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Error estimates of finite element methods for nonlinear fractional stochastic differential equations

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Abstract

In this paper, we consider the Galerkin finite element approximations of the initial value problem for the nonlinear fractional stochastic partial differential equations with multiplicative noise. We study a spatial semidiscrete scheme with the standard Galerkin finite element method and a fully discrete scheme based on the Goreno–Mainardi–Moretti–Paradisi (GMMP) scheme. We establish strong convergence error estimates for both semidiscrete and fully discrete schemes.

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Keywords: Nonlinear fractional stochastic differential equations; Finite element method; Error estimates; Strong convergence; Initial value problem

1 Introduction

In the last few years, fractional calculus has attracted lots of attention. The increasing interest in fractional equations is motivated by their applications in various fields of science such as fluid mechanics, heat conduction in materials with memory, physics, chemistry, and engineering [1–5]. As we know, fractional differential equations are highly effective mathematical tools to describe complex behaviors and phenomena of memory processes because of the convolution integral with the power-law memory kernel introduced in the fractional derivatives [6–8]. On the other hand, stochastic perturbations cannot be avoided in physical systems and sometimes even cannot be ignored, so that the corresponding stochastic terms need to be added to the deterministic governing equations. Hence stochastic differential equations with fractional time derivatives have been proposed, which are a more realistic mathematical model of the real-world situations [9], just like the equations (1.1) we are going to discuss in this paper naturally arise from the consideration of the heat equation in a material with thermal memory [10].

In this paper, we consider the following initial value problem for the nonlinear fractional stochastic partial differential equation (SPDE) with multiplicative noise:

$$\begin{cases} D_t^\alpha u(t) + Au(t) = f(u(t)) + g(u(t))\frac{dW(t)}{dt}, & \alpha \in (0,1), t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{1.1}$$

The random process $\{u(t)\}_{t\in[0,T]}$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ with normal filtration $\{\mathcal{F}_t\}_{t\geq 0}$, takes values in a separable Hilbert space H with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. The initial value u_0 is an H-valued and $\mathcal{F}_0$-measurable random variable. The operator $A : \mathcal{D}(A) \subset H \to H$ is not necessarily a bounded, linear, densely defined, and selfadjoint operator with compact inverse. The nonlinear operators f and g are Lipschitz continuous in an appropriate sense. The process W with values in some separable Hilbert space U is a nuclear Q-Wiener process with respect to the filtration. The covariance operator Q is assumed to be selfadjoint and positive semidefinite with finite trace. Here, we denote the Caputo fractional derivative of order α $(0 < \alpha < 1)$ with respect to t by D_t^α and define it as [11, 12]

$$D_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{d}{ds} u(s)\, ds.$$

It is known that the fractional derivative D_t^α recovers the canonical first-order derivative $\frac{d}{dt}u(t)$ for the fractional order $\alpha = 1$, and thus model (1.1) evolves into the standard stochastic partial differential equation (SPDE), whose numerical approximation has been extensively discussed in the literature; see, for example, [13–16].

Stochastic partial differential equations have been applied in many fields such as viscoelasticity, turbulence, electromagnetic theory, heterogeneous flows, and materials [17–22], so the study of stochastic partial differential equations has recently attracted a lot of attention. In particular, as in [10, 23–26], equations of type (1.1) can be used to model random effects on transport of particles in medium with thermal memory. In [10], a class of SPDEs with time-fractional derivatives was introduced, and the existence and uniqueness of solutions to these equations was proved. The existence of mild solutions for a class of nonlinear fractional stochastic partial differential equations has been discussed in [24]. Foondun and Nane [23] studied asymptotic properties of space–time fractional SPDEs. In [25], the existence and uniqueness of mild solutions for a class of nonlinear fractional Sobolev-type stochastic differential equations under non-Lipschitz conditions was discussed by employing Picard-type approximate sequences. The approximate controllability problem for fractional stochastic differential inclusions with nonlocal conditions and infinite delay has been researched in [26]. Since the random effects on transport of particles in medium with thermal memory can be exactly modeled by fractional stochastic differential systems, it is important and necessary to discuss numerical schemes and error estimation for stochastic fractional equations. However, numerical methods for these kinds of fractional SPDEs are rarely studied, and we only note [27–30]. To the authors' knowledge, no result has been reported on the error estimation of nonlinear fractional stochastic partial differential equations with multiplicative noise based on the form of mild solutions proposed in [24], so the motivation of this paper is to fill this gap.

The main difficulty in the analysis is estimation of nonlinear terms; see Lemmas 3.6 and 3.7. Estimation of a discrete solution operator with limited smoothing properties is also a challenge; see Lemma 4.3. Our main results are as follows. First, in Theorem 3.1, denoting by $u_h(t)$ and $u(t)$ the mild solutions to (3.2) and (1.1), we derive a strong convergence error bound for the semidiscrete scheme:

$$\|u(t) - u_h(t)\|_{L_2(\Omega,H)} \leq Ch^2.$$

Second, for $\alpha \in (0,1)$, we obtain am $L_{2(\Omega,H)}$-norm error estimate for the fully discrete scheme in Theorem 4.1:

$$\left\|u(t_n) - u_h^n\right\|_{L_{2(\Omega;H)}} \le C\left[k^\alpha + h^2\right],$$

where u_h^n denotes an approximation of the mild solution $u(t)$ at time t_n. The parameters h and k, which will be detailed in Sects. 3 and 4, represent the maximal meshsize and time step, respectively.

The rest of the paper is organized as follows: In Sect. 2, we introduce some basic notation, present the Laplace transform, and give a representation of the mild solution of equation (1.1) by using basic properties of the Mittag–Leffler function. In Sect. 3, we first give a short review of Galerkin finite element methods and then study the space semidiscrete scheme and derive error estimates for the standard Galerkin finite element method with smooth initial data. Finally, in Sect. 4, using the GMMP scheme, we prove strong error estimates for the fully discrete scheme.

2 Preliminaries

In this section, we recall some useful properties on the Mittag–Leffler function, introduce the Laplace transform and present a representation of the mild solution of problem (1.1). Besides, we use the letter C to denote a constant that may vary from one occurrence to another and denote by $L(U,H)$ the space of bounded linear operators from U to H, where U and H are real separable Hilbert spaces with inner product $(\cdot,\cdot)$ and norms $\|\cdot\|_U$ and $\|\cdot\|_H$.

2.1 Mittag–Leffler function

The Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad z \in \mathbb{C},$$

where $\Gamma(\cdot)$ is the standard gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \quad \Re(z) > 0.$$

We give important properties of the Mittag–Leffler function $E_{\alpha,\beta}(z)$ essential in our analysis.

Lemma 2.1 ([31]) *Let $0 < \alpha < 2$ and $\beta \in \mathbb{R}$ be arbitrary, and let $\frac{\pi\alpha}{2} < \mu < \min(\pi, \alpha\pi)$. Then there exists a constant $C = C(\alpha,\beta,\mu) > 0$ such that, for $\mu \le |\arg(z)| \le \pi$,*

$$\left|E_{\alpha,\beta}(z)\right| \le \begin{cases} \frac{C}{1+|z|^2}, & \beta - \alpha \in \mathbb{Z}^- \cup \{0\}, \\ \frac{C}{1+|z|} & \text{otherwise.} \end{cases}$$

Moreover, for $\lambda > 0$, $\alpha > 0$, and $t > 0$, we have

$$D_t^\alpha E_{\alpha,1}\left(-\lambda t^\alpha\right) = -\lambda E_{\alpha,1}\left(-\lambda t^\alpha\right) \quad \text{and} \quad \frac{d}{dt} E_{\alpha,1}\left(-\lambda t^\alpha\right) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda t^\alpha\right).$$

In our analysis, we also use the Laplace transform. Let $\pi : \mathbb{R}_+ \to H$ be subexponential, that is, for any $\varepsilon > 0$, the function $t \to \pi(t)e^{-\varepsilon t}$ belongs to $L^1(\mathbb{R}_+, H)$. The Laplace transform of $\hat{\pi} : \mathbb{C}_+ \to H$ is denoted by

$$\hat{\pi}(z) = \int_0^{+\infty} \pi(t)e^{-zt}\,dt, \quad \Re(z) > 0,$$

where the same notation H represents the complexification of H. Further, we denote by $*$ the Laplace convolution product on $[0, t]$ of two locally integrable subexponential functions $\pi, \sigma \in L^1_{\mathrm{loc}}(\mathbb{R}_+, H)$, that is,

$$(\pi * \sigma)(t) = \int_0^t \pi(t-s)\sigma(s)\,ds.$$

It is well known that $\pi * \sigma \in L^1_{\mathrm{loc}}(\mathbb{R}_+, H)$ is subexponential and

$$\widehat{\pi * \sigma} = \hat{\pi}(z)\hat{\sigma}(z).$$

2.2 Solution representation

In order to study the representation of the solution of (1.1), we introduce some notation.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. By $L_2(\Omega, H)$ we denote the space of H-valued square-integrable random variables with norm

$$\|v\|_{L_2(\Omega,H)} = \left(E\|v\|_H^2\right)^{\frac{1}{2}} = \left(\int_\Omega \left\|v(w)\right\|_H^2 \mathbf{P}(w)\right)^{\frac{1}{2}},$$

where E stands for expected value. Let $Q \in \mathcal{L}(U)$ be a selfadjoint positive semidefinite operator with $\operatorname{Tr}(Q) < \infty$, where $\operatorname{Tr}(Q)$ is the trace of Q. Let $\{(\gamma_j, e_j)\}_{j=1}^\infty$ be the eigenpairs of Q with orthonormal eigenvectors. The U-valued Q-Wiener process $W(t)$, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$, has the orthogonal expansion

$$W(t) = \sum_{j=1}^\infty \gamma_j^{1/2}\beta_j(t)e_j,$$

where $\{\beta_j(t)\}_{j=1}^\infty$ are real-valued mutually independent standard Brownian motions. Further, the set $L_2^0 = HS(Q^{1/2}(H), H)$ expresses the space of all Hilbert–Schmidt operators from $Q^{1/2}(H)$ to H with norm $\|\psi\|_{L_2^0} = (\sum_{j=1}^\infty \|\psi Q^{1/2}e_j\|^2)^{1/2}$, and the subset $L_{2,r}^0 \subset L_2^0, r \geq 0$ is the subspace of all Hilbert–Schmidt operators from $Q^{1/2}(H)$ to $\dot{H}^r$ with norm $\|\psi\|_{L_{2,r}^0} = \|A^{\frac{r}{2}}\psi\|_{L_2^0}$. It is then possible to define the stochastic integral $\int_0^t \psi(s)\,dW(s)$ together with Itô's isometry

$$E\left\|\int_0^t \psi(s)\,dW(s)\right\|_H^2 = \int_0^t E\left\|\psi(s)\right\|_{L_2^0}^2\,ds. \tag{2.1}$$

In a standard way, we present the fractional powers A^s, $s \in \mathbb{R}$, of A as

$$A^s v = \sum_{j=1}^\infty \lambda_j^s(v, \varphi_j)\varphi_j, \qquad D\left(A^{\frac{s}{2}}\right) = \left\{v \in H : \left\|A^{\frac{s}{2}}v\right\|^2 = \sum_{j=1}^\infty \lambda_j^s(v, \varphi_j)^2 < \infty\right\},$$

where $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$ are respectively the eigenvalues and the orthonormal eigenfunctions of A, that is,

$$A\varphi_j = \lambda_j\varphi_j \quad \text{and} \quad (\varphi_i, \varphi_j) = \delta_{i,j} \quad \text{for } i,j \geq 1.$$

In addition, the sequence $\{\lambda_j\}_{j=1}^{\infty}$ is an increasing sequence of real numbers, that is, $0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. Let $\dot{H}^s = D(A^{\frac{s}{2}})$ with norm

$$\|v\|_s = \left\|A^{\frac{s}{2}} v\right\| = \left(\sum_{j=1}^{\infty} \lambda_j^s (v, \varphi_j)^2\right)^{1/2}, \quad v \in \dot{H}^s.$$

We define the operators $E(t)$ and $\bar{E}(t)$ by

$$E(t)v = \sum_{j=1}^{\infty} E_{\alpha,1}\left(-\lambda_j t^{\alpha}\right)(v, \varphi_j)\varphi_j, \quad v \in \dot{H}^s,$$

$$\bar{E}(t)v = \sum_{j=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}\left(-\lambda_j t^{\alpha}\right)(v, \varphi_j)\varphi_j, \quad v \in \dot{H}^s,$$

where $\alpha \in (0,1)$ indicates the order of Caputo fractional derivative. Then, we present the mild solution $u(t)$ of (1.1) [24]:

$$u(t) = E(t)u_0 + \int_0^t \bar{E}(t-s) f\big(u(s)\big)\, ds + \int_0^t \bar{E}(t-s) g\big(u(s)\big)\, dW(s). \tag{2.2}$$

Next, we impose the following conditions on f, g, and $u(t)$, which are the conditions of existence and uniqueness of the mild solution u [24].

Assumption 2.1 For the nonlinear operator $f : H \to H$, there exists a constant C such that

$$\left\|f(x) - f(y)\right\| \leq C\|x - y\|, \quad \left\|f(x)\right\| \leq C\|x\|. \tag{2.3}$$

Assumption 2.2 For the nonlinear operator $g : H \to L_2^0$, there exists a constant C such that

$$\left\|g(x) - g(y)\right\|_{L_2^0} \leq C\|x - y\|, \quad \left\|g(x)\right\|_{L_2^0} \leq C\|x\|. \tag{2.4}$$

Assumption 2.3 The mild solution $u : [0,T] \times \Omega \to H$ satisfies

$$\sup_{0 \leq t \leq T} E\left(\left\|u(t)\right\|_s^2\right) < +\infty, \tag{2.5}$$

where $s \in [0,2]$.

Some properties of the operators $E(t)$ and $\bar{E}(t)$, which are crucial for the semidiscrete error estimates, will be introduced later.

Lemma 2.2 ([32]) *For $\alpha \in (0,1)$, we have the following estimates*:

$$\left\| \left(D_t^\alpha\right)^\ell E(t)v \right\|_p \leq Ct^{-\alpha(\ell+\frac{p-q}{2})}\|v\|_q, \quad t>0,$$

where $0 \leq q \leq p \leq 2$ for $\ell = 0$, and $0 \leq p \leq q \leq 2$ and $q \leq p+2$ for $\ell = 1$.

Lemma 2.3 ([31]) *For any $t > 0$ and $0 \leq p-q \leq 4$, we have*

$$\left\| \bar{E}(t)v \right\|_p \leq Ct^{-1+\alpha(1+\frac{q-p}{2})}\|v\|_q.$$

3 Error estimates for spatially semidiscrete approximation

In this section, we first review the Galerkin finite element methods and some basic estimates for the finite element projection operators. Then we introduce a representation of the semidiscrete scheme of the mild solution $u(t)$ and some smoothing properties of the operators $E_h(t)$ and $\bar{E}_h(t)$. We close this section with the proof of the semidiscrete error estimates.

3.1 Space discretization

Let $\{\mathcal{T}_h\}_{h\in(0,1]}$ denote a regular family of triangulations of $\mathcal{D}$, where h is the maximal meshsize, and let V_h denote the space of piecewise linear continuous functions with respect to $\mathcal{T}_h$ vanishing on $\partial\mathcal{D}$. Thereby, $V_h \subset H_0^1(\mathcal{D}) = \dot{H}^1 = \{v \in L_2(\mathcal{D}), \nabla v \in L_2(\mathcal{D}), v|_{\partial\mathcal{D}} = 0\}$. Denote by $R_h : \dot{H}^1 \to V_h$ the Ritz projector onto V_h with respect to the inner product

$$a(v,w) = \left(A^{\frac{1}{2}}v, A^{\frac{1}{2}}w\right), \quad v,w \in \dot{H}^1.$$

Thus we obtain

$$a(R_h v, \chi) = a(v,\chi), \quad v \in \dot{H}^1, \chi \in V_h.$$

Meanwhile, the following error estimate is established:

$$\|R_h v - v\| \leq Ch^s\|v\|_s, \quad v \in \dot{H}^s, 1 \leq s \leq 2. \tag{3.1}$$

The semidiscrete problem corresponding to (1.1) is to find a process $u_h(t) \in V_h$ such that

$$D_t^\alpha u_h(t) + A_h u_h(t) = P_h f\big(u_h(t)\big) + P_h g\big(u_h(t)\big)\frac{dW}{dt}, \quad u_h(0) = P_h u_0, \tag{3.2}$$

where the mapping $A_h : V_h \to V_h$ is a discrete version of the operator A defined by

$$a(\varphi,\chi) = (A_h\varphi,\chi), \quad \forall\varphi,\chi \in V_h,$$

and P_h is the orthogonal projector

$$P_h : H \to V_h, \quad (P_h v,\chi) = (v,\chi), \quad \forall v \in H, \forall\chi \in V_h.$$

Depending on the eigenvalues and eigenfunctions $\{\lambda_j^h\}_{j=1}^N$ and $\{\varphi_j^h\}_{j=1}^N$ of the discrete operator A_h, we can introduce a representation of the solution of (3.2). Firstly, we present the discrete analogues of operators $E(t)$ and $\bar{E}(t)$ as follows:

$$E_h(t)v_h = \sum_{j=1}^{N} E_{\alpha,1}\bigl(-\lambda_j^h t^\alpha\bigr)\bigl(v_h,\varphi_j^h\bigr)\varphi_j^h, \tag{3.3}$$

$$\bar{E}_h(t)v_h = \sum_{j=1}^{N} t^{\alpha-1}E_{\alpha,\alpha}\bigl(-\lambda_j^h t^\alpha\bigr)\bigl(v_h,\varphi_j^h\bigr)\varphi_j^h. \tag{3.4}$$

Analogously, the unique solution of the finite element problem (3.2) can be given by

$$u_h(t) = E_h(t)P_h u_0 + \int_0^t \bar{E}_h(t-s)P_h f\bigl(u_h(s)\bigr)\,ds + \int_0^t \bar{E}_h(t-s)P_h g\bigl(u_h(s)\bigr)\,dW(s). \tag{3.5}$$

Then, similarly to Lemmas 2.2 and 2.3, we show some vital properties of $E_h(t)$ and $\bar{E}_h(t)$ in the following:

Lemma 3.1 ([32]) *Let $E_h(t)$ be defined by (3.3), and let $\chi \in V_h$. Then, for $\alpha \in (0,1)$ and $p,q \in [-1,1]$, we have*

$$\bigl\|\bigl(D_t^\alpha\bigr)^\ell E_h(t)\chi\bigr\|_p \le Ct^{-\alpha(\ell+\frac{p-q}{2})}\|\chi\|_q,$$

where $q \le p$ and $0 \le p-q \le 2$ for $\ell = 0$, and $p \le q \le p+2$ for $\ell = 1$.

Lemma 3.2 ([32]) *Let $\bar{E}_h$ be defined by (3.4), and let $\chi \in V_h$. Then, for all $t > 0$,*

$$\bigl\|\bar{E}_h(t)\chi\bigr\|_p \le \begin{cases} Ct^{-1+\alpha(1+\frac{q-p}{2})}\|\chi\|_q, & p-2 \le q \le p, \\ Ct^{-1+\alpha}\|\chi\|_q, & p < q, \end{cases}$$

where $p,q \in [-1,1]$.

Based on this lemma, we have the following conclusion.

Lemma 3.3 *Let $\bar{E}_h$ be defined by (3.4), and let $v \in H, P_h v = v_h$. For all $t > 0$, we have*

$$\bigl\|\bar{E}_h(t)P_h v\bigr\| \le Ct^{\alpha-1}\|v\|.$$

Proof By Lemma 3.2 with $p = q = 0$ we get

$$\bigl\|\bar{E}_h(t)v_h\bigr\| \le Ct^{\alpha-1}\|v_h\|.$$

Since $v_h = P_h v$, we get

$$\bigl\|\bar{E}_h(t)v_h\bigr\| \le Ct^{\alpha-1}\|P_h v\| \le Ct^{\alpha-1}\|v\|,$$

which completes the proof. □

Moreover, we need the following estimate of $u_h(t)$.

Lemma 3.4 *For any $t \in [0,T]$ and $\alpha \in (\frac{1}{2},1)$, let $u_h(t)$ be the mild solution of (3.2). Then there exists a constant $C > 0$ such that*

$$\sup_{0\le t\le T}\|u_h(t)\|^2_{L_2(\Omega;H)} \le C\|u_0\|^2_{L_2(\Omega;H)}.$$

Proof For any $t \in [0,T]$, from (3.5) by Lemma 3.1 with $\ell = p = q = 0$, Lemma 3.3, Assumptions 2.1 and 2.2, and Itô's isometry we obtain

$$\begin{aligned} E\|u_h(t)\|^2 &\le 4E\|E_h(t)P_hu_0\|^2 + 4E\left\|\int_0^t \bar{E}_h(t-s)P_hf(u_h(s))\,ds\right\|^2 \\ &\quad + 4E\left\|\int_0^t \bar{E}_h(t-s)P_hg(u_h(s))\,dW(s)\right\|^2 \\ &\le 4E\|E_h(t)P_hu_0\|^2 + 4\int_0^t E\|\bar{E}_h(t-s)P_hf(u_h(s))\|^2\,ds \\ &\quad + 4\int_0^t E\|\bar{E}_h(t-s)P_hg(u_h(s))\|^2_{L_2^0}\,ds \\ &\le CE\|u_0\|^2 + C\int_0^t (t-s)^{2\alpha-2}E\|u_h(s)\|^2\,ds \\ &\quad + C\int_0^t (t-s)^{2\alpha-2}E\|u_h(s)\|^2\,ds. \end{aligned}$$

Thus, applying the integral version of Gronwall's lemma, we deduce that

$$\sup_{0\le t\le T} E\|u_h(t)\|^2 \le CE\|u_0\|^2 \exp\left(C\int_0^t (t-s)^{2\alpha-2}\,ds\right) \le CE\|u_0\|^2. \qquad \square$$

3.2 Semidiscrete finite element approximation

In this subsection, we first present and prove some lemmas, which are crucial for the derivation of the semidiscrete error estimate for the nonlinear fractional stochastic differential equation. Then we give a detailed proof of the semidiscrete error estimate.

Lemma 3.5 ([28]) *Let $0 \le \nu \le \mu \le 2$ and $F_h(t) = E(t) - E_h(t)P_h$. Then, for $\alpha \in (0,1)$, there exists a constant C such that*

$$\|F_h(t)x\| \le Ch^\mu t^{-\alpha\frac{\mu-\nu}{2}}\|x\|_\nu, \quad x \in \dot{H}^\nu.$$

Lemma 3.6 *Let $1 < q \le 2$ and $\bar{F}_h(t) = \bar{E}(t) - \bar{E}_h(t)P_h$. Then, for $t \in [0,T]$, there exists a constant C such that*

$$\left\|\int_0^T \bar{F}_h(t)h(t)\,dt\right\|^2 \le Ch^{2q}\int_0^T \|h(t)\|^2_{q-2}\,dt, \quad h(t) \in \dot{H}^{q-2}.$$

Proof By the definition of $\bar{F}_h(t)$ we split $\int_0^t \bar{F}_h(t-s)h(t)\,ds$ into two additional terms:

$$\begin{aligned} \int_0^t \bar{F}_h(t-s)h(t)\,ds &= \int_0^t \bar{E}(t-s)h(t)\,ds - \int_0^t \bar{E}_h(t-s)P_hh(t)\,ds \\ &= v(t) - v_h(t) \end{aligned}$$

$$= \big(v(t) - P_h v(t)\big) + \big(P_h v(t) - v_h(t)\big)$$
$$= \eta(t) + \xi(t),$$

where $v(t)$ and $v_h(t)$ are the solutions of the following equations:

$$\begin{cases} D_t^\alpha v(t) + Av(t) = h(t), \\ v(0) = 0, \end{cases} \tag{3.6}$$

$$\begin{cases} D_t^\alpha v_h(t) + A_h v_h(t) = P_h h(t), \\ v_h(0) = P_h v(0) = 0. \end{cases} \tag{3.7}$$

To bound ξ, we note that by our definitions

$$D_t^\alpha \xi(t) + A_h \xi(t) = A_h\big(R_h v(t) - P_h v(t)\big), \quad \xi(0) = 0.$$

By the Laplace transforms of both sides of this equation, we recover

$$z^a \hat{\xi}(z) + A_h \hat{\xi}(z) = A_h(R_h - P_h)\hat{v}(z).$$

Therefore

$$\hat{\xi}(z) = \big(z^\alpha I + A_h\big)^{-1} A_h(R_h - P_h)\hat{v}(z).$$

Since the operator A_h generates an analytic contraction semigroup, there exists a constant C, depending only on ϕ and α, such that

$$\big\| \big(z^\alpha I + A_h\big)^{-1} \big\| \le C z^{-\alpha}, \quad \forall z \in \Sigma_\phi,$$

where $\Sigma_\phi = \{z \in \mathbb{C} : |\arg z| \le \phi\}$. By the identity

$$\big(z^\alpha I + A_h\big)^{-1} A_h = I - z^\alpha \big(z^\alpha I + A_h\big)^{-1}$$

we get

$$\big\| \big(z^\alpha I + A_h\big)^{-1} A_h \big\| \le 1 + \big\| z^\alpha \big(z^\alpha I + A_h\big)^{-1} \big\| \le 1 + C \le C.$$

Using the inverse Laplace transform and inequality (3.1), we obtain

$$\begin{aligned} \big\|\xi(t)\big\| &\le C\big\|(R_h - P_h)v(t)\big\| \\ &\le C\big\|(R_h - I)v(t)\big\| + C\big\|(I - P_h)v(t)\big\| \le Ch^q \big\|v(t)\big\|_q. \end{aligned}$$

Then by Theorem 2.1 of [31] we get

$$\int_0^T \big\|\xi(t)\big\|^2 \, dt \le Ch^{2q} \int_0^T \big\|v(t)\big\|_q^2 \, dt \le Ch^{2q} \int_0^T \big\|h(t)\big\|_{q-2}^2 \, dt.$$

According to inequality (3.1) and Theorem 2.1 of [31], the estimate of η yields

$$\begin{aligned}\int_0^T \|\eta(t)\|^2\,dt &\le C\int_0^T \|(R_h-I)v(t)\|^2\,dt \le Ch^{2q}\int_0^T \|v(t)\|_q^2\,dt\\ &\le Ch^{2q}\int_0^T \|h(t)\|_{q-2}^2\,dt.\end{aligned}$$

Since

$$\begin{aligned}\int_0^T \left\|\int_0^t \bar{F}_h(t-s)h\,ds\right\|^2 dt = C(T)\left\|\int_0^t \bar{F}_h(t)h\,ds\right\|^2 &= \int_0^T \|\eta(t)+\xi(t)\|^2\,dt\\ &\le 2\int_0^T \|\eta(t)\|^2\,dt + 2\int_0^T \|\xi(t)\|^2\,dt\\ &\le Ch^{2q}\int_0^T \|h(t)\|_{q-2}^2\,dt,\end{aligned}$$

we get the conclusion

$$\left\|\int_0^T \bar{F}_h(t)h\,dt\right\|^2 \le Ch^{2q}\int_0^T \|h(t)\|_{q-2}^2\,dt.$$

□

Lemma 3.7 *Let* $1<q\le 2$ *and* $\bar{F}_h(t)=\bar{E}(t)-\bar{E}_h(t)P_h$. *Then, for* $t\in[0,T]$ *and* $\tilde{h}(s)\in\dot{H}^q$, *there exists a constant C such that*

$$E\left\|\int_0^t \bar{F}_h(t-s)\tilde{h}(s)\,dW(s)\right\|^2 \le Ch^{2q}\int_0^t (t-s)^{2\alpha-2}E\|\tilde{h}(s)\|_{L^0_{2,q}}^2\,ds.$$

Proof Just like in the proof of Lemma 3.6, we split $\int_0^t \bar{F}_h(t-s)h(t)\,ds$ into two additional terms:

$$\begin{aligned}\int_0^t \bar{F}_h(t-s)\tilde{h}(s)\,dW(s) &= \int_0^t \bar{E}(t-s)\tilde{h}(s)\,dW(s) - \int_0^t \bar{E}_h(t-s)P_h\tilde{h}(s)\,dW(s)\\ &= \tilde{v}(t)-\tilde{v}_h(t) = \big(\tilde{v}(t)-P_h\tilde{v}(t)\big)+\big(P_h\tilde{v}(t)-\tilde{v}_h(t)\big)\\ &= \varrho(t)+\vartheta(t),\end{aligned}$$

where $\tilde{v}(t)$ and $\tilde{v}_h(t)$ are the solutions of the following equations:

$$\begin{cases} D_t^\alpha \tilde{v}(t) + A\tilde{v}(t) = \tilde{h}(t)\frac{dW(t)}{dt},\\ \tilde{v}(0)=0,\end{cases} \tag{3.8}$$

$$\begin{cases} D_t^\alpha \tilde{v}_h(t) + A_h\tilde{v}_h(t) = P_h\tilde{h}(t)\frac{dW(t)}{dt},\\ \tilde{v}_h(0)=P_h\tilde{v}(0)=0.\end{cases} \tag{3.9}$$

To bound ϑ, we note that by our definitions

$$D_t^\alpha\vartheta(t) + A_h\vartheta(t) = A_h\big(R_h\tilde{v}(t)-P_h\tilde{v}(t)\big),\quad \vartheta(0)=0.$$

As in the proof of Lemma 3.6, taking the Laplace transform and inverse Laplace transform on both sides of this equation, we eventually get

$$\begin{aligned}\|\vartheta(t)\| &\le C\|(R_h-P_h)\tilde{v}(t)\| \\ &\le C\|(R_h-I)\tilde{v}(t)\|+C\|(I-P_h)\tilde{v}(t)\|\le Ch^q\|\tilde{v}(t)\|_q.\end{aligned}$$

Thus by Lemma 2.3 with $p=q\in(1,2]$ and Itô's isometry we derive

$$\begin{aligned}E\|\vartheta(t)\|^2 &\le Ch^{2q}E\|\tilde{v}(t)\|_q^2 = Ch^{2q}E\left\|\int_0^t \bar{E}(t-s)\tilde{h}(s)\,dW(s)\right\|_q^2 \\ &= Ch^{2q}E\left\|\int_0^t A^{\frac{q}{2}}\bar{E}(t-s)\tilde{h}(s)\,dW(s)\right\|^2 \\ &= Ch^{2q}\int_0^t E\|A^{\frac{q}{2}}\bar{E}(t-s)\tilde{h}(s)\|_{L_2^0}^2\,ds \\ &\le Ch^{2q}\int_0^t (t-s)^{2\alpha-2}E\left\|\sum_{j=1}^{\infty}\tilde{h}(s)Q^{\frac{1}{2}}e_j\right\|_q^2\,ds \\ &= Ch^{2q}\int_0^t (t-s)^{2\alpha-2}E\|A^{\frac{q}{2}}\tilde{h}(s)\|_{L_2^0}^2\,ds.\end{aligned}$$

According to inequality (3.1) and Lemma 2.3, the estimate of ϱ yields

$$\begin{aligned}E\|\varrho(t)\|^2 &\le C\|(R_h-I)\tilde{v}(t)\|^2 \le Ch^{2q}E\|\tilde{v}(t)\|_q^2 \\ &\le Ch^{2q}\int_0^t (t-s)^{2\alpha-2}E\|A^{\frac{q}{2}}\tilde{h}(s)\|_{L_2^0}^2\,ds.\end{aligned}$$

Thereby,

$$E\left\|\int_0^t \bar{F}_h(t-s)\tilde{h}(s)\,dW(s)\right\|^2 \le Ch^{2q}\int_0^t (t-s)^{2\alpha-2}E\|\tilde{h}(s)\|_{L_{2,q}^0}^2\,ds. \qquad \square$$

Now, we will give the semidiscrete error estimate in space for the stochastic fractional differential equation (1.1).

Theorem 3.1 *Let $u(t)$ and $u_h(t)$ be the solutions of (1.1) and (3.2), respectively. Then, for $t\ge 0$, $\alpha\in(\frac{1}{2},1)$, and $u_0\in L_2(\Omega,\dot{H}^s)$, $s\in[0,2]$, we have*

$$\|u(t)-u_h(t)\|_{L_2(\Omega,H)}\le Ch^2.$$

Proof For $t\in[0,T]$, by (1.1) and (3.2) we have

$$\begin{aligned}&\|u(t)-u_h(t)\|_{L_2(\Omega,H)} \\ &\quad\le \|(E(t)-E_h(t)P_h)u_0\|_{L_2(\Omega,H)} \\ &\qquad+\left\|\int_0^t \big(\bar{E}(t-s)f(u(s))-\bar{E}_h(t-s)P_hf(u_h(s))\big)\,ds\right\|_{L_2(\Omega,H)}\end{aligned}$$

$$
\begin{aligned}
&+ \left\| \int_0^t \left(\bar{E}(t-s)g\big(u(s)\big) - \bar{E}_h(t-s)P_h g\big(u_h(s)\big)\right) dW(s) \right\|_{L_2(\Omega,H)} \\
&= I + II + III.
\end{aligned}
$$

For I, by Lemma 3.5 with $\nu = \mu = 1 + r$ ($r \in (0,1]$) we have

$$
I \le Ch^{1+r} \|u_0\|_{L_2(\Omega;\dot{H}^{1+r})}.
$$

We dominate II by two additional terms:

$$
\begin{aligned}
II &= \left\| \int_0^t \bar{E}(t-s)f\big(u(s)\big) - \bar{E}_h(t-s)P_h f\big(u_h(s)\big)\, ds \right\|_{L_2(\Omega,H)} \\
&\le \left\| \int_0^t \bar{E}_h(t-s)P_h\big(f\big(u(s)\big) - f\big(u_h(s)\big)\big)\, ds \right\|_{L_2(\Omega,H)} \\
&\quad + \left\| \int_0^t \bar{F}_h(t-s)f\big(u(s)\big)\, ds \right\|_{L_2(\Omega,H)} \\
&= I_1 + I_2.
\end{aligned}
$$

We estimate each term separately. First, note that by Lemma 3.3 and Assumption 2.1 we have

$$
\begin{aligned}
I_1 &\le \int_0^t \left\| \bar{E}_h(t-s)P_h\big(f\big(u(s)\big) - f\big(u_h(s)\big)\big)\right\|_{L_2(\Omega,H)} ds \\
&\le C \int_0^t (t-s)^{\alpha-1} \left\| f\big(u(s)\big) - f\big(u_h(s)\big)\right\|_{L_2(\Omega,H)} ds \\
&\le C \int_0^t (t-s)^{\alpha-1} \left\| u(s) - u_h(s)\right\|_{L_2(\Omega,H)} ds.
\end{aligned}
$$

The term I_2 is reckoned by applying Lemma 3.6, Assumptions 2.1 and 2.3. Then we get

$$
\begin{aligned}
I_2^2 &= E\left\| \int_0^t \bar{F}_h(t-s)f\big(u(s)\big)\, ds \right\|^2 \\
&\le Ch^4 \int_0^t E\left\| f\big(u(s)\big)\right\|^2 ds \\
&\le Ch^4 \int_0^t \sup_{0\le s\le T} E\left\| u(s)\right\|^2 ds \\
&\le Ch^4.
\end{aligned}
$$

A combination of the estimates I_1 and I_2 gives

$$
II^2 \le Ch^4 + C\int_0^t (t-s)^{2(\alpha-1)} \left\| u(s) - u_h(s)\right\|_{L_2(\Omega,H)}^2 ds.
$$

In a similar way as for II, we dominate III by two additional terms:

$$
III = \left\| \int_0^t \bar{E}(t-s)g\big(u(s)\big) - \bar{E}_h(t-s)P_h g\big(u_h(s)\big)\, dW(s) \right\|_{L_2(\Omega,H)}
$$

$$\begin{aligned}
&\leq \left\| \int_0^t \bar{E}_h(t-s)P_h\big(g\big(u(s)\big) - g\big(u_h(s)\big)\big)\,dW(s)\right\|_{L_2(\Omega,H)} \\
&\quad + \left\| \int_0^t \bar{F}_h(t-s)g\big(u(s)\big)\,dW(s)\right\|_{L_2(\Omega,H)} \\
&= I_3 + I_4.
\end{aligned}$$

As in an estimate for I_1, we can get an estimate for I_3 by using Lemma 3.3 together with Assumption 2.2 and Itô's isometry:

$$\begin{aligned}
I_3^2 &= E\left\| \int_0^t \bar{E}_h(t-s)P_h\big(g\big(u(s)\big) - g\big(u_h(s)\big)\big)\,dW(s)\right\|^2 \\
&= \int_0^t E\big\| \bar{E}_h(t-s)P_h\big(g\big(u(s)\big) - g\big(u_h(s)\big)\big)\big\|_{L_2^0}^2\,ds \\
&= \int_0^t E\left(\sum_{j=1}^{\infty} \big\| \bar{E}_h(t-s)P_h\big(g\big(u(s)\big) - g\big(u_h(s)\big)\big)Q^{\frac{1}{2}}e_j\big\|^2\right)ds \\
&\leq C\int_0^t (t-s)^{2\alpha-2}E\big\|g\big(u(s)\big) - g\big(u_h(s)\big)\big\|_{L_2^0}^2\,ds \\
&\leq C\int_0^t (t-s)^{2\alpha-2}\big\|u(s) - u_h(s)\big\|_{L_2(\Omega,H)}^2\,ds.
\end{aligned}$$

For the estimate of term I_4, we apply Lemma 3.7, Assumptions 2.2 and 2.3, and Itô's isometry:

$$\begin{aligned}
I_4^2 &= E\left\| \int_0^t \bar{F}_h(t-s)g\big(u(s)\big)\,dW(s)\right\|^2 \\
&\leq Ch^4\int_0^t (t-s)^{2\alpha-2}E\big\|u(s)\big\|_2^2\,ds \\
&\leq Ch^4\int_0^t (t-s)^{2\alpha-2}\,ds\int_0^t \sup_{0\leq s\leq T} E\big\|u(s)\big\|_2^2\,ds \\
&\leq Ch^4.
\end{aligned}$$

In total, we have by I_3 and I_4 that

$$III^2 \leq Ch^4 + C\int_0^t (t-s)^{2\alpha-2}\big\|u(s) - u(t)\big\|_{L_2(\Omega,H)}^2\,ds.$$

Let $\varphi(t) = \|u(s) - u_h(s)\|_{L_2(\Omega,H)}^2$. Since

$$\begin{aligned}
I^2 &\leq Ch^{2+2r}\|u_0\|_{L_2(\Omega;\dot{H}^{1+r})} = Ch^4\|u_0\|_{L_2(\Omega;\dot{H}^2)}, \\
II^2 &\leq Ch^4 + C\int_0^t (t-s)^{2(\alpha-1)}\big\|u(s) - u_h(s)\big\|_{L_2(\Omega,H)}^2\,ds, \\
III^2 &\leq Ch^4 + C\int_0^t (t-s)^{2\alpha-2}\big\|u(s) - u(t)\big\|_{L_2(\Omega,H)}^2\,ds,
\end{aligned}$$

according to the integral version of Gronwall's lemma, we get

$$\varphi(t) \le Ch^4.$$

Then we have

$$\|u(s) - u_h(s)\|_{L_{2(\Omega,H)}} \le Ch^2. \qquad \square$$

4 Error estimates for fully discrete approximation

In this section, we first introduce the GMMP scheme. Then we give a fully discrete scheme and the corresponding fully discrete error estimate, together with some lemmas, which are significant in the proof of the fully discrete error estimate.

4.1 The GMMP scheme

We denote the time mesh points by $t_n = nk, n = 0, 1, \dots, N$, with a fixed time step $k > 0$, such that $0 \le t_n \le T$ and $k = \frac{T}{N}$. Now let us present the GMMP scheme derived by Gorenflo, Mainardi, Moretti, and Paradisi [33]. The Caputo fractional derivative (when $0 < \alpha < 1$) can be approximated by

$$\begin{aligned} D_t^\alpha u(t_n) &\approx \frac{1}{k^\alpha}\sum_{m=0}^{n} w_m^\alpha\bigl[u(t_{n-m}) - u(0)\bigr] \\ &= \frac{1}{k^\alpha}\left[\sum_{m=0}^{n} w_m^\alpha u(t_{n-m}) - \phi_n u(0)\right], \end{aligned} \tag{4.1}$$

where

$$w_m^\alpha = \frac{\Gamma(m-\alpha)}{\Gamma(-\alpha)\Gamma(m+1)}, \tag{4.2}$$

$$\phi_n = \sum_{m=0}^{n} w_m^\alpha = \frac{\Gamma(n+1-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}, \quad n \ge 0. \tag{4.3}$$

Moreover, w_m^α and ϕ_n have the following properties.

Lemma 4.1 ([34, 35]) *For $\alpha > 0, n = 1, 2, \dots$, we have:*

(1) $w_0^\alpha = 1, w_n^\alpha < 0, |w_{n+1}^\alpha| \le |w_n^\alpha|$, *and* $0 < -\sum_{m=1}^{n} w_m^\alpha < -\sum_{m=1}^{\infty} w_m^\alpha = w_0^\alpha$;

(2) $\phi_n - \phi_{n-1} = w_n^\alpha < 0$, *that is*, $\phi_n < \phi_{n-1} < \phi_{n-2} < \cdots < \phi_0 = 1$.

4.2 Error estimates

By using the GMMP scheme (4.1) we indicate the approximation of $u(t_n)$ by $u^n \approx u(t_n)$. Then the fully discrete scheme for equation (1.1) can be defined by

$$\frac{1}{k^\alpha}\left[\sum_{m=0}^{n} w_m^\alpha u_h^{n-m} - \phi_n u_h^0\right] + A_h u_h^n = P_h f\bigl(u_h^n\bigr) + \frac{1}{k}\int_{t_{n-1}}^{t_n} P_h g\bigl(u_h^{n-1}\bigr)\, dW(s). \tag{4.4}$$

Furthermore, we define $R(\lambda, X) = (\lambda I - X)^{-1}, \lambda > 0$, and $\tilde{E}_{kh} = R(k^{-\alpha}, -A_h) = (k^{-\alpha} I + A_h)^{-1}$. Then scheme (4.4) can be rewritten as

$$\begin{aligned} u_h^n &= k^{-\alpha}\phi_n \tilde{E}_{kh} u_h^0 - k^{-\alpha}\tilde{E}_{kh}\sum_{m=1}^{n} w_m^\alpha u_h^{n-m} + \tilde{E}_{kh}P_h f\big(u_h^n\big) \\ &\quad + \frac{1}{k}\int_{t_{n-1}}^{t_n} \tilde{E}_{kh}P_h g\big(u_h^{n-1}\big)\,dW(s). \end{aligned} \tag{4.5}$$

Besides, the semidiscretized version of mild solution (3.5) at time t_n should be shown:

$$\begin{aligned} u_h(t_n) &= E_h(t_n)P_h u_0 + \int_0^{t_n} \bar{E}_h(t_n - s)P_h f\big(u_h(s)\big)\,ds \\ &\quad + \int_0^{t_n} \bar{E}_h(t_n - s)P_h g\big(u_h(s)\big)\,dW(s). \end{aligned} \tag{4.6}$$

Now let us introduce and prove some lemmas, which will play an important role later on.

Lemma 4.2 ([30]) *For any $k > 0$ and $h \in (0,1)$, there exists a constant $C > 0$ such that*

$$\|\tilde{E}_{kh}v\| \le Ck^\alpha\|v\|, \qquad \|\tilde{E}_{kh}P_h v\| \le Ck^\alpha\|v\|, \quad \forall v \in H.$$

Lemma 4.3 *For any $t > 0$ and $p, q \in [-1, 1]$ such that $0 \le p - q < 2$, we have*

$$\big\|E_h(t)v_h - v_h\big\|_p \le Ct^{\frac{(2+q-p)\alpha}{2}}\|v_h\|_{q+2}, \quad \forall v_h \in V_h.$$

Proof The definition of $E_h(t)v_h$ in (3.3) and Lemma 2.1 yield

$$\begin{aligned} &\big\|E_h(t)v_h - v_h\big\|_p^2 \\ &= \sum_{j=1}^{N}\big(\lambda_j^h\big)^p\big(1 - E_{\alpha,1}\big(-\lambda_j^h t^\alpha\big)\big)^2\big(v_h, \varphi_j^h\big)^2 \\ &= t^{(q-p)\alpha}\sum_{j=1}^{N}\big(\lambda_j^h t^\alpha\big)^{p-q}\big(1 - E_{\alpha,1}\big(-\lambda_j^h t^\alpha\big)\big)^2\big(\lambda_j^h\big)^q\big(v_h, \varphi_j^h\big)^2 \\ &= t^{(q-p)\alpha}\sum_{j=1}^{N}\big(\lambda_j^h t^\alpha\big)^{p-q}\left(\int_0^t \lambda_j^h s^{\alpha-1}E_{\alpha,\alpha}\big(-\lambda_j^h s^\alpha\big)\,ds\right)^2\big(\lambda_j^h\big)^q\big(v_h, \varphi_j^h\big)^2 \\ &\le Ct^{(q-p)\alpha}\sum_{j=1}^{N}\big(\lambda_j^h t^\alpha\big)^{p-q}\left(\int_0^t \lambda_j^h s^{\alpha-1}\frac{1}{1 + (\lambda_j^h s^\alpha)^2}\,ds\right)^2\big(\lambda_j^h\big)^q\big(v_h, \varphi_j^h\big)^2 \\ &= Ct^{(q-p)\alpha}\sum_{j=1}^{N}\big(\lambda_j^h t^\alpha\big)^{p-q}\left(\int_0^t \frac{\lambda_j^h s^{\alpha-1}}{(\lambda_j^h s^\alpha)^{(\frac{p-q}{2})}}\frac{(\lambda_j^h s^\alpha)^{(\frac{p-q}{2})}}{1 + (\lambda_j^h s^\alpha)^2}\,ds\right)^2\big(\lambda_j^h\big)^q\big(v_h, \varphi_j^h\big)^2 \\ &\le Ct^{(q-p)\alpha}\sum_{j=1}^{N}\big(\lambda_j^h t^\alpha\big)^{p-q}\left(\int_0^t \frac{\lambda_j^h s^{\alpha-1}}{(\lambda_j^h s^\alpha)^{(\frac{p-q}{2})}}\,ds\right)^2\big(\lambda_j^h\big)^q\big(v_h, \varphi_j^h\big)^2 \end{aligned}$$

$$
\begin{aligned}
&= Ct^{(q-p)\alpha}\sum_{j=1}^{N}(t^{\alpha})^{p-q}\left(\int_0^t \frac{s^{\alpha-1}}{(s^{\alpha})^{\frac{p-q}{2}}}\,ds\right)^2(\lambda_j^h)^{q+2}(v_h,\varphi_j^h)^2\\
&\le Ct^{(q-p)\alpha}\sum_{j=1}^{N}(t^{\alpha})^{p-q}\cdot t^{2\alpha-(p-q)\alpha}(\lambda_j^h)^{q+2}(v_h,\varphi_j^h)^2\\
&= Ct^{(2+q-p)\alpha}\|v_h\|_{q+2}.
\end{aligned}
$$

□

Lemma 4.4 ([30]) *For any $\lambda > 0$ and $\mu \in R$, there exists a constant C such that*

$$\big\|\big[\mu R(\lambda, A_h) - I\big]P_h v\big\| \le C\lambda^{-1}\|v\|.$$

Based on the previous discussion, we are ready to prove the error estimates for the fully discrete approximation.

Theorem 4.1 *Let u_h^n and $u(t_n)$ be solutions of (4.4) and (1.1), respectively, for $t \ge 0$, $\alpha \in (\frac{1}{2}, 1)$, and $u_0 \in L_2(\Omega, \dot{H}^s)$, $s \in [0,2]$. Then there exists a constant $C > 0$ such that*

$$\big\|u(t_n) - u_h^n\big\|^2_{L_2(\Omega;H)} \le C\big[k^{2\alpha} + h^4\big].$$

Proof By the triangle inequality we have

$$
\begin{aligned}
\big\|u(t_n) - u_h^n\big\|_{L_2(\Omega;H)} &\le \big\|u(t_n) - u_h(t_n)\big\|_{L_2(\Omega;H)} + \big\|u_h(t_n) - u_h^n\big\|_{L_2(\Omega;H)}\\
&= \big\|\rho^n\big\|_{L_2(\Omega;H)} + \big\|\theta^n\big\|_{L_2(\Omega;H)}.
\end{aligned}
$$

Since we have estimated the error of $\|\rho^n\|_{L_2(\Omega;H)}$ in Theorem 3.1, we only need to estimate $\|\theta^n\|_{L_2(\Omega;H)}$. Using equations (4.6) and (4.5), we obtain

$$
\begin{aligned}
\big\|\theta^n\big\|_{L_2(\Omega;H)} \le{}& \big\|E_h(t_n)P_h u_0 - k^{-\alpha}\phi_n\tilde{E}_{kh}P_h u_0\big\|_{L_2(\Omega;H)}\\
&+ \left\|-k^{-\alpha}\tilde{E}_{kh}\sum_{m=1}^{n} w_m^{\alpha}u_h^{n-m}\right\|_{L_2(\Omega;H)}\\
&+ \left\|\int_0^{t_n}\bar{E}_h(t_n - s)P_h f\big(u_h(s)\big)\,ds\right\|_{L_2(\Omega;H)}\\
&+ \big\|-\tilde{E}_{kh}P_h f\big(u_h^n\big)\big\|_{L_2(\Omega;H)}\\
&+ \left\|\int_0^{t_n}\bar{E}_h(t_n - s)P_h g\big(u_h(s)\big)\,dW(s)\right\|_{L_2(\Omega;H)}\\
&+ \left\|-\frac{1}{k}\int_{t_{n-1}}^{t_n}\tilde{E}_{kh}P_h g\big(u_h^{n-1}\big)\,dW(s)\right\|_{L_2(\Omega;H)}\\
={}& I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
\end{aligned}
$$

For I_1, by the triangle inequality, we separate I_1^2 into two additional terms:

$$I_1^2 = E\big\|E_h(t_n)P_h u_0 - k^{-\alpha}\phi_n\tilde{E}_{kh}P_h u_0\big\|^2$$

$$= E\big\|\big[E_h(t_n)P_hu_0 - P_hu_0\big] + \big[P_hu_0 - k^{-\alpha}\phi_n\tilde{E}_{kh}P_hu_0\big]\big\|^2$$
$$\leq 2E\big\|\big[E_h(t_n)P_hu_0 - P_hu_0\big]\big\|^2 + 2E\big\|\big[P_hu_0 - k^{-\alpha}\phi_n\tilde{E}_{kh}P_hu_0\big]\big\|^2$$
$$= I_{11} + I_{12}.$$

For I_{11}, by Lemma 4.3 with $p = q = 0$ we get

$$I_{11} = 2E\big\|\big[E_h(t_n)P_hu_0 - P_hu_0\big]\big\|^2 \leq Ct_n^{2\alpha}E\|P_hAu_0\|^2$$
$$\leq Ck^{2\alpha}E\|u_0\|_2^2.$$

For I_{12}, setting $\mu = k^{-\alpha}\phi_n$ and using Lemma 4.4, we have

$$I_{12} = 2E\|\mu\tilde{E}_{kh}P_hu_0 - P_hu_0\|^2$$
$$= 2E\big\|\big[\mu R\big(k^{-\alpha}, A_h\big) - I\big]P_hu_0\big\|^2$$
$$\leq Ck^{2\alpha}E\|u_0\|^2.$$

By Lemma 4.1 we have $\sum_{m=1}^{n}|w_m^\alpha| < w_0^\alpha = 1$. Together with Lemmas 4.2 and 3.4, we obtain

$$I_2^2 = E\left\|k^{-\alpha}\tilde{E}_{kh}\sum_{m=1}^{n}w_m^\alpha u_h^{n-m}\right\|^2$$
$$= E\left\|k^{-\alpha}\tilde{E}_{kh}\sum_{m=1}^{n}w_m^\alpha\big[\big(u_h^{n-m} - u_h(t_{n-m})\big) + u_h(t_{n-m})\big]\right\|^2$$
$$\leq C\sum_{m=1}^{n}E\big\|\theta^{n-m}\big\|^2 + C\sum_{m=1}^{n}E\big\|\tilde{E}_{kh}u_h(t_{n-m})\big\|^2\big)$$
$$\leq C\sum_{m=1}^{n}E\big\|\theta^{n-m}\big\|^2 + Ck^{2\alpha}\big(E\|u_0\|^2\big).$$

The term I_3 is estimated by applying Lemma 3.2, Assumption 2.1, and Lemma 3.4: for $0 < t_n \leq T = Nk$, we get

$$I_3^2 = E\left\|\int_0^{t_n}\bar{E}_h(t_n - s)P_hf\big(u_h(s)\big)\,ds\right\|^2$$
$$\leq \int_0^{t_n}E\big\|\bar{E}_h(t_n - s)P_hf\big(u_h(s)\big)\big\|^2\,ds$$
$$\leq C\int_0^{t_n}(t_n - s)^{2\alpha-2}E\big\|u_h(s)\big\|^2\,ds$$
$$\leq Ck^{2\alpha}E\|u_0\|^2.$$

By Lemma 4.2, Lemma 3.4, and Assumption 2.1 we get the following estimate for I_4:

$$I_4^2 \leq 2E\big\|\tilde{E}_{kh}P_h\big(f\big(u_h^n\big) - f\big(u_h(t_n)\big)\big)\big\|^2 + 2E\big\|\tilde{E}_{kh}P_hf\big(u_h(t_n)\big)\big\|^2$$
$$\leq Ck^{2\alpha}E\big\|f\big(u_h^n\big) - f\big(u_h(t_n)\big)\big\|^2 + Ck^{2\alpha}E\big\|f\big(u_h(t_n)\big)\big\|^2$$

$$
\begin{aligned}
&\le Ck^{2\alpha}E\|u_h^n - u_h(t_n)\|^2 + Ck^{2\alpha}E\|u_h(t_n)\|^2 \\
&\le Ck^{2\alpha}E\|\theta^n\|^2 + Ck^{2\alpha}E\|u_0\|^2.
\end{aligned}
$$

For I_5, by Lemma 3.2, Assumption 2.2, Lemma 3.4, and Itô's isometry, we obtain

$$
\begin{aligned}
I_5^2 &= E\left\|\int_0^{t_n}\bar{E}_h(t_n - s)P_h g(u_h(s))\,dW(s)\right\|^2 \\
&= \int_0^{t_n} E\|\bar{E}_h(t_n - s)P_h g(u_h(s))\|_{L_2^0}^2\,ds \\
&\le C\int_0^{t_n}(t_n - s)^{2\alpha-2}E\|u_h(s)\|^2\,ds \\
&\le Ck^{2\alpha}\left(E\|u_0\|^2\right).
\end{aligned}
$$

For I_6, by Lemma 4.2, Lemma 3.4, Assumption 2.2, and Itô's isometry we have

$$
\begin{aligned}
I_6^2 &= E\left\|-\frac{1}{k}\int_{t_{n-1}}^{t_n}\tilde{E}_{kh}P_h g(u_h^{n-1})\,dW(s)\right\|^2 \\
&\le 2E\left\|\frac{1}{k}\int_{t_{n-1}}^{t_n}\tilde{E}_{kh}P_h\left(g(u_h^{n-1}) - g(u_h(t_{n-1}))\right)dW(s)\right\|^2 \\
&\quad + 2E\left\|\frac{1}{k}\int_{t_{n-1}}^{t_n}\tilde{E}_{kh}P_h g(u_h(t_{n-1}))\,dW(s)\right\|^2 \\
&= \frac{2}{k}\int_{t_{n-1}}^{t_n}E\|\tilde{E}_{kh}P_h\left(g(u_h^{n-1}) - g(u_h(t_{n-1}))\right)\|_{L_2^0}^2\,ds \\
&\quad + \frac{2}{k}\int_{t_{n-1}}^{t_n}E\|\tilde{E}_{kh}P_h g(u_h(t_{n-1}))\|_{L_2^0}^2\,ds \\
&\le Ck^{2\alpha}E\|\theta^{n-1}\|^2 + Ck^{2\alpha}\left(E\|u_0\|^2\right).
\end{aligned}
$$

Therefore, coming back to $\|\theta^n\|_{L_{2(\Omega;H)}}$, combining I_1, I_2, I_3, I_4, I_5, and I_6 and applying a discrete version of Gronwall's lemma, we have

$$
\|\theta^n\|_{L_{2(\Omega;H)}}^2 \le Ck^{2\alpha}.
$$

By the triangle inequality we obtain

$$
\|u(t_n) - u_h^n\|_{L_{2(\Omega;H)}}^2 \le \|\theta^n\|_{L_{2(\Omega;H)}}^2 + \|\rho^n\|_{L_{2(\Omega;H)}}^2 \le C\left(k^{2\alpha} + h^4\right),
$$

which completes the proof. □

5 Conclusions and discussions

In this paper, we have studied semidiscrete and fully discrete schemes for nonlinear time-fractional SPDEs. The semidiscrete scheme employs a standard Galerkin finite element method, and the time direction of the fully discrete scheme is based on the GMMP scheme. The strong convergence error estimates for the semidiscrete and fully discrete schemes in the L_2-norm are demonstrated. However, there are several possible extensions of the work. First, we only consider the initial value condition in our given problem; the complex boundary condition in our future study will be discussed. Second, numerical investigations on time-space fractional SPDEs are an interesting direction for our future research.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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