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The hybrid power mean of the quartic Gauss sums and the two-term exponential sums

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Abstract

In this paper, we use the analytic method and the properties of classical Gauss sums to study the computational problems of one kind hybrid power mean of quartic Gauss sums and two-term exponential sums, and give an interesting fourth-order linear recurrence formula for it.

MSC: 11L05; 11L07

Keywords: Quartic Gauss sums; Two-term exponential sums; Hybrid power mean; Analytic method

1 Introduction

Let $q \geq 3$ be an integer. For any positive integer $k \geq 2$, the kth Gauss sums $G(m,k;q)$ are defined as

$$G(m,k;q) = \sum_{a=0}^{q-1} e\left(\frac{ma^k}{q}\right),$$

where, as usual, $e(y) = e^{2\pi iy}$.

Recently, some scholars have studied the properties of $G(m,4;p)$ and obtained many interesting results, where p is an odd prime with $p \equiv 1 \bmod 4$. For example, Shimeng Shen and Wenpeng Zhang [1] proved a recurrence formula related to $G(m,4;p)$. The author and Jiayuan Hu [2] studied the computational problem of the hybrid power mean

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{bc+\bar{c}}{p}\right)\right|^2. \tag{1}$$

We proved the identity

$$\sum_{b=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ba^4}{p}\right)\right|^2 \cdot \left|\sum_{c=1}^{p-1} e\left(\frac{bc+\bar{c}}{p}\right)\right|^2$$
$$= \begin{cases} 3p^3-3p^2-3p+p(\tau^2(\overline{\chi}_4)+\tau^2(\chi_4)), & \text{if } p \equiv 5 \bmod 8; \\ 3p^3-3p^2-3p-p\tau^2(\overline{\chi}_4)-p\tau^2(\chi_4)+2\tau^5(\overline{\chi}_4)+2\tau^5(\chi_4), & \text{if } p \equiv 1 \bmod 8, \end{cases}$$

where χ_4 denotes any fourth-order character mod$p$, $\tau(\chi) = \sum_{a=1}^{p-1} \chi(a)e(\frac{a}{p})$ denotes the classical Gauss sums, and $\overline{c}$ denotes the multiplicative inverse of c mod p.

At the same time, the author and Jiayuan Hu [2] also pointed out how to compute the exact value of $\tau^2(\overline{\chi}_4) + \tau^2(\chi_4)$ and $\tau^5(\overline{\chi}_4) + \tau^5(\chi_4)$, these are two meaningful problems.

Zhuoyu Chen and Wenpeng Zhang [3] studied the properties of the Gauss sums

$$G(k,p) = \tau^k(\psi) + \tau^k(\overline{\psi}).$$

By using the analytic method and the properties of classical Gauss sums, they obtained an exact computational formula for $G(k,p)$, which completely solved the problem proposed by the author and Jiayuan Hu in [2]. Some related works can also be found in references [4–11].

Inspired by reference [3], we will consider the following hybrid power mean:

$$M_k(p) = \sum_{m=1}^{p-1} \left(\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right)^k \cdot \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right)\right|^2.$$

For convenience, hereinafter, we always assume that p is a prime with $p \equiv 1 \bmod 4$, $(\frac{*}{p}) = \chi_2$ denotes the Legendre symbol mod$p$, and

$$\alpha = \alpha(p) = \sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + \overline{a}}{p}\right),$$

$\overline{a}$ denotes the solution of the equation $ax \equiv 1 \bmod p$. The number α is closely related to prime p. In fact, we have a very important formula

$$p = \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{a + \overline{a}}{p}\right)\right)^2 + \left(\sum_{a=1}^{\frac{p-1}{2}} \left(\frac{ra + \overline{a}}{p}\right)\right)^2 \equiv \alpha^2 + \beta^2,$$

where r is any integer with $(\frac{r}{p}) = -1$ (see Theorems 4–11 in [12]).

In this paper, by using the analytic method, the properties of the classical Gauss sums, and trigonometric sums, we will study the computational problem of $M_k(p)$, and give an interesting fourth-order linear recurrence formula for it. That is, we will prove the following two results.

Theorem 1 *If p is a prime with $p \equiv 5 \bmod 8$, then for any integer $k \geq 4$, we have the linear recurrence formula*

$$M_k(p) = -2pM_{k-2}(p) + 8p\alpha M_{k-3}(p) - p\bigl(9p - 4\alpha^2\bigr)M_{k-4}(p),$$

where the first four items in the sequence $\{M_k(p)\}$ are: $M_0(p) = p(p-3)$; $M_1(p) = 2p\alpha$; $M_2(p) = -p(p^2 - 3p - 4\alpha^2)$, and $M_3(p) = 2p^2\alpha(3p - 14)$.

Theorem 2 *If p is a prime with $p \equiv 1 \bmod 8$, then for any integer $k \geq 4$, we have the linear recurrence formula*

$$M_k(p) = 6pM_{k-2}(p) + 8p\alpha M_{k-3}(p) - p\bigl(p - 4\alpha^2\bigr)M_{k-4}(p),$$

where the first four items in the sequence $\{M_k(p)\}$ *are*: $M_0(p) = p(p-3)$; $M_1(p) = -6p\alpha$; $M_2(p) = p(3p^2 - 17p - 4\alpha^2)$, *and* $M_3(p) = 6p^2\alpha(p-8)$.

For some special integers $k = 2$ or $k = 4$, from our theorems we may immediately deduce the following three corollaries.

Corollary 1 *If* p *is an odd prime with* $p \equiv 5 \bmod 8$, *then we have*

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{c=0}^{p-1} e\left(\frac{mc^4+c}{p}\right)\right|^2 = p\left(3p^2 - 9p - 4\alpha^2\right).$$

Corollary 2 *If* p *is an odd prime with* $p \equiv 1 \bmod 8$, *then we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^2 \cdot \left|\sum_{c=0}^{p-1} e\left(\frac{mc^4+c}{p}\right)\right|^2 = p\left(3p^2 - 17p - 4\alpha^2\right).$$

Corollary 3 *If* p *is an odd prime with* $p \equiv 1 \bmod 8$, *then we have*

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right)\right|^4 \cdot \left|\sum_{c=0}^{p-1} e\left(\frac{mc^4+c}{p}\right)\right|^2 = p^2\left(17p^2 + 4p\alpha^2 - 99p - 84\alpha^2\right).$$

Notes If $p = 4k + 3$, then $(\frac{-1}{p}) = -1$. Then, in this case, for any integer m with $(m,p) = 1$, we have

$$\begin{aligned}\sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right) &= 1 + \sum_{a=1}^{p-1}\left(1 + \left(\frac{a}{p}\right)\right)e\left(\frac{ma^2}{p}\right)\\ &= \sum_{a=0}^{p-1} e\left(\frac{ma^2}{p}\right) + \sum_{a=1}^{p-1}\left(\frac{a}{p}\right)e\left(\frac{ma^2}{p}\right) = \sum_{a=0}^{p-1} e\left(\frac{ma^2}{p}\right) = \left(\frac{m}{p}\right)i\sqrt{p},\end{aligned}$$

where $i^2 = -1$. Therefore, the hybrid power mean $M_k(p)$ can be easily obtained.

2 Several lemmas

To prove our main results, we need several simple lemmas. Here, we will use many properties of the classical Gauss sums, all of them can be found in reference [13], so they will not be repeated here. First we have the following lemma.

Lemma 1 *If* p *is a prime with* $p \equiv 1 \bmod 4$, *then for any fourth-order character* $\psi \bmod p$, *we have the identity*

$$\sum_{m=1}^{p-1}\psi(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 = -\left(1 + \overline{\psi}(-1)\right)\sqrt{p}\,\tau(\overline{\psi}).$$

Proof First, from the trigonometric identity

$$\sum_{m=1}^{q} e\left(\frac{nm}{q}\right) = \begin{cases} q & \text{if } q \mid n, \\ 0 & \text{if } q \nmid n, \end{cases} \tag{2}$$

the properties of character ψ mod p and noting that $\psi^4 = \chi_0$, the principal character mod$p$, we have

$$\sum_{m=1}^{p-1}\psi(m)\left(\sum_{a=1}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right) = \sum_{a=1}^{p-1}e\left(\frac{a}{p}\right)\sum_{m=1}^{p-1}\psi(m)e\left(\frac{ma^4}{p}\right)$$

$$= \tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}\left(a^4\right)e\left(\frac{a}{p}\right) = \tau(\psi)\sum_{a=1}^{p-1}e\left(\frac{a}{p}\right) = -\tau(\psi). \quad (3)$$

Similarly, we also have

$$\sum_{m=1}^{p-1}\psi(m)\left(\sum_{a=1}^{p-1}e\left(\frac{-ma^4-a}{p}\right)\right) = -\overline{\psi}(-1)\tau(\psi). \quad (4)$$

Since $p \equiv 1 \bmod 4$, so from the properties of the fourth-order character ψ mod p, we have

$$\sum_{m=1}^{p-1}\psi(m)\left|\sum_{a=1}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2 = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=1}^{p-1}\psi(m)e\left(\frac{mb^4(a^4-1)+b(a-1)}{p}\right)$$

$$= \tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}\left(a^4-1\right)\sum_{b=1}^{p-1}e\left(\frac{b(a-1)}{p}\right)$$

$$= -\tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}\left(a^4-1\right)$$

$$= -\tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}(a-1)\left(1+\psi(a)+\psi^2(a)+\overline{\psi}(a)\right)$$

$$= \overline{\psi}(-1)\tau(\psi)+\tau(\psi)$$

$$-\tau(\psi)\sum_{a=1}^{p-1}\chi_2(a)\overline{\psi}(a-1)-\tau(\psi)\sum_{a=1}^{p-1}\overline{\psi}\left(a(a-1)\right), \quad (5)$$

where $\psi^2 = \chi_2 = (\frac{*}{p})$ denotes the Legendre symbol mod$p$.

From the properties of the classical Gauss sums, we have

$$\sum_{a=1}^{p-1}\chi_2(a)\overline{\psi}(a-1) = \frac{1}{\tau(\psi)}\sum_{b=1}^{p-1}\psi(b)\sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{b(a-1)}{p}\right)$$

$$= \frac{\tau(\chi_2)}{\tau(\psi)}\sum_{b=1}^{p-1}\psi(b)\chi_2(b)e\left(\frac{-b}{p}\right) = \frac{\overline{\psi}(-1)\tau(\chi_2)\tau(\overline{\psi})}{\tau(\psi)}; \quad (6)$$

$$\sum_{a=1}^{p-1}\overline{\psi}\left(a(a-1)\right) = \frac{1}{\tau(\psi)}\sum_{b=1}^{p-1}\psi(b)\sum_{a=1}^{p-1}\overline{\psi}(a)e\left(\frac{b(a-1)}{p}\right)$$

$$= \frac{\tau(\overline{\psi})}{\tau(\psi)}\sum_{b=1}^{p-1}\psi^2(b)e\left(\frac{-b}{p}\right) = \frac{\tau(\chi_2)\tau(\overline{\psi})}{\tau(\psi)}. \quad (7)$$

Note that $\tau(\chi_2) = \sqrt{p}$, from (5), (6), and (7) we have

$$\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = \left(1 + \overline{\psi}(-1)\right)\left(\tau(\psi) - \sqrt{p}\tau(\overline{\psi})\right). \tag{8}$$

Combining (3), (4), and (8) and noting the orthogonality properties of characters mod$p$, we have the identity

$$\begin{aligned}
&\sum_{m=1}^{p-1} \psi(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&\quad = \sum_{m=1}^{p-1} \psi(m) \left(\sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) + \sum_{a=1}^{p-1} e\left(\frac{-ma^4 - a}{p}\right) + \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \right) \\
&\quad = -\left(1 + \overline{\psi}(-1)\right)\sqrt{p}\tau(\overline{\psi}).
\end{aligned}$$

This proves Lemma 1. □

Lemma 2 *Let p be an odd prime with $p \equiv 1 \bmod 4$. Then, for the Legendre symbol $\chi_2 \bmod p$, we have the identity*

$$\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = -\psi(-1)\left(\tau^2(\psi) + \tau^2(\overline{\psi})\right).$$

Proof First, from (2) and the method of proving Lemma 1, we have

$$\begin{aligned}
&\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&\quad = \sum_{a=1}^{p-1} \sum_{m=1}^{p-1} \chi_2(m) e\left(\frac{ma^4 + a}{p}\right) \\
&\qquad + \sum_{a=1}^{p-1} \sum_{m=1}^{p-1} \chi_2(m) e\left(\frac{-ma^4 - a}{p}\right) + \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&\quad = -2\sqrt{p} + \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2;
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=1}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&\quad = \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{m=1}^{p-1} \chi_2(m) e\left(\frac{mb^4(a^4 - 1) + b(a - 1)}{p}\right) \\
&\quad = \sqrt{p} \sum_{a=1}^{p-1} \chi_2\left(a^4 - 1\right) \sum_{b=1}^{p-1} e\left(\frac{b(a - 1)}{p}\right) = -\sqrt{p} \sum_{a=1}^{p-1} \chi_2\left(a^4 - 1\right) \\
&\quad = -\sqrt{p} \sum_{a=1}^{p-1} \chi_2(a - 1)\left(1 + \psi(a) + \chi_2(a) + \overline{\psi}(a)\right)
\end{aligned}$$

$$= 2\sqrt{p} - \sqrt{p}\sum_{a=1}^{p-1}\chi_2(a-1)\psi(a) - \sqrt{p}\sum_{a=1}^{p-1}\chi_2(a-1)\overline{\psi}(a)$$
$$= 2\sqrt{p} - \psi(-1)\big(\tau^2(\psi) + \tau^2(\overline{\psi})\big). \tag{10}$$

Combining (9) and (10), we can deduce the identity

$$\sum_{m=1}^{p-1}\chi_2(m)\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2 = -\psi(-1)\big(\tau^2(\psi) + \tau^2(\overline{\psi})\big).$$

This proves Lemma 2. □

Lemma 3 *Let p be an odd prime with* $p \equiv 1 \bmod 4$, ψ *be any fourth-order character* $\bmod p$. *Then we have the identity*

$$\tau^2(\psi) + \tau^2(\overline{\psi}) = \sqrt{p}\cdot\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right) = 2\sqrt{p}\cdot\alpha.$$

Proof See Lemma 2.2 in [3]. □

Lemma 4 *Let p be an odd prime with* $p \equiv 1 \bmod 4$. *Then we have the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2 = p(p-3).$$

Proof Since the congruence equation $x^4 \equiv 1 \bmod p$ has four different solutions in a reduced residue system $\bmod p$, so from (2) we have

$$\sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2$$
$$= \sum_{m=0}^{p-1}\left|\sum_{a=0}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2$$
$$= \sum_{m=0}^{p-1}\left(1 + \sum_{a=1}^{p-1}e\left(\frac{ma^4+a}{p}\right) + \sum_{a=1}^{p-1}e\left(\frac{-ma^4-a}{p}\right) + \left|\sum_{a=1}^{p-1}e\left(\frac{ma^4+a}{p}\right)\right|^2\right)$$
$$= p + \sum_{a=1}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{ma^4+a}{p}\right) + \sum_{a=1}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{-ma^4-a}{p}\right)$$
$$+ \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{mb^4(a^4-1)+b(a-1)}{p}\right)$$
$$= p + p(p-1) + \sum_{a=2}^{p-1}\sum_{b=1}^{p-1}\sum_{m=0}^{p-1}e\left(\frac{mb^4(a^4-1)+b(a-1)}{p}\right)$$
$$= p + p(p-1) - 3p = p(p-3).$$

This proves Lemma 4. □

3 Proofs of the theorems

Now we complete the proofs of our theorems. First we prove Theorem 1. For convenience, we let

$$B(m) = \sum_{a=0}^{p-1} e\left(\frac{ma^4}{p}\right).$$

Then, for any integer m with $(m,p) = 1$, from (2) and the properties of the fourth-order character ψ mod p, we have

$$\begin{aligned}
B(m) &= 1 + \sum_{a=1}^{p-1}\bigl(1 + \psi(a) + \chi_2(a) + \overline{\psi}(a)\bigr)e\left(\frac{ma}{p}\right) \\
&= \sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma}{p}\right) \\
&\quad + \sum_{a=1}^{p-1}\chi_2(a)e\left(\frac{ma}{p}\right) + \sum_{a=1}^{p-1}\overline{\psi}(a)e\left(\frac{ma}{p}\right) \\
&= \chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi}).
\end{aligned} \tag{11}$$

If $p = 8r + 5$, then $\psi(-1) = -1$. In this case, from Lemma 1 we have the identity

$$\sum_{m=1}^{p-1}\psi(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 = \sum_{m=1}^{p-1}\overline{\psi}(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 = 0. \tag{12}$$

It is clear that from Lemma 4 we have

$$M_0(p) = \sum_{m=1}^{p-1}\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 = p(p-3). \tag{13}$$

From (12), Lemma 2, and Lemma 3, we have

$$\begin{aligned}
M_1(p) &= \sum_{m=1}^{p-1} B(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1}\bigl(\chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\bigr)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= \sqrt{p}\bigl(\tau^2(\psi) + \tau^2(\overline{\psi})\bigr) = 2p\alpha.
\end{aligned} \tag{14}$$

Similarly, noting that $\tau(\psi)\tau(\overline{\psi}) = -p$, from (12) and Lemma 4 we also have

$$\begin{aligned}
M_2(p) &= \sum_{m=1}^{p-1} B^2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1}\bigl(\chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\bigr)^2\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2
\end{aligned}$$

$$
\begin{aligned}
&= p^2(p-3) + \left(\tau^2(\psi) + \tau^2(\overline{\psi})\right) \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&\quad + 2\tau(\psi)\tau(\overline{\psi}) \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= p^2(p-3) + \left(\tau^2(\psi) + \tau^2(\overline{\psi})\right)^2 - 2p^2(p-3) \\
&= -p\left(p^2 - 3p - 4\alpha^2\right); \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
M_3(p) &= \sum_{m=1}^{p-1} B^3(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} \left(\chi_2(m)\sqrt{p} + \overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\right)^3 \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= p^{\frac{3}{2}} \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 - 6p^{\frac{3}{2}} \sum_{m=1}^{p-1} \chi_2(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&\quad + 3\sqrt{p}\left(\tau^2(\psi) + \tau^2(\overline{\psi})\right) \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= 2p^2\alpha(3p - 14). \qquad (16)
\end{aligned}
$$

From [1] (see Lemma 3) we have

$$
B^4(m) = -2pB^2(m) + 8p\alpha B(m) - 9p^2 + 4p\alpha^2. \qquad (17)
$$

So, if $k \geq 4$, then we have

$$
\begin{aligned}
M_k(p) &= \sum_{m=1}^{p-1} B^k(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} B^{k-4}(m) B^4(m) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= \sum_{m=1}^{p-1} B^{k-4}(m)\left(-2pB^2(m) + 8p\alpha B(m) - 9p^2 + 4p\alpha^2\right) \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 \\
&= -2pM_{k-2}(p) + 8p\alpha M_{k-3}(p) - p\left(9p - 4\alpha^2\right)M_{k-4}(p). \qquad (18)
\end{aligned}
$$

Combining (13)–(16) and (18), we immediately complete the proof of Theorem 1.

Now we prove Theorem 2. If $p = 8k + 1$, then note that $\psi(-1) = 1$, from Lemma 4 we have

$$
M_0(p) = \sum_{m=1}^{p-1} \left| \sum_{a=0}^{p-1} e\left(\frac{ma^4 + a}{p}\right) \right|^2 = p(p-3). \qquad (19)
$$

From (11), Lemma 1, Lemma 2, and Lemma 3, we have

$$
\begin{aligned}
M_1(p) &= \sum_{m=1}^{p-1} B(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1}\left(\chi_2(m)\sqrt{p}+\overline{\psi}(m)\tau(\psi)+\psi(m)\tau(\overline{\psi})\right)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= -3\sqrt{p}\left(\tau^2(\psi)+\tau^2(\overline{\psi})\right) = -6p\alpha. \qquad (20)
\end{aligned}
$$

Applying Lemma 1, Lemma 2, and Lemma 3, we also have

$$
\begin{aligned}
M_2(p) &= \sum_{m=1}^{p-1} B^2(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1}\left(\chi_2(m)\sqrt{p}+\overline{\psi}(m)\tau(\psi)+\psi(m)\tau(\overline{\psi})\right)^2\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= p^2(p-3)-\left(\tau^2(\psi)+\tau^2(\overline{\psi})\right)^2-8p^2+2p^2(p-3) \\
&= p\left(3p^2-17p-4\alpha^2\right); \qquad (21)\\
M_3(p) &= \sum_{m=1}^{p-1} B^3(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1}\left(\chi_2(m)\sqrt{p}+\overline{\psi}(m)\tau(\psi)+\psi(m)\tau(\overline{\psi})\right)^3\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= -p^{\frac{3}{2}}\left(\tau^2(\psi)+\tau^2(\overline{\psi})\right)-2p^{\frac{3}{2}}\left(\tau^2(\psi)+\tau^2(\overline{\psi})\right)-6p^{\frac{3}{2}}\left(\tau^2(\psi)+\tau^2(\overline{\psi})\right) \\
&\quad +3p^{\frac{3}{2}}\left(\tau^2(\psi)+\tau^2(\overline{\psi})\right)(p-3)-6p^{\frac{3}{2}}\left(\tau^2(\psi)+\tau^2(\overline{\psi})\right) \\
&= 6p^2\alpha(p-8). \qquad (22)
\end{aligned}
$$

From [1] (see Lemma 3) we have

$$
B^4(m) = 6pB^2(m)+8p\alpha B(m)-p^2+4p\alpha^2. \qquad (23)
$$

So if $k \ge 4$, then we have the fourth-order linear recurrence formula

$$
\begin{aligned}
M_k(p) &= \sum_{m=1}^{p-1} B^k(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1} B^{k-4}(m)B^4(m)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= \sum_{m=1}^{p-1} B^{k-4}(m)\left(6pB^2(m)+8p\alpha B(m)-p^2+4p\alpha^2\right)\left|\sum_{a=0}^{p-1} e\left(\frac{ma^4+a}{p}\right)\right|^2 \\
&= 6pM_{k-2}(p)+8p\alpha M_{k-3}(p)-p\left(p-4\alpha^2\right)M_{k-4}(p). \qquad (24)
\end{aligned}
$$

Now Theorem 2 follows from (19)–(22) and (24).

If $p \equiv 5 \bmod 8$, then note that $\overline{\tau(\psi)} = -\tau(\overline{\psi})$, from (11) we have

$$\overline{B(m)} = \chi_2(m)\sqrt{p} - \overline{\psi}(m)\tau(\psi) - \psi(m)\tau(\overline{\psi}). \tag{25}$$

Thus, from (25) and $\tau(\psi)\tau(\overline{\psi}) = -p$, we have

$$\left|B(m)\right|^2 = p - \left(\overline{\psi}(m)\tau(\psi) + \psi(m)\tau(\overline{\psi})\right)^2 = 3p - \chi_2(m)\left(\tau^2(\psi) + \tau^2(\overline{\psi})\right). \tag{26}$$

Applying (26) and Lemma 2, we may immediately deduce Corollary 1.

If $p \equiv 1 \bmod 8$, then note that $B(m)$ is a real number. So Corollary 2 and Corollary 3 follow from Theorem 2.

This completes the proofs of all our results.

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Authors' contributions
The author read and approved the final manuscript.

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