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The study of a mixed problem for one class of third order differential equations

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Abstract

This work is dedicated to the study of existence and uniqueness of almost everywhere solution of a multidimensional mixed problem for one class of third order differential equations with nonlinear operator on the right-hand side. The conception of almost everywhere solution for the mixed problem under consideration is introduced. After applying the Fourier method, the solution of the original problem is reduced to the solution of some countable system of nonlinear integro-differential equations in unknown Fourier coefficients of the sought solution. Besides, existence and uniqueness theorems of almost everywhere solution of the mixed problem under consideration are also proved in this work.

Keywords: Fourier method; Mixed problem; Almost everywhere solution

1 Introduction

This work is dedicated to the study of existence and uniqueness of almost everywhere solution for the following multidimensional mixed problem:

$$\frac{\partial^2 u(t, x)}{\partial t^2} - \frac{\partial}{\partial t}(L(u(t, x))) = \mathfrak{F}(u(t, x)) \quad (t \in [0, T], x \in \Omega), \quad (1)$$

$$u(0, x) = \varphi(x) \quad (x \in \Omega), \quad u_t(0, x) = \psi(x) \quad (x \in \Omega), \quad (2)$$

$$u(t, x)|_{\Gamma} = 0, \quad (3)$$

where $0 < T < +\infty$; $x = (x_1, \dots, x_n)$, Ω is a bounded n -dimensional domain with an enough smooth boundary S ; $\Gamma = [0, T] \times S$;

$$L(u(t, x)) = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) - a(x) \cdot u(t, x), \quad (4)$$

functions $a_{ij}(x)$ ($i, j = \overline{1, n}$) and $a(x)$ are measurable and bounded in Ω and satisfy in Ω the following conditions:

$$a_{ij}(x) = a_{ji}(x), \quad a(x) \geq 0, \quad \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \geq \alpha \cdot \sum_{i=1}^n \xi_i^2 \quad (\alpha = \text{const} > 0),$$

where ξ_i ($i = 1, \dots, n$) are arbitrary real numbers; φ, ψ are the given functions; \mathfrak{S} is some, generally speaking, nonlinear operator; and $u(t, x)$ is a sought function.

It must be noted that many problems in elasticity theory, in particular the problems of longitudinal vibration of the viscoelastic non-homogeneous bar, some wave problems for elastic-viscidal liquid, etc. lead to the problems of type (1)–(3).

In the beginning, we will note some papers related to problem (1)–(3).

Note that the results of this work is complete progression of the results of [1]. In particular in this work we prove new theorems about existence and uniqueness of the almost everywhere solution of problem (1)–(3).

In [10] a mixed problem for the equation below was considered

$$u_{tt}(t, x) - \alpha \Delta u(t, x) - \Delta u_t(t, x) = f(t, x, u(t, x), u_t(t, x), Du(t, x), Du_t(t, x), D^2u(t, x), D^2u_t(t, x)).$$

Under certain special conditions with respect to the nonlinear function f the existence of the solution of the problem under consideration for all $t > 0$ has been proved.

In [9], a mixed problem for the equation

$$u_{tt}(t, x) - \alpha \Delta u_t(t, x) - \Delta u(t, x) = f(t, x, u(t, x))$$

with a nonlinearity of the type $|u|^{p-1}$. u was considered. The conditions for the existence of a global weak solution of this problem are indicated.

Further, in [8] and [2] a special case of problem (1)–(3) is considered, when the operator \mathfrak{S} , appearing on the right-hand side of equation (1), is an operator of the type of a function generated by the function $f(t, x, u, u_t, \nabla u, \nabla u_t, \nabla^2 u)$. In [8] by combining the generalized contracted mappings principle and Schauder’s fixed point principle for any dimensions n , the existence in small theorem (i.e., for sufficiently small values of T) and the uniqueness in large theorem (i.e., for any finite value of T) of almost everywhere solution of problem (1)–(3) was proved, and using the method of a priori estimates for all dimensions n , the existence in large theorem of almost everywhere solution of problem (1)–(3) was proved. But in [2] the authors investigated the existence in small of classical solution of problem (1)–(3), and using the contracted mappings principle for any dimensions n , the existence theorem in small of classical solution of the considered mixed problem was proved.

In the work [7] the existence and uniqueness of the strong global solution of the one special case of problem (1)–(3), when $L = \Delta$, $n \leq 3$, and $\mathfrak{S} = \Delta u + f(u)$, were proved.

Finally, we mention the work [5] in which theorems about existence and uniqueness of the generalized, almost everywhere, and classical solution for one special one-dimensional case of problem (1)–(3), when $n = 1$, $\Omega = (0, 1)$, $Lu = \alpha \cdot u_{xx}$, were proved.

2 Auxiliaries

In this section, we introduce a number of concepts, notations, and facts to be used later.

1. In investigating the almost everywhere solution of problem (1)–(3), we will use the classes of functions \dot{D} and \dot{D}_1 , introduced by Friedrichs [6, p. 38].

We denote by $\dot{D}(\Omega)$ the class of all continuously differentiable functions on Ω which vanished near the boundary of Ω . We denote the closure of $\dot{D}(\Omega)$ with respect to the norm of $W_2^1(\Omega)$ by $\dot{D}(\Omega)$. Hence $\dot{D}(\Omega) \subset W_2^1(\Omega)$.

Denote by $\dot{D}_1(Q_T)$ ($Q_T \equiv [0, T] \times \Omega$) the class of all continuously differentiable functions on the cylinder Q_T equal to zero in the δ -neighborhood of the lateral surface on the cylinder Q_T , having the form $Q_{T,\delta} = [0, T] \times \Omega_\delta$, where Ω_δ is a δ -neighborhood of the boundary of Ω . We denote the closure of $\dot{D}_1(Q_T)$ with respect to the norm of $W_2^1(Q_T)$ by $\mathring{D}_1(Q_T)$. Hence $\mathring{D}_1(Q_T) \subset W_2^1(Q_T)$.

Definition The function $u(t, x) \in \mathring{D}_1(Q_T)$, belonging to the space $L_2(Q_T)$ together with all its derivatives $u_t(t, x)$, $u_{x_i}(t, x)$ ($i = \overline{1, n}$), $u_{tx_i}(t, x)$ ($i = \overline{1, n}$), $u_{x_i x_j}(t, x)$ ($i, j = \overline{1, n}$), $u_{tt}(t, x)$, $u_{tx_i x_j}(t, x)$ ($i, j = \overline{1, n}$), satisfying equation (1) almost everywhere in Q_T and taking initial values (2) almost everywhere in Ω , is called an almost everywhere solution of problem (1)–(3).

2. For investigation of problem (1)–(3), we recall one property of the operator L , generated by the differential expression (4) and boundary condition (3): there are denumerable number of negative eigenvalues

$$0 > -\lambda_1^2 \geq -\lambda_2^2 \geq \dots \geq -\lambda_s^2 \geq \dots \quad (0 < \lambda_s \rightarrow +\infty \text{ as } s \rightarrow \infty)$$

with the corresponding generalized eigenfunctions $v_s(x)$ which are complete and orthonormal in $L_2(\Omega)$. We call the function $v_s(x) \in \mathring{D}(\Omega)$ a generalized eigenfunction of the operator L if it is not identically zero and

$$\int_{\Omega} \left\{ \sum_{i,j=1}^n a_{ij}(x) \frac{\partial v_s(x)}{\partial x_i} \cdot \frac{\partial \Phi(x)}{\partial x_j} + a(x)v_s(x)\Phi(x) \right\} dx = \lambda_s^2 \cdot \int_{\Omega} v_s(x)\Phi(x) dx$$

for any function $\Phi(x) \in \mathring{D}(\Omega)$.

As the system $\{v_s(x)\}_{s=1}^\infty$ is complete orthonormal in $L_2(\Omega)$, then it is evident that every almost everywhere solution of problem (1)–(3) has the following form:

$$u(t, x) = \sum_{s=1}^\infty u_s(t)v_s(x),$$

where

$$u_s(t) = \int_{\Omega} u(t, x)v_s(x) dx \quad (s = 1, 2, \dots).$$

Then, after applying the Fourier method, finding the unknown Fourier coefficients $u_s(t)$ ($s = 1, 2, \dots$), the almost everywhere solution $u(t, x)$ of problem (1)–(3) is reduced to the solution of the following countable system of nonlinear integro-differential equations:

$$u_s(t) = \varphi_s + \frac{1}{\lambda_s^2} (1 - e^{-\lambda_s^2 t}) \psi_s + \frac{1}{\lambda_s^2} \int_0^t \int_{\Omega} \mathfrak{S}(u(\tau, x)) [1 - e^{-\lambda_s^2 (t-\tau)}] v_s(x) dx d\tau \quad (s = 1, 2, \dots; t \in [0, T]), \quad (5)$$

where

$$\varphi_s = \int_{\Omega} \varphi(x)v_s(x) dx, \quad \psi_s = \int_{\Omega} \psi(x)v_s(x) dx \quad (s = 1, 2, \dots).$$

Proceeding from the definition of almost everywhere solution of problem (1)–(3), it is easy to prove the following.

Lemma *If $u(t, x) = \sum_{s=1}^{\infty} u_s(t)v_s(x)$ is any almost everywhere solution of problem (1)–(3) and the generalized derivatives $\frac{\partial}{\partial x_k} a_{ij}(x)$ ($i, j, k = 1, 2, \dots, n$) are bounded on Ω , then functions $u_s(t)$ ($s = 1, 2, \dots$) satisfy system (5).*

Proof Let $u(t, x) = \sum_{s=1}^{\infty} u_s(t)v_s(x)$ be any almost everywhere solution of problem (1)–(3). Then it is evident that

$$\int_0^T \int_{\Omega} \left\{ u_{tt}(t, x) - \frac{\partial}{\partial t} (L(u(t, x))) - \mathfrak{S}(u(t, x)) \right\} \Phi(t, x) \, dx \, dt = 0 \tag{6}$$

for each $\Phi(t, x) \in L_2(Q_T)$.

If, in particular, we take

$$\Phi(t, x) = \begin{cases} (t - \tau)^2 v_s(x) & \text{for } 0 \leq t \leq \tau, x \in \Omega, \\ 0 & \text{for } \tau < t \leq T, x \in \Omega, \end{cases}$$

where $s = 1, 2, \dots$ and $\tau \in [0, T]$, then with the help of integration by parts with respect to t twice in the first term and once in the second term of (6) and taking the initial conditions (2) into consideration, we easily get

$$\begin{aligned} & 2 \int_0^{\tau} u_s(t) \, dt - 2\lambda_s^2 \int_0^{\tau} (t - \tau) u_s(t) \, dt - \int_0^{\tau} (t - \tau)^2 \mathfrak{S}_s(u, t) \, dt \\ & - 2\tau \varphi_s - \tau^2 \psi_s - \lambda_s^2 \tau^2 \varphi_s = 0, \end{aligned} \tag{7}$$

where

$$\mathfrak{S}_s(u, t) \equiv \int_{\Omega} \mathfrak{S}(u(t, x)) v_s(x) \, dx.$$

Differentiating (7) three times with respect to τ , we have the next problem

$$\begin{cases} u_s''(\tau) + \lambda_s^2 u_s'(\tau) = \mathfrak{S}_s(u, \tau) & (s = 1, 2, \dots; \tau \in [0, T]), \\ u_s(0) = \varphi_s, & u_s'(0) = \psi_s, \end{cases}$$

which is obviously equivalent to system (5). The lemma is proved. □

3. We denote by $B_{\beta_0, \dots, \beta_l, T}^{\alpha_0, \dots, \alpha_l}$ a totality of all the functions of the form

$$u(t, x) = \sum_{s=1}^{\infty} u_s(t)v_s(x)$$

considered in $Q_T = [0, T] \times \Omega$, where $u_s(t) \in C^{(l)}([0, T])$ for all s and

$$N_T(u) \equiv \sum_{i=0}^l \left\{ \sum_{s=1}^{\infty} (\lambda_s^{\alpha_i} \cdot \max_{0 \leq t \leq T} |u_s^{(i)}(t)|)^{\beta_i} \right\}^{\frac{1}{\beta_l}} < +\infty,$$

with $\alpha_i \geq 0, 1 \leq \beta_i \leq 2 (i = 0, 1, \dots, n)$. We define the norm in this set as $\|u\| = N_T(u)$. It is evident that all these spaces are Banach spaces [4, p. 50].

4. Let G be a class all functions $u(t, x)$ which have the properties $u(t, x), u_t(t, x), u_{x_i}(t, x) (i = \overline{1, n}), u_{tx_i}(t, x) (i = \overline{1, n}), u_{x_i x_j}(t, x) (i, j = \overline{1, n}), u_{tt}(t, x), u_{tx_i x_j}(t, x) (i, j = \overline{1, n}) \in L_2(Q_T)$.

5. We agree to assume that all the quantities throughout this work are real, all the functions are real-valued, and all the integrals are understood in the sense of Lebesgue.

3 On the existence of almost everywhere solution

In this section, using Zabreyko and Krasnoselskiy’s fixed point principle, the following existence theorem for the almost everywhere solution of problem (1)–(3) is proved for n .

Theorem 1 *Let:*

- $a_{ij}(x) \in C^{(2)}(\bar{\Omega}) (i, j = \overline{1, n}); a(x) \in C^{(1)}(\bar{\Omega}); S \in C^{(3)}$; the eigenfunctions $v_s(x)$ of the operator L under boundary condition $v_s(x)|_S = 0$ be three times continuously differentiable on $\bar{\Omega}$; $\varphi(x) \in W_2^3(\Omega), \varphi(x), L\varphi(x) \in \mathring{D}(\Omega); \psi(x) \in W_2^2(\Omega) \cap \mathring{D}(\Omega)$.

- $\mathfrak{S} = \mathfrak{S}_1 + \mathfrak{S}_2$, where

- the operator \mathfrak{S}_1 acts from the closed ball $K_1(\|u\|_{B_{2,T}^3} \leq \frac{1}{\lambda_1} R)$ into the space $W_{t,x,2}^{0,1}(Q_T)$ continuously, where $R > \|W(t, x)\|_{B_{2,T}^{3,2}}$ and

$$W(t, x) = \sum_{s=1}^{\infty} \left\{ \varphi_s + \frac{1}{\lambda_s^2} [1 - e^{-\lambda_s^2 t}] \psi_s \right\} \cdot v_s(x); \tag{8}$$

- the operator \mathfrak{S}_2 acts from the closed ball $K_2(\|u\|_{B_{2,T}^{3,2}} \leq R)$ into the space $W_{t,x,2}^{0,1}(Q_T)$ and satisfies the Lipschitz condition on this closed ball:

$$\|\mathfrak{S}_2(u) - \mathfrak{S}_2(v)\|_{W_{t,x,2}^{0,1}(Q_T)} \leq q \cdot \|u - v\|_{B_{2,T}^{3,2}}, \tag{9}$$

where

$$\left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \cdot C_0 \cdot q \equiv q_0 < 1, \tag{10}$$

$$C_0 \equiv \max \left\{ n \cdot \max_{i,j=\overline{1,n}} \{ \|a_{ij}(x)\|_{C(\bar{\Omega})} \}, \|a(x)\|_{C(\bar{\Omega})} \right\}^{\frac{1}{2}}.$$

-

$$\|W(t, x)\|_{B_{2,T}^{3,2}} + \left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \cdot C_0 \cdot \sup_{u \in K_2} \{ \|\mathfrak{S}(u)\|_{W_{t,x,2}^{0,1}(Q_T)} \} \leq R. \tag{11}$$

- For any $u \in K_2$, for almost all $t \in [0, T]$,

$$\mathfrak{S}_i(u(t, x)) \in \mathring{D}(\Omega), \quad i = 1, 2.$$

Then problem (1)–(3) has an almost everywhere solution.

Proof From condition 1 of the theorem it follows that $W(t, x) \in B_{2,T}^{3,2}$ since $\sum_{s=1}^{\infty} (\lambda_s^3 \varphi_s)^2 < +\infty, \sum_{s=1}^{\infty} (\lambda_s \psi_s)^2 < +\infty$ and $\sum_{s=1}^{\infty} (\lambda_s^2 \psi_s)^2 < +\infty$.

We consider the operators $Q_1(u) = W + P\mathfrak{S}_1(u)$ and $Q_2(u) = P\mathfrak{S}_2(u)$ in the closed ball $K_2(\|u\|_{B_{2,2,T}^{3,2}} \leq R)$, where the function $W(t, x)$ is defined by (8) and

$$P(u(t, x)) \equiv \sum_{s=1}^{\infty} \frac{1}{\lambda_s^2} \int_0^t \int_{\Omega} u(\tau, \xi) v_s(\xi) \cdot [1 - e^{-\lambda_s^2(t-\tau)}] d\xi d\tau \cdot v_s(x). \tag{12}$$

Using condition 4 of this theorem and the relation $Lv_s(x) = -\lambda_s^2 v_s(x)$, we transform (by integrating by parts) system (5) to the following form:

$$u_s(t) = \varphi_s + \frac{1}{\lambda_s^2} (1 - e^{-\lambda_s^2 t}) \psi_s + \frac{1}{\lambda_s^3} \int_0^t \int_{\Omega} \left[\sum_{i,j=1}^n a_{ij}(x) \frac{\partial}{\partial x_i} \mathfrak{S}(u(\tau, x)) \cdot \frac{\partial}{\partial x_j} \left(\frac{v_s(x)}{\lambda_s} \right) + a(x) \mathfrak{S}(u(\tau, x)) \cdot \frac{v_s(x)}{\lambda_s} \right] \cdot [1 - e^{-\lambda_s^2(t-\tau)}] dx d\tau, \quad s = 1, 2, \dots; t \in [0, T]. \tag{13}$$

Then we have, for every $u \in K_2$,

$$Q_1(u(t, x)) = W(t, x) + \sum_{s=1}^{\infty} \frac{1}{\lambda_s^3} \int_0^t \int_{\Omega} \left[\sum_{i,j=1}^n a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \mathfrak{S}_1(u(\tau, \xi)) \cdot \frac{\partial}{\partial \xi_j} \left(\frac{v_s(\xi)}{\lambda_s} \right) + a(\xi) \mathfrak{S}_1(u(\tau, \xi)) \cdot \frac{v_s(\xi)}{\lambda_s} \right] \cdot [1 - e^{-\lambda_s^2(t-\tau)}] d\xi d\tau \cdot v_s(x),$$

$$Q_2(u(t, x)) = \sum_{s=1}^{\infty} \frac{1}{\lambda_s^3} \int_0^t \int_{\Omega} \left[\sum_{i,j=1}^n a_{ij}(\xi) \frac{\partial}{\partial \xi_i} \mathfrak{S}_2(u(\tau, \xi)) \cdot \frac{\partial}{\partial \xi_j} \left(\frac{v_s(\xi)}{\lambda_s} \right) + a(\xi) \mathfrak{S}_2(u(\tau, \xi)) \cdot \frac{v_s(\xi)}{\lambda_s} \right] \cdot [1 - e^{-\lambda_s^2(t-\tau)}] d\xi d\tau \cdot v_s(x).$$

It is easy to obtain that, for any $u, v \in K_2$,

$$\begin{aligned} & \|Q_1(u) + Q_2(u)\|_{B_{2,2,T}^{3,2}} \\ &= \|W(t, x) + P\mathfrak{S}_1(u(t, x)) + P\mathfrak{S}_2(u(t, x))\|_{B_{2,2,T}^{3,2}} \\ &\leq \|W(t, x)\|_{B_{2,2,T}^{3,2}} + \|P\mathfrak{S}_1(u(t, x)) + P\mathfrak{S}_2(u(t, x))\|_{B_{2,2,T}^{3,2}} \\ &\leq \|W(t, x)\|_{B_{2,2,T}^{3,2}} + \left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \\ &\quad \cdot \left\{ \int_0^T \int_{\Omega} \left[\sum_{i,j=1}^n a_{ij}(\xi) \frac{\partial \mathfrak{S}(u(\tau, \xi))}{\partial \xi_i} \cdot \frac{\partial \mathfrak{S}(u(\tau, \xi))}{\partial \xi_j} + a(\xi) (\mathfrak{S}(u(\tau, \xi)))^2 \right] d\xi d\tau \right\}^{\frac{1}{2}} \\ &\leq \|W(t, x)\|_{B_{2,2,T}^{3,2}} + \left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \cdot C_0 \| \mathfrak{S}(u(t, x)) \|_{W_{t,x,2}^{0,1}(Q_T)} \leq R, \end{aligned} \tag{14}$$

$$\begin{aligned} & \|Q_1(u) - Q_1(v)\|_{B_{2,2,T}^{3,2}} \\ &\leq \left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \cdot \left\{ \int_0^T \int_{\Omega} \left[\sum_{i,j=1}^n a_{ij}(\xi) \frac{\partial}{\partial \xi_i} (\mathfrak{S}_1(u(\tau, \xi)) - \mathfrak{S}_1(v(\tau, \xi))) \right] \right\} \end{aligned}$$

$$\begin{aligned}
 & \times \left. \frac{\partial}{\partial \xi_j} (\mathfrak{S}_1(u(\tau, \xi)) - \mathfrak{S}_1(v(\tau, \xi))) + a(\xi)(\mathfrak{S}_1(u(\tau, \xi)) - \mathfrak{S}_1(v(\tau, \xi)))^2 \right] d\xi d\tau \Bigg\}^{\frac{1}{2}} \\
 & \leq \left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \cdot C_0 \| \mathfrak{S}_1(u(t, x)) - \mathfrak{S}_1(v(t, x)) \|_{W_{t,x,2}^{0,1}(Q_T)}, \tag{15} \\
 & \| Q_2(u) - Q_2(v) \|_{B_{2,2,T}^{3,2}} \\
 & \leq \left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \cdot C_0 \| \mathfrak{S}_2(u(t, x)) - \mathfrak{S}_2(v(t, x)) \|_{W_{t,x,2}^{0,1}(Q_T)} \\
 & \leq \left(\sqrt{T} + \frac{1}{\sqrt{2}} \right) \cdot C_0 \cdot q \cdot \| u - v \|_{B_{2,2,T}^{3,2}} = q_0 \| u - v \|_{B_{2,2,T}^{3,2}}, \tag{16}
 \end{aligned}$$

where $\mathfrak{S}(u(t, x)) = \mathfrak{S}_1(u(t, x)) + \mathfrak{S}_2(u(t, x))$ and the number C_0 is defined by (10).

From inequality (15), which holds not only for $u, v \in K_2$, but also for all $u, v \in K_1$, follows that the operator Q_1 acts continuously from the closed ball K_1 into $B_{2,2,T}^{3,2}$. Since the closed ball K_2 is imbedded into the closed ball K_1 compactly by virtue of [4, Theorem 1.1, p. 51], then the operator Q_1 acts compactly from K_2 into $B_{2,2,T}^{3,2}$. Further, since $q_0 < 1$, then from inequality (16) it is seen that the operator satisfies a Lipschitz condition on K_2 with the coefficient which is less than of unity. From inequality (14) it is seen that the operator $Q = Q_1 + Q_2$ acts from the closed ball K_2 into itself. Hence, by virtue of the principle of Zabreyko and Krasnoselskiy about fixed point [11, p. 1234], the operator Q has at least one fixed point $u(x, t)$ in the closed ball K_2 , which is easy to verify (in absolutely the same way as in the proof of Theorem of [1]), which is an almost everywhere solution of problem (1)–(3). The theorem is proved. \square

4 On the existence and uniqueness of almost everywhere solution

In this section, using the successive approximations method, the following existence and uniqueness theorem for the almost everywhere solution of problem (1)–(3) is proved for n .

Theorem 2 *Let:*

1. Condition 1 of Theorem 1 be satisfied.
2. The operator \mathfrak{S} act from $B_{2,2,T}^{3,2} \cup (G \cap B_{2,2,T}^{2,1})$ into $W_{t,x,2}^{0,1}(Q_T)$ so that, for all $u \in B_{2,2,T}^{3,2}$ and $t \in [0, T]$,

$$\| \mathfrak{S}(u(t, x)) \|_{W_2^1(\Omega)} \leq a(t) + b(t) \cdot \| u \|_{B_{2,2,T}^{3,2}}, \tag{17}$$

where $a(t), b(t) \in L_2(0, T)$.

3. For each $u \in B_{2,2,T}^{3,2} \cup (G \cap B_{2,2,T}^{2,2})$ for almost all $t \in [0, T]$ $\mathfrak{S}(u(t, x)) \in \mathring{D}(\Omega)$.
4. For any $u, v \in K_0(\| u \|_{B_{2,2,T}^{3,2}} \leq a_0)$ and $t \in [0, T]$,

$$\| \mathfrak{S}(u(t, x)) - \mathfrak{S}(v(t, x)) \|_{W_2^1(\Omega)} \leq C(t) \cdot \| u - v \|_{B_{2,2,T}^{3,2}}, \tag{18}$$

where $c(t) \in L_2(0, T)$,

$$\begin{aligned}
 a_0 \equiv & \left\{ [2 \| W(t, x) \|_{B_{2,2,T}^{3,2}}^2 + 4(2T + 1) \cdot C_0^2 \cdot \| a(t) \|_{L_2(0,T)}^2] \right. \\
 & \left. \cdot \exp[4(2T + 1) \cdot C_0^2 \cdot \| b(t) \|_{L_2(0,T)}^2] \right\}^{\frac{1}{2}}, \tag{19}
 \end{aligned}$$

the function $W(t, x)$ is defined by (8) and the number C_0 is defined by (10).

Then problem (1)–(3) has a unique almost everywhere solution $u(t, x)$ in $B_{2,2,T}^{3,2}$, which can be found by the method of successive approximation. Moreover, the distance between the solution $u(t, x)$ and the k th approximation $u_k(t, x)$ is estimated by the following formula:

$$\|u_k(t, x) - u(t, x)\|_{B_{2,2,T}^{3,2}} \leq a_0 \cdot \frac{\{\sqrt{2T+1} \cdot C_0 \cdot \|C(t)\|_{L_2(0,T)}\}^k}{\sqrt{k!}} \quad (k = 0, 1, 2, \dots). \tag{20}$$

Proof We consider the following operator Q :

$$Q(u(t, x)) = W(t, x) + P\mathfrak{S}(u(t, x)), \tag{21}$$

where the function $W(t, x)$ is defined by (8) and

$$P\mathfrak{S}(u(t, x)) \equiv \sum_{s=1}^{\infty} \frac{1}{\lambda_s^2} \int_0^t \int_{\Omega} \mathfrak{S}(u(\tau, \xi)) \cdot v_s(\xi) [1 - e^{-\lambda_s^2(t-\tau)}] d\xi d\tau \cdot v_s(x).$$

Then it is obvious that the operator Q acts in the space $B_{2,2,T}^{3,2}$ and satisfies the condition

$$\|Q(u) - Q(v)\|_{B_{2,2,T}^{3,2}} \leq \sqrt{2T+1} \cdot C_0 \cdot \|C(t)\|_{L_2(0,T)} \cdot \|u - v\|_{B_{2,2,T}^{3,2}} \tag{22}$$

in the ball K_0 , where C_0 is defined by (10).

Consider in $B_{2,2,T}^{3,2}$ the sequence $u_k(t, x) = Q(u_{k-1}(t, x))$, where $u_0(t, x) \equiv 0$ in Q_T . Using inequality (17) and applying the inductions method, it is easy to have, for any k ($k = 1, 2, \dots$) and $t \in [0, T]$,

$$\begin{aligned} \|u_k\|_{B_{2,2,T}^{3,2}}^2 &= \|Q(u_{k-1})\|_{B_{2,2,T}^{3,2}}^2 = \|W + P\mathfrak{S}(u_{k-1}(t, x))\|_{B_{2,2,T}^{3,2}}^2 \\ &\leq 2\|W\|_{B_{2,2,T}^{3,2}}^2 + 2\|P\mathfrak{S}(u_{k-1}(t, x))\|_{B_{2,2,T}^{3,2}}^2 \\ &\leq 2\|W\|_{B_{2,2,T}^{3,2}}^2 + 2(2T+1) \cdot C_0^2 \cdot \int_0^t \|\mathfrak{S}(u_{k-1}(\tau, x))\|_{W_2^1(\Omega)}^2 d\tau \\ &\leq 2\|W\|_{B_{2,2,T}^{3,2}}^2 + 4(2T+1) \cdot C_0^2 \cdot \int_0^t \{a^2(\tau) + b^2(\tau) \cdot \|u_{k-1}\|_{B_{2,2,T}^{3,2}}^2\} d\tau \\ &\leq A^2 + A^2 \cdot \int_0^t B^2(\tau) d\tau + \dots + A^2 \frac{\{\int_0^t B^2(\tau) d\tau\}^k}{k!}, \end{aligned} \tag{23}$$

where

$$\begin{aligned} A^2 &\equiv 2\|W\|_{B_{2,2,T}^{3,2}}^2 + 4(2T+1) \cdot C_0^2 \cdot \|a(t)\|_{L_2(0,T)}^2, \\ B^2(t) &\equiv 4(2T+1) \cdot C_0^2 \cdot b^2(t). \end{aligned}$$

From (23) it follows that, for any k ($k = 0, 1, 2, \dots$),

$$\|u_k\|_{B_{2,2,T}^{3,2}}^2 \leq A^2 \exp\left\{\int_0^T B^2(t) dt\right\} \equiv a_0^2, \tag{24}$$

that is, all the approximations $u_k(t, x)$ are contained in the ball K_0 .

Next, using condition 4 of this theorem and taking (24) into account, we have, for every k ($k = 1, 2, \dots$) and $t \in [0, T]$,

$$\begin{aligned}
 \|u_{k+1} - u_k\|_{B_{2,2,T}^{3,2}}^2 &= \|P\mathfrak{S}(u_k(t, x)) - P\mathfrak{S}(u_{k-1}(t, x))\|_{B_{2,2,T}^{3,2}}^2 \\
 &\leq (2T + 1) \cdot C_0^2 \cdot \int_0^t \|\mathfrak{S}(u_k(\tau, x)) - \mathfrak{S}(u_{k-1}(\tau, x))\|_{W_2^1(\Omega)}^2 d\tau \\
 &\leq (2T + 1) \cdot C_0^2 \cdot \int_0^t C^2(\tau) \cdot \|u_k - u_{k-1}\|_{B_{2,2,\tau}^{3,2}}^2 d\tau \\
 &\leq \|u_1 - u_0\|_{B_{2,2,\tau}^{3,2}}^2 \cdot \frac{\{(2T + 1) \cdot C_0^2 \int_0^t C^2(\tau) d\tau\}^k}{k!} \\
 &= \|u_1\|_{B_{2,2,T}^{3,2}}^2 \cdot \frac{\{(2T + 1) \cdot C_0^2 \int_0^t C^2(\tau) d\tau\}^k}{k!}. \tag{25}
 \end{aligned}$$

Consequently,

$$\|u_{k+1}(t, x) - u_k(t, x)\|_{B_{2,2,T}^{3,2}}^2 \leq a_0^2 \cdot \frac{\{(2T + 1) \cdot C_0^2 \|c(t)\|_{L_2(0,T)}^2\}^k}{k!} \quad (k = 0, 1, 2, \dots). \tag{26}$$

From this it follows that the sequence $\{u_k(t, x)\}_{k=1}^\infty$ forms the Cauchy sequence in $B_{2,2,T}^{3,2}$. Due to the completeness of the space $B_{2,2,T}^{3,2}$, we have

$$u_k(t, x) \xrightarrow{B_{2,2,T}^{3,2}} u(t, x) \in K_0 \quad \text{as } k \rightarrow \infty.$$

Since the operator Q is continuous in the ball K_0 (see inequality (22)), from the relations $u_k(t, x) = Q(u_{k-1}(t, x))$ it follows that

$$u(t, x) = Q(u(t, x)).$$

It is easy to verify (in absolutely the same way as in the proof of Theorem of [1]) that the function $u(t, x)$ is the almost everywhere solution of problem (1)–(3). Inequality (20) can be proved similarly to (25) and (26).

Now, we prove the uniqueness of the almost everywhere solution of problem (1)–(3) in $B_{2,2,T}^{3,2}$. Let $u(t, x) = \sum_{s=1}^\infty u_s(t) v_s(x)$ be any almost everywhere solution of problem (1)–(3) in $B_{2,2,T}^{3,2}$. Then $\mathfrak{S}(u(t, x)) \in L_2(Q_T)$. By virtue of the lemma in Sect. 2, the functions $u_s(t)$ ($s = 1, 2, \dots$) satisfy system (5). From system (5) we obtain

$$\|u(t, x)\|_{B_{2,2,T}^{2,1}} \leq \|W(t, x)\|_{B_{2,2,T}^{2,1}} + \left(\sqrt{T} + \frac{1}{\sqrt{2}}\right) \cdot \|\mathfrak{S}(u(t, x))\|_{L_2(Q_T)} < +\infty.$$

Consequently, $u(t, x) \in B_{2,2,T}^{2,1}$. As $u(t, x) \in G \cap B_{2,2,T}^{2,1}$, then due to condition 2 of this theorem $\mathfrak{S}(u(t, x)) \in W_{t,x,2}^{0,1}(Q_T)$, and due to condition 3 of this theorem for almost all $t \in [0, T]$ $\mathfrak{S}(u(t, x)) \in \dot{D}(\Omega)$. Using this, from system (13) equivalent to system (5), we have

$$\|u(t, x)\|_{B_{2,2,T}^{3,2}} \leq \|W(t, x)\|_{B_{2,2,T}^{3,2}} + \left(\sqrt{T} + \frac{1}{\sqrt{2}}\right) \cdot C_0 \|\mathfrak{S}(u(t, x))\|_{W_{t,x,2}^{0,1}(Q_T)} < +\infty.$$

Consequently, $u(t, x) \in B_{2,2,T}^{3,2}$. Then, using inequality (17), similar to (23) and (24), $\forall t \in [0, T]$ we have

$$\|u\|_{B_{2,2,T}^{3,2}}^2 \leq a_0^2.$$

Thus, all the possible almost everywhere solutions of problem (1)–(3) belong to the sphere K_0 , and there is a fixed point in $B_{2,2,T}^{3,2}$ of the operator Q defined by (21). Let $u(t, x)$ and $v(t, x)$ be two arbitrary almost everywhere solutions of problem (1)–(3). Then, using the fact that $u(t, x), v(t, x) \in K_0$ and condition 4 of the present theorem, it is easy to obtain that, $\forall t \in [0, T]$,

$$\begin{aligned} \|u - v\|_{B_{2,2,t}^{3,2}}^2 &\leq (2T + 1)C_0^2 \cdot \int_0^t \|\mathfrak{S}(u(\tau, x)) - \mathfrak{S}(v(\tau, x))\|_{W_2^1(\Omega)}^2 d\tau \\ &\leq (2T + 1) \cdot C_0^2 \cdot \int_0^t c^2(\tau) \cdot \|u - v\|_{B_{2,2,\tau}^{3,2}}^2 d\tau. \end{aligned}$$

From here, on applying Bellman's inequality [3, pp. 188, 189], we obtain that $\forall t \in [0, T]$ $\|u - v\|_{B_{2,2,T}^{3,2}} \equiv 0$. Hence, $u = v$. The theorem is proved. \square

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Authors' contributions

All authors conceived of the study, participated in its design and coordination, read and approved the final manuscript.

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