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Multi-parameter second-order impulsive indefinite boundary value problems

Lishuai Jiao[1] and Xuemei Zhang[1*]

*Correspondence: zxm74@sina.com
[1]Department of Mathematics and Physics, North China Electric Power University, Beijing, People's Republic of China

Abstract

We consider the solvable intervals of three positive parameters λ_i ($i = 1, 2, 3$) in which the second-order impulsive boundary value problem

$$\begin{cases} -x'' = a(t)xy + \lambda_1 g(t)f(x), & 0 < t < 1, t \neq t_k, \\ -y'' = \lambda_2 b(t)x, & 0 < t < 1, \\ \Delta x|_{t=t_k} = \lambda_3 I_k(x(t_k)), & k = 1, 2, \dots, n, \\ x(0) = 0, \qquad x'(1) = 0, \\ y(0) = y(1) = 0 \end{cases}$$

admits at least two positive solutions. The main interest is that the weight functions $a(t)$, $b(t)$, and $g(t)$ change sign on $[0, 1]$, λ_i ($i = 1, 2, 3$) $\not\equiv 1$, and $I_k \not\equiv 0$ ($k = 1, 2, \dots, n$). We will obtain several interesting results: there exist positive constants $\lambda^*, \lambda_*, \lambda_i^*$ ($i = 1, 3$), λ_i^{**} ($i = 1, 2, 3$) and α with $\alpha \neq 1$ such that: (i) if $\alpha > 1$, then for $\lambda_i \in [\lambda_i^*, +\infty)$ ($i = 1, 3$) and $\lambda_2 \in [\lambda_*, \lambda^*]$, the above boundary value problem admits at least two positive solutions; (ii) if $0 < \alpha < 1$, then for $\lambda_i \in (0, \lambda_i^{**}]$ ($i = 1, 2, 3$), the above boundary value problem admits at least two positive solutions.

Keywords: Solvable intervals of three parameters; Positive solutions; Indefinite impulsive boundary value problem; Fixed point technique

1 Introduction

It is well established that impulsive differential equations can provide a natural description of observed evolution processes. Therefore it is regarded as an important mathematical tool to better understand many real world problems in applied sciences. On details and explanations, we refer the readers to the texts [1–4]. Specially, Nieto and O'Regan [5] pointed out that in a second order differential equation $u'' = f(t, u, u')$, one usually considers impulses in the position u and the velocity u'. However, in the motion of spacecraft one has to consider instantaneous impulses depending on the position that result in jump discontinuities in velocity, but with no change in position [6]. The impulses only on velocity occur also in impulsive mechanics [7]. Recently, a great interest has been shown by many authors in the subject of impulsive boundary value problems (IBVPs) by applying different approaches, for example, see [8–28] and the references cited therein.

At the same time, we notice that a class of indefinite weight problems have attracted the attention of Ma and Han [29], López-Gómez and Tellini [30], Boscaggin and Zanolin [31,

32], Sovrano and Zanolin [33], Bravo and Torres [34], Wang and An [35], and Yao [36]. In [29], Ma and Han considered the following boundary value problem:

$$
\begin{cases}
u'' + \lambda a(t)f(u) = 0, & 0 < t < 1, \\
u(0) = u(1) = 0,
\end{cases}
$$

where $a \in C[0,1]$ may change sign, λ is a parameter. They proved the existence, multiplicity, and stability of positive solutions for the above problem by applying bifurcation techniques.

In [33], applying the shooting method, Sovrano and Zanolin presented a multiplicity result of positive solutions for the Neumann problem

$$
\begin{cases}
u'' + a(t)f(u) = 0, & 0 < t < 1, \\
u(t) > 0, & t \in [0, T], \\
u'(0) = u'(T) = 0,
\end{cases}
$$

where the weight function $a \in C[0,1]$ has indefinite sign.

Recently, Wang and An in [35] dealt with the existence and multiplicity of positive solutions for the second-order differential system

$$
\begin{cases}
-u'' = a(t)\varphi u + h(t)f(u), & 0 < t < 1, \\
-\varphi'' = b(t)u, & 0 < t < 1, \\
u(0) = u(1) = 0, \\
\varphi(0) = \varphi(1) = 0,
\end{cases}
$$

where $a(t)$, $b(t)$, $g(t)$ are allowed to change sign on $[0,1]$.

Very recently, López-Gómez, Omari, and Rivetti [37, 38] studied a class of quasilinear indefinite problems

$$
\begin{cases}
-\left(\frac{u'}{\sqrt{1+u'^2}}\right)' = \lambda a(x)f(u), \\
u'(0) = u'(1) = 0,
\end{cases}
$$

where $\lambda \in \mathcal{R}$ is a parameter, $a \in L^1[0,1]$ changes sign. They derived many results on the existence and the multiplicity of positive (regular) solutions by applying topological degree and variational approach, respectively.

For all we know, in literature there are no articles on multiple positive solutions for the analogous of indefinite impulsive problems with multiple parameters. More precisely, the study of λ_i $(i = 1, 2, 3) \not\equiv 1$, and $a(t)$, $b(t)$, and $g(t)$ changing sign on $[0,1]$ is still open for the second order impulsive boundary value problem

$$
\begin{cases}
-x'' = a(t)xy + \lambda_1 g(t)f(x), & 0 < t < 1, t \neq t_k, \\
-y'' = \lambda_2 b(t)x, & 0 < t < 1, \\
\Delta x|_{t=t_k} = \lambda_3 I_k(x(t_k)), & k = 1, 2, \dots, n, \\
x(0) = 0, \quad x'(1) = 0, \\
y(0) = y(1) = 0,
\end{cases}
\tag{1.1}
$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$ are three parameters, $a(t)$, $b(t)$, $g(t)$ change sign on $[0, 1]$, t_k $(k = 1, 2, \ldots, n)$ (where n is a fixed positive integer) are fixed points with $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_n < t_{n+1} = 1$, $\Delta x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e., $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right-hand limit and left-hand limits of $x(t)$ at $t = t_k$, respectively. Moreover, throughout this paper, we assume that a, g, b, f and I_k $(k = 1, 2, \ldots, n)$ satisfy

(H_1) $a, g, b : [0, 1] \to (-\infty, +\infty)$ and continuous, and there exists a constant $\xi \in (0, 1)$ such that

$$\begin{cases} a(t), & g(t), & b(t) \geq 0, & \forall t \in [0, \xi], \\ a(t)), & g(t), & b(t) \leq 0, & \forall t \in [\xi, 1]. \end{cases}$$

Moreover, $a(t)$, $g(t)$, $b(t)$ do not vanish identically on any subinterval of $[0, 1]$.

(H_2) $f \in C(\mathcal{R}^+, \mathcal{R}^+)$, and $f(s) > 0$ for $s > 0$, where $\mathcal{R}^+ = [0, +\infty)$.

(H_3) $I_k \in C(\mathcal{R}^+, \mathcal{R}^+)$, and $I_k(s) > 0$ for $s > 0$, $k = 1, 2, \ldots, n$.

We denote

$a^+(t) = \max\{a(t), 0\}$, $a^-(t) = -\min\{a(t), 0\}$ such that $a(t) = a^+(t) - a^-(t)$, $\forall t \in [0, 1]$,

$g^+(t) = \max\{g(t), 0\}$, $g^-(t) = -\min\{g(t), 0\}$ such that $g(t) = g^+(t) - g^-(t)$, $\forall t \in [0, 1]$,

$b^+(t) = \max\{b(t), 0\}$, $b^-(t) = -\min\{b(t), 0\}$ such that $b(t) = b^+(t) - b^-(t)$, $\forall t \in [0, 1]$.

Some special cases of problem (1.1) have been investigated. For example, Sovrano and Zanolin [33] considered problem (1.1) when $\lambda_1 \equiv 1$, $\lambda_2 = 0$, and $I_k = 0$ $(k = 1, 2, \ldots, n)$. By applying the shooting method, they presented a multiplicity result of positive solutions for problem (1.1). In [35], Wang and An studied problem (1.1) with $\lambda_1 \equiv 1$, $\lambda_2 \equiv 1$, and $I_k = 0$ $(k = 1, 2, \ldots, n)$. Using a fixed point technique, they showed the existence of positive solutions for problem (1.1).

Remark 1.1 Comparing with other related indefinite problems [29–32], the main features of this paper are as follows.

(i) $I_k \neq 0$ $(k = 1, 2, \ldots, n)$ is considered.

(ii) $\lambda_i > 0$ is available, not only $\lambda_i \equiv 1$, $i = 1, 2, 3$.

(iii) The method used in the present paper is completely different from those in [29–32].

Remark 1.2 It is very difficult to obtain the solvable intervals of three positive parameters λ_i $(i = 1, 2, 3)$, especially for parameter λ_2. For details, see the proof of part I of Theorem 3.1.

It is well accepted that fixed point theorems in cones have been instrumental in showing the existence, multiplicity of positive solutions of various boundary value problems for differential equations. See, for instance, [39–46] and the references therein. In this paper, we will use Krasnoselskii's fixed point theorem in a cone to investigate the existence and multiplicity of positive solutions of problem (1.1). We remark that this is probably the first time that multiple positive solutions of impulsive differential system with indefinite weight and multiple parameters have been studied.

The rest of the present article is organized as follows. In Sect. 2 we give some preliminary results to be used in the subsequent sections. Section 3 is devoted to stating and proving the main results. Several related remarks are given in Sect. 4, and we give an example to illustrate the main results in the final section.

2 Preliminaries

In this section, we give some preliminary results for the convenience of later use and reference.

Let $J = [0,1]$, $J' = J \setminus \{t_1, t_2, \ldots, t_n\}$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$, $k = 1, 2, \ldots, n$, and

$$PC[0,1] = \{x : x \text{ is continuous at } t \neq t_k, x(t_k^-) = x(t_k) \text{ and } x(t_k^+) \text{ exists}, k = 1, 2, \ldots, n\}.$$

Then $PC[0,1]$ is a real Banach space with the norm

$$\|x\|_{PC} = \max_{t \in J} |x(t)|.$$

By a positive solution of problem (1.1), we understand a pair of functions (x, y) with $x \in C^2(J') \cap PC[0,1]$, $y \in C^2(J) \cap C[0,1]$, if (x, y) satisfies problem (1.1), and $x \geq 0$, $t \in J'$, $y \geq 0$, $t \in J$, $x, y \not\equiv 0$.

It is clear that problem (1.1) is equivalent to the following two boundary value problems:

$$\begin{cases} -x'' = a(t)xy + \lambda_1 g(t)f(x), & 0 < t < 1, t \neq t_k, \\ \Delta x|_{t=t_k} = \lambda_3 I_k(x(t_k)), & k = 1, 2, \ldots, n, \\ x(0) = 0, \quad x'(1) = 0, \end{cases} \tag{2.1}$$

and

$$\begin{cases} -y'' = \lambda_2 b(t)x, & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases} \tag{2.2}$$

Lemma 2.1 *Assume that* (H_1)–(H_3) *hold. Then problem* (2.1) *has a solution* x, *and* x *can be expressed in the form*:

$$x(t) = \int_0^1 G(t,s)a(s)x(s)y(s)\, ds + \lambda_1 \int_0^1 G(t,s)g(s)f(x(s))\, ds$$

$$+ \lambda_3 \sum_{k=1}^n G_s'(t,t_k)I_k(x(t_k)), \tag{2.3}$$

where

$$G(t,s) = \begin{cases} t, & 0 \leq t \leq s \leq 1, \\ s, & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.4}$$

It is obvious that

$$G_s'(t,s) = \begin{cases} 0, & 0 \leq t \leq s \leq 1, \\ 1, & 0 \leq s \leq t \leq 1, \end{cases} \tag{2.5}$$

and

$$\max_{t,s \in J, t \neq s} |G_s'(t,s)| = 1.$$

Proof The proof of Lemma 2.1 is similar to that of Lemma 2.6 in [47]. □

It follows from the definition of $G(t, s)$ that Proposition 2.1 holds.

Proposition 2.1 *Let $G(t, s)$ be given as in* (2.4). *Then we have the following results*:

$$G(t, s) > 0, \quad \forall t, s \in (0, 1),$$

$$tG(s, s) \leq G(t, s) \leq G(s, s) = s \leq 1, \quad \forall t, s \in J, \tag{2.6}$$

$$G(t, s) \geq \xi G(s, s), \quad \forall t \in [\xi, 1], s \in J.$$

Proof By the definition of $G(t, s)$, it is not difficult to see that $G(t, s) > 0, \forall t, s \in (0, 1)$.

Next, turning to (2.6), if $t \leq s$, then $G(t, s) = t \leq s$; if $t \geq s$, then $G(t, s) = s$. So we get that $G(t, s) \leq s = G(s, s), \forall t, s \in J$.

On the other hand, noticing that $t, s \in J$, if $t \leq s$, then $G(t, s) = t \geq ts$; if $t \geq s$, then $G(t, s) = s \geq st$. Therefore, we obtain that $G(t, s) \geq ts = tG(s, s), \forall t, s \in J$. This gives the proof of (2.6).

If $t \in [\xi, 1]$, it is easy to see that by (2.6), $G(t, s) \geq \xi G(s, s), \forall s \in J$. □

Remark 2.1 Noticing that $t_k \in (0, 1)$, we get $G(t_k, t_k) > 0$.

Lemma 2.2 *Assume that (H_1)–(H_3) hold. Then problem* (2.2) *has a solution y given by*

$$y(t) = \lambda_2 \int_0^1 G_1(t, s)b(s)x(s)\, ds, \tag{2.7}$$

where

$$G_1(t, s) = \begin{cases} t(1 - s), & 0 \leq t \leq s \leq 1, \\ s(1 - t), & 0 \leq s \leq t \leq 1. \end{cases} \tag{2.8}$$

Proof The proof of Lemma 2.2 is similar to that of Lemma 2.1 in [48]. □

Similar to the proof of Proposition 2.1, we can get Proposition 2.2.

Proposition 2.2 *Let $G_1(t, s)$ be given as in* (2.8). *Then we have the following results*:

$$G_1(t, s) > 0, \quad \forall t, s \in (0, 1),$$

$$G_1(t, t)G_1(s, s) \leq G_1(t, s) \leq G_1(s, s) = s(1 - s) \leq \frac{1}{4}, \quad \forall t, s \in J, \tag{2.9}$$

$$G_1(t, s) \geq \xi^2 G_1(s, s), \quad \forall t \in [\xi, 1 - \xi], s \in J.$$

Remark 2.2 Letting (x, y) be a solution of problem (1.1), from Lemma 2.1 and Lemma 2.2, we have

$$x(t) = \lambda_2 \int_0^1 \int_0^1 G(t, s)G_1(s, \tau)a(s)b(\tau)x(s)x(\tau)\, d\tau\, ds$$

$$+ \lambda_1 \int_0^1 G(t, s)g(s)f(x(s))\, ds + \lambda_3 \sum_{k=1}^n G_s'(t, t_k)I_k(x(t_k)), \tag{2.10}$$

and y is defined in (2.7).

To obtain the existence of positive solution of problem (1.1), we make the following hypotheses:

(H_4) There exists $0 < \sigma_1 < \xi$ such that

$$\sigma_1 \int_{\sigma_1}^{\xi} G_1(t,s)b^+(s)\,ds \geq \xi \int_{\xi}^{1} G_1(t,s)b^-(s)\,ds;$$

(H_5) There exists $0 < \sigma_2 < \xi$ such that

$$\sigma_2 \int_{\sigma_2}^{\xi} G(t,s)G_1(s,s)a^+(s)\,ds \geq \xi \int_{\xi}^{1} G(t,s)a^-(s)\,ds;$$

(H_6) There exists $0 < \mu \leq 1$ such that

$$f(\omega) \geq \mu\varphi(\omega), \quad \omega \in [0,+\infty),$$

where $\varphi(\omega) = \max\{f(\rho) : 0 \leq \rho \leq \omega\}$;

(H_7) There exist $0 < \alpha < +\infty$ with $\alpha \neq 1$ and $k_1, k_2, l_1, l_2 > 0$ such that

$$k_1 x^{\alpha} \leq f(x) \leq k_2 x^{\alpha}, \qquad l_1 x^{\alpha} \leq I_k(x) \leq l_2 x^{\alpha}, \quad x \in [0,+\infty);$$

(H_8) There exists $0 < \sigma_3 < \xi$ satisfying $\frac{\sigma_3}{2} < t_1 < \sigma_3$ such that

$$\sigma_3^{\alpha} \mu^2 k_1 \int_{\sigma_3}^{\xi} G(t,s)g^+(s)\,ds \geq k_2 \xi^{\alpha} \int_{\xi}^{1} G(t,s)g^-(s)\,ds.$$

We denote

$$C_0^+[0,1] = \left\{ x \in PC[0,1] : \min_{t \in J} x(t) \geq 0, x(0) = 0, x'(1) = 0 \right\},$$

$$K = \left\{ x \in C_0^+[0,1] : x \text{ is concave on } [0,\xi], \text{ and convex on } [\xi,1] \right\}.$$

If $x \in K$, it is not difficult to see that $\|x\|_{PC} = \max_{0 \leq t \leq \xi} x(t)$. Also, for a positive number r, we define Ω_r by $\Omega_r = \{x \in K : \|x\|_{PC} < r\}$, and then we get $\partial\Omega_r = \{x \in K : \|x\|_{PC} = r\}$.

Define $T : K \to PC[0,1]$ by

$$(Tx)(t) = \lambda_2 \int_0^1 \int_0^1 G(t,s)G_1(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$+ \lambda_1 \int_0^1 G(t,s)g(s)f(x(s))\,ds + \lambda_3 \sum_{k=1}^{n} G_s'(t,t_k)I_k(x(t_k)). \tag{2.11}$$

Remark 2.3 It follows from Lemmas 2.1–2.2 and Remark 2.2 that (x,y) is a solution of problem (1.1) if and only if x is a fixed point of operator T and y satisfies (2.7).

Lemma 2.3 *Assume that* (H_1)–(H_8) *hold. Then* $T(K) \subset K$ *and* $T : K \to K$ *is completely continuous.*

Proof Define a function $e : [0, 1] \rightarrow [0, 1]$ as follows:

$$\text{if } x(1) = 0, \text{ then } e(t) = \min\left\{\frac{t}{\xi}, \frac{1-t}{1-\xi}\right\}, \quad \forall t \in J;$$

$$\text{if } x(1) > 0, \text{ then } e(t) = \min\left\{\frac{t}{\xi}, 1\right\}, \quad \forall t \in J.$$

So, we get $\min_{\sigma \leq t \leq \xi} e(t) = \frac{\sigma}{\xi}$, $\max_{\xi \leq t \leq 1} e(t) = 1$.

Letting $x \in K$, then x is concave on $[0, \xi]$ and convex on $[\xi, 1]$. Noticing that $x(0) = 0$, $x'(1) = 0$, we get $x(t) \geq x(\xi)e(t)$, $t \in [0, \xi]$, $x(t) \leq x(\xi)e(t)$, $t \in [\xi, 1]$.

Firstly, for any $x \in K$, we show that

$$\int_0^1 G_1(t, s)b(s)x(s)\,ds \geq \int_0^{\sigma_1} G_1(t, s)b^+(s)x(s)\,ds, \quad t \in J. \tag{2.12}$$

In fact, for $x \in K$, noticing (H_4), we get

$$\int_0^1 G_1(t, s)b(s)x(s)\,ds - \int_0^{\sigma_1} G_1(t, s)b^+(s)x(s)\,ds$$

$$= \int_{\sigma_1}^{\xi} G_1(t, s)b^+(s)x(s)\,ds - \int_{\xi}^1 G_1(t, s)b^-(s)x(s)\,ds$$

$$\geq \int_{\sigma_1}^{\xi} G_1(t, s)b^+(s)e(s)x(\xi)\,ds - \int_{\xi}^1 G_1(t, s)b^-(s)e(s)x(\xi)\,ds$$

$$\geq x(\xi)\left[\min_{s \in [\sigma_1, \xi]} e(s) \int_{\sigma_1}^{\xi} G_1(t, s)b^+(s)\,ds - \max_{s \in [\xi, 1]} e(s) \int_{\xi}^1 G_1(t, s)b^-(s)\,ds\right]$$

$$= x(\xi)\left[\frac{\sigma_1}{\xi} \int_{\sigma_1}^{\xi} G_1(t, s)b^+(s)\,ds - \int_{\xi}^1 G_1(t, s)b^-(s)\,ds\right]$$

$$\geq 0,$$

which shows that (2.12) holds.

Next, for any $x \in K$, we prove

$$\int_0^1 \int_0^1 G(t, s)G_1(s, \tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$\geq \int_0^{\sigma_2} \int_0^1 G(t, s)G_1(s, \tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds, \quad t \in J. \tag{2.13}$$

Noticing $\int_0^1 G_1(t, s)b(s)x(s)\,ds \geq 0$, $\forall t \in J$, (2.9) and (H_5), we obtain

$$\int_0^1 \int_0^1 G(t, s)G_1(s, \tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$- \int_0^{\sigma_2} \int_0^1 G(t, s)G_1(s, \tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$= \int_{\sigma_2}^{\xi} \int_0^1 G(t, s)G_1(s, \tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$- \int_\xi^1 \int_0^1 G(t,s)G_1(s,\tau)a^-(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$\geq \int_{\sigma_2}^\xi G(t,s)a^+(s)e(s)x(\xi) \int_0^1 G_1(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$

$$- \int_\xi^1 G(t,s)a^-(s)e(s)x(\xi) \int_0^1 G_1(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$

$$\geq \int_{\sigma_2}^\xi G(t,s)a^+(s) \min_{s\in[\sigma_2,\xi]} e(s)x(\xi) \int_0^1 G_1(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$

$$- \int_\xi^1 G(t,s)a^-(s) \max_{s\in[\xi,1]} e(s)x(\xi) \int_0^1 G_1(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$

$$\geq x(\xi)\frac{\sigma_2}{\xi} \int_{\sigma_2}^\xi G(t,s)a^+(s) \int_0^1 G_1(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$

$$- x(\xi) \int_\xi^1 G(t,s)a^-(s) \int_0^1 G_1(s,\tau)b(\tau)x(\tau)\,d\tau\,ds$$

$$\geq x(\xi)\frac{\sigma_2}{\xi} \int_{\sigma_2}^\xi G(t,s)a^+(s) \int_0^1 G_1(s,s)G_1(\tau,\tau)b(\tau)x(\tau)\,d\tau\,ds$$

$$- x(\xi) \int_\xi^1 G(t,s)a^-(s) \int_0^1 G_1(\tau,\tau)b(\tau)x(\tau)\,d\tau\,ds$$

$$= x(\xi) \int_0^1 G_1(\tau,\tau)b(\tau)x(\tau)\,d\tau \left[\frac{\sigma_2}{\xi} \int_{\sigma_2}^\xi G(t,s)G_1(s,s)a^+(s)\,ds - \int_\xi^1 G(t,s)a^-(s)\,ds\right],$$

which proves that (2.13) holds.

Thirdly, for any $x \in K$, we prove

$$\int_0^1 G(t,s)g(s)f\big(x(s)\big)\,ds \geq \int_0^{\sigma_3} G(t,s)g^+(s)f\big(x(s)\big)\,ds, \quad t \in J. \tag{2.14}$$

In fact, for any $t \in J$ and $x \in K$, noticing (2.6), it is obvious that by (H_6)–(H_8)

$$\int_0^1 G(t,s)g(s)f\big(x(s)\big)\,ds - \int_0^{\sigma_3} G(t,s)g^+(s)f\big(x(s)\big)\,ds$$

$$= \int_{\sigma_3}^\xi G(t,s)g^+(s)f\big(x(s)\big)\,ds - \int_\xi^1 G(t,s)g^-(s)f\big(x(s)\big)\,ds$$

$$\geq \mu \int_{\sigma_3}^\xi G(t,s)g^+(s)\varphi\big(u(s)\big)\,ds - \int_\xi^1 G(t,s)g^-(s)\varphi\big(u(s)\big)\,ds$$

$$\geq \mu \int_{\sigma_3}^\xi G(t,s)g^+(s)\varphi\big(e(s)u(\xi)\big)\,ds - \int_\xi^1 G(t,s)g^-(s)\varphi\big(e(s)u(\xi)\big)\,ds$$

$$\geq \mu \int_{\sigma_3}^\xi G(t,s)g^+(s)f\big(e(s)u(\xi)\big)\,ds - \frac{1}{\mu} \int_\xi^1 G(t,s)g^-(s)f\big(e(s)u(\xi)\big)\,ds$$

$$\geq \mu \int_{\sigma_3}^\xi G(t,s)g^+(s)k_1 e^\alpha(s)u^\alpha(\xi)\,ds - \frac{1}{\mu} \int_\xi^1 G(t,s)g^-(s)k_2 e^\alpha(s)u^\alpha(\xi)\,ds$$

$$\geq \mu \left[\min_{s\in[\sigma_3,\xi]} e(s)\right]^\alpha \int_{\sigma_3}^\xi G(t,s)g^+(s)k_1 u^\alpha(\xi)\,ds$$

$$- \frac{1}{\mu} \Big[\min_{s \in [\xi,1]} e(s) \Big]^{\alpha} \int_{\xi}^{1} G(t,s) g^{-}(s) k_2 u^{\alpha}(\xi) \, ds$$

$$\geq u^{\alpha}(\xi) \Big[\mu k_1 \frac{\sigma_3^{\alpha}}{\xi^{\alpha}} \int_{\sigma_3}^{\xi} G(t,s) g^{+}(s) \, ds$$

$$- \frac{1}{\mu} k_2 \int_{\xi}^{1} G(t,s) g^{-}(s) \, ds \Big]$$

$$\geq 0.$$

This proves that (2.14) holds. Thus, for $x \in K$, we get

$$(Tx)(t) = \lambda_2 \int_0^1 \int_0^1 G(t,s) G_1(s,\tau) a(s) b(\tau) x(s) x(\tau) \, d\tau \, ds$$

$$+ \lambda_1 \int_0^1 G(t,s) g(s) f\big(x(s)\big) \, ds + \lambda_3 \sum_{k=1}^{n} G_s'(t,t_k) I_k\big(x(t_k)\big)$$

$$\geq \lambda_2 \int_0^{\sigma_2} G(t,s) a^{+}(s) x(s) \int_0^1 G_1(s,\tau) b(\tau) x(\tau) \, d\tau \, ds$$

$$+ \lambda_1 \int_0^{\sigma_3} G(t,s) g^{+}(s) f\big(x(s)\big) \, ds + \lambda_3 \sum_{k=1}^{n} G_s'(t,t_k) I_k\big(x(t_k)\big)$$

$$\geq 0.$$

Moreover, by direct calculation, we obtain

$$(Tx)(0) = 0, \qquad (Tx)'(1) = 0,$$

$$(Tx)''(t) = -\lambda_2 a^{+}(t) x(t) \int_0^1 G_1(t,s) b(s) x(s) \, ds - \lambda_1 g^{+}(t) f(x) \leq 0, \quad t \in [0,\xi],$$

$$(Tx)''(t) = \lambda_2 a^{-}(t) x(t) \int_0^1 G_1(t,s) b(s) x(s) \, ds + \lambda_1 g^{-}(t) f(x) \geq 0, \quad t \in [\xi,1],$$

which shows that Tx is concave on $[0,\xi]$ and convex on $[\xi,1]$. It follows that $T : K \to K$.

Finally, by standard methods and Arzelà–Ascoli theorem, one can prove T is completely continuous. So it is omitted. $\qquad \square$

Remark 2.4 The idea of the proof of Lemma 2.3 comes from Theorem 3.1 of [36].

Lemma 2.4 (Theorem 2.3.4 of [49], Fixed point theorem of cone expansion and compression of norm type) *Let Ω_1 and Ω_2 be two bounded open sets in a real Banach space E such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Let the operator $T : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ be completely continuous, where P is a cone in E. Suppose that one of the two conditions*

(i) $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial \Omega_1$ and $\|Tx\| \geq \|x\|, \forall x \in P \cap \partial \Omega_2$,
and

(ii) $\|Tx\| \geq \|x\|, \forall x \in P \cap \partial \Omega_1$, and $\|Tx\| \leq \|x\|, \forall x \in P \cap \partial \Omega_2$,
is satisfied. Then T has at least one fixed point in $P \cap (\bar{\Omega}_2 \backslash \Omega_1)$.

3 Main results

In this part, applying Lemma 2.4, we get the optimal intervals of positive parameters λ_i $(i = 1, 2, 3)$ in which problem (1.1) admits at least two positive solutions.

Theorem 3.1 *Assume that* (H_1)–(H_8) *hold. Then we have*:

(I) *If* $\alpha > 1$, *then there exist* $\lambda_1^*, \lambda_3^*, \lambda^*, \lambda_* > 0$ *such that, for* $\lambda_i \in [\lambda_i^*, \infty)$ $(i = 1, 3)$ *and* $\lambda_2 \in [\lambda_*, \lambda^*]$, *problem* (1.1) *admits at least two positive solutions*.

(II) *If* $0 < \alpha < 1$, *then there exist* $\lambda_1^{**}, \lambda_2^{**}, \lambda_3^{**} > 0$ *such that, for* $\lambda_i \in (0, \lambda_i^*]$, *problem* (1.1) *admits at least two positive solutions*.

Proof Part (I). Considering the case $\alpha > 1$, it follows from (H_7) that

$$\lim_{x \to 0} \frac{f(x)}{x} \leq \lim_{x \to 0} \frac{k_2 x^\alpha}{x} = 0, \qquad \lim_{x \to 0} \frac{I_k(x)}{x} \leq \lim_{x \to 0} \frac{l_2 x^\alpha}{x} = 0,$$

which shows that there exists $r' > 0$ such that

$$f(x) \leq \varepsilon_1 x, \qquad I_k(x) \leq \varepsilon_2 x, \quad k = 1, 2, \ldots, n, \ x \in [0, r'],$$

where ε_1, ε_2 satisfy

$$3\lambda_1 \varepsilon_1 \int_0^\xi g^+(s)\,ds < 1, \qquad 3n\lambda_3 \varepsilon_2 < 1. \tag{3.1}$$

Let

$$A = \lambda_2 \int_0^\xi \int_0^\xi G_1(s, \tau) a^+(s) b^+(\tau)\,d\tau\,ds, \tag{3.2}$$

$$A^* = \int_0^\xi \int_0^\xi G_1(s, \tau) a^+(s) b^+(\tau)\,d\tau\,ds, \tag{3.3}$$

and choose $r = \min\{(3A)^{-1}, \frac{r'}{2}\}$. This implies $\lambda_2 \leq \frac{1}{3A^* r}$.

Then, for any $x \in K \cap \partial\Omega_r$, noticing (2.6), we get

$$\begin{aligned}
(Tx)(t) &= \lambda_2 \int_0^1 \int_0^1 G(t, s) G_1(s, \tau) a(s) b(\tau) x(s) x(\tau)\,d\tau\,ds \\
&\quad + \lambda_1 \int_0^1 G(t, s) g(s) f\big(x(s)\big)\,ds + \lambda_3 \sum_{k=1}^n G_s'(t, t_k) I_k\big(x(t_k)\big) \\
&= \lambda_2 \int_0^\xi \int_0^1 G(t, s) G_1(s, \tau) a^+(s) b(\tau) x(s) x(\tau)\,d\tau\,ds \\
&\quad - \lambda_2 \int_\xi^1 \int_0^1 G(t, s) G_1(s, \tau) a^-(s) b(\tau) x(s) x(\tau)\,d\tau\,ds \\
&\quad + \lambda_1 \int_0^\xi G(t, s) g^+(s) f\big(x(s)\big)\,ds - \lambda_1 \int_\xi^1 G(t, s) g^-(s) f\big(x(s)\big)\,ds \\
&\quad + \lambda_3 \sum_{k=1}^n G_s'(t, t_k) I_k\big(x(t_k)\big)
\end{aligned}$$

$$\le \lambda_2 \int_0^{\xi} \int_0^1 G(t,s)G_1(s,\tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$+ \lambda_1 \int_0^{\xi} G(t,s)g^+(s)f\big(x(s)\big)\,ds + \lambda_3 \sum_{k=1}^{n} G'_s(t,t_k)I_k\big(x(t_k)\big)$$

$$= \lambda_2 \int_0^{\xi} \int_0^{\xi} G(t,s)G_1(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$- \lambda_2 \int_0^{\xi} \int_0^{\xi} G(t,s)G_1(s,\tau)a^+(s)b^-(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$+ \lambda_1 \int_0^{\xi} G(t,s)g^+(s)f\big(x(s)\big)\,ds + \lambda_3 \sum_{k=1}^{n} G'_s(t,t_k)I_k\big(x(t_k)\big)$$

$$\le \lambda_2 \int_0^{\xi} \int_0^{\xi} G(s,s)G_1(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$+ \lambda_1 \int_0^{\xi} G(s,s)g^+(s)f\big(x(s)\big)\,ds + \lambda_3 \sum_{k=1}^{n} I_k\big(x(t_k)\big)$$

$$\le \lambda_2 \xi \int_0^{\xi} \int_0^{\xi} G_1(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds + \lambda_1 \xi \int_0^{\xi} g^+(s)f\big(x(s)\big)\,ds$$

$$+ \lambda_3 \sum_{k=1}^{n} I_k\big(x(t_k)\big)$$

$$< \lambda_2 \int_0^{\xi} \int_0^{\xi} G_1(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds + \lambda_1 \int_0^{\xi} g^+(s)\varepsilon_1 x(s)\,ds$$

$$+ \lambda_3 \sum_{k=1}^{n} \varepsilon_2 x(t_k)$$

$$\le \lambda_2 \int_0^{\xi} \int_0^{\xi} G_1(s,\tau)a^+(s)b^+(\tau)\,d\tau\,ds\|x\|_{PC}^2 + \lambda_1 \varepsilon_1 \int_0^{\xi} g^+(s)\,ds\|x\|_{PC}$$

$$+ \lambda_3 n\varepsilon_2\|x\|_{PC}$$

$$< \frac{1}{3}\|x\|_{PC} + \frac{1}{3}\|x\|_{PC} + \frac{1}{3}\|x\|_{PC}$$

$$= \|x\|_{PC}, \tag{3.4}$$

which shows that

$$\|Tx\|_{PC} < \|x\|_{PC}, \quad \forall x \in K \cap \partial\Omega_r. \tag{3.5}$$

On the other hand, noticing that $\alpha > 1$, then it follows from (H_7) that

$$\lim_{x\to\infty} \frac{f(x)}{x} \ge \lim_{x\to\infty} \frac{k_1 x^{\alpha}}{x} = \infty, \qquad \lim_{x\to\infty} \frac{I_k(x)}{x} \ge \lim_{x\to\infty} \frac{l_1 x^{\alpha}}{x} = \infty,$$

which shows that there exists $R' > 0$ such that

$$f(x) \ge \varepsilon_3 x, \qquad I_k(x) \ge \varepsilon_4 x, \quad x \ge R',$$

where ε_3, ε_4 satisfy

$$\frac{3\sigma_2}{2}\lambda_1\varepsilon_3\Gamma\int_{\frac{\sigma_3}{2}}^{\sigma_3}G(s,s)g^+(s)\,ds > 1, \quad 3\lambda_3\varepsilon_4\Gamma > 1,$$

$$\Gamma = \min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t) > 0, \qquad \delta(t) = \min\left\{\frac{t}{\xi}, \frac{\xi-t}{\xi}\right\}, \quad t\in[0,\xi].$$

(3.6)

If $x\in K$, then from the concavity on $[0,\xi]$, it follows that

$$x(t) \geq \delta(t)\|x\|_{PC}, \quad t\in[0,\xi].$$

(3.7)

Let B and B^* denote the following constants:

$$B = \lambda_2\frac{\sigma_2}{2}\min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\delta(t)\min_{\frac{\sigma_1}{2}\leq t\leq\sigma_1}\delta(t)\int_{\frac{\sigma_2}{2}}^{\sigma_2}\int_{\frac{\sigma_1}{2}}^{\sigma_1}G(s,s)G_1(s,\tau)a^+(s)b^+(\tau)\,d\tau\,ds > 0,$$

(3.8)

$$B^* = \frac{\sigma_2}{2}\min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\delta(t)\min_{\frac{\sigma_1}{2}\leq t\leq\sigma_1}\delta(t)\int_{\frac{\sigma_2}{2}}^{\sigma_2}\int_{\frac{\sigma_1}{2}}^{\sigma_1}G(s,s)G_1(s,\tau)a^+(s)b^+(\tau)\,d\tau\,ds > 0.$$

(3.9)

Noticing that, for all $d\in(0,\xi)$, $\min_{\frac{d}{2}\leq t\leq d}\delta(t) = \min\{\frac{d}{2\xi}, \frac{\xi-d}{\xi}\} > 0$, B and B^* are defined well.

Setting $R = \max\{(3B)^{-1}, \frac{R'}{\Gamma}, r'\} + 1$, it follows that $\lambda_2 > \lambda_2' = \frac{1}{3B^*R}$, and then for any $x\in K\cap\partial\Omega_R$, we get

$$\min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}x(t) \geq \min_{\frac{\sigma_3}{2}\leq t\leq\sigma_3}\delta(t)\|x\|_{PC} \geq \Gamma R > R'.$$

Then it follows from (2.6), (2.9), (3.8), and (3.9) that

$$\begin{aligned}
\|Tx\|_{PC} &= \max_{t\in J}\left\{\lambda_2\int_0^1\int_0^1 G(t,s)G_1(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds\right.\\
&\quad\left. + \lambda_1\int_0^1 G(t,s)g(s)f(x(s))\,ds + \lambda_3\sum_{k=1}^n G'_s(t,t_k)I_k(x(t_k))\right\}\\
&\geq \max_{t\in J}\left\{\lambda_2\int_0^{\sigma_2}G(t,s)a^+(s)x(s)\int_0^1 G_1(s,\tau)b(\tau)x(\tau)\,d\tau\,ds\right.\\
&\quad\left. + \lambda_1\int_0^{\sigma_3}G(t,s)g^+(s)f(x(s))\,ds + \lambda_3\sum_{k=1}^n G'_s(t,t_k)I_k(x(t_k))\right\}\\
&\geq \lambda_2\min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\int_{\frac{\sigma_2}{2}}^{\sigma_2}G(t,s)a^+(s)x(s)\int_{\frac{\sigma_1}{2}}^{\sigma_1}G_1(s,\tau)b^+(\tau)x(\tau)\,d\tau\,ds\\
&\quad + \min_{\frac{\sigma_2}{2}\leq t\leq\sigma_2}\lambda_1\int_{\frac{\sigma_3}{2}}^{\sigma_3}G(t,s)g^+(s)f(x(s))\,ds + \lambda_3\min_{\sigma_3\leq t\leq\xi}\sum_{\frac{t}{2}<t_k<t}I_k(x(t_k))\\
&\geq \lambda_2\frac{\sigma_2}{2}\int_{\frac{\sigma_2}{2}}^{\sigma_2}G(s,s)a^+(s)\delta(s)\|x\|_{PC}\int_{\frac{\sigma_1}{2}}^{\sigma_1}G_1(s,\tau)b^+(\tau)\delta(\tau)\|x\|_{PC}\,d\tau\,ds\\
&\quad + \lambda_1\frac{\sigma_2}{2}\int_{\frac{\sigma_3}{2}}^{\sigma_3}G(s,s)g^+(s)\varepsilon_3 x(s)\,ds + \lambda_3\sum_{\frac{\sigma_3}{2}<t_k<\sigma_3}\varepsilon_4 x(t_k)
\end{aligned}$$

$$\geq \lambda_2 \frac{\sigma_2}{2} \int_{\frac{\sigma_2}{2}}^{\sigma_2} G(s,s) a^+(s) \delta(s) \|x\|_{PC} \int_{\frac{\sigma_1}{2}}^{\sigma_1} G_1(s,\tau) b^+(\tau) \delta(\tau) \|x\|_{PC} \, d\tau \, ds$$

$$+ \lambda_1 \frac{\sigma_2}{2} \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s) g^+(s) \varepsilon_3 x(s) \, ds + \lambda_3 \varepsilon_4 \sum_{\frac{\sigma_3}{2} < t_1 < \sigma_3} x(t_1)$$

$$\geq B \|x\|_{PC}^2 + \lambda_1 \frac{\sigma_2}{2} \varepsilon_3 \min_{\frac{\sigma_3}{2} \leq t \leq \sigma_3} \delta(t) \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s) g^+(s) \, ds \|x\|_{PC}$$

$$+ \lambda_3 \varepsilon_4 \min_{\frac{\sigma_3}{2} \leq t \leq \sigma_3} \delta(t) \|x\|_{PC}$$

$$> \frac{1}{3} \|x\|_{PC} + \frac{1}{3} \|x\|_{PC} + \frac{1}{3} \|x\|_{PC}$$

$$= \|x\|_{PC}.$$

Consequently,

$$\|Tx\|_{PC} > \|x\|_{PC}, \quad x \in K \cap \partial \Omega_R. \tag{3.10}$$

Let

$$\lambda_1^* = \frac{(6DA^* - 2B^*)\eta}{3DA^* \sigma_2 \sigma_\eta \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s) g^+(s) \, ds}, \qquad \lambda_2^* = \frac{\frac{1}{2}}{3DA^*\eta}, \qquad \lambda_3^* = \frac{\frac{B^*}{2}\eta}{3DA^* \sigma_\eta^*},$$

where $0 < \eta < r$, $D = \frac{r^2}{\eta^2}$, A^* is defined in (3.3), and B^* is defined in (3.9).
We define σ_η and σ_η^* by

$$\sigma_\eta = \min_{\Gamma\eta \leq x \leq \eta} \{f(x)\} > 0, \qquad \sigma_\eta^* = \min\{m_k, k = 1, 2, \ldots, n\} > 0,$$

$$m_k = \min_{\Gamma\eta \leq x \leq \eta} \{I_k(x)\}, \quad k = 1, 2, \ldots, n,$$

where Γ is defined in (3.6). Noticing (H_2) and (H_3), then σ_η and σ_η^* are defined well and $\sigma_\eta > 0$, $\sigma_\eta^* > 0$. So, for any $x \in K \cap \partial \Omega_\eta$, we get

$$\Gamma\eta \leq \Gamma \|x\|_{PC} \leq \min_{\frac{\sigma_3}{2} \leq t \leq \sigma_3} x(t) \leq x(t) \leq \eta, \quad \frac{\sigma_3}{2} \leq t \leq \sigma_3,$$

and hence, it follows from (2.6), (2.7), (3.7), and (3.9) that

$$\|Tx\|_{PC} = \max_{t \in J} \left\{ \lambda_2 \int_0^1 \int_0^1 G(t,s) G_1(s,\tau) a(s) b(\tau) x(s) x(\tau) \, d\tau \, ds \right.$$

$$\left. + \lambda_1 \int_0^1 G(t,s) g(s) f(x(s)) \, ds + \lambda_3 \sum_{k=1}^n G_s'(t,t_k) I_k(x(t_k)) \right\}$$

$$\geq \max_{t \in J} \left\{ \lambda_2 \int_0^{\sigma_2} G(t,s) a^+(s) x(s) \int_0^1 G_1(s,\tau) b(\tau) x(\tau) \, d\tau \, ds \right.$$

$$\left. + \lambda_1 \int_0^{\sigma_3} G(t,s) g^+(s) f(x(s)) \, ds + \lambda_3 \sum_{k=1}^n G_s'(t,t_k) I_k(x(t_k)) \right\}$$

$$\geq \lambda_2 \min_{\frac{\sigma_2}{2} \leq t \leq \sigma_2} \int_{\frac{\sigma_2}{2}}^{\sigma_2} G(t,s) a^+(s) x(s) \int_{\frac{\sigma_1}{2}}^{\sigma_1} G_1(s,\tau) b^+(\tau) x(\tau) \, d\tau \, ds$$

$$+ \min_{\frac{\sigma_2}{2} \leq t \leq \sigma_2} \lambda_1 \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(t,s) g^+(s) f(x(s)) \, ds + \lambda_3 \min_{\sigma_3 \leq t \leq \xi} \sum_{\frac{t}{2} < t_k < t} I_k(x(t_k))$$

$$\geq \lambda_2 \frac{\sigma_2}{2} \int_{\frac{\sigma_2}{2}}^{\sigma_2} G(s,s) a^+(s) \delta(s) \|x\|_{PC} \int_{\frac{\sigma_1}{2}}^{\sigma_1} G_1(s,\tau) b^+(\tau) \delta(\tau) \|x\|_{PC} \, d\tau \, ds$$

$$+ \lambda_1 \frac{\sigma_2}{2} \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s) g^+(s) \sigma_\eta \, ds + \lambda_3 \sum_{\frac{\sigma_3}{2} < t_k < \sigma_3} \sigma_\eta^*$$

$$\geq \lambda_2 \frac{\sigma_2}{2} \int_{\frac{\sigma_2}{2}}^{\sigma_2} G(s,s) a^+(s) \delta(s) \|x\|_{PC} \int_{\frac{\sigma_1}{2}}^{\sigma_1} G_1(s,\tau) b^+(\tau) \delta(\tau) \|x\|_{PC} \, d\tau \, ds$$

$$+ \lambda_1 \frac{\sigma_2}{2} \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s) g^+(s) \sigma_\eta \, ds + \lambda_3 \sigma_\eta^*$$

$$\geq \lambda_2^* B^* \|x\|_{PC}^2 + \lambda_1^* \frac{\sigma_2}{2} \sigma_\eta \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s) g^+(s) \, ds + \lambda_3^* \sigma_\eta^*$$

$$= \frac{\frac{B^*}{2}}{3DA^*} \|x\|_{PC} + \frac{3DA^* - B^*}{3DA^*} \|x\|_{PC} + \frac{\frac{B^*}{2}}{3DA^*} \|x\|_{PC}$$

$$= \|x\|_{PC}.$$

This shows

$$\|Tx\|_{PC} > \|x\|_{PC}, \quad x \in K \cap \partial\Omega_\eta. \tag{3.11}$$

Let $\lambda_* = \max\{\lambda_2', \lambda_2^*\}$, $\lambda^* = \frac{1}{3A^*r}$, then, for $\lambda_i \in [\lambda_i^*, +\infty)$, $i = 1, 3$, and $\lambda_2 \in [\lambda_*, \lambda^*]$, applying Lemma 2.4 to (3.5), (3.10), and (3.11) yields that T has two fixed points x_1 and x_2 satisfying $x_1 \in K \cap (\bar{\Omega}_R \setminus \bar{\Omega}_r)$, $x_2 \in K \cap (\Omega_r \setminus \Omega_\eta)$. Thus it follows from Remark 2.3 that problem (1.1) admits at least two positive solutions (x_i, y_i) $(i = 1, 2)$ with $x_1 \in K \cap (\bar{\Omega}_R \setminus \bar{\Omega}_r)$, $x_2 \in K \cap (\Omega_r \setminus \Omega_\eta)$, and

$$y_1(t) = \lambda_2 \int_0^1 G_1(t,s) b(s) x_1(s) \, ds,$$

$$y_2(t) = \lambda_2 \int_0^1 G_1(t,s) b(s) x_2(s) \, ds.$$

Part (II). Next turning to $0 < \alpha < 1$, it follows from (H_7) that

$$\lim_{x \to 0} \frac{f(x)}{x} \geq \lim_{x \to 0} \frac{k_1 x^\alpha}{x} = +\infty,$$

$$\lim_{x \to 0} \frac{I_k(x)}{x} \geq \lim_{x \to 0} \frac{l_1 x^\alpha}{x} = +\infty,$$

which shows that there exists a positive constant $r > 0$ such that

$$f(x) \geq \varepsilon_5 x, \qquad I_k(x) \geq \varepsilon_6 x, \quad k = 1, 2, \ldots, n, 0 \leq x \leq r,$$

where ε_5, ε_6 satisfy

$$\lambda_1\sigma_2\varepsilon_5 \min_{\frac{\sigma_3}{2} \leq t \leq \sigma_3} \delta(t) \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s)g^+(s)\,ds > 1, \qquad 2\lambda_3\varepsilon_6 \min_{\frac{\sigma_3}{2} \leq t \leq \sigma_3} \delta(t) > 1,$$

where $\delta(t)$ is defined in (3.6), and A is defined in (3.2).

Therefore, for any $x \in K \cap \partial\Omega_r$, noticing (3.7), we get

$$
\begin{aligned}
\|Tx\|_{PC} &= \max_{t\in J}\left\{\lambda_2 \int_0^1 \int_0^1 G(t,s)G_1(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \right.\\
&\qquad \left. + \lambda_1 \int_0^1 G(t,s)g(s)f(x(s))\,ds + \lambda_3 \sum_{k=1}^n G_s'(t,t_k)I_k(x(t_k))\right\}\\
&\geq \max_{t\in J}\left\{\lambda_2 \int_0^{\sigma_2} G(t,s)a^+(s)x(s)\int_0^1 G_1(s,\tau)b(\tau)x(\tau)\,d\tau\,ds \right.\\
&\qquad \left. + \lambda_1 \int_0^{\sigma_3} G(t,s)g^+(s)f(x(s))\,ds + \lambda_3 \sum_{k=1}^n G_s'(t,t_k)I_k(x(t_k))\right\}\\
&\geq \lambda_2 \min_{\frac{\sigma_2}{2} \leq t \leq \sigma_2} \int_{\frac{\sigma_2}{2}}^{\sigma_2} G(t,s)a^+(s)x(s)\int_{\frac{\sigma_1}{2}}^{\sigma_1} G_1(s,\tau)b^+(\tau)x(\tau)\,d\tau\,ds\\
&\qquad + \min_{\frac{\sigma_2}{2} \leq t \leq \sigma_2}\lambda_1 \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(t,s)g^+(s)f(x(s))\,ds + \lambda_3 \min_{\sigma_3 \leq t \leq \xi} \sum_{\frac{t}{2} < t_k < t} I_k(x(t_k))\\
&\geq \lambda_2 \frac{\sigma_2}{2} \int_{\frac{\sigma_2}{2}}^{\sigma_2} G(s,s)a^+(s)\delta(s)\|x\|_{PC}\int_{\frac{\sigma_1}{2}}^{\sigma_1} G_1(s,\tau)b^+(\tau)\delta(\tau)\|x\|_{PC}\,d\tau\,ds\\
&\qquad + \lambda_1 \frac{\sigma_2}{2} \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s)g^+(s)\varepsilon_5 x(s)\,ds + \lambda_3 \sum_{\frac{\sigma_3}{2} < t_k < \sigma_3} \varepsilon_6 x(t_k)\\
&\geq \lambda_1 \frac{\sigma_2}{2} \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s)g^+(s)\varepsilon_5 x(s)\,ds + \lambda_3\varepsilon_6 \sum_{\frac{\sigma_3}{2} < t_1 < \sigma_3} x(t_1)\\
&\geq \lambda_1 \frac{\sigma_2}{2}\varepsilon_5 \min_{\frac{\sigma_3}{2} \leq t \leq \sigma_3} \delta(t) \int_{\frac{\sigma_3}{2}}^{\sigma_3} G(s,s)g^+(s)\,ds\|x\|_{PC} + \lambda_3\varepsilon_6 \min_{\frac{\sigma_3}{2} \leq t \leq \sigma_3} \delta(t)\|x\|_{PC}\\
&> \frac{1}{2}\|x\|_{PC} + \frac{1}{2}\|x\|_{PC}\\
&= \|x\|_{PC}.
\end{aligned}
$$

Consequently,

$$\|Tx\|_{PC} > \|x\|_{PC}, \quad x \in K \cap \partial\Omega_r. \tag{3.12}$$

On the other hand, notice that $0 < \alpha < 1$, then it follows from (H_7) that

$$\lim_{x\to\infty} \frac{f(x)}{x} \leq \lim_{x\to\infty} \frac{k_2 x^\alpha}{x} = 0, \qquad \lim_{x\to\infty} \frac{I_k(x)}{x} \leq \lim_{x\to\infty} \frac{l_2 x^\alpha}{x} = 0,$$

which shows that there exists $R' > r$ such that

$$f(x) \le \varepsilon_7 x, \qquad I_k(x) \le \varepsilon_8 x, \quad x \ge R',$$

where ε_7, ε_8 satisfy

$$5\lambda_1 \varepsilon_7 \int_{D(x)} g^+(s)\,ds < 1, \quad 5\lambda_3 n\varepsilon_8 < 1.$$

Let

$$M = \lambda_1 \max_{\|x\|_{PC}=R'} f(x) \int_0^\xi g^+(s)\,ds, \qquad M' = \lambda_1 n \max_{\|x\|_{PC}=R'} I_k(x).$$

Then it is not difficult to see that $M < +\infty$ and $M' < +\infty$.

Choosing $\max\{R', 5M, 5M'\} < R < (5A)^{-1}$, then $M < \frac{R}{5}$, $M' < \frac{R}{5}$.

Now, choosing $x \in K \cap \partial\Omega_R$ arbitrarily and letting $\bar{x}(t) = \min\{x(t), R'\}$, then $\bar{x}(t) \le R'$. Moreover, write $D(x) = \{t \in [0, \xi] : x(t) > R'\}$. Therefore, for $t \in D(x)$, we get $R' < x(t) \le \|x\|_{PC} = R$, $\forall t \in D(x)$. By the choice of R', for $t \in D(x)$, we have $f(x) \le \varepsilon_1 R$.

Similarly, one can define $D'(x) = \{t_k \in J : x(t_k) > R'\}$. Then, for $t_k \in D'(x)$, we get $R' < x(t_k) \le \|x\|_{PC} = R$, $\forall t \in D'(x)$. By the choice of R', for $t_k \in D'(x)$, we get $I_k(x(t_k)) \le \varepsilon_2 R$, $k = 1, 2, \ldots, n$.

Therefore, for any $x \in K \cap \partial\Omega_R$, we get

$$
\begin{aligned}
(Tx)(t) &= \lambda_2 \int_0^1 \int_0^1 G(t,s)G_1(s,\tau)a(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \\
&\quad + \lambda_1 \int_0^1 G(t,s)g(s)f\big(x(s)\big)\,ds + \lambda_3 \sum_{k=1}^n G_s'(t,t_k)I_k\big(x(t_k)\big) \\
&= \lambda_2 \int_0^\xi \int_0^1 G(t,s)G_1(s,\tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \\
&\quad - \lambda_2 \int_\xi^1 \int_0^1 G(t,s)G_1(s,\tau)a^-(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \\
&\quad + \lambda_1 \int_0^\xi G(t,s)g^+(s)f\big(x(s)\big)\,ds - \lambda_1 \int_\xi^1 G(t,s)g^-(s)f\big(x(s)\big)\,ds \\
&\quad + \lambda_3 \sum_{k=1}^n G_s'(t,t_k)I_k\big(x(t_k)\big) \\
&\le \lambda_2 \int_0^\xi \int_0^1 G(t,s)G_1(s,\tau)a^+(s)b(\tau)x(s)x(\tau)\,d\tau\,ds \\
&\quad + \lambda_1 \int_0^\xi G(t,s)g^+(s)f\big(x(s)\big)\,ds + \lambda_3 \sum_{k=1}^n G_s'(t,t_k)I_k\big(x(t_k)\big) \\
&= \lambda_2 \int_0^\xi \int_0^\xi G(t,s)G_1(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds \\
&\quad - \lambda_2 \int_0^\xi \int_0^\xi G(t,s)G_1(s,\tau)a^+(s)b^-(\tau)x(s)x(\tau)\,d\tau\,ds
\end{aligned}
$$

$$+ \lambda_1 \int_0^\xi G(t,s)g^+(s)f(x(s))\,ds + \lambda_3 \sum_{k=1}^n G'_s(t,t_k)I_k(x(t_k))$$

$$\leq \lambda_2 \int_0^\xi \int_0^\xi G(s,s)G_1(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds$$

$$+ \lambda_1 \int_0^\xi G(s,s)g^+(s)f(x(s))\,ds + \lambda_3 \sum_{k=1}^n I_k(x(t_k))$$

$$\leq \lambda_2\xi \int_0^\xi \int_0^\xi G_1(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds + \lambda_1\xi \int_0^\xi g^+(s)f(x(s))\,ds$$

$$+ \lambda_3 \sum_{k=1}^n I_k(x(t_k))$$

$$< \lambda_2 \int_0^\xi \int_0^\xi G_1(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds + \lambda_1 \int_{D(x)} g^+(s)f(x(s))\,ds$$

$$+ \lambda_1 \int_{[0,\xi]\setminus D(x)} g^+(s)f(x(s))\,ds + \lambda_3 \sum_{t_k\in D'(x)} I_k(x(t_k))$$

$$+ \lambda_3 \sum_{t_k\in J\setminus D'(x)} I_k(x(t_k))$$

$$\leq \lambda_2 \int_0^\xi \int_0^\xi G_1(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds + \lambda_1 \int_{D(x)} g^+(s)f(x(s))\,ds$$

$$+ \lambda_1 \int_0^\xi g^+(s)f(\bar{x}(s))\,ds + \lambda_3 \sum_{t_k\in D'(x)} I_k(x(t_k)) + \lambda_3 \sum_{k=1}^n I_k(\bar{x}(t_k))$$

$$\leq \lambda_2 \int_0^\xi \int_0^\xi G_1(s,\tau)a^+(s)b^+(\tau)\,d\tau\,ds\|x\|_{PC}^2 + \lambda_1\varepsilon_7 \int_{D(x)} g^+(s)\,ds\|x\|_{PC}$$

$$+ M + \lambda_3 n\varepsilon_8\|x\|_{PC} + M'$$

$$< \frac{1}{5}\|x\|_{PC} + \frac{1}{5}\|x\|_{PC} + \frac{1}{5}\|x\|_{PC} + \frac{1}{5}\|x\|_{PC} + \frac{1}{5}\|x\|_{PC}$$

$$= \|x\|_{PC}, \tag{3.13}$$

which shows that

$$\|Tx\|_{PC} < \|x\|_{PC}, \quad \forall x \in K \cap \partial\Omega_R. \tag{3.14}$$

Let

$$\lambda_1^{**} = \frac{\eta}{3M_\eta \int_0^\xi g^+(s)\,ds}, \qquad \lambda_2^{**} = \frac{1}{3A^*R} < \frac{1}{3A^*\eta}, \qquad \lambda_3^{**} = \frac{\eta}{3nM_\eta^*},$$

where $0 < \eta < r$, and A^* is defined in (3.3).

We define M_η and M_η^* by

$$M_\eta = \max_{0\leq x\leq \eta} \{f(x)\} > 0, \qquad M_\eta^* = \max\{m_k, \ k = 1,2,\dots,n\} > 0,$$

$$m_k = \min_{0\leq x\leq \eta} \{I_k(x)\}, \quad k = 1,2,\dots,n.$$

Noticing (H_2) and (H_3), then M_η and M_η^* are defined well and $M_\eta > 0$, $M_\eta^* > 0$. So, for any $x \in K \cap \partial\Omega_\eta$, similar to the proof of (3.4), we get

$$
\begin{aligned}
\|Tx\|_{PC} &\leq \lambda_2 \int_0^\xi \int_0^\xi G(s,\tau)a^+(s)b^+(\tau)x(s)x(\tau)\,d\tau\,ds \\
&\quad + \lambda_1 \int_0^\xi g^+(s)f(x(s))\,ds + \lambda_3 \sum_{k=1}^n I_k(x(t_k)) \\
&\leq \lambda_2 \|x\|_{PC}^2 A^* + \lambda_1 \int_0^\xi g^+(s)\,ds M_\eta + n\lambda_3 M_\eta^* \\
&\leq \lambda_2^{**} \|x\|_{PC}^2 A^* + \lambda_1^{**} \int_0^\xi g^+(s)\,ds M_\eta + n\lambda_3^{**} M_\eta^* \\
&\leq \frac{1}{3}\|x\|_{PC} + \frac{1}{3}\|x\|_{PC} + \frac{1}{3}\|x\|_{PC} \\
&= \|x\|_{PC},
\end{aligned}
$$

which shows

$$
\|Tx\|_{PC} \leq \|x\|_{PC}, \quad x \in K \cap \partial\Omega_\eta. \tag{3.15}
$$

Therefore, applying Lemma 2.4 to (3.12), (3.14), and (3.15) yields T has two fixed points x_1 and x_2 satisfying $x_1 \in K \cap (\bar{\Omega}_R \setminus \bar{\Omega}_r)$, $x_2 \in K \cap (\Omega_r \setminus \Omega_\eta)$. Thus it follows from Remark 2.3 that problem (1.1) admits at least two positive solutions (x_i, y_i) $(i = 1, 2)$ with $x_1 \in K \cap (\bar{\Omega}_R \setminus \bar{\Omega}_r)$, $x_2 \in K \cap (\Omega_r \setminus \Omega_\eta)$, and

$$
y_1(t) = \lambda_2 \int_0^1 G_1(t,s)b(s)x_1(s)\,ds,
$$

$$
y_2(t) = \lambda_2 \int_0^1 G_1(t,s)b(s)x_2(s)\,ds.
$$

The proof of Theorem 3.1 is complete. □

4 Remarks and comments

In this section, we offer some remarks and comments on the associated problem (1.1).

Remark 4.1 It is not difficult to see that the details of the proof of (3.4), (3.10) are very different from those of (3.12) and (3.13), respectively. And the idea to prove (3.13) comes from Theorem 3.2 of [50].

Remark 4.2 Some ideas of dealing with parameters in Theorem 3.1 come from Theorems 2.1–2.4 in [51], but there are very few papers in the literature considering the multiplicity of positive solutions for second-order impulsive differential system, especially under the case the weight functions $a(t)$, $b(t)$, and $g(t)$ change sign on $[0, 1]$.

Remark 4.3 It is not difficult to see that conditions (H_2), (H_3), and (H_7) play an important role in the proof of Theorem 3.1, and there are many functions satisfying (H_2), (H_3), and

(H_7), for example,

$$f(x) = x^\alpha, \qquad I_k(x) = x^\alpha, \quad k = 1, 2, \ldots, n, 0 < \alpha < +\infty.$$

Remark 4.4 If we respectively replace conditions (H_2) and (H_3) by

(H_2') $f \in C(\mathcal{R}^+, \mathcal{R}^+)$,

(H_3') $I_k \in C(\mathcal{R}^+, \mathcal{R}^+)$,

then we cannot guarantee $\sigma_\eta > 0$, $\sigma_\eta^* > 0$, $M_\eta >$ and $M_\eta^* > 0$, which are respectively defined in Theorem 3.1. Further, we cannot obtain the results of Theorem 3.1.

Remark 4.5 Similarly, one can consider the multiplicity of positive solutions for the following nonlocal impulsive systems:

$$\begin{cases}
-x'' = a(t)xy + \lambda_1 g(t)f(x), & 0 < t < 1, t \neq t_k, \\
-y'' = \lambda_2 b(t)x, & 0 < t < 1, \\
\Delta x|_{t=t_k} = \lambda_3 I_k(x(t_k)), & \\
\Delta x'|_{t=t_k} = 0, & k = 1, 2, \ldots, n, \\
x(0) = 0, \quad x'(1) = 0, & \\
y(0) = 0, \quad y'(1) = 0, &
\end{cases} \tag{4.1}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$ are three parameters, $a(t)$, $b(t)$, $g(t)$ change sign on $[0, 1]$.

On the other hand, we conjecture that the conclusion in Theorem 3.1 holds for the following systems:

$$\begin{cases}
-x'' = a(t)xy + \lambda_1 g(t)f(x), & 0 < t < 1, t \neq t_k, \\
-y'' = \lambda_2 b(t)x, & 0 < t < 1, \\
-\Delta x'|_{t=t_k} = \lambda_3 I_k(x(t_k)), & k = 1, 2, \ldots, n, \\
x(0) = 0, \quad x'(1) = 0, & \\
y(0) = y(1) = 0, &
\end{cases} \tag{4.2}$$

and

$$\begin{cases}
-x'' = a(t)xy + \lambda_1 g(t)f(x), & 0 < t < 1, t \neq t_k, \\
-y'' = \lambda_2 b(t)x, & 0 < t < 1, \\
-\Delta x'|_{t=t_k} = \lambda_3 I_k(x(t_k)), & k = 1, 2, \ldots, n, \\
x(0) = 0, \quad x'(1) = 0, & \\
y(0) = 0, \quad y'(1) = 0, &
\end{cases} \tag{4.3}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_3 > 0$ are three parameters, $a(t)$, $b(t)$, $g(t)$ change sign on $[0, 1]$.

5 An example

Example 5.1 Let $n = 1$, $t_1 = \frac{1}{5}$. Consider the following system:

$$
\begin{cases}
-x'' = a(t)xy + \lambda_1 g(t)\sqrt{x}, & 0 < t < 1, t \neq \frac{1}{5}, \\
-y'' = \lambda_2 b(t)x, & 0 < t < 1, \\
\Delta x|_{t=\frac{1}{5}} = \lambda_3 I_1(x(\frac{1}{5})), \\
\Delta x'|_{t=\frac{1}{5}} = 0, \\
x(0) = 0, \qquad x'(1) = 0, \\
y(0) = y(1) = 0,
\end{cases}
\tag{5.1}
$$

where $I_1(x) = \sqrt{x}$ and

$$
b(t) = \begin{cases}
\frac{128}{3}(\frac{1}{3} - t), & t \in [0, \frac{1}{3}], \\
-\frac{1}{16}(t - \frac{1}{3}), & t \in [\frac{1}{3}, 1],
\end{cases}
$$

$$
a(t) = \begin{cases}
\frac{2048}{3}(\frac{1}{3} - t), & t \in [0, \frac{1}{3}], \\
-\frac{1}{16}(t - \frac{1}{3}), & t \in [\frac{1}{3}, 1],
\end{cases}
$$

$$
g(t) = \begin{cases}
\frac{64}{\sqrt{3}}(\frac{1}{3} - t), & t \in [0, \frac{1}{3}], \\
-\frac{1}{16}(t - \frac{1}{3}), & t \in [\frac{1}{3}, 1].
\end{cases}
$$

Firstly, we give a result which is similar to that of Proposition 2.3 in [36]. Consider the following boundary value problems:

$$
\begin{cases}
-x'' = k(t)x^\alpha, & 0 < t < 1, \\
x(0) = 0, \qquad x'(1) = 0,
\end{cases}
\tag{5.2}
$$

$$
\begin{cases}
-x'' = k(t)x^\alpha, & 0 < t < 1, \\
x(0) = x(1) = 0,
\end{cases}
\tag{5.3}
$$

where $\alpha > 0$ with $\alpha \neq 1$, $k(t)$ satisfies the changing sign condition

$$
\begin{cases}
k(t) \geq 0, & t \in [0, \xi], \\
k(t) \leq 0, & t \in [\xi, 1],
\end{cases}
$$

and

$$
c_1 x^\alpha \leq f(x) = x^\alpha \leq c_2 x^\alpha, \qquad c_1, c_2 > 0.
$$

If there exists $0 < \sigma < \xi$ such that

$$
c_1 \frac{\xi - \sigma}{1 - \xi} \sigma^{\alpha+1} \mu^2 k^+ \left(\xi - \frac{\xi - \sigma}{1 - \xi} \tau \right) \geq c_2 \xi^\alpha k^-(\xi + \tau), \qquad \tau \in [0, 1 - \xi],
\tag{5.4}
$$

then the following inequalities hold:

$$\sigma^\alpha \mu^2 \int_\sigma^\xi G(t,s) k^+(s)\,ds \geq \frac{c_2}{c_1} \xi^\alpha \int_\xi^1 G(t,s) k^-(s)\,ds, \tag{5.5}$$

$$\sigma^\alpha \mu^2 \int_\sigma^\xi G_1(t,s) k^+(s)\,ds \geq \frac{c_2}{c_1} \xi^\alpha \int_\xi^1 G_1(t,s) k^-(s)\,ds. \tag{5.6}$$

Next, we study the multiplicity of positive solutions of problem (5.1). From definitions of $a(t)$, $b(t)$, and $g(t)$, we know that $\xi = \frac{1}{3}$.

Step 1. We show that (H_4) holds. For fixed $c_1 = c_2 = 1$, $\sigma_1 = \frac{1}{4}$, $\mu = 1$, and $\alpha = 1$, (5.4) is equivalent to the following inequality:

$$\frac{3}{128} b^+ \left(\frac{1}{3} - \frac{1}{8}\tau \right) \geq b^- \left(\frac{1}{3} + \tau \right), \quad \tau \in \left[0, \frac{2}{3} \right].$$

Letting $\frac{1}{3} - \frac{1}{8}\tau = \zeta$, then the above inequality is equivalent to

$$\frac{3}{128} b^+ (\zeta) \geq b^- (3 - 8\zeta), \quad \zeta \in \left[\frac{1}{4}, \frac{1}{3} \right].$$

By the definition of $b(t)$, the above inequality holds obviously. It is clear that by (5.6), (H_4) is reasonable.

Step 2. We show that (H_5) holds. Similar to the arguments in Step 1, letting $c_1 = 1$, $c_2 = 16$, $\sigma_2 = \frac{1}{4}$, $\mu = 1$, and $\alpha = 1$, then by (5.5) we get

$$\frac{1}{4} \int_{\frac{1}{4}}^{\frac{1}{3}} G(t,s) a^+(s)\,ds \geq \frac{16}{3} \int_{\frac{1}{3}}^1 G(t,s) a^-(s)\,ds.$$

Further, from the above inequality, it follows that

$$\frac{1}{4} \int_{\frac{1}{4}}^{\frac{1}{3}} G(t,s) \frac{3}{16} a^+(s)\,ds \geq \int_{\frac{1}{3}}^1 G(t,s) a^-(s)\,ds$$

$$\Leftrightarrow \quad \frac{1}{4} \int_{\frac{1}{4}}^{\frac{1}{3}} G(t,s) \left(\min_{s \in [\frac{1}{4}, \frac{1}{3}]} G_1(s,s) \right) a^+(s)\,ds \geq \int_{\frac{1}{3}}^1 G(t,s) a^-(s)\,ds$$

$$\Rightarrow \quad \frac{1}{4} \int_{\frac{1}{4}}^{\frac{1}{3}} G(t,s) G_1(s,s) a^+(s)\,ds \geq \int_{\frac{1}{3}}^1 G(t,s) a^-(s)\,ds$$

$$\Rightarrow \quad \frac{1}{4} \int_{\frac{1}{4}}^{\frac{1}{3}} G(t,s) G_1(s,s) a^+(s)\,ds > \frac{1}{3} \int_{\frac{1}{3}}^1 G(t,s) a^-(s)\,ds.$$

So, (H_5) holds.

Step 3. Similar to the arguments in Step 1, letting $c_1 = c_2 = 1$, $\sigma_3 = \frac{1}{4}$, $\mu = 1$, and $\alpha = \frac{1}{2}$, then (H_8) holds.

Step 4. It is easy to see by direct calculation that

$$\int_0^\xi g^+(s)\,ds = \int_0^{\frac{1}{3}} \frac{64}{\sqrt{3}} \left(\frac{1}{3} - s \right) ds = \frac{32}{9\sqrt{3}},$$

$$A^* = \int_0^\xi \int_0^\xi G_1(s,\tau) a^+(s) b^+(\tau) \, d\tau \, ds = \int_0^{\frac{1}{3}} a^+(s) \int_0^{\frac{1}{3}} G_1(s,\tau) b^+(\tau) \, d\tau \, ds$$

$$= \int_0^{\frac{1}{3}} a^+(s) \left[\int_0^s \tau(1-s) b^+(\tau) \, d\tau + \int_s^{\frac{1}{3}} s(1-\tau) b^+(\tau) \, d\tau \right] ds = \frac{4,325,376}{885,735}.$$

Let $\eta = 1$, $r = 2$, and $R = 10$. Then we have

$$M_\eta = \max_{0 \le x \le 1} \{\sqrt{x}\} = 1, \qquad M_\eta^* = \max\{m_k^*, k = 1, 2, \ldots, n\} = 1,$$

$$m_k = \max_{0 \le x \le 1} \{\sqrt{x}\} = 1, \quad k = 1, 2, \ldots, n.$$

Therefore, we get

$$\lambda_1^{**} = \frac{\eta}{3M_\eta \int_0^\xi g^+(s) \, ds} \approx 0.1623, \qquad \lambda_2^{**} = \frac{1}{3A^* R} \approx 0.0068,$$

$$\lambda_3^{**} = \frac{\eta}{3nM_\eta^*} \approx 0.3333.$$

Hence, all conditions of (II) of Theorem 3.1 hold. Thus by (II) of Theorem 3.1, problem (5.1) admits at least two positive solutions for $0 < \lambda_1 \le 0.1623$, $0 < \lambda_2 \le 0.0068$, and $0 < \lambda_3 \le 0.3333$.

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