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On multiplicity of solutions to nonlinear partial difference equations with delay

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Abstract

In this paper, we present an existence criterion for multiple positive solutions of nonlinear neutral delay partial difference equations. Such equations can be regarded as a discrete analog of neutral delay partial differential equations. Our main result relies on fixed point index theory. An example is constructed to show the applicability of the obtained result.

MSC: 35R10; 39A11

Keywords: Partial difference equations; Multiple positive solutions; Fixed point index

1 Introduction

Partial difference equations constitute an important and interesting area of research in mathematics. For some classical results concerning the solvability of some classes of partial difference equations, see [1]. The qualitative analysis of partial difference equations has been studied later, especially in recent years; see [2, 3].

Many researchers recently investigated solvability and oscillation criteria for partial difference equations with two variables. For some solvability results, we refer the reader to a series of papers [4–10] and the references therein, while some recent work on the oscillation and nonoscillation criteria for partial difference equations can be found in the articles [11–18]. However, to the best of our knowledge, the topic of existence of multiple positive solutions for partial difference equations has yet to be addressed.

The goal of this paper is to discuss the multiplicity of positive solutions of nonlinear neutral partial difference equation with the aid of the fixed point index theory. Precisely, we consider the following neutral partial difference equation:

$$\triangle_n^h \triangle_m^r (y_{m,n} - c_{m,n} y_{m-k,n-l}) + (-1)^{h+r+1} P_{m,n} f(y_{m-\sigma,n-\tau}) = 0, \tag{1.1}$$

where $h, r \in \mathbf{N}^+$, $k, l, \sigma, \tau \in \mathbf{N}(0)$; $\{P_{m,n}\}_{m=m_0,n=n_0}^{\infty}{}^{\infty}$ and $\{c_{m,n}\}_{m=m_0,n=n_0}^{\infty}{}^{\infty}$ are nonnegative sequences; $f(x)$ is a real-valued continuous function of x.

Equation (1.1) can be considered as a discrete analog of neutral delay partial differential equations. Such equations appear frequently in random walk problems, molecular orbit structures, dynamical systems, economics, biology, population dynamics, and other fields. Finite difference methods applied to partial differential equations also give rise to an equation of the form (1.1).

The forward differences $\triangle_m$ and $\triangle_n$ are defined in the usual manner as

$$\triangle_m y_{m,n} = y_{m+1,n} - y_{m,n} \quad \text{and} \quad \triangle_n y_{m,n} = y_{m,n+1} - y_{m,n}.$$

The higher order forward differences for positive integers r and h are given by

$$\begin{aligned} &\triangle_m^r y_{m,n} = \triangle_m\big(\triangle_m^{r-1} y_{m,n}\big), \qquad \triangle_m^0 y_{m,n} = y_{m,n}, \\ &\triangle_n^h y_{m,n} = \triangle_n\big(\triangle_n^{h-1} y_{m,n}\big), \qquad \triangle_n^0 y_{m,n} = y_{m,n}. \end{aligned}$$

In the sequel, we denote by $\mathbf{N} = \{0, 1, \ldots\}$ the set of integers and by $\mathbf{N}^+ = \{1, 2, \ldots\}$ the set of positive integers; $\mathbf{N}(a) = \{a, a+1, \ldots\}$, where $a \in \mathbf{N}$, $\mathbf{N}(a,b) = \{a, a+1, \ldots, b\}$ with $a < b < \infty$ and $a, b \in \mathbf{N}$. Any one of these three sets will be denoted by $\overline{\mathbf{N}}$. For $t \in R$, we define the usual factorial expression $(t)^{(m)} = t(t-1)\cdots(t-m+1)$ with $(t)^0 = 1$.

The space $l^\infty_{m=m_0,n=n_0}{}^\infty$ is the set of double real sequences defined on the set of positive integer pairs, where any individual double sequence is bounded with respect to the usual supremum norm, that is,

$$\|y\| = \sup_{m \in \mathbf{N}(m_0), n \in \mathbf{N}(n_0)} |y_{m,n}| < \infty.$$

It is well known that $l^\infty_{m=m_0,n=n_0}{}^\infty$ is a Banach space under the supremum norm.

Let

$$P = \big\{ y \in l^\infty_{m=m_0,n=n_0}{}^\infty \mid y_{m,n} \geq 0, m \in \mathbf{N}(m_0), n \in \mathbf{N}(n_0) \big\}.$$

Then it is easy to see that P is a cone. We define a partial order $\leq$ in $l^\infty_{m=m_0,n=n_0}{}^\infty$ as follows:

$$\text{for any } x, y \in l^\infty_{m=m_0,n=n_0}{}^\infty, \quad x \leq y \quad \Leftrightarrow \quad y - x \in P.$$

Definition 1 ([16]) A set Ω of double sequences in $l^\infty_{m=m_0,n=n_0}{}^\infty$ is uniformly Cauchy (or equi-Cauchy) if for every $\varepsilon > 0$, there exist positive integers m_1 and n_1 such that, for any $x = \{x_{m,n}\}$ in Ω,

$$|x_{m,n} - x_{m',n'}| < \varepsilon$$

holds whenever $(m,n) \in D', (m',n') \in D'$, where $D' = D_1' \cup D_2' \cup D_3'$, $D_1' = \{(m,n) \mid m > m_1, n > n_1\}$, $D_2' = \{(m,n) \mid m_0 \leq m \leq m_1, n > n_1\}$, $D_3' = \{(m,n) \mid m > m_1, n_0 \leq n \leq n_1\}$.

Definition 2 ([19]) An operator $A : D \to E$ is called a k-set-contraction ($k \geq 0$) if it is continuous, bounded and

$$\gamma\big(A(S)\big) < k\gamma(S)$$

for any bounded set $S \subset D$, where $\gamma(S)$ denotes the measure of noncompactness of S. A k-set-contraction is called a strict set contraction if $k < 1$.

Definition 3 Let K be a retract of a Banach space X, $\Omega \subset K$ an open set and $f : \overline{\Omega} \to K$ a compact map such that $f(x) \neq x$ on $\partial\Omega$. If $r : X \to K$ is a retraction, then $\deg(I - fr, r^{-1}(\Omega), \theta)$ is defined, where deg denotes the Leray–Schauder degree, this number is called the fixed point index of f over Ω with respect to K, $i(f, \Omega, K)$ for short.

The fixed point index $i(f, \Omega, K)$ has the following properties:

(i) *Normalization*: for every constant map f mapping $\overline{\Omega}$ into Ω, $i(f, \Omega, K) = 1$.

(ii) *Additivity*: for every pair of disjoint open subsets Ω_1, Ω_2 of Ω such that f has no fixed points on $\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$,

$$i(f, \overline{\Omega}, K) = i(A, \Omega_1, K) + i(f, \Omega_2, K),$$

where $i(f, \Omega_n, K) = i(f|_{\overline{\Omega}_n}, \Omega_n, K)$ for $n = 1, 2$.

(iii) *Homotopy invariance*: for every compact interval $[a, b] \subset \mathbb{R}$ and every compact map $h : [a, b] \times \Omega \to K$ such that $h(\lambda, x) \neq x$ for $(\lambda, x) \in [a, b] \times \partial\Omega$, $i(h(\lambda, \cdot), \Omega, K)$ is well defined and independent of $\lambda \in [a, b]$.

Now we state some well-known lemmas which will be used in the next section.

Lemma 1 ([16] (Discrete Arzela–Ascoli's theorem)) *A bounded, uniformly Cauchy subset* Ω *of* $l^{\infty}_{m=m_0}{}^{\infty}_{n=n_0}$ *is relatively compact.*

Lemma 2 ([19]) *Let P be a cone in a real Banach space X and Ω be a nonempty bounded open convex subset of P. Suppose that $T : \overline{\Omega} \to P$ is a strict set contraction operator and $T(\Omega) \subset \Omega$, where $\overline{\Omega}$ denotes the closure of Ω in P. Then the fixed point index $i(T, \Omega, P) = 1$.*

2 Main result

Theorem 1 *Assume that*

(R_1) *there exists a constant c such that* $0 \leq c_{m,n} \leq c < 1$, $m \in \mathbf{N}(m_0)$, $n \in \mathbf{N}(n_0)$;

(R_2) *for any* $m \in \mathbf{N}(m_0)$, $n \in \mathbf{N}(n_0)$, $P_{m,n} > 0$, $xf(x) > 0$ $(x \neq 0)$ *with*

$$\lim_{x \to 0+} \frac{f(x)}{x} = 0, \qquad \lim_{x \to +\infty} \frac{f(x)}{x} = 0;$$

(R_3) *for* $\delta_1 = \max\{k, \sigma\}$, $\delta_2 = \min\{k, \sigma\}$, $\eta_1 = \max\{l, \tau\}$, $\eta_2 = \min\{l, \tau\}$, *there exist positive integers* m_1, n_1 *satisfying* $m_1 - \delta_1 \in \mathbf{N}(m_0)$ *and* $n_1 - \eta_1 \in \mathbf{N}(n_0)$ *such that*

$$0 < c_0 \triangleq \sum_{i=m_1}^{\infty} \sum_{j=n_1}^{\infty} \frac{(i + r - 1)^{(r-1)}(j + h - 1)^{(h-1)}}{(r - 1)!(h - 1)!} P_{i,j} < +\infty;$$

(R_4) *there exist constants* c_1 *and* $u_0 > 0$ *such that* $f(x) \geq c_1 u_0$ *for* $x \geq u_0$, *and furthermore there exist positive integers* b_1, b_2 *satisfying* $b_1 > m_1$, $b_2 > n_1$ *such that*

$$c_1 c_2 > 1,$$

where

$$c_2 \triangleq \sum_{i=b_1}^{b_1+\delta_2} \sum_{j=b_2}^{b_2+\eta_2} \frac{(i - b_1 + r - 1)^{(r-1)}(j - b_2 + h - 1)^{(h-1)}}{(r - 1)!(h - 1)!} P_{i,j} > 0.$$

Then Eq. (1.1) *has at least two positive solutions* x^* *and* y^* *satisfying the relation*:

$$\inf_{\substack{m\in\mathbf{N}(a_1,b_1)\\ n\in\mathbf{N}(a_2,b_2)}} x^*_{m,n} < u_0 < \inf_{\substack{m\in\mathbf{N}(a_1,b_1)\\ n\in\mathbf{N}(a_2,b_2)}} y^*_{m,n},$$

where a_1 *and* a_2 *are positive integers with* $a_1 \in [m_1-\delta_1, b_1-\delta_1)$, $a_2 \in [n_1-\eta_1, b_2-\eta_1)$.

Proof Set

$$\begin{aligned}
&D = \{(m,n) \mid m \ge m_0, n \ge n_0\},\\
&D_1 = \{(m,n) \mid m \ge m_1, n \ge n_1\},\\
&D_2 = \{(m,n) \mid m_0 \le m < m_1, n \ge n_1\},\\
&D_3 = \{(m,n) \mid m \ge m_1, n_0 \le n < n_1\},\\
&D_4 = \{(m,n) \mid m_0 \le m < m_1, n_0 \le n < n_1\}.
\end{aligned}$$

For any $y \in P$, define operators T_1 and T_2 as follows:

$$(T_1y)_{m,n} = \begin{cases} c_{m,n}y_{m-k,n-l}, & (m,n) \in D_1;\\ (T_1y)_{m_1,n}, & (m,n) \in D_2;\\ (T_1y)_{m,n_1}, & (m,n) \in D_3;\\ (T_1y)_{m_1,n_1}, & (m,n) \in D_4; \end{cases}$$

$$(T_2y)_{m,n} = \begin{cases} \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \frac{(i-m+r-1)^{(r-1)}}{(r-1)!}\frac{(j-n+h-1)^{(h-1)}}{(h-1)!}P_{i,j}f(y_{i-\sigma,j-\tau}), & (m,n) \in D_1;\\ (T_2y)_{m_1,n}, & (m,n) \in D_2;\\ (T_2y)_{m,n_1}, & (m,n) \in D_3;\\ (T_2y)_{m_1,n_1}, & (m,n) \in D_4. \end{cases}$$

Fixing $T = T_1 + T_2$, one can observe that $T : P \to P$. First we show that T is a strict set contraction operator in P.

(i) T_1 is a contraction operator on P.

For any $x, y \in P$, $x = \{x_{m,n}\}_{m=m_0,n=n_0}^{\infty\ \ \infty}$, $y = \{y_{m,n}\}_{m=m_0,n=n_0}^{\infty\ \ \infty}$, we have

$$\begin{aligned}
&(T_1x)_{m,n} = c_{m,n}x_{m-k,n-l}, \quad m \in \mathbf{N}(m_1), n \in \mathbf{N}(n_1),\\
&(T_1y)_{m,n} = c_{m,n}y_{m-k,n-l}, \quad m \in \mathbf{N}(m_1), n \in \mathbf{N}(n_1),
\end{aligned}$$

so that

$$\begin{aligned}
\|T_1x - T_1y\| &= \sup_{(m,n)\in\mathbf{N}(m_0)\times\mathbf{N}(n_0)} \left|(T_1x)_{m,n} - (T_1y)_{m,n}\right|\\
&= \sup_{(m,n)\in\mathbf{N}(m_0)\times\mathbf{N}(n_0)} \left|(T_1x)_{m,n} - (T_1y)_{m,n}\right|\\
&= \sup_{(m,n)\in\mathbf{N}(m_0)\times\mathbf{N}(n_0)} |c_{m,n}x_{m-k,n-l} - c_{m,n}y_{m-k,n-l}|\\
&= \sup_{(m,n)\in\mathbf{N}(m_0)\times\mathbf{N}(n_0)} c_{m,n}|x_{m-k,n-l} - y_{m-k,n-l}|
\end{aligned}$$

$$\begin{aligned} &< c \sup_{(m,n)\in \mathbf{N}(m_0)\times \mathbf{N}(n_0)} |x_{m-k,n-l} - y_{m-k,n-l}| \\ &= c\|x-y\|. \end{aligned} \tag{2.1}$$

From (R_1), we know that $c < 1$, therefore T_1 is a contraction operator.

(ii) T_2 is completely continuous.

From (R_3) and the continuity of f, it follows that $T_2 : P \to P$ is continuous. Thus we just need to establish that T_2 is a compact operator in P. For any bounded subset $Q \subset P$, without loss of generality, we may assume $Q = \{x \in P \mid \|x\| \leq r'\}$. Now it suffices to show that T_2Q is relatively compact.

According to $\lim_{x\to+\infty} \frac{f(x)}{x} = 0$, we know that there exists an $r'' > 0$ such that

$$0 < f(x) \leq \frac{1-c}{4c_0}x, \quad x \geq r''.$$

Thus

$$0 < f(x) \leq \frac{1-c}{4c_0}x + M, \quad x \in R_+, \tag{2.2}$$

where $M = \max_{0\leq x\leq r''} f(x)$. Let

$$\overline{r} = \max\left\{r', r'', \frac{4c_0M}{1-c}\right\}. \tag{2.3}$$

Define $[\alpha,\beta] = \{x \in P \mid \alpha \leq x \leq \beta\}$, where $\alpha = (0,0,\ldots)$, $\beta = (\overline{r},\overline{r},\ldots)$. Obviously, $Q \subset [\alpha,\beta]$. From (R_1), (R_3), (2.2) and (2.3), for any $x \in [\alpha,\beta]$, we have $T_2x \geq \alpha$ and, when $(m,n) \in \mathbf{N}(m_1) \times \mathbf{N}(n_1)$,

$$\begin{aligned} (T_2x)_{m,n} &= \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \frac{(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}}{(r-1)!(h-1)!} P_{i,j} f(x_{i-\sigma,j-\tau}) \\ &\leq \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \frac{(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}}{(r-1)!(h-1)!} P_{i,j}\left(\frac{1-c}{4c_0}x_{i-\sigma,j-\tau} + M\right) \\ &\leq \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \frac{(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}}{(r-1)!(h-1)!} P_{i,j}\left(\frac{1-c}{4c_0}\overline{r} + M\right) \\ &\leq \frac{1-c}{4}\overline{r} + c_0M \\ &\leq \frac{1-c}{2}\overline{r} \\ &< \overline{r}. \end{aligned}$$

This means that $T_2x < \beta$; in particular, $T_2\beta < \beta$. Hence, $T_2 : [\alpha,\beta] \to [\alpha,\beta]$, which implies that $T_2[\alpha,\beta]$ is bounded.

Next we show that $T_2[\alpha,\beta]$ is uniformly Cauchy. For any given $\varepsilon > 0$, by the condition (R_3), there exist sufficiently large positive integers $m_2 \in \mathbf{N}(m_1)$, $n_2 \in \mathbf{N}(n_1)$ such that

$$\sum_{i=m_2}^{\infty}\sum_{j=n_2}^{\infty} \frac{(i+r-1)^{(r-1)}(j+h-1)^{(h-1)}}{(r-1)!(h-1)!} P_{i,j} < \frac{\varepsilon}{4}\left(\frac{1-c}{4c_0}\overline{r} + M\right)^{-1}. \tag{2.4}$$

By the condition (R_3), we have

$$\sum_{i=m_1}^{\infty}\sum_{j=n_1}^{n_2}\frac{(i+r-1)^{(r-1)}(j+h-1)^{(h-1)}}{(r-1)!(h-1)!}P_{i,j}<\infty.$$

Hence, there exists an $m_3 \geq m_2$ such that

$$\sum_{i=m_3}^{\infty}\sum_{j=n_1}^{n_2}\frac{(i+r-1)^{(r-1)}(j+h-1)^{(h-1)}}{(r-1)!(h-1)!}P_{i,j}<\frac{\varepsilon}{4}\left(\frac{1-c}{4c_0}\overline{r}+M\right)^{-1}. \tag{2.5}$$

Similarly, there exists an $n_3 \geq n_2$ such that

$$\sum_{i=m_1}^{m_2}\sum_{j=n_3}^{\infty}\frac{(i+r-1)^{(r-1)}(j+h-1)^{(h-1)}}{(r-1)!(h-1)!}P_{i,j}<\frac{\varepsilon}{4}\left(\frac{1-c}{4c_0}\overline{r}+M\right)^{-1}. \tag{2.6}$$

For any $x=\{x_{m,n}\}\in[\alpha,\beta]$, when $(m,n),(m',n')\in\mathbf{N}(m_2)\times\mathbf{N}(n_2)$, from (2.4) we have

$$\begin{aligned}
&\left|(T_2x)_{m,n}-(T_2x)_{m',n'}\right| \\
&\quad\leq\frac{1}{(r-1)!(h-1)!}\left[\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}P_{i,j}f(x_{i-\sigma,j-\tau})\right. \\
&\qquad\left.+\sum_{i=m'}^{\infty}\sum_{j=n'}^{\infty}\left(i-m'+r-1\right)^{(r-1)}\left(j-n'+h-1\right)^{(h-1)}P_{i,j}f(x_{i-\sigma,j-\tau})\right] \\
&\quad\leq\frac{1}{(r-1)!(h-1)!}\left(\frac{1-c}{4c_0}\overline{r}+M\right)\left[\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}P_{i,j}\right. \\
&\qquad\left.+\sum_{i=m'}^{\infty}\sum_{j=n'}^{\infty}\left(i-m'+r-1\right)^{(r-1)}\left(j-n'+h-1\right)^{(h-1)}P_{i,j}\right] \\
&\quad\leq\frac{2}{(r-1)!(h-1)!}\left(\frac{1-c}{4c_0}\overline{r}+M\right)\sum_{i=m_2}^{\infty}\sum_{j=n_2}^{\infty}(i+r-1)^{(r-1)}(j+h-1)^{(h-1)}P_{i,j} \\
&\quad<2\left(\frac{1-c}{4c_0}\overline{r}+M\right)\cdot\frac{\varepsilon}{4}\left(\frac{1-c}{4c_0}\overline{r}+M\right)^{-1} \\
&\quad<\varepsilon.
\end{aligned}$$

When $(m,n),(m',n')\in\{(m,n)\mid m\geq m_3, n_1\leq n<n_2\}$, from (2.4) and (2.5) we have

$$\begin{aligned}
&\left|(T_2x)_{m,n}-(T_2x)_{m',n'}\right| \\
&\quad=\frac{1}{(r-1)!(h-1)!}\left|\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}P_{i,j}f(x_{i-\sigma,j-\tau})\right. \\
&\qquad\left.-\sum_{i=m'}^{\infty}\sum_{j=n'}^{\infty}\left(i-m'+r-1\right)^{(r-1)}\left(j-n'+h-1\right)^{(h-1)}P_{i,j}f(x_{i-\sigma,j-\tau})\right| \\
&\quad\leq\frac{1}{(r-1)!(h-1)!}\left[\sum_{i=m}^{\infty}\sum_{j=n}^{\infty}(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}P_{i,j}f(x_{i-\sigma,j-\tau})\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=m'}^{\infty}\sum_{j=n'}^{\infty}\bigl(i-m'+r-1\bigr)^{(r-1)}\bigl(j-n'+h-1\bigr)^{(h-1)}P_{i,j}f(x_{i-\sigma,j-\tau})\Biggr] \\
&\leq \frac{2}{(r-1)!(h-1)!}\Biggl[\sum_{i=m_3}^{\infty}\sum_{j=n_1}^{\infty}(i-m_3+r-1)^{(r-1)}(j-n_1+h-1)^{(h-1)}P_{i,j}f(x_{i-\sigma,j-\tau})\Biggr] \\
&\leq \frac{2}{(r-1)!(h-1)!}\biggl(\frac{1-c}{4c_0}\overline{r}+M\biggr)\Biggl[\sum_{i=m_3}^{\infty}\sum_{j=n_2}^{\infty}(i+r-1)^{(r-1)}(j+h-1)^{(h-1)}P_{i,j} \\
&+\sum_{i=m_3}^{\infty}\sum_{j=n_1}^{n_2}(i+r-1)^{(r-1)}(j+h-1)^{(h-1)}P_{i,j}\Biggr] \\
&< 4\biggl(\frac{1-c}{4c_0}\overline{r}+M\biggr)\frac{\varepsilon}{4}\biggl(\frac{1-c}{4c_0}\overline{r}+M\biggr)^{-1} \\
&= \varepsilon.
\end{aligned}
$$

Similarly, when $(m,n),(m',n') \in \{(m,n) \mid m_1 \leq m < m_2, n \geq n_3\}$, from (2.4) and (2.6) we have

$$\bigl|(T_2x)_{m,n}-(T_2x)_{m',n'}\bigr| < \varepsilon.$$

Let $D' = D'_1 \cup D'_2 \cup D'_3$, where $D'_1 = \{(m,n) \mid m \geq m_3, n \geq n_3\}$, $D'_2 = \{(m,n) \mid m \geq m_3, n_1 \leq n < n_3\}$, $D'_3 = \{(m,n) \mid m_1 \leq m < m_3, n \geq n_3\}$.

Then, for any given ε, there exist positive integers $(m_3,n_3) \in \mathbf{N}(m_2) \times \mathbf{N}(n_2)$ such that, for all $x = \{x_{m,n}\} \in [\alpha,\beta]$,

$$\bigl|(T_2x)_{m,n}-(T_2x)_{m',n'}\bigr| < \varepsilon$$

holds for all $(m,n),(m',n') \in D'$, which implies $T_2[\alpha,\beta]$ is uniformly Cauchy.

Hence, $T_2[\alpha,\beta]$ is relatively compact. Since $Q \subset [\alpha,\beta]$ is any bounded subset of P, T_2Q is relatively compact. Thus T_2 is a compact operator in P. Hence T_2 is completely continuous in P. Then $T = T_1 + T_2 : P \to P$ is a strict set contraction operator.

Next, from condition (R_2), there exist positive constants $0 < r_1 < u_0 < r_2$ such that

$$0 < f(x) \leq \frac{1-c}{4c_0}x, \quad \text{for } 0 < x \leq r_1, \text{or } x \geq r_2, \tag{2.7}$$

and

$$0 < f(x) \leq \frac{1-c}{4c_0}x + \overline{M}, \quad \text{for } x \in R_+, \tag{2.8}$$

where $\overline{M} = \max_{0<x\leq r_2} f(x)$.

Set

$$r_3 = \max\biggl\{r_2, \frac{4c_0}{1-c}\overline{M}\biggr\}, \qquad \Omega_1 = \bigl\{x \in P \mid \|x\| < r_1\bigr\},$$

$$\Omega_2 = \bigl\{x \in P \mid \|x\| < r_3\bigr\}, \qquad \Omega_3 = \Bigl\{x \in P \mid \|x\| < r_3, \inf_{\substack{m\in\mathbf{N}(a_1,b_1)\\ n\in\mathbf{N}(a_2,b_2)}} x_{m,n} > u_0\Bigr\}.$$

Then Ω_1, Ω_2 and Ω_3 are nonempty bounded open convex subsets of P such that

$$\begin{aligned}
&\Omega_1 \subset \Omega_2, \qquad \Omega_3 \subset \Omega_2, \qquad \Omega_1 \cap \Omega_3 = \emptyset, \\
&\overline{\Omega}_1 = \bigl\{x \in P \mid \|x\| \le r_1\bigr\}, \qquad \overline{\Omega}_2 = \bigl\{x \in P \mid \|x\| \le r_3\bigr\}, \\
&\overline{\Omega}_3 = \Bigl\{x \in P \Bigm| \|x\| \le r_3, \inf_{\substack{m\in\mathbf{N}(a_1,b_1)\\ n\in\mathbf{N}(a_2,b_2)}} x_{m,n} \ge u_0\Bigr\}.
\end{aligned}$$

Let $l = 1, 2, 3$. For any $x = \{x_{m,n}\}, y = \{y_{m,n}\} \in \overline{\Omega}_l \subset P$, from (2.1), we have

$$\|T_1x - T_1y\| \le c\|x - y\|,$$

where $c < 1$, thus $T_1 : \overline{\Omega}_l \to P$ is a contraction operator.

Notice that any bounded subset D of $\overline{\Omega}_l$ is also a bounded subset of P. Thus it follows from the above conclusion that T_2D is relatively compact. Also $T_2 : \overline{\Omega}_l \to P$ is continuous. In consequence, we deduce that $T_2 : \overline{\Omega}_l \to P$ is completely continuous. Thus, $T = T_1 + T_2 : \overline{\Omega}_l \to P$ $(l = 1, 2, 3)$ is a strict set contraction operator.

Next we show that $T(\Omega) \subset \Omega$.

(i) For $x \in \Omega_1$, when $(m, n) \in \mathbf{N}(m_1) \times \mathbf{N}(n_1)$, we get

$$\begin{aligned}
0 &\le (Tx)_{m,n} \\
&\le cx_{m-k,n-l} + \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \frac{(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}}{(r-1)!(h-1)!} P_{i,j} f(x_{i-\sigma,j-\tau}).
\end{aligned}$$

From (2.7), we have

$$\begin{aligned}
\|Tx\| &\le c\|x\| + \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \frac{(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}}{(r-1)!(h-1)!} P_{i,j} \frac{1-c}{4c_0}\|x\| \\
&< cr_1 + c_0\frac{1-c}{4c_0} r_1 \\
&< r_1.
\end{aligned}$$

Thus $T(\Omega_1) \subset \Omega_1$.

(ii) For $x \in \Omega_2$, from (2.8), we also have

$$\begin{aligned}
\|Tx\| &\le c\|x\| + \sum_{i=m}^{\infty}\sum_{j=n}^{\infty} \frac{(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}}{(r-1)!(h-1)!} P_{i,j} \biggl(\frac{1-c}{4c_0}\|x\| + \overline{M}\biggr) \\
&< cr_3 + c_0\biggl(\frac{1-c}{4c_0} r_3 + \overline{M}\biggr) \\
&\le cr_3 + \frac{1-c}{4} r_3 + \frac{1-c}{4} r_3 \\
&< r_3.
\end{aligned}$$

Thus, $T(\Omega_2) \subset \Omega_2$.

(iii) For any $x \in \Omega_3$, we have $\|Tx\| < r_3$ and $\inf_{\substack{m\in \mathbf{N}(a_1,b_1)\\ n\in\mathbf{N}(a_2,b_2)}} x_{m,n} > u_0$. For any $(m,n) \in \mathbf{N}(a_1,b_1)\times\mathbf{N}(a_2,b_2)$, from condition (R_4), we have

$$\begin{aligned}
(Tx)_{m,n} &\ge \sum_{i=m}^{\infty}\sum_{j=n}^{\infty}\frac{(i-m+r-1)^{(r-1)}(j-n+h-1)^{(h-1)}}{(r-1)!(h-1)!}P_{i,j}f(x_{i-\sigma,j-\tau})\\
&\ge \sum_{i=b_1}^{\infty}\sum_{j=b_2}^{\infty}\frac{(i-b_1+r-1)^{(r-1)}(j-b_2+h-1)^{(h-1)}}{(r-1)!(h-1)!}P_{i,j}f(x_{i-\sigma,j-\tau})\\
&\ge \sum_{i=b_1}^{b_1+\delta_2}\sum_{j=b_2}^{b_2+\eta_2}\frac{(i-b_1+r-1)^{(r-1)}(j-b_2+h-1)^{(h-1)}}{(r-1)!(h-1)!}P_{i,j}f(x_{i-\sigma,j-\tau})\\
&\ge c_2c_1u_0\\
&> u_0.
\end{aligned}$$

Thus, for any $x \in \Omega_3$, we have

$$\inf_{\substack{m\in \mathbf{N}(a_1,b_1)\\ n\in\mathbf{N}(a_2,b_2)}} (Tx)_{m,n} > u_0.$$

Hence $T(\Omega_3) \subset \Omega_3$. From (i), (ii), (iii) and Lemma 2, we obtain

$$i(T,\Omega_l,P) = 1, \quad l = 1,2,3.$$

Hence

$$i\bigl(T,\Omega_2/(\overline{\Omega}_1\cup\overline{\Omega}_3),P\bigr) = i(T,\Omega_2,P) - i(T,\Omega_1,P) - i(T,\Omega_3,P) = -1.$$

Thus, T has fixed points x^* and y^* such that $x^* \in \Omega_2/(\overline{\Omega}_1\cup\overline{\Omega}_3)$, $y^* \in \Omega_3$, and

$$\inf_{\substack{m\in \mathbf{N}(a_1,b_1)\\ n\in\mathbf{N}(a_2,b_2)}} x^*_{m,n} < u_0 < \inf_{\substack{m\in \mathbf{N}(a_1,b_1)\\ n\in\mathbf{N}(a_2,b_2)}} y^*_{m,n}.$$

It is easy to prove that the fixed points of T are exactly the positive solutions of Eq. (1.1). The proof is complete. □

Example 2.1 Consider a nonlinear partial difference equation given by

$$\Delta_m^2\Delta_n^3\left(x_{m,n} - \frac{1}{4}x_{m-1,n-2}\right) + \frac{a^{15}}{30\ln 2}a^{-m-n}x_{m-2,n-3}^{\frac{1}{2}}\ln(1+x_{m-2,n-3}) = 0, \tag{2.9}$$

where $(m,n) \in \mathbf{N}(0)\times\mathbf{N}(0)$ and $a > 1$ is a constant.

Let us fix $c = \frac{1}{2}$ so that $c_{m,n} = \frac{1}{4} < c < 1$. Thus (R_1) is satisfied. Also we have

$$\delta_1 = \max\{1,2\} = 2, \qquad \delta_2 = \min\{1,2\} = 1,$$

$$\eta_1 = \max\{2,3\} = 3, \qquad \eta_2 = \min\{2,3\} = 2,$$

$$P_{m,n} = \frac{a^{15}}{30\ln 2}a^{-m-n} > 0, \qquad f(x) = x^{\frac{1}{2}}\ln(1+x).$$

It is easy to see that $xf(x) > 0$ ($x \neq 0$) and

$$\lim_{x\to 0+}\frac{f(x)}{x} = \lim_{x\to 0+}\frac{\ln(1+x)}{x^{\frac{1}{2}}} = \lim_{x\to 0+}\frac{(\ln(1+x))'}{(x^{\frac{1}{2}})'} = \lim_{x\to 0+}\frac{2x^{\frac{1}{2}}}{1+x} = 0,$$

$$\lim_{x\to +\infty}\frac{f(x)}{x} = \lim_{x\to +\infty}\frac{\ln(1+x)}{x^{\frac{1}{2}}} = \lim_{x\to +\infty}\frac{(\ln(1+x))'}{(x^{\frac{1}{2}})'} = \lim_{x\to +\infty}\frac{2x^{\frac{1}{2}}}{1+x} = 0.$$

Thus, Eq. (2.9) satisfies the condition (R_2).

Let $m_1 = 2$, $n_1 = 3$. Then we consider a series of positive terms

$$\sum_{i=2}^{\infty}\sum_{j=3}^{\infty}\frac{(i+1)^{(1)}(j+2)^{(2)}}{1!2!}P_{i,j} = \frac{a^{15}}{30\ln 2}\sum_{i=2}^{\infty}(i+1)a^{-i}\sum_{j=3}^{\infty}\frac{(j+2)(j+1)}{2}a^{-j}.$$

Setting

$$\sum_{i=2}^{\infty}u_i = \sum_{i=2}^{\infty}(i+1)a^{-i}, \tag{2.10}$$

$$\sum_{j=3}^{\infty}v_j = \sum_{j=3}^{\infty}\frac{(j+1)(j+2)}{2}a^{-j}, \tag{2.11}$$

we get

$$\lim_{i\to\infty}\frac{u_{i+1}}{u_i} = \lim_{i\to\infty}\frac{(i+2)a^{-i-1}}{(i+1)a^{-i}} = \lim_{i\to\infty}\frac{i+2}{a(i+1)} = \frac{1}{a} < 1,$$

$$\lim_{j\to\infty}\frac{v_{j+1}}{v_j} = \lim_{j\to\infty}\frac{(j+2)(j+3)}{2a^{j+1}}\frac{2a^j}{(j+1)(j+2)} = \lim_{j\to\infty}\frac{j+3}{a(j+1)} = \frac{1}{a} < 1.$$

According to the D'Alembert comparison test, the series of positive terms (2.10) and (2.11) are convergent and consequently, we get

$$0 < c_0 \triangleq \sum_{i=2}^{\infty}\sum_{j=3}^{\infty}\frac{(i+1)^{(1)}(j+2)^{(2)}}{1!2!}P_{i,j} < +\infty.$$

Thus the condition (R_3) is satisfied.

Next, we check that the condition (R_4) holds true. Letting $u_0 = 1$, $c_1 = \ln 2$, $\mathbf{N}(a_1, b_1) = \mathbf{N}(2,5) = \{2,3,4,5\}$, $\mathbf{N}(a_2, b_2) = \mathbf{N}(3,7) = \{3,4,5,6,7\}$, we have

$$f(x) = x^{\frac{1}{2}}\ln(1+x) \geq \ln 2 = c_1 u_0, \quad \text{for } x \geq u_0,$$

and

$$\begin{aligned} c_2 &= \sum_{i=b_1}^{b_1+\delta_2}\sum_{j=b_2}^{b_2+\eta_2}\frac{(i-b_1+1)^{(1)}(j-b_2+2)^{(2)}}{1!2!}P_{i,j} \\ &= \sum_{i=5}^{5+1}\sum_{j=7}^{7+2}\frac{(i-5+1)^{(1)}(j-7+2)^{(2)}}{1!2!}\frac{a^{15}}{30\ln 2}a^{-i-j} \end{aligned}$$

$$
\begin{aligned}
&= \frac{a^{15}}{30\ln 2}\sum_{i=5}^{6}(i-4)a^{-i}\sum_{j=7}^{9}\frac{(j-5)(j-6)}{2}a^{-j} \\
&= \frac{a^{15}}{30\ln 2}\left(a^{-5}+2a^{-6}\right)\left(a^{-7}+3a^{-8}+6a^{-9}\right) \\
&> \frac{a^{15}}{30\ln 2}30a^{-15} \\
&= \frac{1}{\ln 2}.
\end{aligned}
$$

Clearly $c_1c_2 > 1$, which shows that the condition (R_4) is satisfied. Consequently, the conclusion of Theorem 1 is applied and hence Eq. (2.9) has at least two positive solutions x^* and y^* such that

$$
\inf_{\substack{m\in\mathbf{N}(2,5)\\ n\in\mathbf{N}(3,7)}} x^*_{m,n} < 1 < \inf_{\substack{m\in\mathbf{N}(2,5)\\ n\in\mathbf{N}(3,7)}} y^*_{m,n}.
$$

3 Conclusions

In the past years, the qualitative theory of partial difference equations has been developed by means of different tools such as comparison principle, Schauder type fixed point theorem, Banach's contraction principle, method of upper and lower solutions, the method of positive operators, etc. However, the issue of existence of multiple positive solutions for neutral delay partial difference equations has yet to be addressed. Here we have investigated this topic with the aid of the fixed point index theory and obtained a criterion ensuring the existence of multiple positive solutions to Eq. (1.1). Thus the present work opens a new avenue in the field of partial difference equations and contributes significantly to the existing literature on the subject.

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