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Limit cycles in a quartic system with a third-order nilpotent singular point

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Abstract

In this paper, limit cycles bifurcating from a third-order nilpotent critical point in a class of quartic planar systems are studied. With the aid of computer algebra system MAPLE, the first 12 Lyapunov constants are deduced by the normal form method. As a result, sufficient and necessary center conditions are derived, and the fact that there exist 12 or 13 limit cycles bifurcating from the nilpotent critical point is proved by different perturbations. The result in [Qiu et al. in *Adv. Differ. Equ.* 2015(1):1, 2015] is improved.

Keywords: Quartic system; Nilpotent critical point; Lyapunov constants; Bifurcation of limit cycles

1 Introduction

An isolated critical point $O(0,0)$ is called a nilpotent singular point if the linear part of systems has double zero eigenvalues but the matrixes of the linearized systems at the origin are not identically null. Over the past twenty years, many different kinds of topological phase portraits around a nilpotent critical point have been given in [2]. Some results before 1980 were also introduced in [3]. It is well known that it is more difficult to study a phase portrait around a nilpotent critical point than around an element critical point. However, more and more attention has been paid to the center problem and bifurcation of limit cycles of a system with a nilpotent critical point recently.

By a proper linear transformation, the planar autonomous analytic system with a nilpotent critical point can always be changed into

$$\begin{aligned}\frac{dx}{dt} &= \Phi(x, y) = y + \sum_{k+j=2}^{\infty} a_{kj}x^k y^j, \\ \frac{dy}{dt} &= \Psi(x, y) = \sum_{k+j=2}^{\infty} b_{kj}x^k y^j,\end{aligned}\tag{1.1}$$

where $\Phi(x, y)$ and $\Psi(x, y)$ are analytic functions in the neighborhood of origin.

On the basis of the discussion in [3], the origin of system (1.1) is a monodromic critical point if and only if

$$\begin{aligned} \Psi(x, f(x)) &= \alpha x^{2n-1} + o(x^{2n-1}), \quad \alpha \neq 0, \\ \left[\frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y} \right]_{y=f(x)} &= \beta x^{n-1} + o(x^{n-1}), \\ \beta^2 + 4n\alpha &< 0, \end{aligned} \quad (1.2)$$

where n is a positive integer. The origin is called a $(2n - 1)$ -order nilpotent critical point.

In [4], local behavior of an isolated nilpotent critical point for polynomial Hamiltonian systems was investigated. They proved that there are exactly three cases: a center, a cusp, or a saddle. Then, for quadratic and cubic Hamiltonian systems, they obtained necessary and sufficient conditions for a nilpotent critical point to be a center, a cusp, or a saddle. Some special systems were also studied in [5, 6] by this method. Furthermore, limit cycle bifurcation near a double homoclinic loop passing through a nilpotent saddle by studying the analytical property of the first-order Melnikov functions for general near-Hamiltonian systems was studied in [7]. They obtained the conditions for the perturbed system to have 8, 10, or 12 limit cycles in a neighborhood of the loop with seven different distributions. Recently, the expansion of the first Melnikov function appearing by perturbing an integrable and reversible system with a homoclinic loop passing through a nilpotent singular point was investigated in [8], and the authors got the formulas for computing the first coefficients of the expansion. Some other methods were also developed recently. For example, in [9], it was proved that all the nilpotent centers are limit of linear type centers; and consequently, the Poincaré–Lyapunov method to find linear type centers can be also used to find the nilpotent centers. Quasi-homogeneous vector fields with a nilpotent and monodromic isolated singular point were investigated in [10]. They also proved for this system the existence of a Lyapunov function, and they solved theoretically the center problem for such a system. Bifurcation theory for finitely smooth planar autonomous differential systems was considered in [11]. Theory of rotated equations was discussed in [12] and was applied to a population model.

The origin is called a third-order nilpotent critical point when $n = 2$ in (1.2). For the planar systems with a third-order nilpotent critical point, the center problem has been completely solved in [13] where the author proved that the system has always an integrating factor, but this method is not valid for any nilpotent singularity. The inverse integrating factor method for the planar systems with a third-order nilpotent critical point was also discussed in [14, 15]. By using those methods, the fact that there exist eight small amplitude limit cycles created from nilpotent critical points was proved in [16] for a class of cubic systems by their method. Furthermore, a new kind of bifurcation phenomenon was discussed in [17]. They proved that a third-order nilpotent focus of the planar dynamical systems could be broken into two element foci and an element saddle, limit cycles could bifurcate out from two-element focus. As an example, a class of cubic systems with 3-multiple nilpotent focus were investigated. They proved that nine limit cycles could bifurcate from the origin when the origin is a weak focus of order eight. For third-order nilpotent critical points of a planar dynamical system, the analytic center problem was completely solved by using the integrating factor method [18]. By using this method, some special systems were investigated in [1, 19, 20].

In fact, Theorem 19.10 in [3] shows when condition (1.2) holds, there exist formal transformations

$$\begin{aligned}
 u &= x + \sum_{k+j=2}^{\infty} a'_{kj} x^k y^j, \\
 v &= y + \sum_{k+j=2}^{\infty} b'_{kj} x^k y^j, \\
 \frac{dt}{d\tau} &= 1 + \sum_{k+j=1}^{\infty} c'_{kj} x^k y^j,
 \end{aligned} \tag{1.3}$$

such that (1.1) becomes the Liénard equation

$$\frac{du}{d\tau} = v + F(u), \quad \frac{dv}{d\tau} = \alpha u^{2n-1}, \tag{1.4}$$

where

$$F(u) = \frac{1}{n} \beta u^n + o(u^n). \tag{1.5}$$

Furthermore, if condition (1.2) holds, there exist analytic transformations in the neighborhood of origin of (1.3) to make (1.1) be changed into the Liénard equation

$$\frac{du}{d\tau} = v, \quad \frac{dv}{d\tau} = \alpha u^{2n-1} + v \sum_{k=n-1}^{\infty} B_k u^k, \quad B_{n-1} = \beta, \tag{1.6}$$

so B_{2k} could be thought of as Lyapunov constants of (1.1).

From then, the normal form theory was applied to solve the center-focus problem for monodromic planar nilpotent singularities. In [3, 21–24], by considering the normal forms of (1.1), the authors tried to study the computation problem of focal values. In [25], by means of a recursive algorithm well suited to symbolic computation of normal form, authors achieved the expressions for its coefficients and investigated the possibilities of simplifying the classical normal form, obtaining simpler and higher order normal forms than in previous works. Furthermore, in [26], authors showed that normal form can also be applied to generate limit cycles from nilpotent singularities. In [27], Hamiltonian linear type centers and nilpotent centers of linear plus cubic polynomial vector fields were considered. They provided twelve normal forms for all the Hamiltonian planar polynomial vector fields having linear plus cubic homogeneous terms which possess a linear type center or a nilpotent center at the origin, and found their global phase portraits on the Poincare disk. Last year, the normal form method was used to investigate the nilpotent critical point again by Pei Yu. The authors gave an efficient program by MAPLE in [28]. Ten limit cycles bifurcating from a nilpotent critical point in a cubic system are obtained. For a quartic system with third-order critical point, 11 limit cycles were obtained in [1] by using the inverse integrating factor method. In this paper, a class of quartic systems with

third-order nilpotent singular point

$$\begin{aligned} \frac{dx}{dt} &= y - b_{12}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4, \\ \frac{dy}{dt} &= -2x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{40}x^4 - 4a_{40}x^3y - \frac{3}{2}a_{31}x^2y^2 \\ &\quad + b_{13}xy^3 + b_{04}y^4 \end{aligned} \tag{1.7}$$

are studied by using the normal form method, one more limit cycle is found.

The rest of this paper is organized as follows. In Sect. 2, with the aid of computer algebra system MAPLE, the first 12 Lyapunov constants are computed by the normal form method. Sufficient and necessary center conditions are derived. In Sect. 3, the fact that there exist 12 or 13 limit cycles bifurcating from the nilpotent critical point is proved. Henceforth, we give a lower bound of cyclicity of a nilpotent critical point for quartic polynomial systems. The result in [1] is improved.

2 Lyapunov constants and center conditions

According to normal form theory, for system (1.1), we can find analytic transformations in the neighborhood of origin to transform system (1.7) into (1.6). Applying the recursive formulae in [28] to carry out calculations, we have the following theorem.

Theorem 2.1 *The first 12 quasi-Lyapunov constants at the origin of system (1.7) are given as follows:*

$$\begin{aligned} B_2 &= \frac{b_{21}}{3}, \\ B_4 &= \frac{2(a_{12} + 3b_{03})}{5}, \\ B_6 &= \frac{b_{40}(2a_{22} + 3b_{13})}{35}, \\ B_8 &= -\frac{(2a_{22} + 3b_{13})a_{31}}{15}, \\ B_{10} &= \frac{10}{77}(3a_{40}b_{03} - 2b_{04})(2a_{22} + 3b_{13}), \\ B_{12} &= \frac{-2}{3003}b_{03}(342a_{22} + 539a_{40}b_{12} - 26b_{13})(2a_{22} + 3b_{13}), \\ B_{14} &= -\frac{1}{83,655}b_{03}(36,504a_{04} + 17,745a_{40}^3 - 20,744a_{22}b_{12} - 31,116a_{40}b_{12}^2) \\ &\quad (2a_{22} + 3b_{13}), \\ B_{16} &= \frac{14}{1,680,705}b_{03}(217,464a_{03}a_{22} + 398,853a_{22}a_{40}^2 + 334,620a_{40}b_{03}^2 \\ &\quad + 326,196a_{03}a_{40}b_{12} + 561,522a_{40}^3b_{12} + 12,244a_{22}b_{12}^2 + 18,366a_{40}b_{12}^3) \\ &\quad \times (2a_{22} + 3b_{13}). \end{aligned} \tag{2.1}$$

Case 1: $2a_{22} + 3a_{40}b_{12} \neq 0$

$$\begin{aligned}
 B_{18} = & \frac{b_{03}(2a_{22} + 3b_{13})}{133,545,457,890(2a_{22} + 3a_{40}b_{12})} (409,339,646,784a_{22}^3a_{40} \\
 & - 76,483,842,435a_{22}a_{40}^5 \\
 & + 265,048,602,624a_{22}^2b_{03}^2 + 100,943,142,300a_{40}^4b_{03}^2 \\
 & + 2,055,888,104,472a_{22}^2a_{40}^2b_{12} \\
 & - 125,814,214,890a_{40}^6b_{12} + 1,353,439,930,752a_{22}a_{40}b_{03}^2b_{12} \\
 & + 3,384,047,982,384a_{22}a_{40}^3b_{12}^2 + 1,433,800,540,224a_{40}^2b_{03}^2b_{12}^2 \\
 & - 3,536,917,504a_{22}^2b_{12}^3 + 1,831,845,046,410a_{40}^4b_{12}^3 \\
 & - 10,610,752,512a_{22}a_{40}b_{12}^4 - 7,958,064,384a_{40}^2b_{12}^5), \\
 B_{20} = & -b_{03}(2a_{22} + 3b_{13})f_1, \\
 B_{22} = & b_{03}(2a_{22} + 3b_{13})f_2, \\
 B_{24} = & b_{03}(2a_{22} + 3b_{13})f_3.
 \end{aligned} \tag{2.2}$$

Case 2: $2a_{22} + 3a_{40}b_{12} = 0$

$$\begin{aligned}
 B_{16} = & \frac{7}{663}a_{40}b_{03}(264b_{03}^2 - 29a_{40}^2b_{12})(2a_{22} + 3b_{13}), \\
 B_{18} = & -\frac{1}{627}a_{40}^3b_{03}(154a_{03} + 462a_{40}^2 - 87b_{12}^2)(2a_{22} + 3b_{13}), \\
 B_{20} = & \frac{1}{70,686}a_{40}^3b_{03}b_{12}(9779a_{40}^2 - 1740b_{12}^2)(2a_{22} + 3b_{13}), \\
 B_{22} = & \frac{4,589,527}{6,273,135}a_{40}^7b_{03}(2a_{22} + 3b_{13}).
 \end{aligned} \tag{2.3}$$

For each $k = 2, \dots, 11$, $B_2 = B_4 = \dots = B_{2k-2} = 0$ have been set. $f_i, i = 1, 2, 3$, are given in the Appendix. They have 26, 30, 48 terms, respectively.

Theorem 2.1 directly shows the following assertion.

Proposition 2.1 *For system (1.7), the first twelve Lyapunov constants at the origin are zero if and only if one of the following three conditions is satisfied:*

$$b_{21} = a_{12} = b_{03} = b_{40} = a_{31} = b_{04} = 0; \tag{2.4}$$

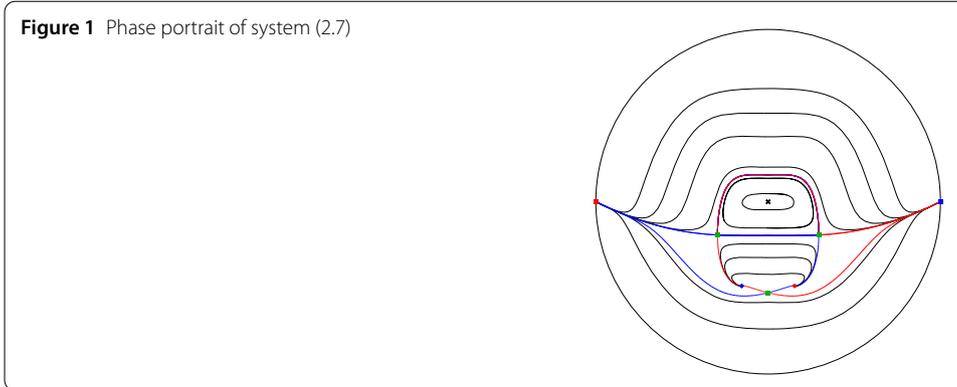
$$b_{21} = 0, \quad a_{12} = -3b_{03}, \quad a_{22} = -\frac{3b_{13}}{2}; \tag{2.5}$$

$$b_{21} = a_{31} = b_{40} = b_{04} = b_{13} = a_{04} = a_{22} = a_{40} = 0, \quad a_{12} = -3b_{03}. \tag{2.6}$$

According to Proposition 2.1, we have the following theorem.

Theorem 2.2 *The origin of system (1.7) is a center if and only if the first twelve Lyapunov constants are zero, that is, one of the conditions in Proposition 2.1 is satisfied.*

Figure 1 Phase portrait of system (2.7)



Proof When condition (2.4) is satisfied, system (1.7) can be brought to

$$\begin{aligned} \frac{dx}{dt} &= y - b_{12}x^2y + a_{03}y^3 + a_{40}x^4 + a_{22}x^2y^2 + a_{04}y^4, \\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 - 4a_{40}x^3y + b_{13}xy^3, \end{aligned} \tag{2.7}$$

whose vector field is symmetric with respect to the y -axis. An example when all parameters are equal to 1 is given in Fig. 1.

When condition (2.5) holds, system (1.7) can be rewritten as follows:

$$\begin{aligned} \frac{dx}{dt} &= y - b_{12}x^2y - 3b_{03}xy^2 + a_{03}y^3 + a_{40}x^4 + a_{31}x^3y \\ &\quad - \frac{3}{2}b_{13}x^2y^2 - 4b_{04}xy^3 + a_{04}y^4, \\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 + b_{03}y^3 + b_{40}x^4 - 4a_{40}x^3y - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4, \end{aligned} \tag{2.8}$$

which has an analytic first integral

$$\begin{aligned} H_1(x, y) &= \frac{1}{2}y^2 + \frac{1}{2}x^4 + \left(\frac{a_{03}}{4}y^4 + \frac{a_{04}}{5}y^5 - \frac{b_{40}}{5}x^5 - \frac{b_{12}}{2}x^2y^2 \right. \\ &\quad \left. - \frac{a_{31}}{2}x^3y^2 - b_{03}xy^3 - \frac{b_{13}}{2}x^2y^3 - b_{04}xy^4 \right). \end{aligned} \tag{2.9}$$

An example when all parameters are equal to 1 is given in Fig. 2.

When condition (2.6) holds, system (1.7) becomes

$$\begin{aligned} \frac{dx}{dt} &= y - b_{12}x^2y - 3b_{03}xy^2 + a_{03}y^3, \\ \frac{dy}{dt} &= -2x^3 + b_{12}xy^2 + b_{03}y^3, \end{aligned} \tag{2.10}$$

which has an analytic first integral

$$H_2(x, y) = \frac{1}{2}y^2 + \frac{1}{2}x^4 - \left(\frac{b_{12}}{2}x^2y^2 + b_{03}xy^3 \right).$$

An example when all parameters are equal to 1 is given in Fig. 3. □

Figure 2 Phase portrait of system (2.8)

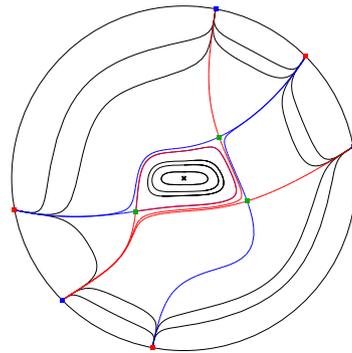
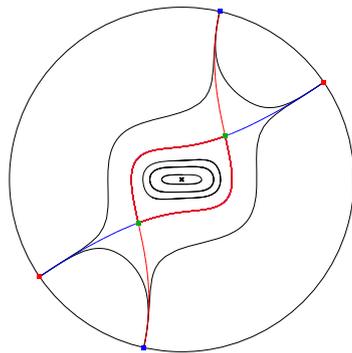


Figure 3 Phase portrait of system (2.10)



Remark 2.1 The center condition of system (1.7) is the same as the center condition of system in [1].

3 Bifurcation of a limit cycle from the origin of system (1.7)

In this section, two perturbation methods are given in order to obtain more limit cycles. The first method is called perturbation method of small parameters by perturbing the coefficients. The second method is double bifurcation.

3.1 Perturbation method of small parameters

At first, we will prove that the perturbed system of (1.7) can generate twelve limit cycles enclosing an elementary node at the origin of unperturbed system (1.7) when the third-order nilpotent critical point $O(0, 0)$ is a 12th-order weak focus by perturbing the coefficients.

$B_2 = B_4 = B_6 = B_8 = B_{10} = B_{12} = B_{14} = B_{16} = B_{18} = B_{20} = B_{22} = 0, = B_{24} \neq 0$ show the following.

Theorem 3.1 For system (1.7), the origin is a 12th-order weak focus if and only if $a_{22}b_{03}b_{12}(2a_{22} + 3b_{13}) \neq 0$ and

$$b_{21} = b_{40} = a_{31} = 0,$$

$$a_{12} = -3b_{03},$$

$$b_{04} = \frac{3a_{40}b_{03}}{2},$$

$$\begin{aligned}
 b_{13} &= \frac{342a_{22} + 539a_{40}b_{12}}{26}, \\
 a_{04} &= -\frac{28,665a_{40}^3 + 56,568a_{22}b_{12} + 91,700a_{40}b_{12}^2 - 6848b_{12}b_{13}}{58,968}, \\
 a_{03} &= -(398,853a_{22}a_{40}^2 + 334,620a_{40}b_{03}^2 + 561,522a_{40}^3b_{12} + 12,244a_{22}b_{12}^2 \\
 &\quad + 18,366a_{40}b_{12}^3)/(108,732(2a_{22} + 3a_{40}b_{12})), \\
 b_{03}^2 &= f_4/(468(566,343,168a_{22}^2 + 215,690,475a_{40}^4 + 2,891,965,664a_{22}a_{40}b_{12} \\
 &\quad + 3,063,676,368a_{40}^2b_{12}^2)),
 \end{aligned}
 \tag{3.1}$$

where

$$\begin{aligned}
 f_4 &= -409,339,646,784a_{22}^3a_{40} + 76,483,842,435a_{22}a_{40}^5 \\
 &\quad - 2,055,888,104,472a_{22}^2a_{40}^2b_{12} \\
 &\quad + 125,814,214,890a_{40}^6b_{12} - 3,384,047,982,384a_{22}a_{40}^3b_{12}^2 \\
 &\quad + 3,536,917,504a_{22}^2b_{12}^3 \\
 &\quad - 1,831,845,046,410a_{40}^4b_{12}^3 + 10,610,752,512a_{22}a_{40}b_{12}^4 \\
 &\quad + 7,958,064,384a_{40}^2b_{12}^5.
 \end{aligned}
 \tag{3.2}$$

Proof From $B_2 = B_4 = B_6 = B_8 = B_{10} = B_{12} = B_{14} = B_{16} = B_{18} = 0$, we obtain

$$\begin{aligned}
 b_{21} &= b_{40} = a_{31} = 0, \\
 a_{12} &= -3b_{03}, \quad b_{04} = \frac{3a_{40}b_{03}}{2}, \\
 b_{13} &= \frac{342a_{22} + 539a_{40}b_{12}}{26}, \\
 a_{04} &= -\frac{28,665a_{40}^3 + 56,568a_{22}b_{12} + 91,700a_{40}b_{12}^2 - 6848b_{12}b_{13}}{58,968}, \\
 a_{03} &= -(398,853a_{22}a_{40}^2 + 334,620a_{40}b_{03}^2 + 561,522a_{40}^3b_{12} + 12,244a_{22}b_{12}^2 \\
 &\quad + 18,366a_{40}b_{12}^3)/(108,732(2a_{22} + 3a_{40}b_{12})), \\
 b_{03}^2 &= f_4/(468(566,343,168a_{22}^2 + 215,690,475a_{40}^4 + 2,891,965,664a_{22}a_{40}b_{12} \\
 &\quad + 3,063,676,368a_{40}^2b_{12}^2)),
 \end{aligned}
 \tag{3.3}$$

and let

$$f_{12} = \text{Res}[f_1, f_2, a_{22}], \quad f_{13} = \text{Res}[f_1, f_3, a_{22}],$$

f_{12} and f_{13} have 20, 23 terms, respectively.

$$\text{Res}[f_{12}, f_{13}, b_{12}] = 6.48339335810581 \times 10^{15,074} a_{40}^{1672}.$$

If $a_{40} = 0$, we get $a_{22} = 0$, which contradicts $2a_{22} + 3a_{40}b_{12} \neq 0$. So, when condition (3.3) holds, the origin of system (1.7) is a 12th-order weak focus. □

Next, we study the perturbed system of (1.7) given by

$$\begin{aligned} \frac{dx}{dt} &= \delta x + y - b_{12}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{40}x^4 + a_{31}x^3y + a_{22}x^2y^2 \\ &\quad - 4b_{04}xy^3 + a_{04}y^4, \\ \frac{dy}{dt} &= \delta y - 2x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + b_{40}x^4 - 4a_{40}x^3y \\ &\quad - \frac{3}{2}a_{31}x^2y^2 + b_{13}xy^3 + b_{04}y^4. \end{aligned} \tag{3.4}$$

When conditions in (3.3) hold, using relationships $B_2 = B_4 = B_6 = B_8 = B_{10} = B_{12} = B_{14} = B_{16} = B_{18} = B_{20} = B_{22} = 0$, we can determine the values of

$$b_{21}, a_{12}, b_{40}, a_{31}, b_{04}, b_{13}, a_{04}, a_{03}, b_{03}.$$

Hence, when the conditions in (3.3) are satisfied, we have

$$\begin{aligned} &\frac{\partial(B_2, B_4, B_6, B_8, B_{10}, B_{12}, B_{14}, B_{16}, B_{18}, B_{20}, B_{22})}{\partial(b_{21}, a_{12}, b_{40}, a_{31}, b_{04}, b_{13}, a_{04}, a_{03}, b_{03}, a_{22})} \\ &= f_5. \end{aligned} \tag{3.5}$$

Denote $f_{15} = \text{Res}[f_1, f_5, a_{22}]$, it has 51 terms.

$$\text{Res}[f_{12}, f_{15}, b_{12}] = 6.48339335810581 \times 10^{15,074} a_{40}^{1672} \neq 0.$$

So, when condition (3.3) holds and $f_1 = f_2 = 0$, we can conclude that $f_5 \neq 0$.

Further, Theorem 3.1.3 in [15] yields that if the origin of system (3.4) $_{|\delta=\varepsilon=0}$ is a weak focus of order m , then, when $0 < \delta, \varepsilon \ll 1$, (3.4) has at most m limit cycles in a neighborhood of the origin. Namely the following theorem holds.

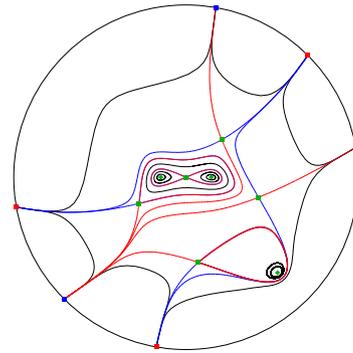
Theorem 3.2 *If the origin of system (1.1) is a 12th-order weak focus, for $0 < \delta, \varepsilon \ll 1$, then, for system (3.4), in a small neighborhood of the origin, there exist exactly 12 small amplitude limit cycles enclosing the origin $O(0, 0)$, which is an elementary node.*

3.2 Double perturbation method

In this section, an interesting bifurcation of limit cycles, which is different from the first kind of bifurcation discussed in previous section, will be considered. One more small-amplitude limit cycles near the origin can be found. Consider the following perturbed system of (1.1):

$$\begin{aligned} \frac{dx}{dt} &= a_{40}x^2(x^2 - \varepsilon^2) + y(1 + x(-b_{12}x + a_{31}x^2)) + (a_{12}x + a_{03}y + a_{22}x^2 \\ &\quad - 4b_{04}xy + a_{04}y^2)y^2, \\ \frac{dy}{dt} &= (4\delta\varepsilon - 2a_{40}\varepsilon^2x)y - (x^2 - \varepsilon^2)\left(2x\left(1 - \frac{b_{40}}{2}x\right) - (b_{21} - 4a_{40}x)y\right) \\ &\quad + \left(b_{12}x + b_{03}y - \frac{3}{2}a_{31}x^2 + b_{13}xy + b_{04}y^2\right)y^2. \end{aligned} \tag{3.6}$$

Figure 4 Phase portrait of system (3.6)



System (3.6) is called double perturbed system of system (1.7). When $0 < |\varepsilon| \ll 1$, system (3.6) has three real singular points in the neighborhood of the origin, namely $O(0, 0)$ and $P_{1,2}(\pm\varepsilon; 0)$.

By transformation

$$\begin{aligned} x &= \varepsilon(u \pm 1), \\ y &= 2\varepsilon^2 \frac{(\delta \pm a_{40}\varepsilon^2 u) - \rho v}{1 \pm \varepsilon(-b_{12}\varepsilon + a_{31}\varepsilon^2)}, \\ t &= \frac{\tau}{2\rho\varepsilon}, \\ \rho &= \sqrt{(1 \pm \varepsilon(\mp b_{12}\varepsilon + a_{31}\varepsilon^2)) \left(1 \mp \frac{b_{40}}{2}\varepsilon\right) - (\delta \mp a_{40}\varepsilon^2)^2}, \end{aligned}$$

$P_{1,2}(\pm\varepsilon, 0)$ of system (3.6) can be shifted to the origin, and we obtain a new system in the form of

$$\begin{aligned} \frac{d\xi}{d\tau} &= \Phi(\xi, \eta, \varepsilon, \delta) = \frac{\delta\xi}{\rho} - \eta + \sum_{k+j=2}^{\infty} A_{kj}(\varepsilon, \delta)\xi^k\eta^j, \\ \frac{d\eta}{d\tau} &= \Psi(\xi, \eta, \varepsilon, \delta) = \xi + \frac{\delta\eta}{\rho} + \sum_{k+j=2}^{\infty} B_{kj}(\varepsilon, \delta)\xi^k\eta^j, \end{aligned} \tag{3.7}$$

where $\Phi(\xi, \eta, \varepsilon, \delta)$ and $\Psi(\xi, \eta, \varepsilon, \delta)$ are power series in $(u, v, \varepsilon, \delta)$ with nonzero convergence radius. So $P_{1,2}(\pm\varepsilon, 0)$ of (3.6) are fine foci when $\delta \neq 0$, and weak foci or centers when $\delta = 0$.

When the origin of system (1.7) is a 12th-order weak focus, the first Lyapunov constant of system (3.7) at origin is

$$V_1 = -\frac{1}{4}b_{21}\varepsilon + o(\varepsilon) \neq 0$$

when $\varepsilon \rightarrow 0$. So it leads to bifurcations of two small-amplitude limit cycles around the two symmetric singular points $P_{1,2}(\pm\varepsilon, 0)$ of (3.6). Similarly, summarizing the above results yields the following theorem.

Theorem 3.3 *If the origin of system (1.1) is a 12-order weak focus, choosing proper coefficients in system (1.1), when $0 < |\varepsilon| \ll 1$, there exist thirteen limit cycles with the distribution*

of one limit cycle enclosing each of $P_{1,2}(\pm\varepsilon, 0)$, and eleven limit cycles enclosing both $(\varepsilon, 0)$ and $(-\varepsilon, 0)$ in the neighborhood of origin.

So, an example which can generate 13 limit cycles by perturbing the quartic system with a nilpotent critical point is given in this section. The following result is easy to obtain from the above discussion.

Theorem 3.4 *If $\delta = 0, b_{21} = 0, a_{12} = -3b_{03}, a_{22} = -\frac{3b_{13}}{2}$, there exist two centers $P(\pm\varepsilon, 0)$ and a saddle $(0, 0)$ in (3.6).*

The Phase portrait of system (3.6) with two centers $P(\pm\varepsilon, 0)$ and a saddle $(0, 0)$ was given in Fig. 4.

4 Conclusion

Let $M(n)$ denote the maximal number of small-amplitude limit cycles bifurcating from either an elementary focus or a center. There have been many results in the literature. $M(2) = 3$ was obtained by Bautin in 1952 [29]. For $n = 3$, the existence of 11 small-amplitude limit cycles around a singular point was proved by [30], which is the best result so far for cubic systems, namely $M(3) = 11$. For a quartic system, using the techniques developed in [31], 21 limit cycles around an elementary focus or a center were found in [32]. For the result to quintic systems, 26 limit cycles around an elementary focus or a center were found in [32]. The authors have improved the bounds known up to now for quartic and quintic systems. Regarding the maximal number of small-amplitude limit cycles bifurcating from a nilpotent singular point, nine limit cycles were found in [28]. In [1], 12 limit cycles were obtained for a quartic system with a nilpotent singular point. In this paper, we obtained 13 limit cycles which can be bifurcated from a third-order nilpotent singular point by the normal form method. We also check our computation by the inverse integrating factor method. The result in [1] is improved.

Appendix

$$\begin{aligned}
 f_1 = & 232,806,990,642,173,234,650,476,969,984a_{22}^7 \\
 & - 477,044,769,321,151,747,627,094,876,160a_{22}^5a_{40}^4 \\
 & - 82,502,158,763,349,110,891,893,788,000a_{22}^3a_{40}^8 \\
 & + 13,425,916,805,077,472,438,495,878,125a_{22}a_{40}^12 \\
 & + 3,626,132,661,875,872,721,780,739,145,728a_{22}^6a_{40}b_{12} \\
 & - 4,874,662,820,265,947,537,539,453,516,800a_{22}^4a_{40}^5b_{12} \\
 & - 432,703,859,977,355,762,471,434,258,500a_{22}^2a_{40}^9b_{12} \\
 & + 19,056,777,587,139,908,577,974,512,500a_{40}^13b_{12} \\
 & + 23,278,576,817,946,630,221,224,920,219,648a_{22}^5a_{40}^2b_{12}^2 \\
 & - 19,532,242,491,602,183,368,669,540,915,200a_{22}^3a_{40}^6b_{12}^2 \\
 & - 752,717,192,943,025,488,830,965,062,750a_{22}a_{40}^10b_{12}^2 \\
 & + 80,182,154,821,647,691,393,757,541,171,200a_{22}^4a_{40}^3b_{12}^3
 \end{aligned}$$

$$\begin{aligned}
 & - 38,225,551,446,390,544,749,673,769,232,000a_{22}^2a_{40}^7b_{12}^3 \\
 & - 433,744,350,169,117,045,973,492,828,250a_{40}^1b_{12}^3 \\
 & - 763,796,971,844,160,152,666,112,000a_{22}^5b_{12}^4 \\
 & + 160,998,064,918,174,797,651,106,928,670,720a_{22}^3a_{40}^4b_{12}^4 \\
 & - 36,568,732,582,402,041,283,807,931,942,400a_{22}^8a_{40}^8b_{12}^4 \\
 & - 5,773,134,335,673,254,169,280,512,000a_{22}^4a_{40}b_{12}^5 \\
 & + 189,440,538,958,857,143,945,451,460,248,576a_{22}^2a_{40}^5b_{12}^5 \\
 & - 13,711,504,717,817,841,782,955,386,017,440a_{40}^9b_{12}^5 \\
 & - 14,167,203,178,857,126,540,705,792,000a_{22}^3a_{40}^2b_{12}^6 \\
 & + 121,457,742,579,096,223,411,904,427,643,392a_{22}^6a_{40}^6b_{12}^6 \\
 & - 11,593,248,573,008,543,300,370,432,000a_{22}^2a_{40}^3b_{12}^7 \\
 & + 32,837,212,576,741,882,904,483,077,930,752a_{40}^7b_{12}^7 \\
 & + 2,245,173,056,476,346,827,726,848,000a_{22}a_{40}^4b_{12}^8 \\
 & + 5,064,667,464,745,198,665,216,000,000a_{40}^5b_{12}^9, \\
 f_2 = & - 1,024,947,112,912,004,777,224,251,017,134,080a_{22}^6a_{40}^3 \\
 & - 266,474,629,928,468,613,669,557,286,240,000a_{22}^4a_{40}^7 \\
 & + 2,872,149,434,420,042,500,705,386,405,000a_{22}^2a_{40}^{11} \\
 & - 445,566,812,245,372,227,872,915,578,125a_{40}^{15} \\
 & + 291,969,434,543,132,541,842,261,954,002,944a_{22}^7b_{12} \\
 & - 12,365,037,948,610,082,121,455,696,104,857,600a_{22}^5a_{40}^4b_{12} \\
 & - 1,807,720,964,924,211,050,490,834,741,684,000a_{22}^3a_{40}^8b_{12} \\
 & + 7,826,499,497,633,136,164,923,512,183,750a_{22}a_{40}^{12}b_{12} \\
 & + 4,469,135,964,829,014,341,045,756,192,882,688a_{22}^6a_{40}b_{12}^2 \\
 & - 60,534,442,756,814,886,093,718,953,699,609,600a_{22}^4a_{40}^5b_{12}^2 \\
 & - 4,533,006,178,962,510,464,756,656,493,703,000a_{22}^2a_{40}^9b_{12}^2 \\
 & + 4,340,383,679,701,089,115,056,480,723,750a_{40}^{13}b_{12}^2 \\
 & + 28,237,918,683,246,180,995,584,983,497,637,888a_{22}^5a_{40}^2b_{12}^3 \\
 & - 154,765,499,379,216,156,315,309,044,374,560,000a_{22}^3a_{40}^6b_{12}^3 \\
 & - 4,995,407,878,226,678,735,912,107,001,119,500a_{22}a_{40}^{10}b_{12}^3 \\
 & + 95,810,653,920,815,635,406,440,880,887,398,400a_{22}^4a_{40}^3b_{12}^4 \\
 & - 218,742,243,627,424,283,250,241,241,440,195,200a_{22}^2a_{40}^7b_{12}^4 \\
 & - 2,045,744,749,175,487,458,544,170,912,128,500a_{40}^{11}b_{12}^4 \\
 & - 813,586,646,290,453,924,746,048,307,200a_{22}^5b_{12}^5 \\
 & + 189,555,525,106,594,928,420,150,364,255,006,720a_{22}^3a_{40}^4b_{12}^5
 \end{aligned}$$

$$\begin{aligned}
 & - 162,486,461,423,769,223,728,413,597,878,998,720a_{22}^8a_{40}^5b_{12}^5 \\
 & - 6,641,496,500,058,207,978,636,233,932,800a_{22}^4a_{40}^6b_{12}^6 \\
 & + 219,763,449,413,443,714,186,877,891,390,908,416a_{22}^2a_{40}^5b_{12}^6 \\
 & - 49,659,806,636,672,871,688,527,633,292,144,800a_{40}^9b_{12}^6 \\
 & - 19,064,228,662,213,309,723,664,174,284,800a_{22}^3a_{40}^2b_{12}^7 \\
 & + 138,807,072,047,279,379,408,768,873,696,863,232a_{22}^6a_{40}^6b_{12}^7 \\
 & - 23,587,375,541,476,906,005,078,756,556,800a_{22}^2a_{40}^3b_{12}^8 \\
 & + 36,965,323,313,333,940,486,460,749,189,299,712a_{40}^7b_{12}^8 \\
 & - 11,144,873,921,049,570,121,957,903,564,800a_{22}^4a_{40}^4b_{12}^9 \\
 & - 543,085,211,944,693,588,070,611,353,600a_{40}^5b_{12}^{10}, \\
 f_3 = & 50,981,886,359,079,612,587,706,382,255,372,752,829,487,775,744a_{22}^9a_{40}^2 \\
 & - 37,067,692,293,092,445,441,750,012,511,920,751,446,131,343,360a_{22}^7a_{40}^6 \\
 & + 10,870,813,246,986,673,638,486,304,086,018,078,167,585,088,000a_{22}^5a_{40}^{10} \\
 & + 1,865,468,139,566,284,196,816,322,839,970,936,660,955,650,000a_{22}^3a_{40}^{14} \\
 & - 603,923,588,989,755,945,512,889,950,992,874,035,584,843,750a_{22}a_{40}^{18} \\
 & + 1,022,486,657,553,039,896,608,738,191,617,360,223,834,828,963,840a_{22}^8a_{40}^3b_{12} \\
 & - 458,337,148,013,787,047,780,469,938,840,107,759,739,495,546,880a_{22}^6a_{40}^7b_{12} \\
 & + 127,105,557,085,114,016,587,071,383,065,327,336,213,508,880,000a_{22}^4a_{40}^{11}b_{12} \\
 & + 10,293,584,307,015,442,523,985,281,679,548,906,547,040,893,750a_{22}^2a_{40}^{15}b_{12} \\
 & - 819,782,869,838,484,626,774,512,854,042,791,435,270,625,000a_{40}^{19}b_{12} \\
 & - 27,151,592,389,373,137,926,236,904,088,746,183,800,075,059,200a_{22}^9b_{12}^2 \\
 & + 8,940,700,083,625,325,227,481,046,577,354,689,218,587,001,880,576a_{22}^7a_{40}^4b_{12}^2 \\
 & - 2,327,073,899,335,537,014,385,148,389,219,002,199,314,773,084,160a_{22}^5a_{40}^8b_{12}^2 \\
 & + 576,812,223,109,715,545,609,263,444,945,637,012,697,548,425,000a_{22}^3a_{40}^{12}b_{12}^2 \\
 & + 19,397,906,489,183,486,274,296,855,420,984,668,056,392,746,875a_{22}a_{40}^{16}b_{12}^2 \\
 & - 555,569,150,979,780,182,547,784,105,215,654,496,407,493,017,600a_{22}^8a_{40}^3b_{12}^3 \\
 & + 44,829,701,780,109,113,211,300,344,287,213,556,558,697,675,620,352a_{22}^6a_{40}^5b_{12}^3 \\
 & - 6,211,455,356,481,472,181,416,216,543,937,020,596,156,227,558,400a_{22}^4a_{40}^9b_{12}^3 \\
 & + 1,248,196,196,928,988,776,133,983,498,657,830,652,209,486,940,000a_{22}^2a_{40}^{13}b_{12}^3 \\
 & + 12,244,662,313,483,580,417,405,784,575,032,932,936,001,725,000a_{40}^{17}b_{12}^3 \\
 & - 4,916,346,647,424,871,457,169,961,006,248,839,236,974,280,704,000a_{22}^7a_{40}^2b_{12}^4 \\
 & + 142,279,577,432,706,147,170,857,397,343,977,295,082,151,327,531,008a_{22}^5a_{40}^6b_{12}^4 \\
 & - 9,290,899,349,623,044,173,008,296,104,827,167,661,839,731,769,600a_{22}^3a_{40}^{10}b_{12}^4 \\
 & + 1,292,017,920,275,009,356,925,165,982,242,445,151,371,946,916,250a_{22}a_{40}^4b_{12}^4
 \end{aligned}$$

$$\begin{aligned}
& - 24,711,797,748,969,205,614,634,795,454,823,111,714,629,117,542,400a_{22}^6a_{40}^3b_{12}^5 \\
& + 296,723,701,499,161,287,704,516,622,860,637,906,314,000,798,846,976a_{22}^4a_{40}^7b_{12}^5 \\
& - 7,598,375,984,298,187,054,214,113,602,180,794,604,878,607,085,440a_{22}^2a_{40}^{11}b_{12}^5 \\
& + 515,403,389,037,848,152,893,742,601,009,301,446,964,737,813,250a_{40}^{15}b_{12}^5 \\
& + 89,151,402,261,964,717,348,593,902,889,295,518,105,600,000a_{22}^7b_{12}^6 \\
& - 77,871,378,889,486,757,955,374,362,879,501,321,944,798,907,596,800a_{22}^5a_{40}^4b_{12}^6 \\
& + 406,912,841,296,344,786,726,551,685,152,696,660,334,511,488,737,280a_{22}^3a_{40}^8b_{12}^6 \\
& - 2,959,508,856,221,408,195,975,754,548,025,003,389,634,647,181,120a_{22}a_{40}^{14}b_{12}^6 \\
& + 1,212,459,455,348,222,745,203,216,678,389,937,329,930,240,000a_{22}^6a_{40}b_{12}^7 \\
& - 159,874,291,856,994,698,343,598,397,136,909,959,783,750,585,548,800a_{22}^4a_{40}^5b_{12}^7 \\
& + 354,030,364,127,034,481,692,680,776,563,146,077,303,717,016,684,544a_{22}^2a_{40}^9b_{12}^7 \\
& - 338,017,002,725,212,903,867,985,105,175,332,778,568,798,931,040a_{40}^{13}b_{12}^7 \\
& + 6,672,226,780,950,021,057,789,431,484,431,911,931,084,800,000a_{22}^5a_{40}^2b_{12}^8 \\
& - 214,346,332,995,384,744,374,932,391,608,747,042,625,686,988,390,400a_{22}^3a_{40}^6b_{12}^8 \\
& + 177,421,389,360,763,691,749,439,937,259,376,019,693,424,578,797,568a_{22}a_{40}^{10}b_{12}^8 \\
& + 19,201,252,172,312,399,449,409,674,516,768,771,434,086,400,000a_{22}^4a_{40}^3b_{12}^9 \\
& - 181,375,047,489,804,263,451,726,975,105,980,803,657,328,001,024,000a_{22}^2a_{40}^7b_{12}^9 \\
& + 39,043,027,707,216,401,602,019,440,086,182,542,407,958,295,162,112a_{40}^{11}b_{12}^9 \\
& + 31,126,909,610,212,084,842,208,982,324,018,330,311,065,600,000a_{22}^3a_{40}^4b_{12}^{10} \\
& - 88,077,082,183,082,163,309,945,475,832,601,170,419,458,062,745,600a_{22}a_{40}^8b_{12}^{10} \\
& + 28,187,565,565,757,221,229,117,444,517,422,578,375,065,600,000a_{22}^2a_{40}^5b_{12}^{11} \\
& - 18,735,560,247,850,408,809,771,660,270,641,909,467,182,488,371,200a_{40}^9b_{12}^{11} \\
& + 12,916,473,356,332,131,362,657,092,325,292,496,398,581,760,000a_{22}a_{40}^6b_{12}^{12} \\
& + 2,179,454,745,027,373,896,621,572,921,424,926,638,080,000,000a_{40}^7b_{12}^{13}.
\end{aligned}$$

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