \geq \delta\|v^*\|, \quad \textit{and}$$

$$v^* = \lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0, \quad \textit{where } v_0(t) = \frac{b}{\gamma}\bigl(\beta + (\gamma-\beta)t\bigr), 0 \leq t \leq 1,$$

where $(Tu)(t)$ *is defined by* (2.3).

The successive iterative schemes in the theorem are $w_0(t) = a$, $w_{n+1} = Tw_n = T^n w_0$, $n = 0, 1, 2, \ldots$, which starts off with the constant function, and $v_0(t) = \frac{b}{\gamma}(\beta + (\gamma - \beta)t)$, $v_{n+1} = Tv_n = T^n v_0$, $n = 0, 1, 2, \ldots$, which starts off with a known simple linear function.

Proof We denote $P[b, a] = \{u \in P \mid b \le \|u\| \le a\}$.

In the following content of proof, we will firstly prove that $T : P[b, a] \to P[b, a]$.

If $u \in P[b, a]$, then

$$0 \le u(t) \le \max_{0 \le t \le 1} u(t) = \|u\| \le a, \quad \text{and} \quad \min_{\eta \le t \le 1} u(t) \ge \delta\|u\| \ge \delta b.$$

Therefore, under assumptions (H_5) and (H_6), we get

$$0 \le f\bigl(t, u(t)\bigr) \le f(t, a) \le \sup_{0 \le t \le 1} f(t, a) \le \frac{a}{A} \quad \text{for } 0 \le t \le 1,$$

$$f\bigl(t, u(t)\bigr) \ge f(t, \delta b) \ge \inf_{\eta \le t \le 1} f(t, \delta b) \ge \frac{b}{B} \quad \text{for } \eta \le t \le 1.$$

For any $u \in P[b, a]$, by Lemma 2.2, we know that $Tu \in P$ and

$$\begin{aligned}
\bigl\|(Tu)(t)\bigr\| &= \max_{0 \le t \le 1} (Tu)(t) \\
&\le \frac{1 - (\gamma - \beta)\eta + (\gamma - \beta)}{1 - \beta - (\gamma - \beta)\eta} \lambda[u] \\
&\quad + \frac{\beta + (\gamma - \beta)}{1 - \beta - (\gamma - \beta)\eta} \int_0^1 F(\eta, s) q(s) f\bigl(s, u\bigl(\alpha(s)\bigr)\bigr)\, ds \\
&\quad + \int_0^1 \frac{1}{8}(1 + s)^2 (1 - s)^2 q(s) f\bigl(s, u\bigl(\alpha(s)\bigr)\bigr)\, ds \\
&\le a \frac{1 + (\gamma - \beta)(1 - \eta)}{1 - \beta - (\gamma - \beta)\eta} \int_0^1 d\Lambda(t) + \frac{a}{A} \frac{\gamma}{1 - \beta - (\gamma - \beta)\eta} \int_0^1 F(\eta, s) q(s)\, ds \\
&\quad + \frac{a}{A} \int_0^1 \frac{1}{8}(1 + s)^2 (1 - s)^2 q(s)\, ds \\
&\le a \biggl(\frac{1 + (\gamma - \beta)(1 - \eta)}{1 - \beta - (\gamma - \beta)\eta} \int_0^1 d\Lambda(t) + \frac{1}{A} \frac{\gamma}{1 - \beta - (\gamma - \beta)\eta} \int_0^1 F(\eta, s) q(s)\, ds \\
&\quad + \frac{1}{A} \int_0^1 \frac{1}{8}(1 + s)^2 (1 - s)^2 q(s)\, ds \biggr) \\
&= a,
\end{aligned}$$

and

$$\begin{aligned}
\bigl\|(Tu)(t)\bigr\| &= \max_{0 \le t \le 1} (Tu)(t) \\
&\ge \frac{1}{1 - \beta - (\gamma - \beta)\eta} \lambda[u] + \frac{\beta + (\gamma - \beta)\eta}{1 - \beta - (\gamma - \beta)\eta} \int_0^1 F(\eta, s) q(s) f\bigl(s, u\bigl(\alpha(s)\bigr)\bigr)\, ds \\
&\quad + \int_0^1 F(\eta, s) q(s) f\bigl(s, u\bigl(\alpha(s)\bigr)\bigr)\, ds \\
&\ge \frac{1}{1 - \beta - (\gamma - \beta)\eta} \int_\eta^1 F(\eta, s) q(s) f\bigl(s, u\bigl(\alpha(s)\bigr)\bigr)\, ds
\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{b}{B}\frac{1}{1-\beta-(\gamma-\beta)\eta}\int_\eta^1 F(\eta,s)q(s)\,ds \\
&\geq b.
\end{aligned}
$$

Thus, we get $b \leq \|Tu\| \leq a$. So, we get that $T : P[b,a] \to P[b,a]$.

Let $w_0(t) = a$, $0 \leq t \leq 1$, then $w_0(t) \in P[b,a]$. Let $w_1 = Tw_0$, then $w_1 \in P[b,a]$. We denote $w_{n+1} = Tw_n$, $n = 0,1,2,\ldots$. Then we have $w_n \subseteq P[b,a]$, $n = 1,2,\ldots$. Since T is completely continuous, we assert that $\{w_n\}_{n=1}^{\infty}$ is a sequentially compact set.

Then we will search the convergence property of the iterative scheme since

$$
\begin{aligned}
&w_1(t) = Tw_0(t) \leq a = w_0(t), \\
&w_2(t) = Tw_1(t) \leq Tw_0(t) = w_1(t).
\end{aligned}
$$

After calculation, the iterative scheme is clear, then

$$
w_{n+1} \leq w_n, \quad 0 \leq t \leq 1, n = 0,1,2,\ldots.
$$

Thus, we can get that there exists $w^* \in P[b,a]$ such that $w_n \to w^*$. Combining with the continuity of T and $w_{n+1} = Tw_n$, we obtain that $Tw^* = w^*$.

On the other hand, another way to approach this is to start off with a linear function. Let $v_0(t) = \frac{b}{\gamma}(\beta + (\gamma-\beta)t)$, $0 \leq t \leq 1$, then $v_0(t) \in P[b,a]$. Let $v_1 = Tv_0$, then $v_1 \in P[b,a]$. We denote $v_{n+1} = Tv_n$, $n = 0,1,2,\ldots$. Then we have $v_n \subseteq P[b,a]$, $n = 1,2,\ldots$. Since T is completely continuous, we assert that $\{v_n\}_{n=1}^{\infty}$ is a sequentially compact set.

In a similar way, since $v_1 = Tv_0 \in P[b,a]$ and

$$
\begin{aligned}
v_1 &= Tv_0(t) \\
&= \frac{1-(\gamma-\beta)\eta+(\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\lambda[u] + \frac{\beta+(\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\int_0^1 F(\eta,s)q(s)f\bigl(s,v_0\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\quad + \int_0^1 F(t,s)q(s)f\bigl(s,v_0\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\geq \frac{\beta+(\gamma-\beta)t}{1-\beta-(\gamma-\beta)\eta}\int_\eta^1 F(\eta,s)q(s)f\bigl(s,v_0\bigl(\alpha(s)\bigr)\bigr)\,ds \\
&\geq \frac{1}{\gamma}b\bigl(\beta+(\gamma-\beta)t\bigr) \\
&= v_0(t).
\end{aligned}
$$

Through a similar calculation argument, we can easily get that

$$
v_{n+1} \geq v_n, \quad 0 \leq t \leq 1, n = 0,1,2,\ldots.
$$

Hence there exists $v^* \in P[b,a]$ such that $v_n \to v^*$. Combining with the continuity of T and $v_{n+1} = Tv_n$, we get $Tv^* = v^*$.

Assumption (H_7) indicates that $f(t,0) \not\equiv 0$, $0 \leq t \leq 1$, then the zero function is not the solution of (1.3). Thus we have $v^* > 0$ for $0 < t < 1$.

It is well known that each fixed point of T in P is a solution of (1.3). Hence, we assert that the boundary value problem (1.3) has at least two positive concave solutions w^* and v^*.

The proof is completed. □

Remark 3.1 If $\lim_{n\to\infty} w_n \neq \lim_{n\to\infty} v_n$, then w^* and v^* are two positive concave solutions of problem (1.3). And if $\lim_{n\to\infty} w_n = \lim_{n\to\infty} v_n$, then $w^* = v^*$ is a positive concave solution of problem (1.3).

The following corollary can be obtained easily.

Corollary 3.1 *Assume that* (H_1)–(H_5) *and* (H_7) *hold, and there exists* $a > 0$ *such that*
(H_8): $\overline{\lim}_{l\to 0} \inf_{\eta\leq t\leq 1} \frac{f(t,l)}{l} > \frac{1}{\delta B}$, $\underline{\lim}_{l\to+\infty} \sup_{0\leq t\leq 1} \frac{f(t,l)}{l} < \frac{1}{A}$
(*particularly*, $\overline{\lim}_{l\to 0} \inf_{\eta\leq t\leq 1} \frac{f(t,l)}{l} = +\infty$, $\underline{\lim}_{l\to+\infty} \sup_{0\leq t\leq 1} \frac{f(t,l)}{l} = 0$.)
Then the boundary value problem (1.3) *has at least two positive symmetric concave solutions* w^* *and* v^* *such that the conclusion of Theorem* 3.1 *holds.*

4 Example

In the following part, we will discuss an example and simulations. Then we will get a perfect result by using the method above.

Example 4.1 Let $\eta = \frac{1}{2}$ and $q(t) = 1$, we consider the following boundary value problem:

$$\begin{cases} u'''(t) + f(t, u(\alpha(t))) = 0, & 0 < t < 1, \\ u(0) = \frac{1}{4}u(\frac{1}{2}) + \lambda[u], \qquad u''(0) = 0, & u(1) = \frac{1}{2}u(\frac{1}{2}) + \lambda[u], \end{cases} \tag{4.1}$$

where

$$\lambda[u] = \int_0^1 (2t-1)u(t)\,dt,$$

$$f\big(t, u\big(\alpha(t)\big)\big) = t + 130\big(u(\sqrt{t})\big)^{\frac{1}{16}}.$$

From calculation we can get that

$$\delta = \frac{2}{7}, \qquad A = \frac{7}{60}, \qquad B = \frac{1}{120}.$$

The verification of conditions (H_1)–(H_5), (H_7) is very easy to complete, then we set $b = 1$, $a = 21$, then it also satisfies

$$\sup_{0\leq t\leq 1} f(t,a) \leq \frac{a}{A}, \qquad \inf_{\frac{1}{2}\leq t\leq 1} f\left(t, \frac{2}{7}b\right) \geq \frac{b}{B}.$$

So all the hypotheses of Theorem 3.1 are fulfilled, and we can obtain that the boundary value problem (4.1) has at least two positive concave solutions w^* and v^* such that

$$1 \leq \|w^*\| \leq 21, \qquad \min_{\frac{1}{2}\leq t\leq 1} w^*(t) \geq \frac{2}{7}\|w^*\|, \quad \text{and}$$

$$w^* = \lim_{n\to\infty} w_n = \lim_{n\to\infty} T^n w_0, \quad \text{where } w_0(t) = 21, 0 \leq t \leq 1,$$

$$1 \leq \|v^*\| \leq 21, \qquad \min_{\frac{1}{2}\leq t\leq 1} v^*(t) \geq \frac{2}{7}\|v^*\|, \quad \text{and}$$

$$v^* = \lim_{n\to\infty} v_n = \lim_{n\to\infty} T^n v_0, \quad \text{where } v_0(t) = \frac{1}{2} + \frac{1}{2}t, 0 \le t \le 1,$$

where $(Tu)(t)$ is defined by (2.3).

For $n = 0, 1, 2, \ldots$, the two iterative schemes are as follows:

$$\begin{aligned}
w_0(t) &= 21,\\
w_1(t) &= -\frac{1}{24}t^4 - \frac{65 \times 7^{\frac{1}{16}}}{3^{\frac{15}{16}}}t^3 + \left(\frac{47}{960} + \frac{299 \times 7^{\frac{1}{16}}}{4 \times 3^{\frac{15}{16}}}\right)t + \frac{7}{960} + \frac{13 \times 21^{\frac{1}{16}}}{4},\\
&\ldots,\\
w_{n+1}(t) &= (Tw_n)(t)\\
&= \frac{1}{5}(7 + 2t)\int_0^1 (2t-1)w_n(t)\,dt + \frac{2}{5}(1+t)\int_0^1 F\left(\frac{1}{2}, s\right)\left(s + 130\left(w_n(\sqrt{s})\right)^{\frac{1}{16}}\right)ds\\
&\quad + \int_0^1 F(t,s)\left(s + 130\left(w_n(\sqrt{s})\right)^{\frac{1}{16}}\right)ds,
\end{aligned}$$

which starts off with a constant function, and

$$\begin{aligned}
v_0(t) &= \frac{1}{2} + \frac{1}{2}t,\\
v_1(t) &= -\frac{1}{24}t^4 - \frac{32{,}768 \times 2^{\frac{15}{16}}(1+\sqrt{t})^{\frac{1}{16}}}{3201}t^3 - \frac{95{,}977{,}472 \times 2^{\frac{15}{16}}(1+\sqrt{t})^{\frac{1}{16}}}{71{,}993{,}691}t^{\frac{5}{2}}\\
&\quad + \frac{2{,}078{,}769{,}152 \times 2^{\frac{15}{16}}(1+\sqrt{t})^{\frac{1}{16}}}{3201}t^2 - \frac{16{,}640 \times 2^{\frac{15}{16}}}{561}t^2\\
&\quad + \frac{113{,}410{,}048 \times 2^{\frac{15}{16}}(1+\sqrt{t})^{\frac{1}{16}}}{71{,}993{,}691}t^{\frac{3}{2}} - \frac{2{,}037{,}383{,}168 \times 2^{\frac{15}{16}}(1+\sqrt{t})^{\frac{1}{16}}}{71{,}993{,}691}t\\
&\quad + \frac{22{,}847{,}488 \times 2^{\frac{7}{8}}(2+\sqrt{2})^{\frac{1}{16}}}{51{,}424{,}065}t - \frac{376{,}832 \times 2^{\frac{3}{8}}(2+\sqrt{2})^{\frac{1}{16}}}{7{,}346{,}295}t\\
&\quad + \frac{2{,}885{,}841{,}536 \times 2^{\frac{15}{16}}}{71{,}993{,}691}t + \frac{70{,}488{,}247}{41{,}065{,}920}t - \frac{41{,}943{,}040 \times 2^{\frac{15}{16}}(1+\sqrt{t})^{\frac{1}{16}}}{71{,}993{,}691}\sqrt{t}\\
&\quad + \frac{671{,}088{,}640 \times 2^{\frac{15}{16}}(1+\sqrt{t})^{\frac{1}{16}}}{71{,}993{,}691} + \frac{22{,}847{,}488 \times 2^{\frac{7}{8}}(2+\sqrt{2})^{\frac{1}{16}}}{51{,}424{,}065}\\
&\quad - \frac{376{,}832 \times 2^{\frac{3}{8}}(2+\sqrt{2})^{\frac{1}{16}}}{7{,}346{,}295} - \frac{197{,}254{,}528 \times 2^{\frac{15}{16}}}{23{,}997{,}897} + \frac{48{,}825{,}821}{123{,}197{,}760},\\
&\ldots,\\
v_{n+1}(t) &= (Tv_n)(t)\\
&= \frac{1}{5}(7 + 2t)\int_0^1 (2t-1)v_n(t)\,dt + \frac{2}{5}(1+t)\int_0^1 F\left(\frac{1}{2}, s\right)\left(s + 130\left(v_n(\sqrt{s})\right)^{\frac{1}{16}}\right)ds\\
&\quad + \int_0^1 F(t,s)\left(s + 130\left(v_n(\sqrt{s})\right)^{\frac{1}{16}}\right)ds,
\end{aligned}$$

which starts off with a known simple linear function.

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