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The stability analysis of an epidemic model with saturating incidence and age-structure in the exposed and infectious classes

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Abstract

In this paper, an HBV epidemic model that incorporates saturating incidence and age-structure in the exposed and infectious classes is proposed. We study the asymptotic smoothness of semi-flow generated by the model. By calculating the basic reproduction number and analyzing the characteristic equations, the local stability of disease-free and endemic steady states is studied. We investigate the global dynamics of this model by using Lyapunov functionals and LaSalle's invariance principle and prove that, if the basic reproduction number is less than unity, the disease-free steady state is globally asymptotically stable; if the basic reproduction number is greater than unity, the endemic steady state is globally asymptotically stable.

Keywords: Age-structured model; Saturating incidence; Asymptotic smoothness; Lyapunov functional; Global stability

1 Introduction

Hepatitis B is very contagious and it is hard to control its transmission. Thus studying the law of its infection has become the focus of attention. Researchers have presented an HBV epidemic model to study hepatitis B transmission and some results of the investigation have been given. The model's dynamics is determined by the basic reproduction number (the average number of secondary infections caused by one infectious individual in its duration of infection). When the basic reproduction number is less than unity, the disease-free steady state is globally asymptotically stable and all hepatitis B patients will recover; when the basic reproduction number is larger than unity, there exists a unique endemic steady state and it is globally asymptotically stable.

Although there exist many epidemic models, some are not appropriate to show the hepatitis B transmission rules. In [1], Liu divided hepatitis B patients into acute and chronic patients and presented an ordinary differential model. However, the scaled probability of hepatitis B virus infection is connected with the age of infection and the risk per unit time of activation appears to be higher in the early stages of infection than in later stages.

Therefore, in [2], Li presented the following model:

$$
\begin{aligned}
&\dot{S}(t) = \Lambda - (\mu + p)S(t) - S(t)\int_0^\infty \beta_1(a)i(t,a)\,\mathrm{d}a - S(t)\int_0^\infty \beta_3(a)j(t,a)\,\mathrm{d}a,\\
&\dot{V}(t) = pS(t) - (\mu + \rho)V(t) - V(t)\int_0^\infty \beta_2(a)i(t,a)\,\mathrm{d}a\\
&\qquad - V(t)\int_0^\infty \beta_4(a)j(t,a)\,\mathrm{d}a,\\
&\frac{\partial e(t,a)}{\partial a} + \frac{\partial e(t,a)}{\partial t} = -\theta_1(a)e(t,a),\\
&\frac{\partial i(t,a)}{\partial a} + \frac{\partial i(t,a)}{\partial t} = -\theta_2(a)i(t,a),\\
&\frac{\partial j(t,a)}{\partial a} + \frac{\partial j(t,a)}{\partial t} = -\theta_3(a)j(t,a),\\
&\dot{R}(t) = \rho V(t) + \int_0^\infty \delta_1(a)i(t,a)\,\mathrm{d}a + \int_0^\infty \delta_2(a)j(t,a)\,\mathrm{d}a - \mu R(t),
\end{aligned}
\tag{1.1}
$$

with boundary conditions

$$
\begin{aligned}
&e(t,0) = S(t)\int_0^\infty \bigl(\beta_1(a)i(t,a) + \beta_3(a)j(t,a)\bigr)\,\mathrm{d}a\\
&\qquad + V(t)\int_0^\infty \bigl(\beta_2(a)i(t,a) + \beta_4(a)j(t,a)\bigr)\,\mathrm{d}a,\\
&i(t,0) = \int_0^\infty \gamma_1(a)e(t,a)\,\mathrm{d}a,\\
&j(t,0) = \int_0^\infty \gamma_2(a)e(t,a)\,\mathrm{d}a + \int_0^\infty \xi(a)i(t,a)\,\mathrm{d}a.
\end{aligned}
$$

It seems that vaccination has been the most effective prevention measure against hepatitis B. However, clinic evidence shows that some vaccines gradually lost immunity soon after vaccination. In other words, even though all individuals have been vaccinated, many of them may still be infected. Therefore, it is necessary to consider that a small part of vaccines will be susceptible individuals after a period of time. Furthermore, acute hepatitis B patients are highly infectious and they have apparent symptoms, the government and hospital have to take strategies as soon as possible, such like quarantine measures, therapeutic measures and so on. Considering the real transmission rules of hepatitis B, it is necessary to introduce saturating incidence to describe the law of hepatitis B's infection. Therefore, we propose an epidemic model as follows:

$$
\begin{aligned}
&\dot{S}(t) = \Lambda - (\mu + p)S(t) + \eta V(t) - S(t)\int_0^\infty \left(\frac{\beta_1(a)i(t,a)}{1 + \alpha i(t,a)} + \beta_3(a)j(t,a)\right)\mathrm{d}a,\\
&\dot{V}(t) = pS(t) - (\mu + \rho + \eta)V(t) - V(t)\int_0^\infty \left(\frac{\beta_2(a)i(t,a)}{1 + \alpha i(t,a)} + \beta_4(a)j(t,a)\right)\mathrm{d}a,\\
&\frac{\partial e(t,a)}{\partial a} + \frac{\partial e(t,a)}{\partial t} = -\theta_1(a)e(t,a),\\
&\frac{\partial i(t,a)}{\partial a} + \frac{\partial i(t,a)}{\partial t} = -\theta_2(a)i(t,a),
\end{aligned}
\tag{1.2}
$$

$$\frac{\partial j(t,a)}{\partial a}+\frac{\partial j(t,a)}{\partial t}=-\theta_3(a)j(t,a),$$
$$\dot{R}(t)=\rho V(t)+\int_0^\infty \delta_1(a)i(t,a)\,\mathrm{d}a+\int_0^\infty \delta_2(a)j(t,a)\,\mathrm{d}a-\mu R(t),$$

where

$$\theta_1(a)=\mu+\gamma_1(a)+\gamma_2(a),\qquad \theta_2(a)=\mu+\delta_1(a)+\varepsilon_1(a)+\xi(a),$$
$$\theta_3(a)=\mu+\delta_2(a)+\varepsilon_2(a),$$

with boundary conditions

$$\begin{aligned}
e(t,0)&=S(t)\int_0^\infty\left(\frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)}+\beta_3(a)j(t,a)\right)\mathrm{d}a\\
&\quad+V(t)\int_0^\infty\left(\frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)}+\beta_4(a)j(t,a)\right)\mathrm{d}a,\\
i(t,0)&=\int_0^\infty \gamma_1(a)e(t,a)\,\mathrm{d}a,\\
j(t,0)&=\int_0^\infty \gamma_2(a)e(t,a)\,\mathrm{d}a+\int_0^\infty \xi(a)i(t,a)\,\mathrm{d}a,
\end{aligned}\tag{1.3}$$

and initial conditions

$$\begin{aligned}
&S(0)=\varphi_S\ge 0,\qquad V(0)=\varphi_V\ge 0,\qquad e(0,a)=\varphi_e(a)\in L^1_+(0,\infty),\\
&i(0,a)=\varphi_i(a)\in L^1_+(0,\infty),\qquad j(0,a)=\varphi_j(a)\in L^1_+(0,\infty),\qquad R(0)=R_0\ge 0,
\end{aligned}\tag{1.4}$$

where $e(t,a)$ represents the density of exposed individuals with age of latency a at time t. $i(t,a)$, $j(t,a)$ represent the density of patients with acute hepatitis B and chronic hepatitis B with age of infection a at time t, respectively. α means the saturating incidence coefficient. The parameters of model (1.2) are biologically explained in Table 1.

Table 1 Parameters and their biological meaning in model (1.2)

Parameter	Interpretation
Λ	constant recruitment rate
μ	natural death rate
p	the rate for susceptible individuals to be vaccinated
ρ	the rate for vaccinees to obtain immunity and move into recovered population
η	the proportion of vaccines that lose efficacy
$\beta_1(a)$	the rate for acute hepatitis B patients infecting susceptible individuals at age a
$\beta_2(a)$	the rate for acute hepatitis B patients infecting vaccinees at age a
$\beta_3(a)$	the rate for chronic hepatitis B patients infecting susceptible individuals at age a
$\beta_4(a)$	the rate for chronic hepatitis B patients infecting vaccinees at age a
$\gamma_1(a)$	the rate for exposed individuals being acute hepatitis B patients at age a
$\gamma_2(a)$	the rate for exposed individuals being chronic hepatitis B patients at age a
$\varepsilon_1(a)$	acute hepatitis B death rate at age a
$\varepsilon_2(a)$	chronic hepatitis B death rate at age a
$\delta_1(a)$	the rate for acute hepatitis B patients being recovered population at age a
$\delta_2(a)$	the rate for chronic hepatitis B patients being recovered population at age a

Since the variable $R(t)$ does not appear in the first five equations of (1.2), in this paper, we consider the following subsystem:

$$
\begin{aligned}
&\dot{S}(t) = \Lambda - (\mu + p)S(t) + \eta V(t) - S(t)\int_0^\infty \left(\frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)} + \beta_3(a)j(t,a)\right)\mathrm{d}a,\\
&\dot{V}(t) = pS(t) - (\mu + \rho + \eta)V(t) - V(t)\int_0^\infty \left(\frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)} + \beta_4(a)j(t,a)\right)\mathrm{d}a,\\
&\frac{\partial e(t,a)}{\partial a} + \frac{\partial e(t,a)}{\partial t} = -\theta_1(a)e(t,a),\\
&\frac{\partial i(t,a)}{\partial a} + \frac{\partial i(t,a)}{\partial t} = -\theta_2(a)i(t,a),\\
&\frac{\partial j(t,a)}{\partial a} + \frac{\partial j(t,a)}{\partial t} = -\theta_3(a)j(t,a),\\
&e(t,0) = S(t)\int_0^\infty \left(\frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)} + \beta_3(a)j(t,a)\right)\mathrm{d}a\\
&\qquad + V(t)\int_0^\infty \left(\frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)} + \beta_4(a)j(t,a)\right)\mathrm{d}a,\\
&i(t,0) = \int_0^\infty \gamma_1(a)e(t,a)\,\mathrm{d}a,\\
&j(t,0) = \int_0^\infty \gamma_2(a)e(t,a)\,\mathrm{d}a + \int_0^\infty \xi(a)i(t,a)\,\mathrm{d}a.
\end{aligned}
\tag{1.5}
$$

This paper is organized as follows. In Sect. 2, we introduce some basic results of system (1.5), including state space, assumptions and boundedness of the solutions. Asymptotic smoothness of the semi-flow is analyzed in Sect. 3, which is generated by the system (1.5). Then we study the existence of equilibria and obtain the expression of the basic reproduction number R_0 in Sect. 4. The local stability of equilibria is proved in Sect. 5, while the uniform persistence of the system (1.5) is verified in Sect. 6. In Sect. 7, we give a proof of the global stability of equilibria. A brief remark is given in Sect. 8 to conclude this work.

More details concerning the global stability analysis of epidemic model, we refer the reader to [3–23].

2 Preliminaries

To make the model be biologically significant, we list the assumption as follows:

Assumption 1 We assume that

(i) $\beta_1(a)$, $\beta_2(a)$, $\beta_3(a)$, $\beta_4(a)$, $\theta_1(a)$, $\theta_2(a)$, $\xi(a)$ are non-negative and belong to $L_+^\infty(0,\infty)$ with respective essential upper bound $\bar{\beta}_1$, $\bar{\beta}_2$, $\bar{\beta}_3$, $\bar{\beta}_4$, $\bar{\theta}_1$, $\bar{\theta}_2$, $\bar{\xi} \in (0,\infty)$;

(ii) $\beta_1(a)$, $\beta_2(a)$, $\beta_3(a)$, $\beta_4(a)$, $\xi(a)$ are Lipschitz continuous on R_+ with coefficients M_{β_1}, M_{β_2}, M_{β_3}, M_{β_4}, M_ξ, respectively;

(iii) there exists a positive constant $\mu_0 \in (0,\mu]$ such that $|\theta_1(a) - \gamma_1(a) - \gamma_2(a)| \geq \mu_0$, $|\theta_2(a) - \xi(a)| \geq \mu_0$, $\theta_3(a) \geq \mu_0$, for all $a > 0$.

2.1 State space

Define the space of functions X as

$$X = R_+ \times R_+ \times L_+^1(0,\infty) \times L_+^1(0,\infty) \times L_+^1(0,\infty),$$

equipped with the norm

$$\left\|(x_1,x_2,x_3,x_4,x_5)\right\|_{\mathcal{X}} = |x_1| + |x_2| + \int_0^\infty \left|x_3(a)\right| \mathrm{d}a + \int_0^\infty \left|x_4(a)\right| \mathrm{d}a + \int_0^\infty \left|x_5(a)\right| \mathrm{d}a.$$

Then the initial values (1.4) of system (1.5) are taken to be included in X:

$$\big(S(0), V(0), e(0,a), i(0,a), j(0,a)\big) = \big(S_0, V_0, \varphi_e(a), \varphi_i(a), \varphi_j(a)\big) \in X.$$

By the standard theory of functional differential equations [24], it can be verified that system (1.5) with initial conditions (1.4) has a unique non-negative solution. Thus, we have a continuous semi-flow associated with system (1.5), that is,

$$\Phi_t(X_0) := X(t) = \big(S(t), V(t), e(t,\cdot), i(t,\cdot), j(t,\cdot)\big) \in \mathcal{X}, \quad t \geq 0, \tag{2.1}$$

with

$$\begin{aligned}\left\|\Phi_t(X_0)\right\|_{\mathcal{X}} &= \left\|S(t), V(t), e(t,\cdot), i(t,\cdot), j(t,\cdot)\right\|_{\mathcal{X}} \\ &= \left|S(t)\right| + \left|V(t)\right| + \int_0^\infty \left|e(t,a)\right| \mathrm{d}a + \int_0^\infty \left|i(t,a)\right| \mathrm{d}a + \int_0^\infty \left|j(t,a)\right| \mathrm{d}a.\end{aligned}$$

Finally, we define the state space for system (1.5) as

$$\begin{aligned}\Upsilon := \Bigg\{&\big(S(t), V(t), e(t,\cdot), i(t,\cdot), j(t,\cdot)\big) \in \mathcal{X} : 0 \leq S(t) + V(t) + \int_0^\infty e(t,a)\,\mathrm{d}a \\ &+ \int_0^\infty i(t,a)\,\mathrm{d}a + \int_0^\infty j(t,a)\,\mathrm{d}a \leq \frac{\Lambda}{\mu_0}\Bigg\},\end{aligned}$$

which can be proved to be positive invariant by the following proposition.

2.2 Boundedness

The last three equations of system (1.5) can be reformulated as Volterra equations by use of Volterra formulation. In order to be convenient for computation, we denote

$$\begin{aligned}&B_1(a) = \exp\left(-\int_0^a \theta_1(\tau)\,\mathrm{d}\tau\right), \qquad B_2(a) = \exp\left(-\int_0^a \theta_2(\tau)\,\mathrm{d}\tau\right), \\ &B_3(a) = \exp\left(-\int_0^a \theta_3(\tau)\,\mathrm{d}\tau\right).\end{aligned}$$

From the expressions of $B_1(a)$, $B_2(a)$ and $B_3(a)$, according to Assumption 1, it is easy to see that, for all $a \geq 0$,

$$\begin{aligned}&0 \leq B_1(a), B_2(a), B_3(a) \leq e^{-\mu_0 a}, \\ &B_1'(a) = -\theta_1(a)B_1(a), \qquad B_2'(a) = -\theta_2(a)B_2(a), \qquad B_3'(a) = -\theta_3(a)B_3(a).\end{aligned}$$

By integrating the terms $e(t,a)$, $i(t,a)$ and $j(t,a)$ along the characteristic line $t - a = \text{const.}$, respectively, we get the following expressions:

$$e(t,a) = \begin{cases} e(t-a,0)B_1(a) & \text{for } 0 \le a \le t, \\ \varphi_e(a-t)\frac{B_1(a)}{B_1(a-t)} & \text{for } 0 \le t \le a, \end{cases} \tag{2.2}$$

$$i(t,a) = \begin{cases} i(t-a,0)B_2(a) & \text{for } 0 \le a \le t, \\ \varphi_i(a-t)\frac{B_2(a)}{B_2(a-t)} & \text{for } 0 \le t \le a, \end{cases} \tag{2.3}$$

$$j(t,a) = \begin{cases} j(t-a,0)B_3(a) & \text{for } 0 \le a \le t, \\ \varphi_j(a-t)\frac{B_3(a)}{B_3(a-t)} & \text{for } 0 \le t \le a. \end{cases} \tag{2.4}$$

Before analyzing the boundedness of system (1.5), we first show the non-negative of the solution. From (2.2)–(2.4), it is not difficult to verify that $e(t,a)$, $i(t,a)$ and $j(t,a)$ are non-negative, due to the non-negativity of $B_1(a)$, $B_2(a)$, $B_3(a)$, and the non-negative initial conditions (1.4). From the first equation of (1.3), we see that $S(t) > 0$ holds or $V(t) > 0$ holds. For the sake of contradiction, if $V(t) > 0$ and $S(t) \le 0$, from the second equation of (1.5), we know that $\dot{V}(t) < 0$. Then $V(t)$ is a monotone decreasing function with respect to t and there exists at least one zero solution, which contradicts $V(t) > 0$. If $S(t) > 0$ and $V(t) \le 0$, $V(t)$ is a monotone increasing function with respect to t, which contradicts the later boundedness analysis. Thus, all the solutions of system (1.5) remain non-negative.

In order to imply the boundedness of system (1.5), we have the following proposition.

Proposition 2.1 *Consider system* (1.5) *and Eq.* (2.1), *we have*

(i) Υ *is positively invariant for* Φ_t, *that is*, $\Phi_t(X_0) \in \Upsilon$, *for* $\forall t \ge 0$, $X_0 \in \Upsilon$;

(ii) Φ_t *is point dissipative*: *there is a bounded set that attracts all points in* $\mathcal{X}$.

Proof Note that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left\| \Phi_t(X_0) \right\|_{\mathcal{X}} &= \frac{\mathrm{d}S(t)}{\mathrm{d}t} + \frac{\mathrm{d}V(t)}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty e(t,a)\,\mathrm{d}a \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\,\mathrm{d}a + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty j(t,a)\,\mathrm{d}a. \end{aligned} \tag{2.5}$$

By Eq. (2.2), we get

$$\int_0^\infty e(t,a)\,\mathrm{d}a = \int_0^t e(t-a,0)B_1(a)\,\mathrm{d}a + \int_t^\infty \varphi_e(a-t)\frac{B_1(a)}{B_1(a-t)}\,\mathrm{d}a.$$

Taking the substitution $\sigma = t - a$ and $\tau = a - t$ in the first and second integral, respectively, and differentiating by t, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty e(t,a)\,\mathrm{d}a &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t e(\sigma,0)B_1(t-\sigma)\,\mathrm{d}\sigma + \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty \varphi_e(\tau)\frac{B_1(t+\tau)}{B_1(\tau)}\,\mathrm{d}\tau \\ &= e(t,0)B_1(0) + \int_0^\infty \varphi_e(\tau)\frac{B_1'(t+\tau)}{B_1(\tau)}\,\mathrm{d}\tau + \int_0^t e(\sigma-a)B_1'(t-\sigma)\,\mathrm{d}\sigma. \end{aligned}$$

Noting that $B_1(0) = 1$ and $B_1'(a) = -\theta_1(a)B_1(a)$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty e(t,a)\,\mathrm{d}a = e(t,0) - \int_0^\infty \theta_1(a)e(t,a)\,\mathrm{d}a. \tag{2.6}$$

Similarly, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\,\mathrm{d}a = i(t,0) - \int_0^\infty \theta_2(a)i(t,a)\,\mathrm{d}a, \tag{2.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty j(t,a)\,\mathrm{d}a = j(t,0) - \int_0^\infty \theta_3(a)j(t,a)\,\mathrm{d}a. \tag{2.8}$$

By (2.6), (2.7) and (2.8), Eq. (2.5) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\Phi_t(X_0)\big\|_{\mathcal{X}} &= \Lambda - \mu S(t) - (\mu+\rho)V(t) - S(t)\int_0^\infty\left(\frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)} + \beta_3(a)j(t,a)\right)\mathrm{d}a \\ &\quad - V(t)\int_0^\infty\left(\frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)} + \beta_4(a)j(t,a)\right)\mathrm{d}a \\ &\quad + S(t)\int_0^\infty\left(\frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)} + \beta_3(a)j(t,a)\right)\mathrm{d}a \\ &\quad + V(t)\int_0^\infty\left(\frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)} + \beta_4(a)j(t,a)\right)\mathrm{d}a \\ &\quad - \int_0^\infty \theta_1(a)e(t,a)\,\mathrm{d}a + \int_0^\infty \gamma_1(a)e(t,a)\,\mathrm{d}a - \int_0^\infty \theta_2(a)i(t,a)\,\mathrm{d}a \\ &\quad + \int_0^\infty \gamma_2(a)e(t,a)\,\mathrm{d}a + \int_0^\infty \xi(a)i(t,a)\,\mathrm{d}a - \int_0^\infty \theta_3(a)j(t,a)\,\mathrm{d}a \\ &= \Lambda - \mu S(t) - (\mu+\rho)V(t) - \int_0^\infty\big(\theta_1(a)-\gamma_1(a)-\gamma_2(a)\big)e(t,a)\,\mathrm{d}a \\ &\quad - \int_0^\infty\big(\theta_2(a)-\xi(a)\big)i(t,a)\,\mathrm{d}a - \int_0^\infty \theta_3(a)j(t,a)\,\mathrm{d}a.\end{aligned}$$

Thus, from (iii) of Assumption 1, we can get

$$\begin{aligned}\frac{d}{dt}\big\|\Phi_t(X_0)\big\|_{\mathcal{X}} &\le \Lambda - \mu S(t) - (\mu+\rho)V(t) \\ &\quad - \mu_0\left(\int_0^\infty e(t,a)\,\mathrm{d}a + \int_0^\infty i(t,a)\,\mathrm{d}a + \int_0^\infty j(t,a)\,\mathrm{d}a\right) \\ &\le \Lambda - \mu_0\big\|\Phi_t(X_0)\big\|_{\mathcal{X}}.\end{aligned}$$

Hence, it follows from the variation of constants formula that, for $t \ge 0$,

$$\big\|\Phi_t(X_0)\big\|_{\mathcal{X}} \le \frac{\Lambda}{\mu_0} - e^{-\mu_0 t}\left(\frac{\Lambda}{\mu_0} - \big\|\Phi_t(X_0)\big\|_{\mathcal{X}}\right), \tag{2.9}$$

which implies that $\Phi_t(X_0) \in \Upsilon$ for any solution of (1.5) satisfying $X_0 \in \Upsilon$ and all $t \ge 0$. Thus, the positive invariance of set Υ for semi-flow Φ can be verified.

Moreover, by (2.9) we can make inferences that $\limsup_{t\to\infty}\|\Phi_t(X_0)\|_{\mathcal{X}} \le \Lambda/\mu_0$ for any $X_0 \in \mathcal{X}$. Therefore, Φ is point dissipative and Υ attracts all points in $\mathcal{X}$. This completes the proof. □

Proposition 2.2 *If $X_0 \in \mathcal{X}$ and $\|X_0\|_{\mathcal{X}} \leq M$ for some constant $M \geq \Lambda/\mu_0$, then the following statements hold for $t \geq 0$:*

(i) $0 \leq S(t)$, $V(t)$, $\int_0^\infty e(t,a)\,\mathrm{d}a$, $\int_0^\infty \frac{i(t,a)}{1+\alpha i(t,a)}\,\mathrm{d}a$, $\int_0^\infty j(t,a)\,\mathrm{d}a \leq M$;

(ii) $e(t,0) \leq (\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3 + \bar{\beta}_4)M^2$, $i(t,0) \leq \bar{\gamma}_1 M$, $j(t,0) \leq (\bar{\gamma}_2 + \bar{\xi})M$.

Proposition 2.3 *Let $C \in \mathcal{X}$ be bounded, then:*

(i) *$\Phi_t(C)$ is bounded;*

(ii) *Φ_t is eventually bounded on C.*

3 Asymptotic smoothness

In order to obtain global properties of the semi-flow $\Phi(t)_{t\geq 0}$, it is necessary to prove that the semi-flow is asymptotically smooth. Before giving the results, we first introduce some lemmas for later use.

Lemma 3.1 ([8]) *Let $D \subseteq R$. For $j = 1,2$, suppose $f_j : D \to R$ is a bounded Lipschitz continuous function with bound K_j and Lipschitz coefficient M_j. Then the product function $f_1 f_2$ is Lipschitz with coefficient $K_1 M_2 + K_2 M_1$.*

The definition of asymptotic smoothness is as follows:

Definition 3.1 ([25]) A semi-flow $\Phi(t,X_0) := R^+ \times \mathcal{X} \to \mathcal{X}$ is said to be asymptotically smooth, if, for any nonempty, closed bounded set $B \subset \mathcal{X}$ for which $\Phi(t,B) \subset B$, there is a compact set $B_0 \subset B$ such that B_0 attracts B.

In order to prove the asymptotic smoothness of the semi-flow, we will apply the following results, which is based on Lemma 3.2.3 in [25].

Lemma 3.2 ([21, 22]) *If the following two conditions hold then the semi-flow $\Phi(t,X_0) = \phi(t,X_0) + \varphi(t,X_0) : R^+ \times \mathcal{X} \to \mathcal{X}$ is asymptotically smooth in $\mathcal{X}$.*

(i) *There exists a continuous function $w : R^+ \times R^+ \to R^+$ such that $w(t,h) \to 0$ as $t \to \infty$ and $\|\varphi(t,X_0)\|_{\mathcal{X}} \leq w(t,h)$ if $\|X_0\|_{\mathcal{X}} \leq h$;*

(ii) *For $t \geq 0$, $\phi(t,X_0)$ is completely continuous.*

To verify that the two conditions are fulfilled for system (1.5), we decompose $\Phi : R^+ \times \mathcal{X} \to \mathcal{X}$ into the following two operators $\phi(t,X_0), \varphi(t,X_0) : R^+ \times X \to X$. Let $\phi(t,X_0) := (S(t), V(t), \tilde{e}(t,\cdot), \tilde{i}(t,\cdot), \tilde{j}(t,\cdot))$ and $\varphi(t,X_0) := (0,0,\tilde{\varphi}_e(t,\cdot), \tilde{\varphi}_i(t,\cdot), \tilde{\varphi}_j(t,\cdot))$ where

$$\tilde{\varphi}_e(t,a) := \begin{cases} 0 & \text{for } 0 \leq a \leq t, \\ e(t,a) & \text{for } 0 \leq t \leq a \end{cases} \quad \text{and} \quad \tilde{e}(t,a) := \begin{cases} e(t,a) & \text{for } 0 \leq a \leq t, \\ 0 & \text{for } 0 \leq t \leq a, \end{cases} \tag{3.1}$$

$$\tilde{\varphi}_i(t,a) := \begin{cases} 0 & \text{for } 0 \leq a \leq t, \\ i(t,a) & \text{for } 0 \leq t \leq a \end{cases} \quad \text{and} \quad \tilde{i}(t,a) := \begin{cases} i(t,a) & \text{for } 0 \leq a \leq t, \\ 0 & \text{for } 0 \leq t \leq a, \end{cases} \tag{3.2}$$

$$\tilde{\varphi}_j(t,a) := \begin{cases} 0 & \text{for } 0 \leq a \leq t, \\ j(t,a) & \text{for } 0 \leq t \leq a \end{cases} \quad \text{and} \quad \tilde{j}(t,a) := \begin{cases} j(t,a) & \text{for } 0 \leq a \leq t, \\ 0 & \text{for } 0 \leq t \leq a. \end{cases} \tag{3.3}$$

Then we have $\Phi(t,X_0) = \varphi(t,X_0) + \phi(t,X_0)$ for all $t \geq 0$. In order to verify that condition (i) of Lemma 3.2 holds true, we turn to a proof of the following proposition.

Proposition 3.1 ([26]) *For $h > 0$, let $w(t,h) = he^{-\mu_0 t}$. Then $\lim_{t\to\infty} w(t,h) = 0$ and $\|\varphi(t,X_0)\|_X \le w(t,h)$ if $\|X_0\|_X \le h$.*

Proof Obviously, $\lim_{t\to\infty} w(t,h) = 0$. For $X_0 \in \Upsilon$ and $\|X_0\|_X \le h$, we have

$$\begin{aligned}
\bigl\|\varphi(t,X_0)\bigr\|_X &= |0| + |0| + \int_0^\infty \bigl|\tilde{\varphi}_e(t,a)\bigr|\,\mathrm{d}a + \int_0^\infty \bigl|\tilde{\varphi}_i(t,a)\bigr|\,\mathrm{d}a + \int_0^\infty \bigl|\tilde{\varphi}_j(t,a)\bigr|\,\mathrm{d}a \\
&= \int_t^\infty \left|\varphi_e(a-t)\frac{B_1(a)}{B_1(a-t)}\right|\mathrm{d}a + \int_t^\infty \left|\varphi_i(a-t)\frac{B_2(a)}{B_2(a-t)}\right|\mathrm{d}a \\
&\quad + \int_t^\infty \left|\varphi_j(a-t)\frac{B_3(a)}{B_3(a-t)}\right|\mathrm{d}a \\
&= \int_0^\infty \left|\varphi_e(\tau)\frac{B_1(t+\tau)}{B_1(\tau)}\right|\mathrm{d}\tau + \int_0^\infty \left|\varphi_i(\tau)\frac{B_2(t+\tau)}{B_2(\tau)}\right|\mathrm{d}\tau \\
&\quad + \int_0^\infty \left|\varphi_j(\tau)\frac{B_3(t+\tau)}{B_3(\tau)}\right|\mathrm{d}\tau \\
&= \int_0^\infty \left|\varphi_e(\tau)\exp\biggl(-\int_\tau^{t+\tau}\theta_1(\upsilon)\,\mathrm{d}\upsilon\biggr)\right|\mathrm{d}\tau \\
&\quad + \int_0^\infty \left|\varphi_i(\tau)\exp\biggl(-\int_\tau^{t+\tau}\theta_2(\upsilon)\,\mathrm{d}\upsilon\biggr)\right|\mathrm{d}\tau \\
&\quad + \int_0^\infty \left|\varphi_j(\tau)\exp\biggl(-\int_\tau^{t+\tau}\theta_3(\upsilon)\,\mathrm{d}\upsilon\biggr)\right|\mathrm{d}\tau.
\end{aligned}$$

By (iii) of Assumption 1, $\theta_1(a), \theta_2(a), \theta_2(a) \ge \mu_0$ for $a \ge 0$, we have

$$\begin{aligned}
\bigl\|\varphi(t,X_0)\bigr\|_X &\le e^{-\mu_0 t}\biggl(|0| + |0| + \int_0^\infty \bigl|\varphi_e(\tau)\bigr|\,\mathrm{d}\tau + \int_0^\infty \bigl|\varphi_i(\tau)\bigr|\,\mathrm{d}\tau + \int_0^\infty \bigl|\varphi_j(\tau)\bigr|\,\mathrm{d}\tau\biggr) \\
&= e^{-\mu_0 t}\|X_0\|_X \le he^{-\mu_0 t} \equiv w(t,h).
\end{aligned}$$

This completes the proof. □

To verify (ii) of Lemma 3.2, we need to give the following lemma.

Lemma 3.3 ([27]) *Let $K \subset L^p(0,\infty)$ be closed and bounded where $p \ge 1$. Then K is compact if the following conditions hold true.*

(1) $\sup_{f\in K}\int_0^\infty f(a)\,\mathrm{d}a < \infty$.
(2) $\lim_{r\to\infty}\int_r^\infty f(a)\,\mathrm{d}a \to 0$ *uniformly in* $f \in K$.
(3) $\lim_{h\to 0^+}\int_0^\infty |f(a+h) - f(a)|\,\mathrm{d}a \to 0$ *uniformly in* $f \in K$.
(4) $\lim_{h\to 0^+}\int_0^h f(a)\,\mathrm{d}a \to 0$ *uniformly in* $f \in K$.

Proposition 3.2 ([27]) *For $t \ge 0$, $\phi(t,X_0)$ is completely continuous.*

Proof From Lemma 3.3, for any closed and bounded set $B \subset X$, we see that $\phi(t,B)$ is compact. According to Proposition 2.2, $S(t)$ and $V(t)$ remain in the compact set $[0,\Lambda/\mu_0] \subset [0,M]$, where $M \ge \Lambda/\mu_0$ is the bound for B. Thus, it is only to show that $\tilde{e}(t,a)$, $\tilde{i}(t,a)$ and $\tilde{j}(t,a)$ remain in a precompact subset of $L^1_+(0,\infty)$, which is independent of $X_0 \in \Upsilon$.

Now, from (2.2) and (3.1) we have

$$0 \le \tilde{e}(t,a) = \begin{cases} e(t-a,0)B_1(a) & \text{for } 0 \le a < t, \\ 0 & \text{for } 0 \le t \le a. \end{cases}$$

Then combining (i) of Proposition 2.2, we have

$$\tilde{e}(t,a) \le (\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3 + \bar{\beta}_4)M^2 e^{-\mu_0 a},$$

which implies that (1), (2) and (4) in Lemma 3.3 are satisfied. To check condition (3), for sufficiently small $h \in (0,t)$, we have

$$\begin{aligned} \int_0^\infty \bigl|\tilde{e}(t,a+h) - \tilde{e}(t,a)\bigr| \,\mathrm{d}a &= \int_0^t \bigl|e(t,a+h) - e(t,a)\bigr| \,\mathrm{d}a \\ &= \int_0^{t-h} \bigl|e(t-a-h,0)B_1(a+h) - e(t-a,0)B_1(a)\bigr| \,\mathrm{d}a \\ &\quad + \int_{t-h}^t \bigl|0 - e(t-a,0)B_1(a)\bigr| \,\mathrm{d}a \\ &\le \int_0^{t-h} e(t-a-h,0)\bigl|B_1(a+h) - B_1(a)\bigr| \,\mathrm{d}a \\ &\quad + \int_0^{t-h} B_1(a)\bigl|e(t-a-h,0) - e(t-a,0)\bigr| \,\mathrm{d}a \\ &\quad + \int_{t-h}^t \bigl|e(t-a,0)B_1(a)\bigr| \,\mathrm{d}a. \end{aligned}$$

Recall that $0 \le B_1(a) \le e^{-\mu_0 a} \le 1$ and $B_1(a)$ is non-increasing function with respect to a, it follows that

$$\begin{aligned} \int_0^{t-h} \bigl|B_1(a+h) - B_1(a)\bigr| \,\mathrm{d}a &= \int_0^{t-h} B_1(a) \,\mathrm{d}a - \int_0^{t-h} B_1(a+h) \,\mathrm{d}a \\ &= \int_0^{t-h} B_1(a) \,\mathrm{d}a - \int_h^t B_1(a) \,\mathrm{d}a \\ &= \int_0^{t-h} B_1(a) \,\mathrm{d}a - \int_h^{t-h} B_1(a) \,\mathrm{d}a - \int_{t-h}^t B_1(a) \,\mathrm{d}a \\ &= \int_0^h B_1(a) \,\mathrm{d}a - \int_{t-h}^t B_1(a) \,\mathrm{d}a \le h. \end{aligned}$$

Hence, from (ii) of Proposition 2.2, we have

$$\int_0^\infty \bigl|\tilde{e}(t,a+h) - \tilde{e}(t,a)\bigr| \,\mathrm{d}a \le 2(\bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3 + \bar{\beta}_4)M^2 h + \Delta, \tag{3.4}$$

where

$$\Delta = \int_0^{t-h} B_1(a)\bigl|e(t-a-h,0) - e(t-a,0)\bigr| \,\mathrm{d}a.$$

From (i) of Proposition 2.2, we find that $|\mathrm{d}S(t)/\mathrm{d}t|$ is bounded by $M_S = \Lambda + (\mu + p)M + \bar{\beta}_1 M^2 + \bar{\beta}_3 M^2$ and $|\mathrm{d}V(t)/\mathrm{d}t|$ is bounded by $M_V = (\mu + p)M + \bar{\beta}_2 M^2 + \bar{\beta}_4 M^2$. Therefore, $S(\cdot)$ and $V(\cdot)$ are Lipschitz on $[0,\infty)$ with coefficients M_S and M_V. By Lemma 3.1 of [28], there exist two Lipschitz coefficients $M_{I_1}, M_{I_2}, M_{J_1}, M_{J_2}$ for $\int_0^\infty \beta_1(a)i(\cdot,a)\,\mathrm{d}a$, $\int_0^\infty \beta_2(a)i(\cdot,a)\,\mathrm{d}a$, $\int_0^\infty \beta_3(a)j(\cdot,a)\,\mathrm{d}a$, $\int_0^\infty \beta_4(a)j(\cdot,a)\,\mathrm{d}a$, respectively. Thus, $\int_0^\infty \beta_1(a)i(\cdot,a)S(\cdot)\,\mathrm{d}a$, $\int_0^\infty \beta_2(a)i(\cdot,a)\,V(\cdot)\,\mathrm{d}a$, $\int_0^\infty \beta_3(a)j(\cdot,a)S(\cdot)\,\mathrm{d}a$, $\int_0^\infty \beta_4(a)j(\cdot,a)V(\cdot)\,\mathrm{d}a$ are Lipschitz continuous on $[0,\infty)$ with coefficients $M_{SI} = KM_{I_1} + KM_S$, $M_{SJ} = KM_{J_1} + KM_S$, $M_{VI} = KM_{I_2} + KM_V$, $M_{VJ} = KM_{J_2} + KM_V$, respectively. Denote $M = M_{SI} + M_{SJ} + M_{VI} + M_{VJ}$. Then

$$\Delta \le Mh \int_0^{t-h} e^{-\mu_0 a}\,\mathrm{d}a \le \frac{Mh}{\mu_0}. \tag{3.5}$$

Finally, by (3.4) and (3.5), we have

$$\int_0^\infty \left|\tilde{e}(t,a+h) - \tilde{e}(t,a)\right|\mathrm{d}a \le \left(2\bar{\beta}M^2 + \frac{M}{\mu_0}\right)h, \tag{3.6}$$

where $\bar{\beta} = \bar{\beta}_1 + \bar{\beta}_2 + \bar{\beta}_3 + \bar{\beta}_4$. The right hand of (3.6) converges uniformly to 0 as $h \to 0$ and condition (3) is proved for $\tilde{e}(t,a)$. Noting that (3.4) holds for any $X_0 \in B$, thus, $\tilde{e}(t,a)$ remains in a precompact subset $B_{\tilde{e}}$ of $L^1_+(0,\infty)$. Similarly, $\tilde{i}(t,a)$ and $\tilde{j}(t,a)$ remain in a precompact subset $B_{\tilde{i}}$, $B_{\tilde{j}}$ of $L^1_+(0,\infty)$, respectively. Thus, the proof is completed. □

From Propositions 3.1 and 3.2, we apply Lemma 3.2 and conclude that the following theorem holds.

Theorem 3.1 *The semi-flow $\Phi(t)_{t\ge 0}$ generated by system* (1.5) *is asymptotically smooth.*

4 The existence of steady states

System (1.5) always has the steady state $E_0 = (S_0, V_0, 0, 0, 0)$, where

$$S_0 = \frac{\Lambda(\mu + \rho + \eta)}{(\mu + p)(\mu + \rho + \eta) - p\eta}, \qquad V_0 = \frac{p\Lambda}{(\mu + p)(\mu + \rho + \eta) - p\eta}.$$

Define the basic reproduction number as follows

$$\begin{aligned} R_0 := S_0\bigg[&\int_0^\infty \gamma_1(a)B_1(a)\,\mathrm{d}a \int_0^\infty \beta_1(a)B_2(a)\,\mathrm{d}a \\ &+ \int_0^\infty \beta_3(a)B_3(a)\,\mathrm{d}a\bigg(\int_0^\infty \gamma_2(a)B_1(a)\,\mathrm{d}a \\ &+ \int_0^\infty \gamma_1(a)B_1(a)\,\mathrm{d}a \int_0^\infty \xi(a)B_2(a)\,\mathrm{d}a\bigg)\bigg] \\ &+ V_0\bigg[\int_0^\infty \gamma_1(a)B_1(a)\,\mathrm{d}a \int_0^\infty \beta_2(a)B_2(a)\,\mathrm{d}a \\ &+ \int_0^\infty \beta_4(a)B_3(a)\,\mathrm{d}a\bigg(\int_0^\infty \gamma_2(a)B_1(a)\,\mathrm{d}a \\ &+ \int_0^\infty \gamma_1(a)B_1(a)\,\mathrm{d}a \int_0^\infty \xi(a)B_2(a)\,\mathrm{d}a\bigg)\bigg]. \end{aligned}$$

Now we consider the existence of positive steady state of system (1.5). The steady state $(S^*, V^*, e^*(\cdot), i^*(\cdot), j^*(\cdot))$ of system (1.5) satisfies the following equalities:

$$\begin{aligned}
&\Lambda - (\mu + p)S^* + \eta V^* - S^* \int_0^\infty \left(\frac{\beta_1(a)i^*(a)}{1 + \alpha i^*(a)} + \beta_3(a)j^*(a) \right) \mathrm{d}a = 0, \\
&pS^* - (\mu + \rho + \eta)V^* - V^* \int_0^\infty \left(\frac{\beta_2(a)i^*(a)}{1 + \alpha i^*(a)} + \beta_4(a)j^*(a) \right) \mathrm{d}a = 0, \\
&\frac{de^*(a)}{\mathrm{d}a} = -\theta_1(a)e^*(a), \\
&\frac{di^*(a)}{\mathrm{d}a} = -\theta_2(a)i^*(a), \\
&\frac{dj^*(a)}{\mathrm{d}a} = -\theta_3(a)j^*(a), \\
&e^*(0) = S^* \int_0^\infty \left(\frac{\beta_1(a)i^*(a)}{1 + \alpha i^*(a)} + \beta_3(a)j^*(a) \right) \mathrm{d}a \\
&\qquad + V^* \int_0^\infty \left(\frac{\beta_2(a)i^*(a)}{1 + \alpha i^*(a)} + \beta_4(a)j^*(a) \right) \mathrm{d}a, \\
&i^*(0) = \int_0^\infty \gamma_1(a)e^*(a)\,\mathrm{d}a, \\
&j^*(0) = \int_0^\infty \gamma_2(a)e^*(a)\,\mathrm{d}a + \int_0^\infty \xi(a)i^*(a)\,\mathrm{d}a.
\end{aligned} \tag{4.1}$$

Solving the third, fourth and fifth equations of (4.1) yields

$$e^*(a) = e^*(0)B_1(a), \qquad i^*(a) = i^*(0)B_2(a), \qquad j^*(a) = j^*(0)B_3(a).$$

From the first and second equations of (4.1), it is easy to get

$$\begin{aligned}
&\Lambda - \mu S^* - (\mu + \rho)V^* - S^*\bigl(\Lambda - \mu S^* - (\mu + \rho)V^*\bigr)f_1\bigl(S^*, V^*\bigr) \\
&\quad - V^*\bigl(\Lambda - \mu S^* - (\mu + \rho)V^*\bigr)f_2\bigl(S^*, V^*\bigr) = 0,
\end{aligned}$$

where

$$\begin{aligned}
f_1\bigl(S^*, V^*\bigr) &= \int_0^\infty \left(\frac{K_1K_3(a)}{1 + \alpha K_1(\Lambda - \mu S^* - (\mu + \rho)V^*)B_2(a)} + (K_2 + K_1K_7)K_4(a) \right) \mathrm{d}a, \\
f_2\bigl(S^*, V^*\bigr) &= \int_0^\infty \left(\frac{K_1K_5(a)}{1 + \alpha K_1(\Lambda - \mu S^* - (\mu + \rho)V^*)B_2(a)} + (K_2 + K_1K_7)K_6(a) \right) \mathrm{d}a,
\end{aligned}$$

and

$$\begin{aligned}
&K_1 = \int_0^\infty \gamma_1(a)B_1(a)\,\mathrm{d}a, \qquad K_2 = \int_0^\infty \gamma_2(a)B_1(a)\,\mathrm{d}a, \\
&K_3(a) = \beta_1(a)B_2(a), \qquad K_4(a) = \beta_3(a)B_3(a), \qquad K_5(a) = \beta_2(a)B_2(a), \\
&K_6(a) = \beta_4(a)B_3(a), \qquad K_7 = \int_0^\infty \xi(a)B_2(a)\,\mathrm{d}a.
\end{aligned}$$

Let

$$F(S^*, V^*) = \Lambda - \mu S^* - (\mu + \rho)V^* - S^*\big(\Lambda - \mu S^* - (\mu + \rho)V^*\big)f_1(S^*, V^*) \\ - V^*\big(\Lambda - \mu S^* - (\mu + \rho)V^*\big)f_2(S^*, V^*).$$

Let $V^* = V_0$. It is easy to see that $F(0, V_0) = \Lambda - (\mu + \rho)V_0 - V_0(\Lambda - (\mu + \rho)V_0)f_2(0, V_0)$ and $F(S_0, V_0) = 0$. When $0 \le S^* \le S_0$, $\Lambda - \mu S^* - (\mu + \rho)V_0 \neq 0$, then we get

$$S^*f_1(S^*, V_0) + V_0 f_2(S^*, V_0) - 1 = 0.$$

Let $g(S^*) = S^*f_1(S^*, V_0) + V_0f_2(S^*, V_0) - 1$, where $g(0) = V_0f_2(0, V_0) - 1 < V_0f_2(S^*, V_0) - 1 < 0$ and $g(S_0) = S_0f_1(S_0, V_0) + V_0f_2(S_0, V_0) - 1$. Note that

$$g'(S^*) = f_1(S^*, V_0) + S^*\frac{\mathrm{d}f_1(S^*, V_0)}{\mathrm{d}S^*} + V_0\frac{\mathrm{d}f_2(S^*, V_0)}{\mathrm{d}S^*} > 0,$$

for

$$\frac{\mathrm{d}f_1(S^*, V_0)}{\mathrm{d}S^*} = \int_0^\infty \frac{\mu\alpha K_1^2 K_3(a)B_2(a)}{(1 + \alpha K_1(\Lambda - \mu S^* - (\mu + \rho)V_0)B_2(a))^2}\,\mathrm{d}a > 0,$$
$$\frac{\mathrm{d}f_2(S^*, V_0)}{\mathrm{d}S^*} = \int_0^\infty \frac{\mu\alpha K_1^2 K_5(a)B_2(a)}{(1 + \alpha K_1(\Lambda - \mu S^* - (\mu + \rho)V_0)B_2(a))^2}\,\mathrm{d}a > 0.$$

It is easy to show that if $g(S_0) > 0$, $g(S^*) = 0$ has a unique positive root. Define the basic reproduction number as

$$R_0 = g(S_0) + 1 = S_0f_1(S_0, V_0) + V_0f_2(S_0, V_0),$$

which means the number of newly infected individuals produced by one infected individual during its period of disease. Therefore, if $R_0 > 1$, there exists a unique positive steady state E^* of system (1.5), where $E^* = (S^*, V^*, e^*(a), i^*(a), j^*(a))$.

From the above discussions, we have the following theorem.

Theorem 4.1 *System* (1.5) *always has a steady state* $E_0(S_0, V_0, 0, 0, 0)$, *where* $S_0 = \Lambda(\mu + \rho + \eta)/[(\mu + p)(\mu + \rho + \eta) - p\eta]$, $V_0 = p\Lambda/[(\mu + p)(\mu + \rho + \eta) - p\eta]$; *system* (1.5) *has a unique positive steady state* $E^*(S^*, V^*, e^*(\cdot), i^*(\cdot), j^*(\cdot))$ *if and only if* $R_0 > 1$.

5 Local stability

This section is mainly used to prove the local stability of steady states, and to verify that the basic reproduction number is related to the stability of the steady states.

Theorem 5.1 *The steady state* E_0 *is locally asymptotically stable if* $R_0 < 1$.

Proof First, we introduce the change of variables as follows:

$$s_1(t) = S(t) - S_0, \qquad v_1(t) = V(t) - V_0, \qquad e_1(t,a) = e(t,a),$$
$$i_1(t,a) = i(t,a), \qquad j_1(t,a) = j(t,a).$$

Linearizing system (1.5) at the steady state E_0 yields the following system:

$$\begin{aligned}
&\dot{s}_1(t) = -(\mu + p)s_1(t) + \eta v_1(t) - S_0 \int_0^\infty \bigl(\beta_1(a)i_1(t,a) + \beta_3(a)j_1(t,a)\bigr)\,\mathrm{d}a,\\
&\dot{v}_1(t) = ps_1(t) - (\mu + \rho + \eta)v_1(t) - V_0 \int_0^\infty \bigl(\beta_2(a)i_1(t,a) + \beta_4(a)j_1(t,a)\bigr)\,\mathrm{d}a,\\
&\frac{\partial e_1(t,a)}{\partial a} + \frac{\partial e_1(t,a)}{\partial t} = -\theta_1(a)e_1(t,a),\\
&\frac{\partial i_1(t,a)}{\partial a} + \frac{\partial i_1(t,a)}{\partial t} = -\theta_2(a)i_1(t,a),\\
&\frac{\partial j_1(t,a)}{\partial a} + \frac{\partial j_1(t,a)}{\partial t} = -\theta_3(a)j_1(t,a),\\
&e_1(t,0) = S_0 \int_0^\infty \bigl(\beta_1(a)i_1(t,a) + \beta_3(a)j_1(t,a)\bigr)\,\mathrm{d}a\\
&\qquad\qquad + V_0 \int_0^\infty \bigl(\beta_2(a)i_1(t,a) + \beta_4(a)j_1(t,a)\bigr)\,\mathrm{d}a,\\
&i_1(t,0) = \int_0^\infty \gamma_1(a)e_1(t,a)\,\mathrm{d}a,\\
&j_1(t,0) = \int_0^\infty \gamma_2(a)e_1(t,a)\,\mathrm{d}a + \int_0^\infty \xi(a)i_1(t,a)\,\mathrm{d}a.
\end{aligned}$$

Set

$$\begin{aligned}
&s_1(t) = s_1^0 e^{\lambda t}, \qquad v_1(t) = v_1^0 e^{\lambda t}, \qquad e_1(t,a) = e_1^0(a)e^{\lambda t},\\
&i_1(t,a) = i_1^0(a)e^{\lambda t}, \qquad j_1(t,a) = j_1^0(a)e^{\lambda t},
\end{aligned}$$

where s_1^0, v_1^0, $e_1^0(a)$, $i_1^0(a)$, $j_1^0(a)$ will be determined later. We can get

$$\begin{aligned}
&\lambda s_1^0 = -(\mu + p)s_1^0 + \eta v_1^0 - S_0 \int_0^\infty \bigl(\beta_1(a)i_1^0(a) + \beta_3(a)j_1^0(a)\bigr)\,\mathrm{d}a,\\
&\lambda v_1^0 = ps_1^0 - (\mu + \rho + \eta)v_1^0 - V_0 \int_0^\infty \bigl(\beta_2(a)i_1^0(a) + \beta_4(a)j_1^0(a)\bigr)\,\mathrm{d}a,\\
&\frac{\mathrm{d}e_1^0(a)}{\mathrm{d}a} = -\bigl(\lambda + \theta_1(a)\bigr)e_1^0(a),\\
&\frac{\mathrm{d}i_1^0(a)}{\mathrm{d}a} = -\bigl(\lambda + \theta_2(a)\bigr)i_1^0(a),\\
&\frac{\mathrm{d}j_1^0(a)}{\mathrm{d}a} = -\bigl(\lambda + \theta_3(a)\bigr)j_1^0(a),\\
&e_1^0(0) = S_0 \int_0^\infty \bigl(\beta_1(a)i_1^0(a) + \beta_3(a)j_1^0(a)\bigr)\,\mathrm{d}a\\
&\qquad\qquad + V_0 \int_0^\infty \bigl(\beta_2(a)i_1^0(a) + \beta_4(a)j_1^0(a)\bigr)\,\mathrm{d}a\\
&i_1^0(0) = \int_0^\infty \gamma_1(a)e_1^0(a)\,\mathrm{d}a,\\
&j_1^0(0) = \int_0^\infty \gamma_2(a)e_1^0(a)\,\mathrm{d}a + \int_0^\infty \xi(a)i_1^0(a)\,\mathrm{d}a.
\end{aligned} \tag{5.1}$$

Integrating the third, fourth and fifth equations of (5.1) from 0 to a yields

$$\begin{aligned} e_1^0(a) &= e_1^0(0)\exp\biggl(-\int_0^a \bigl(\lambda+\theta_1(\tau)\bigr)\,\mathrm{d}\tau\biggr), \\ i_1^0(a) &= i_1^0(0)\exp\biggl(-\int_0^a \bigl(\lambda+\theta_2(\tau)\bigr)\,\mathrm{d}\tau\biggr), \\ j_1^0(a) &= j_1^0(0)\exp\biggl(-\int_0^a \bigl(\lambda+\theta_3(\tau)\bigr)\,\mathrm{d}\tau\biggr). \end{aligned} \tag{5.2}$$

If $e_1^0(0)=0$, then $i_1^0(0)=0$, $j_1^0(0)=0$. Plugging it into (5.1), we have

$$(\lambda+\mu+p)s_1^0-\eta v_1^0=0, \qquad (\lambda+\mu+\rho+\eta)v_1^0-ps_1^0=0.$$

For $s_1^0\neq 0$ and $v_1^0\neq 0$, it is easy to get

$$\lambda^2+(2\mu+p+\rho+\eta)\lambda+(\mu+p)(\mu+\rho+\eta)-p\eta=0.$$

Then

$$\lambda=\frac{-b_1\pm\sqrt{b_1^2-4b_2}}{2},$$

where $b_1=2\mu+p+\rho+\eta$, $b_2=(\mu+p)(\mu+\rho+\eta)-p\eta$.

If $e_1^0(0)\neq 0$,

$$(\lambda+\mu)s_1^0+(\lambda+\mu+\rho)v_1^0+e_1^0(0)=0,$$

where

$$\begin{aligned} s_1^0 &= \frac{S_0(\lambda+\mu+\rho+\eta)f_3(\lambda)+\eta V_0f_4(\lambda)}{p\eta-(\lambda+\mu+p)(\lambda+\mu+\rho+\eta)}e_1^0(0), \\ v_1^0 &= \frac{pS_0f_3(\lambda)+V_0(\lambda+\mu+p)f_4(\lambda)}{p\eta-(\lambda+\mu+p)(\lambda+\mu+\rho+\eta)}e_1^0(0), \end{aligned}$$

where

$$\begin{aligned} f_3(\lambda) &= \int_0^\infty \bigl(\beta_1(a)B_2(\lambda,a)U_1(\lambda)+\beta_3(a)B_3(\lambda,a)\bigl(U_2(\lambda)+U_1(\lambda)U_7(\lambda)\bigr)\bigr)\,\mathrm{d}a, \\ f_4(\lambda) &= \int_0^\infty \bigl(\beta_2(a)B_2(\lambda,a)U_1(\lambda)+\beta_4(a)B_3(\lambda,a)\bigl(U_2(\lambda)+U_1(\lambda)U_7(\lambda)\bigr)\bigr)\,\mathrm{d}a, \end{aligned}$$

where

$$\begin{aligned} &U_1(\lambda)=\int_0^\infty \gamma_1(a)B_1(\lambda,a)\,\mathrm{d}a, \qquad U_2(\lambda)=\int_0^\infty \gamma_2(a)B_1(\lambda,a)\,\mathrm{d}a, \\ &U_7(\lambda)=\int_0^\infty \xi(a)B_2(\lambda,a)\,\mathrm{d}a, \qquad B_1(\lambda,a)=\exp\biggl(-\int_0^a \bigl(\lambda+\theta_1(\tau)\bigr)\,\mathrm{d}\tau\biggr), \\ &B_2(\lambda,a)=\exp\biggl(-\int_0^a \bigl(\lambda+\theta_2(\tau)\bigr)\,\mathrm{d}\tau\biggr), \qquad B_3(\lambda,a)=\exp\biggl(-\int_0^a \bigl(\lambda+\theta_3(\tau)\bigr)\,\mathrm{d}\tau\biggr). \end{aligned}$$

That is,

$$S_0 f_3(\lambda) + V_0 f_4(\lambda) = 1. \tag{5.3}$$

Assume that $\operatorname{Re}\lambda \geq 0$, then $|f_3(\lambda)| \leq f_1(S_0, V_0)$ and $|f_4(\lambda)| \leq f_2(S_0, V_0)$ hold. Hence, the modulus of the left-hand side of Eq. (5.3) satisfies

$$\begin{aligned}
&\left|S_0 f_3(\lambda) + V_0 f_4(\lambda)\right| \\
&\quad \leq \left|S_0 f_3(\lambda)\right| + \left|V_0 f_4(\lambda)\right| < S_0 f_1(S_0, V_0) + V_0 f_2(S_0, V_0) = R_0.
\end{aligned}$$

It follows from (5.3) that there is a contradiction. Thus, all the roots of Eq. (5.3) have a negative real part if and only if $R_0 < 1$. Therefore, the steady state E_0 is locally asymptotically stable if $R_0 < 1$. This completes the proof. □

Theorem 5.2 *The steady state E^* is locally asymptotically stable if $R_0 > 1$.*

Proof Linearizing system (1.5) at the steady state E^* under introducing the perturbation variables

$$\begin{aligned}
&s_2(t) = S(t) - S^*, \qquad v_2(t) = V(t) - V^*, \qquad e_2(t,a) = e(t,a) - e^*(a), \\
&i_2(t,a) = i(t,a) - i^*(a), \qquad j_2(t,a) = j(t,a) - j^*(a),
\end{aligned}$$

we obtain the following system:

$$\begin{aligned}
\dot{s}_2(t) &= -(\mu + p)s_2(t) + \eta v_2(t) - s_2(t)\int_0^\infty \frac{\beta_1(a)i^*(a)}{1+\alpha i^*(a)}\,\mathrm{d}a - S^*\int_0^\infty \frac{\beta_1(a)i_2(t,a)}{(1+\alpha i^*(a))^2}\,\mathrm{d}a \\
&\quad - s_2(t)\int_0^\infty \beta_3(a)j^*(a)\,\mathrm{d}a - S^*\int_0^\infty \beta_3(a)j_2(t,a)\,\mathrm{d}a, \\
\dot{v}_2(t) &= ps_2(t) - (\mu + \rho + \eta)v_2(t) - v_2(t)\int_0^\infty \frac{\beta_2(a)i^*(a)}{1+\alpha i^*(a)}\,\mathrm{d}a - V^*\int_0^\infty \frac{\beta_2(a)i_2(t,a)}{(1+\alpha i^*(a))^2}\,\mathrm{d}a \\
&\quad - v_2(t)\int_0^\infty \beta_4(a)j^*(a)\,\mathrm{d}a - V^*\int_0^\infty \beta_4(a)j_2(t,a)\,\mathrm{d}a, \\
&\frac{\partial e_2(t,a)}{\partial a} + \frac{\partial e_2(t,a)}{\partial t} = -\theta_1(a)e_2(t,a), \\
&\frac{\partial i_2(t,a)}{\partial a} + \frac{\partial i_2(t,a)}{\partial t} = -\theta_2(a)i_2(t,a), \\
&\frac{\partial j_2(t,a)}{\partial a} + \frac{\partial j_1(t,a)}{\partial t} = -\theta_3(a)j_2(t,a), \\
e_2(t,0) &= s_2(t)\int_0^\infty \frac{\beta_1(a)i^*(a)}{1+\alpha i^*(a)}\,\mathrm{d}a + S^*\int_0^\infty \frac{\beta_1(a)i_2(t,a)}{(1+\alpha i^*(a))^2}\,\mathrm{d}a + s_2(t)\int_0^\infty \beta_3(a)j^*(a)\,\mathrm{d}a \\
&\quad + S^*\int_0^\infty \beta_3(a)j_2(t,a)\,\mathrm{d}a + v_2(t)\int_0^\infty \frac{\beta_2(a)i^*(a)}{1+\alpha i^*(a)}\,\mathrm{d}a + V^*\int_0^\infty \frac{\beta_2(a)i_2(t,a)}{(1+\alpha i^*(a))^2}\,\mathrm{d}a \\
&\quad + v_2(t)\int_0^\infty \beta_4(a)j^*(a)\,\mathrm{d}a + V^*\int_0^\infty \beta_4(a)j_2(t,a)\,\mathrm{d}a, \\
i_2(t,0) &= \int_0^\infty \gamma_1(a)e_2(t,a)\,\mathrm{d}a,
\end{aligned}$$

$$j_2(t,0) = \int_0^\infty \gamma_2(a)e_2(t,a)\,\mathrm{d}a + \int_0^\infty \xi(a)i_2(t,a)\,\mathrm{d}a.$$

Set

$$\begin{aligned}
&s_2(t) = s_2^0 e^{\lambda t}, \qquad v_2(t) = v_2^0 e^{\lambda t}, \qquad e_2(t,a) = e_2^0(a)e^{\lambda t},\\
&i_2(t,a) = i_2^0(a)e^{\lambda t}, \qquad j_2(t,a) = j_2^0(a)e^{\lambda t},
\end{aligned}$$

where s_2^0, v_2^0, $e_2^0(a)$, $i_2^0(a)$, $j_2^0(a)$ will be determined later. We get

$$\begin{aligned}
\lambda s_2^0 &= -(\mu + p)s_2^0 + \eta v_2^0 - S^* \int_0^\infty \frac{\beta_1(a)i_2^0(a)}{(1+\alpha i^*(a))^2}\,\mathrm{d}a - s_2^0 \int_0^\infty \frac{\beta_1(a)i^*(a)}{1+\alpha i^*(a)}\,\mathrm{d}a\\
&\quad - S^* \int_0^\infty \beta_3(a)j_2^0(a)\,\mathrm{d}a - s_2^0 \int_0^\infty \beta_3(a)j^*(a)\,\mathrm{d}a,\\
\lambda v_2^0 &= p s_2^0 - (\mu + \rho + \eta)v_2^0 - V^* \int_0^\infty \frac{\beta_2(a)i_2^0(a)}{(1+\alpha i^*(a))^2}\,\mathrm{d}a - v_2^0 \int_0^\infty \frac{\beta_2(a)i^*(a)}{1+\alpha i^*(a)}\,\mathrm{d}a\\
&\quad - V^* \int_0^\infty \beta_4(a)j_2^0(a)\,\mathrm{d}a - v_2^0 \int_0^\infty \beta_4(a)j^*(a)\,\mathrm{d}a,\\
\frac{\mathrm{d}e_2^0(a)}{\mathrm{d}a} &= -\big(\lambda + \theta_1(a)\big)e_2^0(a),\\
\frac{\mathrm{d}i_2^0(a)}{\mathrm{d}a} &= -\big(\lambda + \theta_2(a)\big)i_2^0(a),\\
\frac{\mathrm{d}j_2^0(a)}{\mathrm{d}a} &= -\big(\lambda + \theta_3(a)\big)j_2^0(a),\\
e_2^0(0) &= s_2^0 \int_0^\infty \frac{\beta_1(a)i^*(a)}{1+\alpha i^*(a)}\,\mathrm{d}a + S^* \int_0^\infty \frac{\beta_1(a)i_2^0(a)}{(1+\alpha i^*(a))^2}\,\mathrm{d}a\\
&\quad + s_2^0 \int_0^\infty \beta_3(a)j^*(a)\,\mathrm{d}a + S^* \int_0^\infty \beta_3(a)j_2^0(a)\,\mathrm{d}a\\
&\quad + v_2^0 \int_0^\infty \frac{\beta_2(a)i^*(a)}{1+\alpha i^*(a)}\,\mathrm{d}a + V^* \int_0^\infty \frac{\beta_2(a)i_2^0(a)}{(1+\alpha i^*(a))^2}\,\mathrm{d}a\\
&\quad + v_2^0 \int_0^\infty \beta_4(a)j^*(a)\,\mathrm{d}a + V^* \int_0^\infty \beta_4(a)j_2^0(a)\,\mathrm{d}a,\\
i_2^0(0) &= \int_0^\infty \gamma_1(a)e_2^0(a)\,\mathrm{d}a,\\
j_2^0(0) &= \int_0^\infty \gamma_2(a)e_2^0(a)\,\mathrm{d}a + \int_0^\infty \xi(a)i_2^0(a)\,\mathrm{d}a.
\end{aligned} \tag{5.4}$$

Integrating the third, fourth and fifth equations of (5.4) from 0 to a yields

$$\begin{aligned}
e_2^0(a) &= e_2^0(0)\exp\biggl(-\int_0^a \big(\lambda + \theta_1(\tau)\big)\,\mathrm{d}\tau\biggr),\\
i_2^0(a) &= i_2^0(0)\exp\biggl(-\int_0^a \big(\lambda + \theta_2(\tau)\big)\,\mathrm{d}\tau\biggr),\\
j_2^0(a) &= j_2^0(0)\exp\biggl(-\int_0^a \big(\lambda + \theta_3(\tau)\big)\,\mathrm{d}\tau\biggr).
\end{aligned} \tag{5.5}$$

The characteristic equation is

$$(\lambda + \mu)s_2^0 + (\lambda + \mu + \rho)v_2^0 + e_2^0(0) = 0,$$

where

$$\begin{aligned}
s_2^0 &= \frac{S^*(\lambda + \mu + \rho + \eta + f_6(e^*))f_3(e^*, \lambda) + \eta V^* f_4(e^*, \lambda)}{p\eta - (\lambda + \mu + p + f_5(e^*))(\lambda + \mu + \rho + \eta + f_6(e^*))} e_2^0, \\
v_2^0 &= \frac{pS^* f_3(e^*, \lambda) + V^*(\lambda + \mu + p + f_5(e^*))f_4(e^*, \lambda)}{p\eta - (\lambda + \mu + p + f_5(e^*))(\lambda + \mu + \rho + \eta + f_6(e^*))} e_2^0,
\end{aligned}$$

where

$$\begin{aligned}
f_3(e^*, \lambda) &= \int_0^\infty \left(\frac{\beta_1(a)B_2(\lambda, a)U_1(\lambda)}{(1 + \alpha K_1 e^*(0)B_2(a))^2} + \beta_3(a)B_3(\lambda, a)\bigl(U_2(\lambda) + U_1(\lambda)U_7(\lambda)\bigr) \right) \mathrm{d}a, \\
f_4(e^*, \lambda) &= \int_0^\infty \left(\frac{\beta_2(a)B_2(\lambda, a)U_1(\lambda)}{(1 + \alpha K_1 e^*(0)B_2(a))^2} + \beta_4(a)B_3(\lambda, a)\bigl(U_2(\lambda) + U_1(\lambda)U_7(\lambda)\bigr) \right) \mathrm{d}a, \\
f_5(e^*) &= \int_0^\infty \left(\frac{K_1\beta_1(a)B_2(a)}{1 + \alpha K_1 e^*(0)B_2(a)} + \beta_3(a)B_3(a)(K_2 + K_1K_7) \right) \mathrm{d}a, \\
f_6(e^*) &= \int_0^\infty \left(\frac{K_1\beta_2(a)B_2(a)}{1 + \alpha K_1 e^*(0)B_2(a)} + \beta_4(a)B_3(a)(K_2 + K_1K_7) \right) \mathrm{d}a.
\end{aligned}$$

That is,

$$\begin{aligned}
&\frac{(\lambda + \mu)(\lambda + \mu + \rho + \eta + f_6(e^*)) + p(\lambda + \mu + \rho)}{(\lambda + \mu + f_5(e^*))(\lambda + \mu + \rho + \eta + f_6(e^*)) + p(\lambda + \mu + \rho + f_6(e^*))} S^* f_3(e^*, \lambda) \\
&\quad + \frac{(\lambda + \mu + p + f_5(e^*))(\lambda + \mu + \rho) + \eta(\lambda + \mu)}{(\lambda + \mu + p + f_5(e^*))(\lambda + \mu + \rho + f_6(e^*)) + \eta(\lambda + \mu + f_5(e^*))} V^* f_4(e^*, \lambda) \\
&\quad = 1.
\end{aligned} \tag{5.6}$$

Assume that $\operatorname{Re}\lambda \geq 0$, then $|f_3(e^*, \lambda)| \leq f_5(e^*)$ and $|f_4(e^*, \lambda)| \leq f_6(e^*)$ hold. Hence, the modulus of the left-hand side of Eq. (5.6) satisfies

$$\begin{aligned}
&\left| \frac{(\lambda + \mu)(\lambda + \mu + \rho + \eta + f_6(e^*)) + p(\lambda + \mu + \rho)}{(\lambda + \mu + f_5(e^*))(\lambda + \mu + \rho + \eta + f_6(e^*)) + p(\lambda + \mu + \rho + f_6(e^*))} S^* f_3(e^*, \lambda) \right. \\
&\quad \left. + \frac{(\lambda + \mu + p + f_5(e^*))(\lambda + \mu + \rho) + \eta(\lambda + \mu)}{(\lambda + \mu + p + f_5(e^*))(\lambda + \mu + \rho + f_6(e^*)) + \eta(\lambda + \mu + f_5(e^*))} V^* f_4(e^*, \lambda) \right| \\
&\quad < \bigl| S^* f_3(e^*, \lambda) + V^* f_4(e^*, \lambda) \bigr| \leq \bigl| S^* f_3(e^*, \lambda) \bigr| + \bigl| V^* f_4(e^*, \lambda) \bigr| \\
&\quad \leq S^* f_5(e^*) + V^* f_6(e^*) = 1.
\end{aligned}$$

It follows from (5.6) that there is a contradiction. Therefore, $\operatorname{Re}\lambda < 0$. This means that all the roots of (5.6) have negative real parts. Consequently, if $R_0 > 1$, the steady state E^* is locally asymptotically stable. This completes the proof. □

6 Uniform persistence

In this section, we investigate the uniform persistence of system (1.5) by using the persistence theory for infinite dimensional dynamics system. Define

$$\begin{aligned}&\bar{a} = \inf\left\{a : \int_a^\infty \theta_1(u)\,\mathrm{d}u = 0\right\}, \qquad \bar{b} = \inf\left\{b : \int_b^\infty \theta_2(u)\,\mathrm{d}u = 0\right\},\\ &\bar{c} = \inf\left\{c : \int_c^\infty \theta_3(u)\,\mathrm{d}u = 0\right\}.\end{aligned}$$

Since $\theta_1(a), \theta_2(a), \theta_3(a) \in L^1_+(0,\infty)$, we have $\bar{a}, \bar{b}, \bar{c} > 0$. Furthermore, let

$$\begin{aligned}&\tilde{X} = L^1_+(0,\infty) \times L^1_+(0,\infty) \times L^1_+(0,\infty),\\ &\tilde{Y} = \left\{\bigl(e(t,\cdot), i(t,\cdot), j(t,\cdot)\bigr)^T \in \tilde{X} : \int_0^{\bar{a}} e(t,x)\,\mathrm{d}x > 0, \int_0^{\bar{b}} i(t,x)\,\mathrm{d}x > 0, \int_0^{\bar{c}} j(t,x)\,\mathrm{d}x > 0\right\},\end{aligned}$$

and

$$Y = R^+ \times R^+ \times \tilde{Y}, \qquad \partial Y = X \backslash Y, \qquad \partial \tilde{Y} = \tilde{X} \backslash \tilde{Y}.$$

It is not difficult to verify the following proposition.

Proposition 6.1 *The subsets Y and ∂Y are both positively invariant under the semi-flow $\{\Phi(t)\}_{t\geq 0}$, namely, $\Phi(t, Y) \subset Y$ and $\Phi(t, \partial Y) \subset \partial Y$ for $t \geq 0$.*

Furthermore, the following result is useful for the proof of uniform persistence.

Theorem 6.1 *The disease-free steady state E_0 of system* (1.5) *is globally asymptotically stable for the semi-flow $\{\Phi(t)\}_{t\geq 0}$ restricted to ∂Y.*

Proof Letting $(S_0, V_0, e_0(\cdot), i_0(\cdot), j_0(\cdot)) \in \partial Y$, namely, $(e_0(\cdot), i_0(\cdot), j_0(\cdot)) \in \partial \tilde{Y}$, we consider the following system:

$$\begin{aligned}&\frac{\partial e(t,a)}{\partial a} + \frac{\partial e(t,a)}{\partial t} = -\theta_1(a) e(t,a),\\ &\frac{\partial i(t,a)}{\partial a} + \frac{\partial i(t,a)}{\partial t} = -\theta_2(a) i(t,a),\\ &\frac{\partial j(t,a)}{\partial a} + \frac{\partial j(t,a)}{\partial t} = -\theta_3(a) j(t,a),\\ &e(t,0) = S(t) \int_0^\infty \left(\frac{\beta_1(a) i(t,a)}{1 + \alpha i(t,a)} + \beta_3(a) j(t,a)\right) \mathrm{d}a\\ &\qquad\quad + V(t) \int_0^\infty \left(\frac{\beta_2(a) i(t,a)}{1 + \alpha i(t,a)} + \beta_4(a) j(t,a)\right) \mathrm{d}a,\\ &i(t,0) = \int_0^\infty \gamma_1(a) e(t,a)\,\mathrm{d}a,\\ &j(t,0) = \int_0^\infty \gamma_2(a) e(t,a)\,\mathrm{d}a + \int_0^\infty \xi(a) i(t,a)\,\mathrm{d}a,\\ &e(0,a) = \varphi_e(a), \qquad i(0,a) = \varphi_i(a), \qquad j(0,a) = \varphi_j(a).\end{aligned}$$

Since $S(t) \le S_0$ and $V(t) \le V_0$ as t tends to infinity, by comparison, we have $e(t,a) \le \tilde{e}(t,a), i(t,a) \le \tilde{i}(t,a), j(t,a) \le \tilde{j}(t,a)$, where $\tilde{e}(t,a)$, $\tilde{i}(t,a)$ and $\tilde{j}(t,a)$ satisfy the following auxiliary system:

$$
\begin{aligned}
&\frac{\partial \tilde{e}(t,a)}{\partial a} + \frac{\partial \tilde{e}(t,a)}{\partial t} = -\theta_1(a)\tilde{e}(t,a),\\
&\frac{\partial \tilde{i}(t,a)}{\partial a} + \frac{\partial \tilde{i}(t,a)}{\partial t} = -\theta_2(a)\tilde{i}(t,a),\\
&\frac{\partial \tilde{j}(t,a)}{\partial a} + \frac{\partial \tilde{j}(t,a)}{\partial t} = -\theta_3(a)\tilde{j}(t,a),\\
&\tilde{e}(t,0) = S_0 \int_0^\infty \left(\frac{\beta_1(a)\tilde{i}(t,a)}{1+\alpha\tilde{i}(t,a)} + \beta_3(a)\tilde{j}(t,a) \right) \mathrm{d}a\\
&\qquad\quad + V_0 \int_0^\infty \left(\frac{\beta_2(a)\tilde{i}(t,a)}{1+\alpha\tilde{i}(t,a)} + \beta_4(a)\tilde{j}(t,a) \right) \mathrm{d}a,\\
&\tilde{i}(t,0) = \int_0^\infty \gamma_1(a)\tilde{e}(t,a)\,\mathrm{d}a,\\
&\tilde{j}(t,0) = \int_0^\infty \gamma_2(a)\tilde{e}(t,a)\,\mathrm{d}a + \int_0^\infty \xi(a)\tilde{i}(t,a)\,\mathrm{d}a,\\
&\tilde{e}(0,a) = \varphi_e(a), \tilde{i}(0,a) = \varphi_i(a), \tilde{j}(0,a) = \varphi_j(a).
\end{aligned}
\tag{6.1}
$$

Similar to (2.2)–(2.4), solving the first three equations of (6.1) yields

$$
\tilde{e}(t,a) = \begin{cases} \tilde{L}_1(t-a)B_1(a), & 0 \le a \le t,\\ \varphi_e(a-t)\frac{B_1(a)}{B_1(a-t)}, & 0 \le t \le a, \end{cases} \tag{6.2}
$$

$$
\tilde{i}(t,a) = \begin{cases} \tilde{L}_2(t-a)B_2(a), & 0 \le a \le t,\\ \varphi_i(a-t)\frac{B_2(a)}{B_2(a-t)}, & 0 \le t \le a, \end{cases} \tag{6.3}
$$

$$
\tilde{j}(t,a) = \begin{cases} \tilde{L}_3(t-a)B_3(a) & \text{for } 0 \le a \le t,\\ \varphi_j(a-t)\frac{B_3(a)}{B_3(a-t)} & \text{for } 0 \le t \le a, \end{cases} \tag{6.4}
$$

where

$$
\begin{aligned}
\tilde{L}_1(t) &= S_0 \int_0^\infty \left(\frac{\beta_1(a)\tilde{i}(t,a)}{1+\alpha\tilde{i}(t,a)} + \beta_3(a)\tilde{j}(t,a) \right) \mathrm{d}a\\
&\quad + V_0 \int_0^\infty \left(\frac{\beta_2(a)\tilde{i}(t,a)}{1+\alpha\tilde{i}(t,a)} + \beta_4(a)\tilde{j}(t,a) \right) \mathrm{d}a,\\
\tilde{L}_2(t) &= \int_0^\infty \gamma_1(a)\tilde{e}(t,a)\,\mathrm{d}a,\\
\tilde{L}_3(t) &= \int_0^\infty \gamma_2(a)\tilde{e}(t,a)\,\mathrm{d}a + \int_0^\infty \xi(a)\tilde{i}(t,a)\,\mathrm{d}a.
\end{aligned}
$$

It follows from (6.2)–(6.4) that

$$\begin{aligned}\tilde{L}_1(t) &= S_0\int_0^t\left(\frac{\beta_1(a)\tilde{L}_2(t-a)B_2(a)}{1+\alpha\tilde{L}_2(t-a)B_2(a)}+\beta_3(a)\tilde{L}_3(t-a)B_3(a)\right)\mathrm{d}a\\&\quad+V_0\int_0^t\left(\frac{\beta_2(a)\tilde{L}_2(t-a)B_2(a)}{1+\alpha\tilde{L}_2(t-a)B_2(a)}+\beta_4(a)\tilde{L}_3(t-a)B_3(a)\right)\mathrm{d}a+G_1(t),\\\tilde{L}_2(t) &= \int_0^t\gamma_1(a)\tilde{L}_1(t-a)B_1(a)\,\mathrm{d}a+G_2(t),\\\tilde{L}_3(t) &= \int_0^t\gamma_2(a)\tilde{L}_1(t-a)B_1(a)\,\mathrm{d}a+\int_0^t\xi(a)\tilde{L}_2(t-a)B_2(a)\,\mathrm{d}a+G_3(t),\end{aligned}\tag{6.5}$$

where

$$\begin{aligned}G_1(t) &= S_0\int_t^\infty\left(\frac{\beta_1(a)\varphi_i(a-t)B_2(a)}{B_2(a-t)+\alpha\varphi_i(a-t)B_2(a)}+\beta_3(a)\varphi_j(a-t)\frac{B_3(a)}{B_3(a-t)}\right)\mathrm{d}a\\&\quad+V_0\int_t^\infty\left(\frac{\beta_2(a)\varphi_i(a-t)B_2(a)}{B_2(a-t)+\alpha\varphi_i(a-t)B_2(a)}+\beta_4(a)\varphi_j(a-t)\frac{B_3(a)}{B_3(a-t)}\right)\mathrm{d}a,\\G_2(t) &= \int_t^\infty\gamma_1(a)\varphi_e(a-t)\frac{B_1(a)}{B_1(a-t)}\,\mathrm{d}a,\\G_3(t) &= \int_t^\infty\gamma_2(a)\varphi_e(a-t)\frac{B_1(a)}{B_1(a-t)}\,\mathrm{d}a+\int_t^\infty\xi(a)\varphi_i(a-t)\frac{B_2(a)}{B_2(a-t)}\,\mathrm{d}a.\end{aligned}$$

Since $(\varphi_e(\cdot),\varphi_i(\cdot),\varphi_j(\cdot))\in\partial\tilde{Y}$, we have $G_i(t)\equiv 0$ $(i=1,2,3)$ for all $t\geq 0$. From (6.5), we obtain

$$\begin{aligned}\tilde{L}_1(t) &= S_0\int_0^t\left(\frac{\beta_1(a)\tilde{L}_2(t-a)B_2(a)}{1+\alpha\tilde{L}_2(t-a)B_2(a)}+\beta_3(a)\tilde{L}_3(t-a)B_3(a)\right)\mathrm{d}a\\&\quad+V_0\int_0^t\left(\frac{\beta_2(a)\tilde{L}_2(t-a)B_2(a)}{1+\alpha\tilde{L}_2(t-a)B_2(a)}+\beta_4(a)\tilde{L}_3(t-a)B_3(a)\right)\mathrm{d}a,\\\tilde{L}_2(t) &= \int_0^t\gamma_1(a)\tilde{L}_1(t-a)B_1(a)\,\mathrm{d}a,\\\tilde{L}_3(t) &= \int_0^t\gamma_2(a)\tilde{L}_1(t-a)B_1(a)\,\mathrm{d}a+\int_0^t\xi(a)\tilde{L}_2(t-a)B_2(a)\,\mathrm{d}a.\end{aligned}\tag{6.6}$$

It is easy to show that system (6.6) has a unique solution $\tilde{L}_i(t)\equiv 0$ $(i=1,2,3)$. From (6.2)–(6.4), we have $\tilde{e}(t,a)=0$, $\tilde{i}(t,a)=0$, $\tilde{j}(t,a)=0$. For $a\geq t$, it follows that

$$\begin{aligned}\|\tilde{e}(t,a)\|_{L^1} &= \left\|\varphi_e(a-t)\frac{B_1(a)}{B_1(a-t)}\right\|_{L^1}\leq e^{-\mu_0 t}\|\varphi_e\|_{L^1},\\\|\tilde{i}(t,a)\|_{L^1} &= \left\|\varphi_i(a-t)\frac{B_2(a)}{B_2(a-t)}\right\|_{L^1}\leq e^{-\mu_0 t}\|\varphi_i\|_{L^1},\\\|\tilde{j}(t,a)\|_{L^1} &= \left\|\varphi_j(a-t)\frac{B_3(a)}{B_3(a-t)}\right\|_{L^1}\leq e^{-\mu_0 t}\|\varphi_j\|_{L^1},\end{aligned}$$

which implies that $\tilde{e}(t,a)=0$, $\tilde{i}(t,a)=0$, $\tilde{j}(t,a)=0$ as $t\to\infty$. Noting that $e(t,a)\leq\tilde{e}(t,a)$, $i(t,a)\leq\tilde{i}(t,a)$ and $j(t,a)\leq\tilde{j}(t,a)$, we have $e(t,a)\to 0$, $i(t,a)\to 0$ and $j(t,a)\to 0$ as $t\to\infty$.

It follows from the first two equations of system (1.5) that $S(t) \to S_0$ and $V(t) \to V_0$ as $t \to \infty$. Thus, E_0 is globally asymptotically stable in ∂Y. □

Theorem 6.2 *If $R_0 > 1$, then the semi-flow $\{\Phi(t)\}_{t\geq 0}$ is uniformly persistent with respect to $(Y, \partial Y)$, i.e., there exists an $\varepsilon > 0$ which is independent of initial values such that $\lim_{t\to\infty} \|\Phi(t,x)\|_X \geq \varepsilon$ for $x \in Y$. Furthermore, there is a compact subset $A_0 \subset Y$ which is a global attractor for $\{\Phi(t,x)\}_{t\geq 0}$ in Y.*

Proof It follows from Theorem 6.1 that E_0 is globally asymptotically stable in ∂Y. Applying Theorem 4.2 in [25], we need only to show that $W^s(E_0) \cap Y = \emptyset$, where

$$W^s(E_0) = \Bigl\{x \in Y : \lim_{t\to\infty} \Phi(t,x) = E_0\Bigr\}.$$

Otherwise, there exists a solution $y \subset Y$ such that $\Phi(t,y) \to E_0$ as $t \to \infty$. In this case, there exists a sequence $\{y_n\} \subset Y$ such that $\|\Phi(t,y_n) - E_0\|_X < 1/n$ for $t \geq 0$. Denote $\Phi(t,y_n) = (S_n(t), V_n(t), e(t,\cdot), i(t,\cdot), j(t,\cdot))$ and $y_n = (S_n(0), V_n(0), e(t,\cdot), i(t,\cdot), j(t,\cdot))$. Since $R_0 > 1$, we can choose n sufficiently large satisfying $S_0 > 1/n$ and $V_0 > 1/n$,

$$\begin{aligned} &\left(S_0 - \frac{1}{n}\right) \int_0^\infty \bigl[K_1K_3(a) + (K_2 + K_1K_7)K_4(a)\bigr]\,\mathrm{d}a \\ &\quad + \left(V_0 - \frac{1}{n}\right) \int_0^\infty \bigl[K_1K_5(a) + (K_2 + K_1K_7)K_6(a)\bigr]\,\mathrm{d}a > 1. \end{aligned} \tag{6.7}$$

For such a $n > 0$, there exists a $T > 0$ such that, for $t > T$, $S_0 - 1/n < S_n(t) < S_0 + 1/n$ and $V_0 - 1/n < V_n(t) < V_0 + 1/n$. Consider the following auxiliary system:

$$\begin{aligned} &\frac{\partial \hat{e}(t,a)}{\partial a} + \frac{\partial \hat{e}(t,a)}{\partial t} = -\theta_1(a)\hat{e}(t,a), \\ &\frac{\partial \hat{i}(t,a)}{\partial a} + \frac{\partial \hat{i}(t,a)}{\partial t} = -\theta_2(a)\hat{i}(t,a), \\ &\frac{\partial \hat{j}(t,a)}{\partial a} + \frac{\partial \hat{j}(t,a)}{\partial t} = -\theta_3(a)\hat{j}(t,a), \\ &\hat{e}(t,0) = \left(S_0 - \frac{1}{n}\right) \int_0^\infty \left(\frac{\beta_1(a)\hat{i}(t,a)}{1 + \alpha\hat{i}(t,a)} + \beta_3(a)\hat{j}(t,a)\right) \mathrm{d}a \\ &\qquad\quad + \left(V_0 - \frac{1}{n}\right) \int_0^\infty \left(\frac{\beta_2(a)\hat{i}(t,a)}{1 + \alpha\hat{i}(t,a)} + \beta_4(a)\hat{j}(t,a)\right) \mathrm{d}a, \\ &\hat{i}(t,0) = \int_0^\infty \gamma_1(a)\hat{e}(t,a)\,\mathrm{d}a, \\ &\hat{j}(t,0) = \int_0^\infty \gamma_2(a)\hat{e}(t,a)\,\mathrm{d}a + \int_0^\infty \xi(a)\hat{i}(t,a)\,\mathrm{d}a. \end{aligned} \tag{6.8}$$

Looking for solutions of system (6.8) of the form $\hat{e}(t,a) = \hat{e}(a)e^{\lambda t}$, $\hat{i}(t,a) = \hat{i}(a)e^{\lambda t}$ and $\hat{j}(t,a) = \hat{j}(a)e^{\lambda t}$, where the functions $\hat{e}(a)$, $\hat{i}(a)$ and $\hat{j}(a)$ will be determined later, we obtain the following linear eigenvalue problem:

$$\frac{\partial \hat{e}(a)}{\partial a} = -\bigl(\lambda + \theta_1(a)\bigr)\hat{e}(a),$$

$$\begin{aligned}
\frac{\partial \hat{i}(a)}{\partial a} &= -\big(\lambda + \theta_2(a)\big)\hat{i}(a), \\
\frac{\partial \hat{j}(a)}{\partial a} &= -\big(\lambda + \theta_3(a)\big)\hat{j}(a), \\
\hat{e}(0) &= \left(S_0 - \frac{1}{n}\right)\int_0^\infty \left(\frac{\beta_1(a)\hat{i}(a)}{1 + \alpha\hat{i}(a)e^{\lambda t}} + \beta_3(a)\hat{j}(a)\right)\mathrm{d}a \\
&\quad + \left(V_0 - \frac{1}{n}\right)\int_0^\infty \left(\frac{\beta_2(a)\hat{i}(a)}{1 + \alpha\hat{i}(a)e^{\lambda t}} + \beta_4(a)\hat{j}(a)\right)\mathrm{d}a, \\
\hat{i}(0) &= \int_0^\infty \gamma_1(a)\hat{e}(a)\,\mathrm{d}a, \\
\hat{j}(0) &= \int_0^\infty \gamma_2(a)\hat{e}(a)\,\mathrm{d}a + \int_0^\infty \xi(a)\hat{i}(a)\,\mathrm{d}a.
\end{aligned} \tag{6.9}$$

Solving the first three equations of system (6.9) yields

$$\begin{aligned}
&\hat{e}(a) = \hat{e}(0)\exp\left[-\int_0^a \big(\lambda + \theta_1(s)\big)\,\mathrm{d}s\right], \qquad \hat{i}(a) = \hat{i}(0)\exp\left[-\int_0^a \big(\lambda + \theta_2(s)\big)\,\mathrm{d}s\right], \\
&\hat{j}(a) = \hat{j}(0)\exp\left[-\int_0^a \big(\lambda + \theta_3(s)\big)\,\mathrm{d}s\right].
\end{aligned} \tag{6.10}$$

Substituting (6.10) into the last three equations of (6.9), we obtain the characteristic equation of system (6.8) at the steady state E_0 as follows:

$$f(\lambda) = 1, \tag{6.11}$$

where

$$\begin{aligned}
f(\lambda) &= \left(S_0 - \frac{1}{n}\right) \\
&\quad \times \int_0^\infty \frac{\beta_1(a)\int_0^\infty \gamma_1(a)\exp[-\int_0^a (\lambda + \theta_1(s))\,\mathrm{d}s]\,\mathrm{d}a\exp[-\int_0^a (\lambda + \theta_2(s))\,\mathrm{d}s]}{1 + \alpha\hat{i}(0)\exp[\lambda t - \int_0^a (\lambda + \theta_2(s))\,\mathrm{d}s]}\,\mathrm{d}a \\
&\quad + \left(S_0 - \frac{1}{n}\right)\int_0^\infty \left\{\beta_3(a)\exp\left[-\int_0^a \big(\lambda + \theta_3(s)\big)\,\mathrm{d}s\right]\right. \\
&\quad \left. \times \int_0^\infty \gamma_2(a)\exp\left[-\int_0^a \big(\lambda + \theta_1(s)\big)\,\mathrm{d}s\right]\mathrm{d}a\right\}\mathrm{d}a \\
&\quad + \left(S_0 - \frac{1}{n}\right)\int_0^\infty \beta_3(a)\exp\left[-\int_0^a \big(\lambda + \theta_3(s)\big)\,\mathrm{d}s\right] \\
&\quad \times \int_0^\infty \xi(a)\exp\left[-\int_0^a \big(\lambda + \theta_2(s)\big)\,\mathrm{d}s\right] \\
&\quad \times \int_0^\infty \gamma_1(a)\exp\left[-\int_0^a \big(\lambda + \theta_1(s)\big)\,\mathrm{d}s\right]\mathrm{d}a\,\mathrm{d}a\,\mathrm{d}a \\
&\quad + \left(V_0 - \frac{1}{n}\right) \\
&\quad \times \int_0^\infty \frac{\beta_2(a)\exp[-\int_0^a (\lambda + \theta_2(s))\,\mathrm{d}s]\int_0^\infty \gamma_1(a)\exp[-\int_0^a (\lambda + \theta_1(s))\,\mathrm{d}s]\,\mathrm{d}a}{1 + \alpha\hat{i}(0)\exp[\lambda t - \int_0^a (\lambda + \theta_2(s))\,\mathrm{d}s]}\,\mathrm{d}a
\end{aligned}$$

$$
\begin{aligned}
&+\left(V_0-\frac{1}{n}\right)\int_0^\infty\left\{\beta_4(a)\exp\left[-\int_0^a\bigl(\lambda+\theta_3(s)\bigr)\,\mathrm{d}s\right]\right.\\
&\times\left.\int_0^\infty\gamma_2(a)\exp\left[-\int_0^a\bigl(\lambda+\theta_1(s)\bigr)\,\mathrm{d}s\right]\mathrm{d}a\right\}\mathrm{d}a\\
&+\left(V_0-\frac{1}{n}\right)\int_0^\infty\beta_4(a)\exp\left[-\int_0^a\bigl(\lambda+\theta_3(s)\bigr)\,\mathrm{d}s\right]\\
&\times\int_0^\infty\xi(a)\exp\left[-\int_0^a\bigl(\lambda+\theta_2(s)\bigr)\,\mathrm{d}s\right]\\
&\times\int_0^\infty\gamma_1(a)\exp\left[-\int_0^a\bigl(\lambda+\theta_1(s)\bigr)\,\mathrm{d}s\right]\mathrm{d}a\,\mathrm{d}a\,\mathrm{d}a.
\end{aligned}
$$

Clearly, we have $\lim_{\lambda\to\infty}f(\lambda)=0$ and

$$
\begin{aligned}
f(0)<{}&\left(S_0-\frac{1}{n}\right)\int_0^\infty\bigl[K_1K_3(a)+(K_2+K_1K_7)K_4(a)\bigr]\,\mathrm{d}a\\
&+\left(V_0-\frac{1}{n}\right)\int_0^\infty\bigl[K_1K_5(a)+(K_2+K_1K_7)K_6(a)\bigr]\,\mathrm{d}a.
\end{aligned}
$$

From (6.7), there exist a $n>0$ and a $T>0$ such that

$$
\begin{aligned}
&\left(S_0-\frac{1}{n}\right)\int_0^\infty\bigl[K_1K_3(a)+(K_2+K_1K_7)K_4(a)\bigr]\,\mathrm{d}a\\
&\quad+\left(V_0-\frac{1}{n}\right)\int_0^\infty\bigl[K_1K_5(a)+(K_2+K_1K_7)K_6(a)\bigr]\,\mathrm{d}a>1.
\end{aligned}
$$

Hence, if $R_0>1$, Eq. (6.11) has at least one positive root. This implies that the solution $(\hat{e}(t,\cdot),\hat{i}(t,\cdot),\hat{j}(t,\cdot))$ of system (6.8) is unbounded. By comparison, the solution $\Phi(t,y_n)$ of system (1.5) is unbounded, which contradicts Proposition 2.1. Therefore, the semi-flow $\Phi(t)_{t\geq 0}$ generated by system (1.5) is uniformly persistent. Furthermore, there is a compact subset $A_0\subset Y$ which is a global attractor for $\Phi(t)_{t\geq 0}$ in Y. This completes the proof. □

7 Global stability

This section is devoted to the global stability of equilibria. Before going into details, we make some preparations.

First, we introduce an important function which is obtained from the linear combination of Volterra-type functions of the form

$$g(x)=x-1-\ln x.$$

Obviously, $g(x)\geq 0$ for $x>0$ and $g'(x)=1-1/x$. Then $g(x)$ has a global minimum at $x=1$ and $g(1)=0$.

Theorem 7.1 *If $R_0<1$, the disease-free steady state E_0 is globally asymptotically stable.*

Proof Define a Lyapunov functional as

$$V_1=V_{11}+V_{12}+V_{13},$$

where

$$V_{11}=S_0g\big(S(t)/S_0\big)+V_0g\big(V(t)/V_0\big),\qquad V_{12}=\int_0^\infty \omega_1(a)e(t,a)\,\mathrm{d}a,$$
$$V_{13}=\int_0^\infty \omega_2(a)i(t,a)\,\mathrm{d}a,\qquad V_{14}=\int_0^\infty \omega_3(a)j(t,a)\,\mathrm{d}a,$$

where

$$\begin{aligned}
\omega_1(a)&=\int_a^\infty \big(\omega_3(0)\gamma_2(x)+\omega_2(0)\gamma_1(x)\big)\exp\biggl(-\int_a^x \theta_1(\tau)d\tau\biggr)dx,\\
\omega_2(a)&=\int_a^\infty \big(S_0\beta_1(x)+V_0\beta_2(x)+\omega_3(0)\xi(x)\big)\exp\biggl(-\int_a^x \theta_2(\tau)d\tau\biggr)dx,\\
\omega_3(a)&=\int_a^\infty \big(S_0\beta_3(x)+V_0\beta_4(x)\big)\exp\biggl(-\int_a^x \theta_3(\tau)d\tau\biggr)dx,
\end{aligned}$$

then

$$\begin{aligned}
&\omega_2(0)\gamma_1(a)+\omega_3(0)\gamma_2(a)+\omega_1'(a)-\theta_1(a)\omega_1(a)=0,\\
&S_0\beta_1(a)+V_0\beta_2+\omega_3(0)\xi(a)+\omega_2'(a)-\theta_2(a)\omega_2(a)=0,\\
&S_0\beta_3(a)+V_0\beta_4+\omega_3'(a)-\theta_3(a)\omega_3(a)=0.
\end{aligned}$$

The derivative of V_{11} along with the solution of system (1.5) can be calculated as

$$\begin{aligned}
\frac{\mathrm{d}V_{11}}{\mathrm{d}t}&=\biggl(1-\frac{S_0}{S(t)}\biggr)\biggl(\Lambda-(\mu+p)S(t)+\eta V(t)-S(t)\int_0^\infty \frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)}\,\mathrm{d}a\\
&\quad+S(t)\int_0^\infty \beta_3(a)j(t,a)\,\mathrm{d}a\biggr)\\
&\quad+\biggl(1-\frac{V_0}{V(t)}\biggr)\biggl(pS(t)-(\mu+\rho+\eta)V(t)-V(t)\int_0^\infty \frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)}\,\mathrm{d}a\\
&\quad-V(t)\int_0^\infty \beta_4(a)j(t,a)\,\mathrm{d}a\biggr)\\
&=\mu S_0\biggl(2-\frac{S(t)}{S_0}-\frac{S_0}{S(t)}\biggr)+(\mu+\rho)V_0\biggl(3-\frac{V(t)}{V_0}-\frac{S_0}{S(t)}-\frac{S(t)V_0}{S_0V(t)}\biggr)\\
&\quad+\eta V_0\biggl(2-\frac{S(t)V_0}{S_0V(t)}-\frac{S_0V(t)}{S(t)V_0}\biggr)\\
&\quad-\big(S(t)-S_0\big)\int_0^\infty \biggl(\frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)}+\beta_3(a)j(t,a)\biggr)\,\mathrm{d}a\\
&\quad-\big(V(t)-V_0\big)\int_0^\infty \biggl(\frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)}+\beta_4(a)j(t,a)\biggr)\,\mathrm{d}a.
\end{aligned}\tag{7.1}$$

The derivative of V_{12} along with the solution of system (1.5) can be calculated as

$$\begin{aligned}
\frac{dV_{12}}{dt}&=\frac{d}{dt}\int_0^t \omega_1(a)e(t-a,0)\exp\biggl(-\int_0^a \theta_1(\tau)d\tau\biggr)\,\mathrm{d}a\\
&\quad+\frac{d}{dt}\int_t^\infty \omega_1(a)\varphi_e(a-t)\exp\biggl(-\int_{a-t}^a \theta_1(\tau)d\tau\biggr)\,\mathrm{d}a.
\end{aligned}$$

Let $r = t - a$, then

$$\begin{aligned}
\frac{\mathrm{d}V_{12}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \omega_1(t-r)e(r,0)\exp\left(-\int_0^{t-r}\theta_1(\tau)d\tau\right)\mathrm{d}r \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty \omega_1(t+r)\varphi_e(r)\exp\left(-\int_r^{t+r}\theta_1(\tau)d\tau\right)\mathrm{d}r \\
&= \omega_1(0)e(t,0) + \int_0^\infty \bigl(\omega'(a) - \theta_1(a)\omega_1(a)\bigr)e(t,a)\,\mathrm{d}a.
\end{aligned} \tag{7.2}$$

Similarly, we can get

$$\begin{aligned}
\frac{\mathrm{d}V_{13}}{\mathrm{d}t} &= \omega_2(0)\int_0^\infty \gamma_1(a)e(t,a)\,\mathrm{d}a + \int_0^\infty \bigl(\omega_2'(a) - \theta_2(a)\omega_2(a)\bigr)i(t,a)\,\mathrm{d}a, \\
\frac{\mathrm{d}V_{14}}{\mathrm{d}t} &= \omega_3(0)\int_0^\infty \bigl(\gamma_2(a)e(t,a) + \xi(a)i(t,a)\bigr)\,\mathrm{d}a \\
&\quad + \int_0^\infty \bigl(\omega_3'(a) - \theta_3(a)\omega_3(a)\bigr)j(t,a)\,\mathrm{d}a.
\end{aligned} \tag{7.3}$$

Combining the (7.1)–(7.3), it is easy to get

$$\begin{aligned}
\frac{\mathrm{d}V_1}{\mathrm{d}t} &= \mu S_0\left(2 - \frac{S_0}{S(t)} - \frac{S(t)}{S_0}\right) + (\mu+\rho)V_0\left(3 - \frac{V(t)}{V_0} - \frac{S_0}{S(t)} - \frac{S(t)V_0}{S_0V(t)}\right) \\
&\quad + \eta V_0\left(2 - \frac{S(t)V_0}{S_0V(t)} - \frac{S_0V(t)}{S(t)V_0}\right) - e(t,0) + \omega_1(0)e(t,0) \\
&\quad + \left(S_0\frac{\beta_1(a)}{1+\alpha i(t,a)} + V_0\frac{\beta_2(a)}{1+\alpha i(t,a)} + \omega_2(0)\xi(a) + \omega_1'(a) - \theta_1(a)\omega_1(a)\right)i(t,a)\,\mathrm{d}a \\
&\quad + \int_0^\infty \bigl(S_0\beta_3(a) + V_0\beta_4(a) + \omega_3'(a) - \theta_3(a)\omega_3(a)\bigr)j(t,a)\,\mathrm{d}a \\
&\quad + \int_0^\infty \bigl(\omega_2(0)\gamma_1(a) + \omega_3(0)\gamma_2(a) + \omega'(a) - \theta_1(a)\omega_1(a)\bigr)e(t,a)\,\mathrm{d}a \\
&\le \mu S_0\left(2 - \frac{S_0}{S(t)} - \frac{S(t)}{S_0}\right) + (\mu+\rho)V_0\left(3 - \frac{V(t)}{V_0} - \frac{S_0}{S(t)} - \frac{S(t)V_0}{S_0V(t)}\right) \\
&\quad + \eta V_0\left(2 - \frac{S(t)V_0}{S_0V(t)} - \frac{S_0V(t)}{S(t)V_0}\right) + (R_0 - 1)e(t,0).
\end{aligned}$$

Therefore, $R_0 \le 1$ ensures that $\mathrm{d}V_1/\mathrm{d}t \le 0$ holds. Furthermore, the strict equality holds if and only if $S = S_0$, $V = V_0$, $e(t,a) = 0$, $i(t,a) = 0$ and $j(t,a) = 0$, simultaneously. Thus, $M_0 = E_0 \subset \Upsilon$ is the largest invariant subset of $\mathrm{d}V_1/\mathrm{d}t = 0$, and by the Lyapunov–LaSalle invariance principle, the steady state E_0 is globally asymptotically stable when $R_0 \le 1$. □

Theorem 7.2 *If $R_0 > 1$, the steady state E^* is globally asymptotically stable.*

Proof Constructing a Lyapunov functional as follows

$$V_2 = V_{21} + V_{22} + V_{23} + V_{24},$$

where

$$\begin{aligned}
V_{21} &= g\big(S(t)/S^*\big) + g\big(V(t)/V^*\big), \\
V_{22} &= \int_0^\infty \frac{S^*\beta_1(a)B_2(a) + V^*\beta_2(a)B_2(a)}{1+\alpha i^*(a)}\,\mathrm{d}a \int_0^\infty q_1(a)g\big(e(t,a)/e^*(a)\big)\,\mathrm{d}a \\
&\quad + \int_0^\infty \big(S^*\beta_3(a)B_3(a) + V^*\beta_4(a)B_3(a)\big)\,\mathrm{d}a\bigg(\int_0^\infty q_2(a)g\big(e(t,a)/e^*(a)\big)\,\mathrm{d}a \\
&\quad + \int_0^\infty q_3(a)g\big(i(t,a)/i^*(a)\big)\,\mathrm{d}a\bigg), \\
V_{23} &= \int_0^\infty q_i(a)g\big(i(t,a)\big(1+\alpha i^*(a)\big)/i^*(a)\big(1+\alpha i(t,a)\big)\big)\,\mathrm{d}a, \\
V_{24} &= \int_0^\infty q_j(a)g\big(j(t,a)/j^*(a)\big)\,\mathrm{d}a,
\end{aligned}$$

where

$$\begin{aligned}
q_1(a) &= \int_a^\infty \gamma_1(\sigma)e^*(\sigma)\,\mathrm{d}\sigma, \qquad q_2(a) = \int_a^\infty \gamma_2(\sigma)e^*(\sigma)\,\mathrm{d}\sigma, \\
q_3(a) &= \int_a^\infty \xi(\sigma)i^*(\sigma)\,\mathrm{d}\sigma, \qquad q_i(a) = \int_a^\infty \frac{\beta_1(\sigma)S^* + \beta_2(\sigma)V^*}{1+\alpha i^*(\sigma)} i^*(\sigma)\,\mathrm{d}\sigma, \\
q_j(a) &= \int_a^\infty \big(\beta_3(\sigma)S^* + \beta_4(\sigma)V^*\big)j^*(\sigma)\,\mathrm{d}\sigma.
\end{aligned}$$

Calculating the derivative of V_{21} along with the solution of system (1.5), we have

$$\begin{aligned}
\frac{\mathrm{d}V_{21}}{\mathrm{d}t} &= \left(1-\frac{S^*}{S(t)}\right)\bigg(\Lambda - (\mu+p)S(t) + \eta V(t) \\
&\quad - S(t)\int_0^\infty \left(\frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)} + \beta_3(a)j(t,a)\right)\mathrm{d}a\bigg) \\
&\quad + \left(1-\frac{V^*}{V(t)}\right)\bigg(pS(t) - (\mu+\rho+\eta)V(t) - V(t)\int_0^\infty \frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)}\,\mathrm{d}a \\
&\quad - V(t)\int_0^\infty \beta_4(a)j(t,a)\,\mathrm{d}a\bigg) \\
&= \mu S^*\left(2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)}\right) + \left(\mu+\rho+\int_0^\infty \left(\frac{\beta_2(a)i^*(a)}{1+\alpha i^*(a)} + \beta_4(a)j^*(a)\right)\mathrm{d}a\right)V^* \\
&\quad \times \left(3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right) + \eta V^*\left(2-\frac{S^*V(t)}{S(t)V^*}-\frac{S(t)V^*}{S^*V(t)}\right) \\
&\quad + S^*\int_0^\infty \left(\frac{\beta_1(a)i^*(a)}{1+\alpha i^*(a)} + \beta_3(a)j^*(a)\right)\mathrm{d}a \\
&\quad - S(t)\int_0^\infty \left(\frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)} + \beta_3(a)j(t,a)\right)\mathrm{d}a \\
&\quad - \frac{S^*}{S(t)}S^*\int_0^\infty \left(\frac{\beta_1(a)i^*(a)}{1+\alpha i^*(a)} + \beta_3(a)j^*(a)\right)\mathrm{d}a \\
&\quad + S^*\int_0^\infty \left(\frac{\beta_1(a)i(t,a)}{1+\alpha i(t,a)} + \beta_3(a)j(t,a)\right)\mathrm{d}a - V^*\int_0^\infty \bigg(\frac{\beta_2 i^*(a)}{1+\alpha i^*(a)}
\end{aligned}$$

$$
\begin{aligned}
&+\beta_4(a)j^*(a)\Big)\,\mathrm{d}a + V^*\int_0^\infty\left(\frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)}+\beta_4(a)j(t,a)\right)\mathrm{d}a\\
&-V(t)\int_0^\infty\left(\frac{\beta_2(a)i(t,a)}{1+\alpha i(t,a)}+\beta_4(a)j(t,a)\right)\mathrm{d}a\\
&+V(t)\int_0^\infty\left(\frac{\beta_2(a)i^*(a)}{1+\alpha i^*(a)}+\beta_4(a)j^*(a)\right)\mathrm{d}a.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{\mathrm{d}V_{21}}{\mathrm{d}t} ={}& \mu S^*\left(2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)}\right)+\left(\mu+\rho+\int_0^\infty\left(\frac{\beta_2(a)i^*(a)}{1+\alpha i^*(a)}\right.\right.\\
&\left.\left.+\beta_4(a)j^*(a)\right)\mathrm{d}a\right)V^*\left(3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right)\\
&+\eta V^*\left(2-\frac{S^*V(t)}{S(t)V^*}-\frac{S(t)V^*}{S^*V(t)}\right)+\int_0^\infty\left(\frac{\beta_1(a)S^*i^*(a)}{1+\alpha i^*(a)}+\beta_3(a)S^*j^*(a)\right)\mathrm{d}a\\
&-\frac{S(t)}{S^*}\int_0^\infty\left(\frac{\beta_1(a)S^*i(t,a)}{1+\alpha i(t,a)}+\beta_3(a)S^*j(t,a)\right)\mathrm{d}a\\
&-\frac{S^*}{S(t)}\int_0^\infty\left(\frac{\beta_1(a)S^*i^*(a)}{1+\alpha i^*(a)}+\beta_3(a)S^*j^*(a)\right)\mathrm{d}a\\
&+\int_0^\infty\left(\frac{\beta_1(a)S^*i(t,a)}{1+\alpha i(t,a)}+\beta_3(a)S^*j(t,a)\right)\mathrm{d}a-\int_0^\infty\left(\frac{\beta_2(a)V^*i^*(a)}{1+\alpha i^*(a)}\right.\\
&\left.+\beta_4(a)V^*j^*(a)\right)\mathrm{d}a+\int_0^\infty\left(\frac{\beta_2(a)V^*i(t,a)}{1+\alpha i^*(a)}+\beta_4(a)V^*j(t,a)\right)\mathrm{d}a\\
&-\frac{V(t)}{V^*}\int_0^\infty\left(\frac{\beta_2(a)V^*i(t,a)}{1+\alpha i(t,a)}+\beta_4(a)V^*j(t,a)\right)\mathrm{d}a\\
&+\frac{V(t)}{V^*}\int_0^\infty\left(\frac{\beta_2(a)V^*i^*(a)}{1+\alpha i^*(a)}+\beta_4(a)V^*j^*(a)\right)\mathrm{d}a. \qquad (7.4)
\end{aligned}
$$

The derivative of V_{22}, V_{23} and V_{24} can be calculated as follows:

$$
\begin{aligned}
\frac{\mathrm{d}V_{22}}{\mathrm{d}t} ={}& \int_0^\infty\frac{S^*\beta_1(a)B_2(a)+V^*\beta_2(a)B_2(a)}{1+\alpha i^*(a)}\,\mathrm{d}a\int_0^\infty\gamma_1(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}-\frac{e(t,a)}{e^*(a)}\right.\\
&\left.+\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a\\
&+\int_0^\infty\left(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\right)\mathrm{d}a\int_0^\infty\gamma_2(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}-\frac{e(t,a)}{e^*(a)}\right.\\
&\left.+\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a\\
&+\int_0^\infty\left(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\right)\mathrm{d}a\int_0^\infty\xi(a)i^*(a)\left\{\frac{i(t,0)}{i^*(0)}-\frac{i(t,a)}{i^*(a)}\right.\\
&\left.+\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}\mathrm{d}a, \qquad (7.5)\\
\frac{\mathrm{d}V_{23}}{\mathrm{d}t} ={}& \int_0^\infty\frac{(\beta_1(a)S^*+\beta_2(a)V^*)i^*(a)}{1+\alpha i^*(a)}\left\{\frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}-\frac{i(t,a)(1+\alpha i^*(a))}{i^*(a)(1+\alpha i(t,a))}\right.\\
&\left.+\ln\frac{i(t,a)(1+\alpha i^*(a))}{i^*(a)(1+\alpha i(t,a))}-\ln\frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\right\}\mathrm{d}a, \qquad (7.6)
\end{aligned}
$$

$$\frac{\mathrm{d}V_{24}}{\mathrm{d}t} = \int_0^\infty \big(\beta_3(a)S^* + \beta_4(a)V^*\big)j^*(a)\left\{\frac{j(t,0)}{j^*(0)} - \frac{j(t,a)}{j^*(a)} + \ln\frac{j(t,a)}{j^*(a)} - \ln\frac{j(t,0)}{j^*(0)}\right\}\mathrm{d}a. \quad (7.7)$$

Combining (7.4)–(7.7), we get

$$\begin{aligned}
\frac{\mathrm{d}V_2}{\mathrm{d}t} &= \mu S^*\left(2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)}\right) + \left(\mu + \rho + \int_0^\infty \left(\frac{\beta_2(a)i^*(a)}{1+\alpha i^*(a)} + \beta_4(a)j^*(a)\right)\mathrm{d}a\right)V^* \\
&\quad \times \left(3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)}\right) + \eta V^*\left(2 - \frac{S^*V(t)}{S(t)V^*} - \frac{S(t)V^*}{S^*V(t)}\right) \\
&\quad + \int_0^\infty \frac{\beta_1(a)S^*i^*(a)}{1+\alpha i^*(a)}\left\{1 - \frac{S(t)i(t,a)(1+\alpha i^*(a))}{S^*i^*(a)(1+\alpha i(t,a))} - \frac{S^*}{S(t)} + \frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\right. \\
&\quad \left. + \ln\frac{i(t,a)(1+\alpha i^*(a))}{i^*(a)(1+\alpha i(t,a))} - \ln\frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\right\}\mathrm{d}a \\
&\quad + \int_0^\infty \frac{\beta_2(a)V^*i^*(a)}{1+\alpha i^*(a)}\left\{-1 - \frac{V(t)i(t,a)(1+\alpha i^*(a))}{V^*i^*(a)(1+\alpha i(t,a))} + \frac{V^*}{V(t)} + \frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\right. \\
&\quad \left. + \ln\frac{i(t,a)(1+\alpha i^*(a))}{i^*(a)(1+\alpha i(t,a))} - \ln\frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\right\}\mathrm{d}a \\
&\quad + \int_0^\infty \beta_3(a)S^*j^*(a)\left\{1 - \frac{S(t)j(t,a)}{S^*j^*(a)} - \frac{S^*}{S(t)} + \frac{j(t,0)}{j^*(0)} + \ln\frac{j(t,a)}{j^*(a)} - \ln\frac{j(t,0)}{j^*(0)}\right\}\mathrm{d}a \\
&\quad + \int_0^\infty \beta_4(a)V^*j^*(a)\left\{-1 - \frac{V(t)j(t,a)}{V^*j^*(a)} + \frac{V^*}{V(t)} + \frac{j(t,0)}{j^*(0)}\right. \\
&\quad \left. + \ln\frac{j(t,a)}{j^*(a)} - \ln\frac{j(t,0)}{j^*(0)}\right\}\mathrm{d}a \\
&\quad + \int_0^\infty \frac{S^*\beta_1(a)B_2(a) + V^*\beta_2(a)B_2(a)}{1+\alpha i^*(a)}\,\mathrm{d}a \int_0^\infty \gamma_1(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)} - \frac{e(t,a)}{e^*(a)}\right. \\
&\quad \left. + \ln\frac{e(t,a)}{e^*(a)} - \ln\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a \\
&\quad + \int_0^\infty \big(S^*\beta_3(a)B_3(a) + V^*\beta_4(a)B_3(a)\big)\,\mathrm{d}a \int_0^\infty \gamma_2(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)} - \frac{e(t,a)}{e^*(a)}\right. \\
&\quad \left. + \ln\frac{e(t,a)}{e^*(a)} - \ln\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a \\
&\quad + \int_0^\infty \big(S^*\beta_3(a)B_3(a) + V^*\beta_4(a)B_3(a)\big)\,\mathrm{d}a \int_0^\infty \xi(a)i^*(a)\left\{\frac{i(t,0)}{i^*(0)} - \frac{i(t,a)}{i^*(a)}\right. \\
&\quad \left. + \ln\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)}\right\}\mathrm{d}a \\
&= \mu S^*\left(2 - \frac{S(t)}{S^*} - \frac{S^*}{S(t)}\right) + \left(\mu + \rho + \int_0^\infty \left(\frac{\beta_2(a)i^*(a)}{1+\alpha i^*(a)} + \beta_4(a)j^*(a)\right)\mathrm{d}a\right)V^* \\
&\quad \times \left(3 - \frac{V(t)}{V^*} - \frac{S^*}{S(t)} - \frac{S(t)V^*}{S^*V(t)}\right) + \eta V^*\left(2 - \frac{S^*V(t)}{S(t)V^*} - \frac{S(t)V^*}{S^*V(t)}\right) \\
&\quad + \int_0^\infty \frac{\beta_1(a)S^*i^*(a)}{1+\alpha i^*(a)}\left\{1 - \frac{S^*}{S(t)} + \ln\frac{i(t,a)(1+\alpha i^*(a))}{i^*(a)(1+\alpha i(t,a))}\right. \\
&\quad \left. - \ln\frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\right\}\mathrm{d}a
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty \frac{\beta_2(a)V^*i^*(a)}{1+\alpha i^*(a)}\left\{-1+\frac{V^*}{V(t)}+\ln\frac{i(t,a)(1+\alpha i^*(a))}{i^*(a)(1+\alpha i(t,a))}\right.\\
&\left.-\ln\frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\right\}\mathrm{d}a\\
&+\int_0^\infty \beta_3(a)S^*j^*(a)\left\{1-\frac{S^*}{S(t)}+\ln\frac{j(t,a)}{j^*(a)}-\ln\frac{j(t,0)}{j^*(0)}\right\}\mathrm{d}a\\
&+\int_0^\infty \beta_4(a)V^*j^*(a)\left\{-1+\frac{V^*}{V(t)}+\ln\frac{j(t,a)}{j^*(a)}-\ln\frac{j(t,0)}{j^*(0)}\right\}\mathrm{d}a\\
&+\int_0^\infty \frac{S^*\beta_1(a)B_2(a)+V^*\beta_2(a)B_2(a)}{1+\alpha i^*(a)}\,\mathrm{d}a\\
&\times\int_0^\infty \gamma_1(a)e^*(a)\left\{\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a\\
&+\int_0^\infty \bigl(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\bigr)\,\mathrm{d}a\\
&\times\int_0^\infty \gamma_2(a)e^*(a)\left\{\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a\\
&+\int_0^\infty \bigl(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\bigr)\,\mathrm{d}a\\
&\times\int_0^\infty \xi(a)i^*(a)\left\{\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}\mathrm{d}a\\
&+\int_0^\infty \frac{S^*\beta_1(a)B_2(a)+V^*\beta_2(a)B_2(a)}{1+\alpha i^*(a)}\,\mathrm{d}a\int_0^\infty \gamma_1(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a\\
&+\int_0^\infty \bigl(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\bigr)\,\mathrm{d}a)\int_0^\infty \gamma_2(a)e^*(a)\left\{\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a\\
&+\int_0^\infty \bigl(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\bigr)\,\mathrm{d}a\int_0^\infty \xi(a)i^*(a)\left\{\frac{i(t,0)}{i^*(0)}\right\}\mathrm{d}a\\
&-\int_0^\infty \frac{\beta_1(a)S^*i^*(a)}{1+\alpha i^*(a)}\frac{S(t)i(t,a)(1+\alpha i^*(a))}{S^*i^*(a)(1+\alpha i(t,a))}\,\mathrm{d}a\\
&-\int_0^\infty \frac{\beta_2V^*(a)i^*(a)}{1+\alpha i^*(a)}\frac{V(t)i(t,a)(1+\alpha i^*(a))}{V^*i^*(a)(1+\alpha i(t,a))}\,\mathrm{d}a\\
&-\int_0^\infty \beta_3(a)S^*j^*(a)\frac{S(t)j(t,a)}{S^*j^*(a)}\,\mathrm{d}a-\int_0^\infty \beta_4(a)V^*j^*(a)\frac{V(t)j(t,a)}{V^*j^*(a)}\,\mathrm{d}a\\
&-\bigl(K_3S^*+K_5V^*\bigr)i(t,0)-\bigl(K_4S^*+K_6V^*\bigr)j(t,0)\\
&+\int_0^\infty \frac{\beta_1(a)S^*+\beta_2(a)V^*}{1+\alpha i^*(a)}i^*(a)\frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\,\mathrm{d}a\\
&+\int_0^\infty \bigl(\beta_3(a)S^*+\beta_4(a)V^*\bigr)j^*(a)\frac{j(t,0)}{j^*(0)}\,\mathrm{d}a.
\end{aligned}
$$

It is easy to see that the last 11 terms of the above equation equal 0. Then we have

$$
\begin{aligned}
\frac{\mathrm{d}V_2}{\mathrm{d}t}\le{}&\mu S^*\left(2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)}\right)+(\mu+\rho)V^*\left(3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right)\\
&+\eta V^*\left(2-\frac{S^*V(t)}{S(t)V^*}-\frac{S(t)V^*}{S^*V(t)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^\infty \left(\frac{\beta_2(a)V^*i^*(a)}{1+\alpha i^*(a)}+\beta_4(a)V^*j^*(a)\right)\left\{3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right\}\mathrm{d}a \\
&+\int_0^\infty \frac{\beta_1(a)S^*i^*(a)}{1+\alpha i^*(a)}\left\{1-\frac{S^*}{S(t)}+\ln\frac{i(t,a)(1+\alpha i^*(a))}{i^*(a)(1+\alpha i(t,a))}\right. \\
&\left.-\ln\frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\right\}\mathrm{d}a \\
&+\int_0^\infty \frac{\beta_2(a)V^*i^*(a)}{1+\alpha i^*(a)}\left\{-1+\frac{V^*}{V(t)}+\ln\frac{i(t,a)(1+\alpha i^*(a))}{i^*(a)(1+\alpha i(t,a))}\right. \\
&\left.-\ln\frac{i(t,0)(1+\alpha i^*(0))}{i^*(0)(1+\alpha i(t,0))}\right\}\mathrm{d}a \\
&+\int_0^\infty \beta_3(a)S^*j^*(a)\left\{1-\frac{S^*}{S(t)}+\ln\frac{j(t,a)}{j^*(a)}-\ln\frac{j(t,0)}{j^*(0)}\right\}\mathrm{d}a \\
&+\int_0^\infty \beta_4(a)V^*j^*(a)\left\{-1+\frac{V^*}{V(t)}+\ln\frac{j(t,a)}{j^*(a)}-\ln\frac{j(t,0)}{j^*(0)}\right\}\mathrm{d}a \\
&+\int_0^\infty \frac{S^*\beta_1(a)B_2(a)+V^*\beta_2(a)B_2(a)}{1+\alpha i^*(a)}\,\mathrm{d}a \\
&\times\int_0^\infty \gamma_1(a)e^*(a)\left\{\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a \\
&+\int_0^\infty \big(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\big)\,\mathrm{d}a \\
&\times\int_0^\infty \gamma_2(a)e^*(a)\left\{\ln\frac{e(t,a)}{e^*(a)}-\ln\frac{e(t,0)}{e^*(0)}\right\}\mathrm{d}a \\
&+\int_0^\infty \big(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\big)\,\mathrm{d}a \\
&\times\int_0^\infty \xi(a)i^*(a)\left\{\ln\frac{i(t,a)}{i^*(a)}-\ln\frac{i(t,0)}{i^*(0)}\right\}\mathrm{d}a.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
\frac{\mathrm{d}V_2}{\mathrm{d}t} &\le \mu S^*\left(2-\frac{S(t)}{S^*}-\frac{S^*}{S(t)}\right)+(\mu+\rho)V^*\left(3-\frac{V(t)}{V^*}-\frac{S^*}{S(t)}-\frac{S(t)V^*}{S^*V(t)}\right) \\
&+\eta V^*\left(2-\frac{S^*V(t)}{S(t)V^*}-\frac{S(t)V^*}{S^*V(t)}\right) \\
&-\int_0^\infty \left[\frac{\beta_1(a)S^*+\beta_2(a)V^*}{1+\alpha i^*(a)}i^*(a)+\big(\beta_3(a)S^*+\beta_4(a)V^*\big)j^*(a)\right]g\left(\frac{S^*}{S(t)}\right)\mathrm{d}a \\
&-\int_0^\infty \frac{S^*\beta_1(a)B_2(a)+V^*\beta_2(a)B_2(a)}{1+\alpha i^*(a)}\,\mathrm{d}a\int_0^\infty \gamma_1(a)e^*(a)g\left(\frac{e(t,a)i^*(0)}{e^*(a)i(t,0)}\right)\mathrm{d}a \\
&-\int_0^\infty \big(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\big)\,\mathrm{d}a\int_0^\infty \gamma_2(a)e^*(a)g\left(\frac{e(t,a)j^*(0)}{e^*(a)j(t,0)}\right)\mathrm{d}a \\
&-\int_0^\infty \big(S^*\beta_3(a)B_3(a)+V^*\beta_4(a)B_3(a)\big)\,\mathrm{d}a\int_0^\infty \xi(a)i^*(a)g\left(\frac{i(t,a)j^*(0)}{i^*(a)j(t,0)}\right)\mathrm{d}a \\
&-\int_0^\infty \frac{\beta_1(a)S^*i^*(a)}{1+\alpha i^*(a)}g\left(\frac{S(t)i(t,a)e^*(0)(1+\alpha i^*(a))}{S^*i^*(a)e(t,0)(1+\alpha i(t,a))}\right)\mathrm{d}a \\
&-\int_0^\infty \frac{\beta_2(a)V^*i^*(a)}{1+\alpha i^*(a)}g\left(\frac{V(t)i(t,a)e^*(0)(1+\alpha i^*(a))}{V^*i^*(a)e(t,0)(1+\alpha i(t,a))}\right)\mathrm{d}a
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^\infty \beta_3(a) S^* j^*(a) g\left(\frac{S(t) j(t,a) e^*(0)}{S^* j^*(a) e(t,0)}\right) \mathrm{d}a \\
&- \int_0^\infty \beta_4(a) V^* j^*(a) g\left(\frac{V(t) j(t,a) e^*(0)}{V^* j^*(a) e(t,0)}\right) \mathrm{d}a \\
&- \int_0^\infty \frac{\beta_2(a) V^* i^*(a)}{1+\alpha i^*(a)} g\left(\frac{S(t) V^*}{S^* V(t)}\right) \mathrm{d}a \\
&- \int_0^\infty \beta_4(a) V^* j^*(a) g\left(\frac{S(t) V^*}{S^* V(t)}\right) \mathrm{d}a.
\end{aligned}
$$

Hence, $\mathrm{d}V_2/\mathrm{d}t \le 0$ holds. Furthermore, the strict equality holds if and only if $S = S^*$, $V = V^*$, $e(t,a) = e^*(a)$, $i(t,a) = i^*(a)$, $j(t,a) = j^*(a)$. Thus, $M^* = \{E^*\} \subset \Omega$ is the largest invariant subset of $\mathrm{d}V_2/\mathrm{d}t = 0$, and by the Lyapunov–LaSalle invariance principle, when $R_0 > 1$, the steady state E^* is globally asymptotically stable. This completes the proof. □

8 Discussion

An age-structured HBV model with saturating incidence has been proposed here to incorporate patients with acute hepatitis B and chronic hepatitis B. By mathematical analysis, the dynamic behavior of system (1.5) was shown to be determined completely by the basic reproduction number R_0: disease free steady state E_0 is locally and globally asymptotically stable if $R_0 < 1$; endemic steady state E^* is locally and globally asymptotically stable if $R_0 > 1$. To place the model on more sound biological grounds, we considered the fact that the vaccines may lose efficacy and the infection may reach a saturating state. Next, we will focus on the numerical simulations of such a complex partial differential equations (PDEs) model.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YL and RX have contributed equally to the writing of this paper except for Sect. 6. JL made major contribution to the revised paper and Sect. 6. All authors read and approved the final manuscript.

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