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# A delayed impulsive food chain system with prey refuge and mutual interference of predator

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## Abstract

In this paper, a three-species delayed food chain system with mutual interference and impulses is established. By utilizing the continuation theorem, the comparison theorem, some analysis techniques, and constructing a suitable Lyapunov functional, some sufficient conditions for the existence of positive periodic solutions, permanence, and global attractivity of the system are obtained. The conditions obtained are related to the mutual interference constant  $m$ , prey refuge constant  $\theta$ , delays, and impulses. An example is given to show the feasibility of the results.

**Keywords:** Impulsive control; Prey refuge; Mutual interference

## 1 Introduction

Impulsive differential equations are used for the mathematical simulation of processes which are subject to impulses during their evolution. Such processes can be observed in numerous fields of science and technology: control theory, population dynamics, biotechnologies, industrial robotics, etc. [1]. In population dynamics, many evolution processes in nature are characterized by the fact that, at certain moments of time, they experience some abrupt changes of state. Examples include annual harvesting and stocking of species as well as annual immigration. The theory of impulsive and delay differential equations is emerging as an important area of investigation [1–10]. For instance, Wang et al. [4] studied a delayed prey–predator system with mutual interference and impulses as follows:

$$\left. \begin{aligned} x'(t) &= x(t)(a_1(t) - b_1(t)x(t - \tau_1)) - c_1(t)x(t)y^m(t), \\ y'(t) &= y(t)(-a_2(t) - b_2(t)y(t - \tau_2)) + c_2(t)x(t)y^m(t), \\ \Delta x(t_k) &= d_{1k}x(t), \\ \Delta y(t_k) &= d_{2k}y(t), \end{aligned} \right\} \begin{aligned} &t \neq t_k, k \in Z^+, \\ &t = t_k, k \in Z^+. \end{aligned} \quad (1.1)$$

In 1971, Hassell [11] proposed the concept of mutual interference of predator species and introduced the mutual interference constant  $m$  ( $0 < m \leq 1$ ). In recent years, mutual interference of predator has been studied extensively by some researchers [11–13]. Wang [12] and He [13] studied several kinds of predator–prey systems with mutual interference

and obtained some sufficient conditions which guaranteed the permanence and globally asymptotic stability of the systems.

However, in the real world, prey species often make use of refuges to decrease predation risk. Here refuges mean some places or situations where predation risk is somehow reduced. Examples include spatial refuges, temporal refuges, and so on. During the last decade, numerous works have been published and much essential progress has been pushed on this direction [14–16]. González-Olivares and Ramos-Jiliberto [15] proposed a predator–prey system with Holling type II functional response and a prey refuge. They showed that there was a trend from limit cycles through non-zero stable points up to predator extinction and prey stabilizing at high densities. In reference [16], Ma et al. were interested in the interplay between the stabilizing effects of mutual interference and the destabilizing effects of prey refuge:

$$\begin{cases} x' = x(a - bx) - c_1(1 - \theta)xy^m, \\ y' = -dy + c_2(1 - \theta)xy^m, \end{cases}$$

where  $m$  ( $0 < m \leq 1$ ) is a constant of mutual interference.  $\theta$  ( $0 < \theta \leq 1$ ) is a constant number of prey using refuges.

Furthermore, there are three rudimentary and important ecological systems that are predator–prey system, competitive system, and cooperative system in mathematical biology. The predator–prey model, such as three-species predator–prey system and food-chain system, is the most popular population model and has extremely rich dynamics [8–10, 12–14]. Recently, the dynamic behaviors of predator–prey system with different kinds of functional responses have been extensively investigated [9, 10, 12–19]. For example, the authors [18] gave the following non-linear functional response:

$$f_i(x, y) = \frac{c_i xy}{\alpha + \beta x + \gamma y},$$

where  $c_i$  is the rate of a predator searching for the prey  $x$ ,  $\alpha$  is a positive constant,  $\beta = h_1 c_1$ ,  $\gamma = h_2 c_2$ ,  $h_i$  represents the expected handling time spent with the prey  $x$  to  $y$ . Obviously, if  $\beta$  or  $\gamma$  tends to zero, it will be Holling type-II functional response [19]. For example, Do proposed and studied a constant periodic releasing for top predator or periodic impulsive immigration for top predator—a three-species food chain system with impulsive control and Beddington–DeAngelis functional response [9].

Motivated by these facts, in this paper, we are concerned with the following delayed three-species Beddington-type food chain system with mutual interference and impulsive control methods. The model is described by the following differential equations:

$$\left. \begin{cases} x'(t) = x(t)(a_1(t) - b_1(t)x(t - \tau_1)) - \frac{c_1(1-\theta(t))x(t)y^m(t)}{\alpha_1(t)+\beta_1(t)x(t)+\gamma_1(t)y(t)}, \\ y'(t) = y(t)(-a_2(t) - b_2(t)y(t - \tau_2)) + \frac{c_2(1-\theta(t))x(t)y^m(t)}{\alpha_1(t)+\beta_1(t)x(t)+\gamma_1(t)y(t)} \\ \quad - \frac{c_3(t)(1-\theta(t))y(t)z^m(t)}{\alpha_2(t)+\beta_2(t)y(t)+\gamma_2(t)z(t)}, \\ z'(t) = z(t)(-a_3(t) - b_3(t)z(t - \tau_3)) + \frac{c_4(t)(1-\theta(t))y(t)z^m(t)}{\alpha_2(t)+\beta_2(t)y(t)+\gamma_2(t)z(t)}, \end{cases} \right\} t \neq t_k, k \in Z^+, \tag{1.2}$$

$$\left. \begin{cases} x(t_k^+) = (1 + d_{1k})x(t_k), \\ y(t_k^+) = (1 + d_{2k})y(t_k), \\ z(t_k^+) = (1 + d_{3k})z(t_k), \end{cases} \right\} t = t_k, k \in Z^+$$

with the initial conditions

$$(x(s), y(s), z(s)) = (\phi_1(s), \phi_2(s), \phi_3(s)) \in C([- \tau, 0], \mathbb{R}_+^3), \quad \phi_i(s) > 0,$$

where  $x(t), y(t), z(t)$  represent the densities of prey and mid-level predator and top predator, respectively.  $\Delta x(t_k) = x(t_k^+) - x(t_k), \Delta y(t_k) = y(t_k^+) - y(t_k)$  and  $\Delta z(t_k) = z(t_k^+) - z(t_k)$  represent the regular harvest or death from spraying pesticide of the populations  $x(t), y(t)$ , and  $z(t)$  at time  $t_k$ , respectively.  $Z^+ = \{1, 2, \dots\}$  and  $0 \leq t_0 < t_1 < t_2 < \dots < t_k < \dots$  are fixed points with  $t_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ .

For the sake of generality and convenience, we assume that the following conditions hold.

(I)  $a_i(t), b_i(t), i = 1, 2, 3, d_i(t), i = 1, 2, c_i(t), i = 1, 2, 3, 4$ , are all positive, continuous, and  $\omega$ -periodic functions, and  $\tau_1, \tau_2$ , and  $\tau_3$  are non-negative constants.

(II)  $-1 < d_{1k} \leq 0, -1 < d_{2k} \leq 0, -1 < d_{3k} \leq 0$  are constants for  $k \in Z^+$  and there exists a positive integer  $q$  such that  $t_{k+q} = t_k + \omega, d_{1,(k+q)} = d_{1k}, d_{2,(k+q)} = d_{2k}, d_{3,(k+q)} = d_{3k}$ , and  $t_k - \tau_1, t_k - \tau_2, t_k - \tau_3 \neq t_m$ .

Without loss of generality, we assume that  $t_k \neq 0, \omega$ , and  $[0, \omega] \cap \{t_k\} = \{t_1, t_2, \dots, t_q\}$ , and let

$$\begin{aligned} f^L &= \min_{t \in [0, T]} f(t), & f^U &= \max_{t \in [0, T]} f(t), & [f] &= \frac{1}{\omega} \int_0^T f(s) ds, \\ \Lambda_{d_1} &= \prod_{t_k < t} (1 + d_{1k}), & \Lambda_{d_2} &= \prod_{t_k < t} (1 + d_{2k}), & \Lambda_{d_3} &= \prod_{t_k < t} (1 + d_{3k}), \\ B_1(t) &= b_1(t) \prod_{t_k < t - \tau_1} (1 + d_{1k}), & B_2(t) &= b_2(t) \prod_{t_k < t - \tau_2} (1 + d_{2k}), \\ B_3(t) &= b_3(t) \prod_{t_k < t - \tau_3} (1 + d_{3k}), & C_1(t) &= c_1(t) \prod_{t_k < t} (1 + d_{2k})^m, \\ C_2(t) &= c_2(t) \prod_{t_k < t} \frac{(1 + d_{1k})}{(1 + d_{2k})^{1-m}}, & C_3(t) &= c_3(t) \prod_{t_k < t} (1 + d_{3k})^m, \\ C_4(t) &= c_4(t) \prod_{t_k < t} \frac{(1 + d_{2k})}{(1 + d_{3k})^{1-m}}. \end{aligned}$$

The rest of this article is organized as follows. In Sect. 2, we present the result on the existence of positive periodic solutions of system (1.2). In Sect. 3, we study the permanence of system (1.2). In Sect. 4, we investigate the global attractivity of system (1.2). In Sect. 5, an example is given to show the feasibility of our results by means of simulation. In the last section, we conclude this paper.

### 2 Existence of periodic solution

In order to prove the existence of positive solutions of system (1.2), we first introduce the following coincidence degree theorem.

Let  $X, Y$  be real Banach spaces,  $L : \text{Dom } L \subset X \rightarrow Y$  be a linear operator. The operator  $L$  will be called a Fredholm operator of index zero if  $\dim \text{Ker } L = \text{co dim Im } L < +\infty$  and  $\text{Im } L$  is closed in  $Y$ . If  $L$  is a Fredholm operator of index zero, there exist continuous projectors  $P : X \rightarrow X$  and  $Q : Y \rightarrow Y$  such that  $\text{Im } P = \text{Ker } L, \text{Ker } Q = \text{Im } L = \text{Im}(I - Q)$ . It follows that

$L|_{\text{Dom } L \cap \text{Ker } P} : (I - P)X \rightarrow \text{Im } L$  is invertible. We denote the inverse of that operator by  $K_P$ . If  $\Omega$  is an open bounded subset of  $X$  and  $N : \bar{\Omega} \rightarrow Y$  is a continuous operator, the operator  $N$  will be called  $L$ -compact on  $\bar{\Omega}$  if  $QN(\bar{\Omega})$  is bounded and  $K_P(I - Q)N : \bar{\Omega} \rightarrow X$  is compact. Since  $\text{Im } Q$  is isomorphic to  $\text{Ker } L$ , there exists isomorphism  $\Phi : \text{Im } Q \rightarrow \text{Ker } L$ .

**Lemma 2.1** (see [20]) *Let  $\Omega \subset X$  be an open bounded set, let  $L$  be a Fredholm operator of index zero and  $N$  be  $L$ -compact on  $\bar{\Omega}$ . Suppose that*

- (i) *for each  $\lambda \in (0, 1)$ , every solution  $x$  of  $Lx = \lambda Nx$  is such that  $x \notin \partial\Omega$ ;*
- (ii)  *$QNx \neq 0$  for each  $x \in \partial\Omega \cap \text{Ker } L$ ;*
- (iii)  *$\text{deg}(\Phi QN, \Omega \cap \text{Ker } L, 0) \neq 0$ .*

*Then the equation  $Lx = Nx$  has at least one solution in  $\bar{\Omega} \cap \text{Dom } L$ .*

**Theorem 2.1** *If  $\int_0^T a_1(t) dt + \sum_{k=1}^q \ln(1 + d_{1k}) > 0$  and the following algebraic equation*

$$\begin{cases} [a_1] + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}) - [b_1]e^u - \left[ \frac{c_1(1-\theta)}{\alpha_1 + \beta_1 e^u + \gamma_1 e^v} \right] e^{mv} = 0, \\ [a_2] - \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}) + [b_2]e^v - \left[ \frac{c_2(1-\theta)}{\alpha_1 + \beta_1 e^u + \gamma_1 e^v} \right] e^{u+(m-1)v} + \left[ \frac{c_3(1-\theta)}{\alpha_2 + \beta_2 e^v + \gamma_2 e^w} \right] e^{mw} = 0, \\ [a_3] - \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{3k}) + [b_3]e^w - \left[ \frac{c_4(1-\theta)}{\alpha_2 + \beta_2 e^v + \gamma_2 e^w} \right] e^{v+(m-1)w} = 0 \end{cases}$$

*has finite solutions  $(u^*, v^*, w^*)$ , then system (2.1) has at least one positive  $\omega$ -periodic solution.*

*Proof* Let  $(x(t), y(t), z(t)) \in R^3$  be an arbitrary positive  $\omega$ -periodic solution of system (1.2). Set  $x(t) = e^{u(t)}$ ,  $y(t) = e^{v(t)}$ ,  $z(t) = e^{w(t)}$ , then it follows from system (1.2) that

$$\left. \begin{cases} \begin{aligned} u'(t) &= a_1(t) - b_1(t)e^{u(t-\tau_1)} - \frac{c_1(t)(1-\theta)e^{mv(t)}}{\alpha_1(t) + \beta_1(t)e^{u(t)} + \gamma_1(t)e^{v(t)}}, \\ v'(t) &= -a_2(t) - b_2(t)e^{v(t-\tau_2)} + \frac{c_2(t)(1-\theta)e^{u(t)+(m-1)v(t)}}{\alpha_1(t) + \beta_1(t)e^{u(t)} + \gamma_1(t)e^{v(t)}} \\ &\quad - \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t) + \beta_2(t)e^{v(t)} + \gamma_2(t)e^{w(t)}}, \\ w'(t) &= -a_3(t) - b_3(t)e^{w(t-\tau_3)} + \frac{c_4(t)(1-\theta)e^{v(t)+(m-1)w(t)}}{\alpha_2(t) + \beta_2(t)e^{v(t)} + \gamma_2(t)e^{w(t)}}, \end{aligned} \\ \Delta u(t_k) = \ln(1 + d_{1k}), \\ \Delta v(t_k) = \ln(1 + d_{2k}), \\ \Delta w(t_k) = \ln(1 + d_{3k}), \end{cases} \right\} \begin{aligned} &t \neq t_k, k \in Z^+, \\ &t = t_k, k \in Z^+. \end{aligned} \tag{2.1}$$

Let  $S = \{x(t) \in PC^1([0, \omega], R) | x(\omega) = x(0)\}$ , where  $PC^1([0, \omega], R) = \{x \in PC([0, \omega], R) | x(t)$  is continuous, differentiable at  $t \neq t_k, x'(t_k^+), x'(t_k^-)$  exist, and  $x'(t_k^-) = x'(t_k), k = 1, 2, \dots, q\}$ ;  $PC([0, \omega], R) = \{x : [0, \omega] \rightarrow R | x(t)$  is continuous at  $t \neq t_k, x(t_k^+), x(t_k^-)$  exist, and  $x(t_k^-) = x(t_k^+), k = 1, 2, \dots, q\}$ . Let  $X = S \times S \times S$ . For  $\xi(t) = (u(t), v(t), w(t)) \in X$ , we define

$$\|\xi\|_X = \sqrt{\left( \max_{t \in [0, \omega]} |u(t)| \right)^2 + \left( \max_{t \in [0, \omega]} |v(t)| \right)^2 + \left( \max_{t \in [0, \omega]} |w(t)| \right)^2}.$$

Let  $Y = X \times R^q \times R^q \times R^q$ . For  $h = (\xi, e_1, e_2, e_3) \in Y, \xi \in X, e_1 \in R^q, e_2 \in R^q, e_3 \in R^q$ , we define  $\|h\|_Y = \|\xi\|_X + \|e_1\| + \|e_2\| + \|e_3\|$ , then  $X$  and  $Y$  are Banach spaces.

For convenience, we denote

$$\begin{aligned}
 F_1(t) &= a_1(t) - b_1(t)e^{\mu(t-\tau_1)} - \frac{c_1(t)(1-\theta)e^{m\nu(t)}}{\alpha_1(t) + \beta_1(t)e^{\mu(t)} + \gamma_1(t)e^{\nu(t)}}, \\
 F_2(t) &= -a_2(t) - b_2(t)e^{\nu(t-\tau_2)} + \frac{c_2(t)(1-\theta)e^{\mu(t)+(m-1)\nu(t)}}{\alpha_1(t) + \beta_1(t)e^{\mu(t)} + \gamma_1(t)e^{\nu(t)}} \\
 &\quad - \frac{c_3(t)(1-\theta)e^{m\nu(t)}}{\alpha_2(t) + \beta_2(t)e^{\nu(t)} + \gamma_2(t)e^{w(t)}}, \\
 F_3(t) &= -a_3(t) - b_3(t)e^{w(t-\tau_3)} + \frac{c_4(t)(1-\theta)e^{\nu(t)+(m-1)w(t)}}{\alpha_2(t) + \beta_2(t)e^{\nu(t)} + \gamma_2(t)e^{w(t)}}.
 \end{aligned}$$

Let  $\text{Dom } L = X, L : \text{Dom } L \rightarrow Y,$

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \rightarrow \left( \begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}, \begin{pmatrix} \Delta u(t_1) \\ \vdots \\ \Delta u(t_q) \end{pmatrix}, \begin{pmatrix} \Delta v(t_1) \\ \vdots \\ \Delta v(t_q) \end{pmatrix}, \begin{pmatrix} \Delta w(t_1) \\ \vdots \\ \Delta w(t_q) \end{pmatrix} \right).$$

Let  $N : X \rightarrow Y,$

$$N \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \left( \begin{pmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{pmatrix}, \begin{pmatrix} \ln(1 + d_{11}) \\ \vdots \\ \ln(1 + d_{1q}) \end{pmatrix}, \begin{pmatrix} \ln(1 + d_{21}) \\ \vdots \\ \ln(1 + d_{2q}) \end{pmatrix}, \begin{pmatrix} \ln(1 + d_{31}) \\ \vdots \\ \ln(1 + d_{3q}) \end{pmatrix} \right),$$

then

$$\begin{aligned}
 \text{Ker } L &= \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \mid \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \in R^3 \text{ is constant vector, } t \in [0, \omega] \right\}, \\
 \text{Im } L &= \left\{ h = \left( \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \begin{pmatrix} D_{11} \\ \vdots \\ D_{1q} \end{pmatrix}, \begin{pmatrix} D_{21} \\ \vdots \\ D_{2q} \end{pmatrix}, \begin{pmatrix} D_{31} \\ \vdots \\ D_{3q} \end{pmatrix} \right) \in Y \mid \right. \\
 &\quad \left. \begin{pmatrix} \int_0^\omega f_1(t) dt + \sum_{k=1}^q D_{1k} = 0 \\ \int_0^\omega f_2(t) dt + \sum_{k=1}^q D_{2k} = 0 \\ \int_0^\omega f_3(t) dt + \sum_{k=1}^q D_{3k} = 0 \end{pmatrix} \right\},
 \end{aligned}$$

$\text{Im } L$  is closed in  $Y, \dim \text{Ker } L = \text{co dim Im } L = 3.$  So  $L$  is a Fredholm operator of index zero.

Let

$$P \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{\omega} \begin{pmatrix} \int_0^\omega u(t) dt \\ \int_0^\omega v(t) dt \\ \int_0^\omega w(t) dt \end{pmatrix},$$

$$\begin{aligned}
 Qh &= Q \left( \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix}, \begin{pmatrix} D_{11} \\ \vdots \\ D_{1q} \end{pmatrix}, \begin{pmatrix} D_{21} \\ \vdots \\ D_{2q} \end{pmatrix}, \begin{pmatrix} D_{31} \\ \vdots \\ D_{3q} \end{pmatrix} \right) \\
 &= \left( \frac{1}{\omega} \begin{pmatrix} \int_0^\omega f_1(t) dt + \sum_{k=1}^q D_{1k} = 0 \\ \int_0^\omega f_2(t) dt + \sum_{k=1}^q D_{2k} = 0 \\ \int_0^\omega f_3(t) dt + \sum_{k=1}^q D_{3k} = 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right).
 \end{aligned}$$

It is easy to know that  $P, Q$  are continuous projectors and  $\text{Im} P = \text{Ker} L, \text{Ker} Q = \text{Im} L = \text{Im}(I - Q)$ .

Therefore, the inverse  $K_P$  exists and it has the following form:  $K_P : \text{Im} L \rightarrow \text{Ker} P \cap \text{Dom} L,$

$$\begin{aligned}
 K_P(h) &= \begin{pmatrix} \int_0^t f_1(s) ds + \sum_{0 < t_k < t} D_{1k} - \frac{1}{\omega} \int_0^\omega \int_0^t f_1(s) ds dt - \sum_{k=1}^q D_{1k} \\ \int_0^t f_2(s) ds + \sum_{0 < t_k < t} D_{2k} - \frac{1}{\omega} \int_0^\omega \int_0^t f_2(s) ds dt - \sum_{k=1}^q D_{2k} \\ \int_0^t f_3(s) ds + \sum_{0 < t_k < t} D_{3k} - \frac{1}{\omega} \int_0^\omega \int_0^t f_3(s) ds dt - \sum_{k=1}^q D_{3k} \end{pmatrix}, \\
 QN \begin{pmatrix} u \\ v \\ w \end{pmatrix} &= \begin{pmatrix} \left( \frac{1}{\omega} \int_0^T F_1(t) dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}) \right) \\ \left( \frac{1}{\omega} \int_0^T F_2(t) dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}) \right) \\ \left( \frac{1}{\omega} \int_0^T F_3(t) dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{3k}) \right) \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}. \\
 K_P(I - Q)N \begin{pmatrix} u \\ v \\ w \end{pmatrix} &= \begin{pmatrix} \int_0^t F_1(s) ds + \sum_{0 < t_k < t} \ln(1 + d_{1k}) \\ \int_0^t F_2(s) ds + \sum_{0 < t_k < t} \ln(1 + d_{2k}) \\ \int_0^t F_3(s) ds + \sum_{0 < t_k < t} \ln(1 + d_{3k}) \end{pmatrix} \\
 &\quad - \begin{pmatrix} \frac{1}{\omega} \int_0^\omega \int_0^t F_1(s) ds dt + \sum_{k=1}^q \ln(1 + d_{1k}) \\ \frac{1}{\omega} \int_0^\omega \int_0^t F_2(s) ds dt + \sum_{k=1}^q \ln(1 + d_{2k}) \\ \frac{1}{\omega} \int_0^\omega \int_0^t F_3(s) ds dt + \sum_{k=1}^q \ln(1 + d_{3k}) \end{pmatrix} \\
 &\quad - \begin{pmatrix} \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega F_1(s) ds + \sum_{k=1}^q \ln(1 + d_{1k}) \\ \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega F_2(s) ds + \sum_{k=1}^q \ln(1 + d_{2k}) \\ \left( \frac{t}{\omega} - \frac{1}{2} \right) \int_0^\omega F_3(s) ds + \sum_{k=1}^q \ln(1 + d_{3k}) \end{pmatrix}.
 \end{aligned}$$

Clearly,  $QN$  and  $K_P(I - Q)N$  are continuous. Let  $\Omega$  be any open bounded set in  $X$ . It is easy to verify that  $QN(\Omega)$  is uniformly bounded,  $K_P(I - Q)N(\bar{\Omega})$  is equicontinuous and uniformly bounded. By Ascoli–Arzela theorem we know that  $K_P(I - Q)N$  is compact on  $\bar{\Omega}$ . Thus,  $N$  is  $L$ -compact on  $\bar{\Omega}$ .

Now, it needs to show that there exists a domain  $\Omega$  that satisfies all the requirements given in Lemma 2.1. Corresponding to operator equation  $L\xi = \lambda N\xi, \lambda \in (0, 1)$ . Assume that  $\xi(t) = (u(t), v(t), w(t)) \in X$  is a solution of this operator equation, then we have

$$\begin{cases} \left. \begin{aligned} u'(t) &= \lambda F_1(t), \\ v'(t) &= \lambda F_2(t), \\ w'(t) &= \lambda F_3(t), \end{aligned} \right\} t \neq t_k, \\ \left. \begin{aligned} \Delta u(t_k) &= \lambda \ln(1 + d_{1k}), \\ \Delta v(t_k) &= \lambda \ln(1 + d_{2k}), \\ \Delta w(t_k) &= \lambda \ln(1 + d_{3k}), \end{aligned} \right\} t = t_k. \end{cases} \tag{2.2}$$

Integrating (2.2) over  $[0, \omega]$ , we get

$$\begin{aligned} & \int_0^\omega a_1(t) dt + \sum_{k=1}^q \ln(1 + d_{1k}) \\ &= \int_0^\omega b_1(t)e^{u(t-\tau_1)} dt + \int_0^\omega \frac{c_1(t)(1-\theta)e^{mv(t)}}{\alpha_1(t) + \beta_1(t)e^{u(t)} + \gamma_1(t)e^{v(t)}} dt, \end{aligned} \tag{2.3}$$

$$\begin{aligned} & \int_0^\omega a_2(t) dt - \sum_{k=1}^q \ln(1 + d_{2k}) + \int_0^\omega b_2(t)e^{v(t-\tau_2)} dt \\ &+ \int_0^\omega \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t) + \beta_2(t)e^{v(t)} + \gamma_2(t)e^{w(t)}} dt \\ &= \int_0^\omega \frac{c_2(t)(1-\theta)e^{u(t)+(m-1)v(t)}}{\alpha_1(t) + \beta_1(t)e^{u(t)} + \gamma_1(t)e^{v(t)}} dt, \end{aligned} \tag{2.4}$$

$$\begin{aligned} & \int_0^\omega a_3(t) dt - \sum_{k=1}^q \ln(1 + d_{3k}) + \int_0^\omega b_3(t)e^{w(t-\tau_3)} dt \\ &= \int_0^\omega \frac{c_4(t)(1-\theta)e^{v(t)+(m-1)w(t)}}{\alpha_2(t) + \beta_2(t)e^{v(t)} + \gamma_2(t)e^{w(t)}} dt. \end{aligned} \tag{2.5}$$

From (2.2), (2.3), we get

$$\begin{aligned} \int_0^\omega |u'(t)| dt &\leq \int_0^\omega a_1(t) dt + \int_0^\omega b_1(t)e^{u(t-\tau_1)} dt + \int_0^\omega \frac{c_1(t)(1-\theta)e^{mv(t)}}{\alpha_1(t) + \beta_1(t)e^{u(t)} + \gamma_1(t)e^{v(t)}} dt \\ &= 2 \int_0^\omega a_1(t) dt + \sum_{k=1}^q \ln(1 + d_{1k}). \end{aligned}$$

From (2.2), (2.4), we get

$$\begin{aligned} & \int_0^\omega |v'(t)| dt \\ &\leq \int_0^\omega a_2(t) dt + \int_0^\omega b_2(t)e^{v(t-\tau_2)} dt + \int_0^\omega \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t) + \beta_2(t)e^{v(t)} + \gamma_2(t)e^{w(t)}} dt \\ &+ \int_0^\omega \frac{c_2(t)(1-\theta)e^{u(t)+(m-1)v(t)}}{\alpha_1(t) + \beta_1(t)e^{u(t)} + \gamma_1(t)e^{v(t)}} dt \\ &= 2 \left( \int_0^\omega a_2(t) dt + \int_0^\omega b_2(t)e^{v(t-\tau_2)} dt + \int_0^\omega \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t) + \beta_2(t)e^{v(t)} + \gamma_2(t)e^{w(t)}} dt \right) \\ &- \sum_{k=1}^q \ln(1 + d_{2k}). \end{aligned}$$

From (2.2), (2.5), we get

$$\begin{aligned} \int_0^\omega |w'(t)| dt &\leq \int_0^\omega a_3(t) dt + \int_0^\omega b_3(t)e^{w(t-\tau_3)} dt + \int_0^\omega \frac{c_4(t)(1-\theta)e^{v(t)+(m-1)w(t)}}{\alpha_2(t) + \beta_2(t)e^{v(t)} + \gamma_2(t)e^{w(t)}} dt \\ &= 2 \left( \int_0^\omega a_3(t) dt + \int_0^\omega b_3(t)e^{w(t-\tau_3)} dt \right) - \sum_{k=1}^q \ln(1 + d_{3k}). \end{aligned}$$

Since  $\xi(t) = (u(t), v(t), w(t)) \in X$ , there exist  $\xi_1, \xi_2, \xi_3 \in [0, \omega]$  such that  $\min_{t \in [0, \omega]} u(t) = u(\xi_1)$ ,  $\min_{t \in [0, \omega]} v(t) = v(\xi_2)$ ,  $\min_{t \in [0, \omega]} w(t) = w(\xi_3)$ .

It follows from (2.3) that

$$\int_0^\omega a_1(t) dt + \sum_{k=1}^q \ln(1 + d_{1k}) \geq \int_0^\omega b_1(t)e^{u(t-\tau_1)} dt \geq \int_0^\omega b_1(t)e^{u(\xi_1)} dt,$$

$$e^{u(\xi_1)} \leq \frac{\int_0^\omega a_1(t) dt + \sum_{k=1}^q \ln(1 + d_{1k})}{\int_0^\omega b_1(t) dt} = \frac{\omega[a_1] + \sum_{k=1}^q \ln(1 + d_{1k})}{\omega[b_1]}.$$

Thus

$$u(t) \leq u(\xi_1 - \tau_1) + \int_0^\omega |u'(t)| dt + \sum_{k=1}^q |\ln(1 + d_{1k})|$$

$$\leq 2\omega[a_1] + \sum_{k=1}^q \ln(1 + d_{1k}) + \sum_{k=1}^q |\ln(1 + d_{1k})| + \ln \left\{ \frac{\omega[a_1] + \sum_{k=1}^q \ln(1 + d_{1k})}{\omega[b_1]} \right\}$$

$$= M_1.$$

Similarly, it follows from (2.4) that

$$\int_0^\omega a_2(t) dt - \sum_{k=1}^q \ln(1 + d_{2k})$$

$$\leq \int_0^\omega \frac{c_2(t)(1 - \theta(t))e^{u(t)+(m-1)v(t)}}{\gamma_1(t)e^{v(t)}} dt = \frac{e^{M_1}}{e^{v(\xi_2)}} \omega \left[ \frac{c_2(1 - \theta)}{\gamma_1} \right],$$

$$e^{v(\xi_2)} \leq \frac{e^{M_1} \omega \left[ \frac{c_2(1 - \theta)}{\gamma_1} \right]}{\omega[a_2] - \sum_{k=1}^q \ln(1 + d_{2k})}.$$

Thus

$$v(t) \leq v(\xi_2) + \int_0^\omega |v'(t)| dt + \sum_{k=1}^q |\ln(1 + d_{2k})|$$

$$\leq 2\omega[a_2] - \sum_{k=1}^q \ln(1 + d_{2k}) + \sum_{k=1}^q |\ln(1 + d_{2k})| + \ln \left\{ \frac{e^{M_1} \omega \left[ \frac{c_2(1 - \theta)}{\gamma_1} \right]}{\omega[a_2] - \sum_{k=1}^q \ln(1 + d_{2k})} \right\}$$

$$= M_2.$$

It follows from (2.5) that

$$\int_0^\omega a_3(t) dt - \sum_{k=1}^q \ln(1 + d_{3k}) \leq \frac{e^{M_2}}{e^{w(\xi_3)}} \omega \left[ \frac{c_4(1 - \theta)}{\gamma_2} \right],$$

$$e^{w(\xi_3)} \leq \frac{e^{M_2} \omega \left[ \frac{c_4(1 - \theta)}{\gamma_2} \right]}{\omega[a_3] - \sum_{k=1}^q \ln(1 + d_{3k})}.$$

Thus

$$\begin{aligned}
 w(t) &\leq w(\xi_2) + \int_0^\omega |w'(t)| dt + \sum_{k=1}^q |\ln(1 + d_{3k})| \\
 &\leq 2\omega[a_3] - \sum_{k=1}^q \ln(1 + d_{3k}) + \sum_{k=1}^q |\ln(1 + d_{3k})| + \ln \left\{ \frac{e^{M_2} \omega \left[ \frac{c_4(1-\theta)}{\gamma_2} \right]}{\omega[a_3] - \sum_{k=1}^q \ln(1 + d_{3k})} \right\} \\
 &= M_3.
 \end{aligned}$$

On the other hand, from the fact of  $\xi(t) = (u(t), v(t), w(t)) \in X$ , we know that  $\sup_{t \in [0, \omega]} u(t)$  exists, and there exists  $\eta_1 \in [0, \omega]$  such that  $u(\eta_1^+) = \sup_{t \in [0, \omega]} u(t)$  ( $u(\eta_1^+) = u(\eta_1)$  for  $\eta_1 \neq t_k$ ;  $u(\eta_1^+) = u(t_k^+)$  for  $\eta_1 = t_k$ ).

Hence, from (2.3) we get

$$\begin{aligned}
 \int_0^\omega a_1(t) dt + \sum_{k=1}^q \ln(1 + d_{1k}) &\leq \int_0^\omega b_1(t) e^{u(\eta_1^+)} dt + \int_0^\omega \frac{c_1(t)(1-\theta) e^{mv(t)}}{\alpha_1(t)} dt \\
 &= \omega[b_1] e^{u(\eta_1^+)} + \left[ \frac{c_1(1-\theta)}{\alpha_1} \right] e^{mM_2}, \\
 e^{u(\eta_1^+)} &\geq \frac{\omega[a_1] + \sum_{k=1}^q \ln(1 + d_{1k}) - \left[ \frac{c_1(1-\theta)}{\gamma_1} \right] e^{mM_2}}{\omega[b_1]}, \\
 u(t) &\geq u(\eta_1^+) - \int_0^\omega |u'(t)| dt - \sum_{k=1}^q |\ln(1 + d_{1k})| \\
 &\geq \ln \left\{ \frac{\omega[a_1] + \sum_{k=1}^q \ln(1 + d_{1k}) - \left[ \frac{c_1(1-\theta)}{\gamma_1} \right] e^{mM_2}}{\omega[b_1]} \right\} \\
 &\quad - 2\omega[a_1] - \sum_{k=1}^q \ln(1 + d_{1k}) - \sum_{k=1}^q |\ln(1 + d_{1k})| \\
 &= M_4.
 \end{aligned}$$

Similarly, there exist  $\eta_2, \eta_3 \in [0, \omega]$ ,  $v(\eta_2^+) = \sup_{t \in [0, \omega]} v(t)$ ,  $w(\eta_3^+) = \sup_{t \in [0, \omega]} w(t)$ . From (2.4) we can get

$$\begin{aligned}
 \int_0^\omega a_2(t) dt - \sum_{k=1}^q \ln(1 + d_{2k}) + \int_0^\omega b_2(t) e^{M_2} dt + \int_0^\omega \frac{c_3(t)(1-\theta)}{\gamma_2(t)} dt \\
 \geq \int_0^\omega \frac{c_2(t)(1-\theta) e^{M_4 + (m-1)v(\eta_2^+)}}{\alpha_1(t) + \beta_1(t) e^{M_1} + \gamma_1(t) e^{M_2}} dt, \\
 \int_0^\omega e^{v(\eta_2^+)} dt \geq \left( \omega(\alpha_1(t) + \beta_1(t) e^{M_1} + \gamma_1(t) e^{M_2}) \right. \\
 \quad \times \left( \int_0^\omega a_2(t) dt - \sum_{k=1}^q \ln(1 + d_{2k}) \right. \\
 \quad \left. \left. + \int_0^\omega b_2(t) e^{M_2} dt + \int_0^\omega \frac{c_3(t)(1-\theta)}{\gamma_2(t)} dt \right) \right. \\
 \quad \left. / \left( \int_0^\omega c_2(t)(1-\theta) e^{M_4} dt \right) \right)^{1/(m-1)}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 v(t) &\geq v(\eta_1^+) - \int_0^\omega |v'(t)| dt - \sum_{k=1}^q |\ln(1 + d_{2k})| \\
 &\geq \ln \left\{ \left( \omega(\alpha_1(t) + \beta_1(t)e^{M_1} + \gamma_1(t)e^{M_2}) \right. \right. \\
 &\quad \times \left( \int_0^\omega a_2(t) dt - \sum_{k=1}^q \ln(1 + d_{2k}) + \int_0^\omega b_2(t)e^{M_2} dt + \int_0^\omega \frac{c_3(t)(1-\theta)}{\gamma_2(t)} dt \right) \\
 &\quad \left. \left. / \left( \int_0^\omega c_2(t)(1-\theta)e^{M_4} dt \right) \right)^{1/(m-1)} \right\} \\
 &\quad - 2\omega[a_2] + \sum_{k=1}^q \ln(1 + d_{2k}) - \sum_{k=1}^q |\ln(1 + d_{2k})| \\
 &= M_5.
 \end{aligned}$$

From (2.5) we can get

$$\begin{aligned}
 \int_0^\omega a_3(t) dt - \sum_{k=1}^q \ln(1 + d_{3k}) + \int_0^\omega b_3(t)e^{M_3} dt &\geq \int_0^\omega \frac{c_4(t)(1-\theta)e^{M_5+(m-1)w(\eta_3^+)}}{\alpha_2(t) + \beta_2(t)e^{M_2} + \gamma_2(t)e^{M_3}} dt, \\
 \int_0^\omega e^{w(\eta_3^+)} dt &\geq \left( \left( \left( \int_0^\omega a_3(t) dt - \sum_{k=1}^q \ln(1 + d_{3k}) + \int_0^\omega b_3(t)e^{M_3} dt \right) \right. \right. \\
 &\quad \left. \left. \times \omega(\alpha_2(t) + \beta_2(t)e^{M_2} + \gamma_2(t)e^{M_3}) \right) \right. \\
 &\quad \left. / (\omega[c_4(1-\theta)e^{M_5}]) \right)^{1/(m-1)}.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 w(t) &\geq w(\eta_1^+) - \int_0^\omega |w'(t)| dt - \sum_{k=1}^q |\ln(1 + d_{2k})| \\
 &\geq \ln \left\{ \left( \left( \left( \int_0^\omega a_3(t) dt - \sum_{k=1}^q \ln(1 + d_{3k}) + \int_0^\omega b_3(t)e^{M_3} dt \right) \right. \right. \right. \\
 &\quad \left. \left. \times \omega(\alpha_2(t) + \beta_2(t)e^{M_2} + \gamma_2(t)e^{M_3}) \right) \right. \\
 &\quad \left. \left. / (\omega[c_4(1-\theta)e^{M_5}]) \right)^{1/(m-1)} \right\} \\
 &\quad - 2\omega[a_3] + \sum_{k=1}^q \ln(1 + d_{3k}) - \sum_{k=1}^q |\ln(1 + d_{3k})| \\
 &= M_6.
 \end{aligned}$$

Hence,

$$\begin{aligned} \max_{t \in [0, T]} |u(t)| &\leq \max\{|M_1|, |M_4|\} = H_1, \\ \max_{t \in [0, T]} |v(t)| &\leq \max\{|M_2|, |M_5|\} = H_2, \\ \max_{t \in [0, T]} |w(t)| &\leq \max\{|M_3|, |M_6|\} = H_3. \end{aligned}$$

This implies that any solution  $\xi(t)$  of the operator equation  $L\xi = \lambda N\xi$  satisfies  $\|\xi\| \leq \sqrt{H_1^2 + H_2^2 + H_3^2} = H$ .

Now we consider the system of algebraic equations with respect to  $u, v, w$  ( $(u, v, w) \in R^3$  is a constant vector):

$$\begin{cases} \int_0^\omega a_1(t) - b_1(t)e^{u(t-\tau_1)} - \mu \frac{c_1(t)(1-\theta)e^{mv(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} dt + \sum_{k=1}^q \ln(1 + d_{1k}) = 0, \\ \int_0^\omega -a_2(t) - b_2(t)e^{v(t-\tau_2)} + \mu \frac{c_2(t)(1-\theta)e^{u(t)+(m-1)v(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} - \mu \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt \\ + \sum_{k=1}^q \ln(1 + d_{2k}) = 0, \\ \int_0^\omega -a_3(t) - b_3(t)e^{w(t-\tau_3)} + \mu \frac{c_4(t)(1-\theta)e^{v(t)+(m-1)w(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt + \sum_{k=1}^q \ln(1 + d_{3k}) = 0, \end{cases} \tag{2.6}$$

where  $\mu \in [0, 1]$ . By the above similar deduction steps, we know

$$\begin{aligned} M_4 \leq u \leq M_1, \quad M_5 \leq v \leq M_2, \quad M_6 \leq w \leq M_3, \\ |u| \leq \max\{|M_1|, |M_4|\} = H_1, \quad |v| \leq \max\{|M_2|, |M_5|\} = H_2, \\ |w| \leq \max\{|M_3|, |M_6|\} = H_3. \end{aligned}$$

This implies that any constant solution  $(u, v, w)$  of (2.6) always satisfies  $\sqrt{u^2 + v^2 + w^2} \leq H$ . (It is independent of  $\mu$ ).

We know that system (2.6) can be changed into the following system (2.7):

$$\begin{cases} [a_1] - [b_1]e^{u(t-\tau_1)} - \mu \frac{1}{\omega} \int_0^\omega \frac{c_1(t)(1-\theta)e^{mv(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}) = 0, \\ -[a_2] - [b_2]e^{v(t-\tau_2)} + \mu \frac{1}{\omega} \int_0^\omega \frac{c_2(t)(1-\theta)e^{u(t)+(m-1)v(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} dt \\ - \mu \frac{1}{\omega} \int_0^\omega \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}) = 0, \\ -[a_3] - [b_3]e^{w(t-\tau_3)} + \mu \frac{1}{\omega} \int_0^T \frac{c_4(t)(1-\theta)e^{v(t)+(m-1)w(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{3k}) = 0. \end{cases} \tag{2.7}$$

For any  $\mu \in [0, 1]$ , any solution  $(u, v, w)$  of (2.7) always satisfies  $\sqrt{u^2 + v^2 + w^2} \leq H$ .

Let  $\mu = 1$ , then any solution  $(u, v, w)$  of the following system (2.8) satisfies  $\sqrt{u^2 + v^2 + w^2} \leq H$ .

$$\begin{cases} [a_1] - [b_1]e^{u(t-\tau_1)} - \frac{1}{\omega} \int_0^T \frac{c_1(t)(1-\theta)e^{mv(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}) = 0, \\ -[a_2] - [b_2]e^{v(t-\tau_2)} + \frac{1}{\omega} \int_0^T \frac{c_2(t)(1-\theta)e^{u(t)+(m-1)v(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} dt \\ - \frac{1}{\omega} \int_0^T \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}) = 0, \\ -[a_3] - [b_3]e^{w(t-\tau_3)} + \frac{1}{\omega} \int_0^T \frac{c_4(t)(1-\theta)e^{v(t)+(m-1)w(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{3k}) = 0. \end{cases} \tag{2.8}$$

Let  $M^* = H + \delta$  ( $\delta > 0$ ,  $\delta$  is sufficiently large). Let  $\Omega = \{\xi = (u(t), v(t), w(t)) \in X \mid \|\xi\| < M^*\}$ , then condition (i) of Lemma 2.1 is satisfied.

For any constant vector  $\xi_0 = (u_0, v_0, w_0) \in \partial\Omega$ ,  $\sqrt{u_0^2 + v_0^2 + w_0^2} = M^*$ .

$QN\xi_0$

$$\begin{aligned}
 &= \left( \begin{aligned} &\frac{1}{\omega} \int_0^\omega (a_1(t) - b_1(t)e^{u(t-\tau_1)} - \frac{c_1(t)(1-\theta)e^{mv(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}}) dt \\ &+ \sum_{k=1}^q \ln(1 + d_{1k}) \\ &\frac{1}{\omega} \int_0^\omega (-a_2(t) - b_2(t)e^{v(t-\tau_2)} + \frac{c_2(t)(1-\theta)e^{u(t)+(m-1)v(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} \\ &- \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}}) dt + \sum_{k=1}^q \ln(1 + d_{2k}) \\ &\frac{1}{\omega} \int_0^\omega (-a_3(t) - b_3(t)e^{w(t-\tau_3)} + \frac{c_4(t)(1-\theta)e^{v(t)+(m-1)w(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}}) dt \\ &+ \sum_{k=1}^q \ln(1 + d_{3k}) \end{aligned} \right), \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\
 &= \left( \begin{aligned} &[a_1] - [b_1]e^{u(t-\tau_1)} - \frac{1}{\omega} \int_0^\omega \frac{c_1(t)(1-\theta)e^{mv(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} dt \\ &+ \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}) \\ &- [a_2] - [b_2]e^{v(t-\tau_2)} + \frac{1}{\omega} \int_0^\omega \frac{c_2(t)(1-\theta)e^{u(t)+(m-1)v(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} dt \\ &- \frac{1}{\omega} \int_0^\omega \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}) \\ &- [a_3] - [b_3]e^{w(t-\tau_3)} + \frac{1}{\omega} \int_0^\omega \frac{c_4(t)(1-\theta)e^{v(t)+(m-1)w(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt \\ &+ \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{3k}) \end{aligned} \right), \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\
 &\neq 0.
 \end{aligned}$$

Hence, condition (ii) of Lemma 2.1 is satisfied.

Let  $\Phi : \text{Im } Q \rightarrow \text{Ker } L$ ,

$$\left( \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right) \rightarrow \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}.$$

Let  $F(\xi, \mu) = \mu \Phi QN\xi + (1 - \mu)\Phi G\xi$ , where  $\mu \in [0, 1]$ ,  $\xi = (u, v, w) \in R^3$ ,

$$\begin{aligned}
 G\xi &= \left( \begin{aligned} &[a_1] - [b_1]e^{u(t-\tau_1)} + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}) \\ &- [a_2] - [b_2]e^{v(t-\tau_2)} + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}) \\ &- [a_3] - [b_3]e^{w(t-\tau_3)} + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{3k}) \end{aligned} \right), \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \\
 \Phi G\xi &= \left( \begin{aligned} &[a_1] - [b_1]e^{u(t-\tau_1)} + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}) \\ &- [a_2] - [b_2]e^{v(t-\tau_2)} + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}) \\ &- [a_3] - [b_3]e^{w(t-\tau_3)} + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{3k}) \end{aligned} \right), \\
 \Phi QN\xi &= \left( \begin{aligned} &[a_1] - [b_1]e^{u(t-\tau_1)} - \frac{1}{\omega} \int_0^T \frac{c_1(t)(1-\theta)e^{mv(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}), \\ &- [a_2] - [b_2]e^{v(t-\tau_2)} + \frac{1}{\omega} \int_0^T \frac{c_2(t)(1-\theta)e^{u(t)+(m-1)v(t)}}{\alpha_1(t)+\beta_1(t)e^{u(t)}+\gamma_1(t)e^{v(t)}} dt \\ &- \frac{1}{\omega} \int_0^T \frac{c_3(t)(1-\theta)e^{mw(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}), \\ &- [a_3] - [b_3]e^{w(t-\tau_3)} + \frac{1}{\omega} \int_0^T \frac{c_4(t)(1-\theta)e^{v(t)+(m-1)w(t)}}{\alpha_2(t)+\beta_2(t)e^{v(t)}+\gamma_2(t)e^{w(t)}} dt + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{3k}) \end{aligned} \right).
 \end{aligned}$$

For  $\xi \in \partial\Omega \cap \text{Ker } L$ ,  $\xi = (u, v, w)$  is a constant vector, and  $\sqrt{u^2 + v^2 + w^2} = M^*$ .

From (2.7) we know  $F(\xi, \mu) \neq 0$  ( $\xi \in \partial\Omega \cap \text{Ker } L$ ).

It is easy to verify that the following system of equations has a unique solution  $\xi^* = (u^*, v^*, w^*)$ .

$$\begin{cases} [a_1] - [b_1]e^{u(t-\tau_1)} + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}) = 0, \\ -[a_2] - [b_2]e^{v(t-\tau_2)} + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}) = 0, \\ -[a_3] - [b_3]e^{w(t-\tau_3)} + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{3k}) = 0. \end{cases}$$

Thus,

$$\begin{aligned} \deg(\Phi QN, \Omega \cap \text{Ker } L, 0) &= \deg(\Phi G, \Omega \cap \text{Ker } L, 0) = \text{sgn } J_{\Phi G}(\xi^*) \\ &= \text{sgn}\{-[b_1]e^{u(t-\tau_1)} \cdot [b_2]e^{v(t-\tau_2)} \cdot [b_3]e^{w(t-\tau_3)}\} \neq 0 \end{aligned}$$

(where  $J_{\Phi G}$  is the Jacobian determinant of  $\Phi G$ ). Hence condition (iii) of Lemma 2.1 is satisfied.

Thus, by Lemma 2.1, we know that the operator equation  $L\xi = N\xi$  has at least one solution in  $\bar{\Omega} \cap \text{Dom } L$ . Therefore system (2.1) has at least one-periodic solution  $(u^*(t), v^*(t), w^*(t))$ . So system (1.2) has at least one  $\omega$ -periodic solution  $(x^*(t), y^*(t), z^*(t)) = (e^{u^*(t)}, e^{v^*(t)}, e^{w^*(t)})$ . The proof of Theorem 2.1 is completed.  $\square$

From Theorem 2.1, if we let  $\theta = 0, \alpha_i = 1, \beta_i = 0, \gamma_i = 0 (i = 1, 2), z(t) = 0$ , then we can get the following corollary.

**Corollary 2.1** *If  $\int_0^T a_1(t) dt + \sum_{k=1}^q \ln(1 + d_{1k}) > 0$  and the following algebraic equation set*

$$\begin{cases} [a_1] + \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{1k}) - [b_1]e^u - c_1 e^{mv} = 0, \\ [a_2] - \frac{1}{\omega} \sum_{k=1}^q \ln(1 + d_{2k}) + [b_2]e^v - c_2 e^{u+(m-1)v} = 0 \end{cases}$$

*has finite solutions  $(u^*, v^*)$ , then system (2.1) has at least one positive  $\omega$ -periodic solution.*

*Remark 2.1* Corollary 2.1 is the same as Theorem 2.1 of [4]. Their model is the special case of (1.2). In this sense, we generalize or improve their results.

### 3 Permanence

**Definition 3.1** System (1.2) is said to be permanent if there exist positive constants  $T, G_i, g_i, i = 1, 2, 3$ , such that any solution  $(x(t), y(t), z(t))^T$  of system (1.2) satisfies  $g_1 \leq x(t) \leq G_1, g_2 \leq y(t) \leq G_2, g_3 \leq z(t) \leq G_3$  for  $t \geq T$ .

**Lemma 3.1** (see [20]) *If  $a > 0, b > 0$ , and  $\xi(t) \geq (\leq) b - a\xi(t)$  for  $t \geq 0, z(0) > 0$ , then  $\xi(t) \geq (\leq) \frac{b}{a} + [\xi(0) - \frac{b}{a}] \exp\{-at\}$ .*

**Lemma 3.2** (see [4, 21]) *If  $a > 0, b > 0, \tau \geq 0$ , then:*

- (1) *If  $y'(t) \leq y(t)[b - ay(t - \tau)]$ , then there exists a constant  $T > 0$  such that  $y(t) \leq \frac{a}{b} \exp\{b\tau\}$  for  $t \geq T$ .*
- (2) *If  $y'(t) \geq y(t)[b - ay(t - \tau)]$ , and there exist positive constants  $T$  and  $G$  such that  $y(t) < G$  for  $t \geq T$ , then for any small constant  $\varepsilon > 0$  there exists a constant  $T^* > T$  such that  $y(t) \geq \min\{\frac{b}{a} \exp\{(b - aM)\tau\}, \frac{b}{a} - \varepsilon\}$  for  $t \geq T^*$ . In [21], the value of the above  $\varepsilon$  is  $b/2a$ .*

Consider the following non-impulsive functional differential system:

$$\begin{cases} u'(t) = u(t)(a_1(t) - B_1(t)u(t - \tau_1) - \frac{C_1(t)(1-\theta(t))v^m(t)}{\alpha_1(t)+\beta_1(t)u(t)+\gamma_1(t)v(t)}), \\ v'(t) = v(t)(-a_2(t) - B_2(t)v(t - \tau_2) + \frac{C_2(t)(1-\theta(t))u(t)v^{m-1}(t)}{\alpha_1(t)+\beta_1(t)u(t)+\gamma_1(t)v(t)} \\ \quad - \frac{C_3(t)(1-\theta(t))w^m(t)}{\alpha_2(t)+\beta_2(t)v(t)+\gamma_2(t)w(t)}), \\ w'(t) = w(t)(-a_3(t) - B_3(t)w(t - \tau_3) + \frac{C_4(t)(1-\theta(t))v(t)w^{m-1}(t)}{\alpha_2(t)+\beta_2(t)v(t)+\gamma_2(t)w(t)}), \end{cases} \tag{3.1}$$

$$(x(s), y(s), z(s)) = (\phi_1(s), \phi_2(s), \phi_3(s)) \in C([- \tau, 0], \mathbb{R}_+^3), \quad \phi_i(s) > 0. \tag{3.2}$$

**Lemma 3.3** *If  $(x(t), y(t), z(t))$  is a solution of system (1.2) on  $[0, +\infty)$ , then  $(u(t), v(t), w(t)) = (\Lambda_{a_1}^{-1}x(t), \Lambda_{d_2}^{-1}y(t), \Lambda_{d_3}^{-1}z(t))$  is a solution of system (3.1) on  $[0, +\infty)$ .*

**Lemma 3.4** *If  $(u(t), v(t), w(t))$  is a solution of system (3.1) on  $[0, +\infty)$ , then  $(x(t), y(t), z(t)) = (\Lambda_{a_1}u(t), \Lambda_{d_2}v(t), \Lambda_{d_3}w(t))$  is a solution of system (1.2) on  $[0, +\infty)$ .*

The proofs of the above two lemmas are similar to that of Theorem 1 in [6], so we omit them here.

**Lemma 3.5** *There exists a positive constant  $T_0$  such that the solution  $(u(t), v(t), w(t))$  of system (3.1) satisfies  $0 < u(t) \leq G_1, 0 < v(t) \leq G_2$ , and  $0 < w(t) \leq G_3$  for  $t \geq T_0$ , where*

$$\begin{aligned} G_1 &= \frac{a_1^U}{B_1^L} \exp\{a_1^U \tau\}, \\ G_2 &= [G_1 C_2^U (1 - \theta^L) / \alpha_1^L (a_2^L + \varepsilon)]^{\frac{1}{1-m}}, \\ G_3 &= [G_2 C_4^U (1 - \theta^L) / \alpha_2^L (a_3^L + \varepsilon)]^{\frac{1}{1-m}}. \end{aligned}$$

*Proof* It follows from system (3.1) that

$$\begin{cases} u'(t) \leq u(t)(a_1^U - B_1^L u(t - \tau_1)), \\ v'(t) \leq v^m(t)(-a_2^L v^{1-m}(t) + \frac{C_2^U(1-\theta^L)u(t)}{\alpha_1^L}), \\ w'(t) \leq w^m(t)(-a_3^L w^{1-m}(t) + \frac{C_4^U(1-\theta^L)v(t)}{\alpha_2^L}). \end{cases} \tag{3.3}$$

The first inequality of (3.3) and Lemma 3.2 yield that there exists a positive constant  $T_1$  such that

$$u(t) \leq \frac{a_1^U}{B_1^L} \exp\{a_1^U \tau_1\} = G_1 \quad \text{for } t \geq T_1,$$

which, together with the second inequality of (3.3), yields

$$\frac{d(v^{1-m}(t))}{dt} \leq (1 - m) \left[ G_1 \frac{C_2^U(1 - \theta^L)}{\alpha_1^L} - a_2^L v^{1-m}(t) \right] \quad \text{for } t \geq T_1,$$

and the third inequality of (3.3) yields

$$\frac{d(w^{1-m}(t))}{dt} \leq (1 - m) \left[ G_2 \frac{C_4^U(1 - \theta^L)}{\alpha_2^L} - a_3^L w^{1-m}(t) \right] \quad \text{for } t \geq T_1.$$

Thus from Lemma 3.1 we know that, for any small positive constant  $\varepsilon$ , there exists a positive constant  $T_0 \geq T_1$  such that  $v(t) \leq G_2, w(t) \leq G_3$  for  $t \geq T_0$ .  $\square$

**Lemma 3.6** *If  $\Delta_1 > 0, \Delta_2 > 0,$  and  $\Delta_3 > 0,$  then there exists a positive constant  $T^*$  such that the solution  $(u(t), v(t), w(t))$  of system (3.1) satisfies*

$$u(t) \geq g_1, \quad v(t) \geq g_2 \quad \text{and} \quad w(t) \geq g_3 \quad \text{for } t \geq T^*,$$

where  $\varepsilon$  is a small enough positive constant, and

$$\begin{aligned} \Delta_1 &= \left[ a_1 - \frac{C_1(1-\theta)}{\alpha_1} G_2^m \right]^L, & \Delta_2 &= \left[ \frac{C_2(1-\theta)g_1}{\alpha_1} - \frac{C_3(1-\theta)G_3^m}{\alpha_2} - B_2 G_2^{2-m} \right]^L, \\ \Delta_3 &= \left[ \frac{C_4(1-\theta)g_2}{\alpha_2} - B_3 G_3^{2-m} \right]^L, \\ g_1 &= \min \left\{ \frac{\Delta_1}{B_1^U} \exp \left\{ [\Delta_1 - G_1 B_1^U] \tau \right\}, \frac{\Delta_1}{B_1^U} - \varepsilon \right\}, & g_2 &= \Delta_2 / a_2^L - \varepsilon, & g_3 &= \Delta_3 / a_3^L - \varepsilon. \end{aligned}$$

*Proof* It follows from Lemma 3.5 and system (3.1) that, for  $t \geq T_0$ ,

$$\begin{cases} u'(t) \geq u(t)(\Delta_1 - B_1^U u(t - \tau_1)), \\ v'(t) \geq v(t)(-a_2(t)v^{1-m}(t) - B_2 M_2^{2-m} + \frac{C_2(t)(1-\theta)u(t)}{\alpha_1(t)} - \frac{C_3(t)(1-\theta)w^m(t)}{\alpha_2(t)}), \\ w'(t) \geq w(t)(-a_3(t)w^{1-m}(t) - B_3 M_3^{2-m} + \frac{C_4(t)(1-\theta)v(t)}{\alpha_2(t)}), \end{cases}$$

which, together with Lemmas 3.1 and 3.2, implies that there exists a positive constant  $T^* \geq T_0$  such that  $u(t) \geq g_1, v(t) \geq g_2,$  and  $w(t) \geq g_3$  for  $t \geq T^*$ .  $\square$

From Lemmas 3.5 and 3.6 we can get the following result on the permanence of system (3.1), which, together with Lemmas 3.3 and 3.4, yields the permanence of system (1.2).

**Theorem 3.1** *If  $\Delta_1 > 0, \Delta_2 > 0,$  and  $\Delta_3 > 0,$  then system (1.2) is permanent and enters eventually into the region*

$$D = \{x(t), y(t), z(t) \mid \Lambda_{d_1} g_1 \leq x(t) \leq \Lambda_{d_1} G_1, \Lambda_{d_2} g_2 \leq x(t) \leq \Lambda_{d_2} G_2, \Lambda_{d_3} g_3 \leq x(t) \leq \Lambda_{d_3} G_3\}.$$

From Theorem 3.1, if we let  $\theta = 0, \alpha_i = 1 (i = 1, 2), z(t) = 0,$  then we can get the following corollary.

**Corollary 3.1** *If  $\Delta_1 > 0$  and  $\Delta_2 > 0,$  then system (1.2) is permanent and enters eventually into the region*

$$D = \{x(t), y(t), z(t) \mid \Lambda_{d_1} g_1 \leq x(t) \leq \Lambda_{d_1} G_1, \Lambda_{d_2} g_2 \leq x(t) \leq \Lambda_{d_2} G_2\}.$$

*Remark 3.1* Corollary 3.1 is the same as Theorem 3.1 of [4]. Their model is the special case of (1.2). In this sense, we generalize or improve their results.

### 4 Stability

**Definition 4.1** System (1.2) is called globally attractive if

$$\lim_{t \rightarrow +\infty} (|x(t) - \hat{x}(t)| + |y(t) - \hat{y}(t)| + |z(t) - \hat{z}(t)|) = 0$$

for any three positive solutions  $(x(t), y(t), z(t))$  and  $(\hat{x}(t), \hat{y}(t), \hat{z}(t))$  of system (1.2).

**Theorem 4.1** If  $\Delta_1 > 0$ ,  $\Delta_2 > 0$ , and  $\Delta_3 > 0$  and

$$\liminf_{t \rightarrow +\infty} \{ \Phi(t), \Psi(t), E(t) \} > 0,$$

then system (1.2) is globally attractive, where

$$\begin{aligned} \Psi(t) &= \Psi_1(t) - \Psi_2(t)mg_2^{m-1} + \frac{C_2(t)}{\alpha_1(t)}(1-m)g_1G_2^{m-2}, \\ \Phi(t) &= \left( \kappa B_1(t) - \frac{C_2(t)}{\alpha_1(t)}g_2^{m-1} \right) - \kappa \tilde{B}_{\tau_{11}}(t) \left[ a_1(t) + G_1B_1(t) + G_2^m \frac{C_1(t)}{\alpha_1(t)} \right] \\ &\quad - \kappa \tilde{B}_{\tau_{12}}(t)G_1B_1(t + \tau_1) - \tilde{B}_{\tau_{21}}(t)G_2^m \frac{C_2(t)}{\alpha_1(t)}, \\ E(t) &= E_1(t) + \frac{C_4(t)}{\alpha_2(t)}g_2(1-m)G_3^{m-2} - E_2(t)mg_3^{m-1}, \\ \Psi_1(t) &= B_2(t) - \tilde{B}_{\tau_{31}}(t)G_3^m \frac{C_4(t)}{\alpha_2(t)} - \tilde{B}_{\tau_{22}}(t)G_2B_2(t + \tau_2) \\ &\quad - \tilde{B}_{\tau_{21}}(t) \left[ a_2(t) + G_2B_2(t) - G_3^m \frac{C_3(t)}{\alpha_2(t)} \right] - \frac{C_4(t)}{\alpha_2(t)}g_3^{m-1}, \\ \Psi_2(t) &= \kappa \frac{C_1(t)}{\alpha_1(t)}(1 + \tilde{B}_{\tau_{11}}(t)G_1) + \tilde{B}_{\tau_{21}}(t)G_1 \frac{C_2(t)}{\alpha_1(t)}, \\ E_1(t) &= B_3(t) - \tilde{B}_{\tau_{31}}(t)[a_3(t) + G_3B_3(t)] - \tilde{B}_{\tau_{32}}(t)G_3B_3(t + \tau_3), \\ E_2(t) &= (1 + \tilde{B}_{\tau_{21}}(t)G_2) \frac{C_3(t)}{\alpha_2(t)} + \tilde{B}_{\tau_{31}}(t)G_2 \frac{C_4(t)}{\alpha_2(t)}, \\ \tilde{B}_{\tau_{11}}(t) &= \int_t^{t+\tau_1} B_1(\sigma) d\sigma, \quad \tilde{B}_{\tau_{21}}(t) = \int_t^{t+\tau_2} B_2(\sigma) d\sigma, \\ \tilde{B}_{\tau_{31}}(t) &= \int_t^{t+\tau_3} B_3(\sigma) d\sigma, \quad \tilde{B}_{\tau_{12}}(t) = \int_{t+\tau_1}^{t+2\tau_1} B_1(\sigma) d\sigma, \\ \tilde{B}_{\tau_{22}}(t) &= \int_{t+\tau_1}^{t+2\tau_2} B_2(\sigma) d\sigma, \quad \tilde{B}_{\tau_{32}}(t) = \int_{t+\tau_1}^{t+2\tau_3} B_3(\sigma) d\sigma. \end{aligned}$$

*Proof* Let  $(\hat{x}(t), \hat{y}(t), \hat{z}(t))$  and  $(x(t), y(t), z(t))$  be any two positive solutions of system (1.2); then  $(\hat{u}(t), \hat{v}(t), \hat{w}(t)) = (\Lambda_{d_1}^{-1}\hat{x}(t), \hat{y}_{d_2}^{-1}(t), \hat{z}_{d_3}^{-1}(t))$  and  $(u(t), v(t), w(t)) = (\Lambda_{d_1}^{-1}x(t), y_{d_2}^{-1}(t), z_{d_3}^{-1}(t))$  are positive solutions of system (3.1). It follows from Lemmas 3.5 and 3.6 that there exist positive constants  $g_i, G_i$ , and  $T^*$  (defined in Lemma 3.5 and Lemma 3.6, respectively) such that

$$g_1 \leq u(t), \hat{u}(t) \leq G_1, \quad g_2 \leq v(t), \hat{v}(t) \leq G_2, \quad g_3 \leq w(t), \hat{w}(t) \leq G_3 \quad \text{for } t \geq T^*.$$

Define a function

$$V_1(t) = \kappa \left| \ln u(t) - \ln \hat{u}(t) \right| + \left| \ln v(t) - \ln \hat{v}(t) \right| + \left| \ln w(t) - \ln \hat{w}(t) \right| \quad \text{for } t \geq T^*.$$

By calculating its upper right derivative along the solution of system (3.1), we get

$$\begin{aligned} D^+ V_1(t) &= \kappa \operatorname{sgn}(u(t) - \hat{u}(t)) \left[ \frac{u'(t)}{u(t)} - \frac{\hat{u}'(t)}{\hat{u}(t)} \right] + \operatorname{sgn}(v(t) - \hat{v}(t)) \left[ \frac{v'(t)}{v(t)} - \frac{\hat{v}'(t)}{\hat{v}(t)} \right] \\ &\quad + \operatorname{sgn}(w(t) - \hat{w}(t)) \left[ \frac{w'(t)}{w(t)} - \frac{\hat{w}'(t)}{\hat{w}(t)} \right] \\ &= \kappa \operatorname{sgn}(u(t) - \hat{u}(t)) \left\{ B_1(t) [\hat{u}(t - \tau_1) - u(t - \tau_1)] \right. \\ &\quad \left. + \frac{C_1(t)(1 - \theta(t))}{\alpha_1(t)} [\hat{v}^m(t) - v^m(t)] \right\} \\ &\quad + \operatorname{sgn}(v(t) - \hat{v}(t)) \left\{ B_2(t) [\hat{v}(t - \tau_2) - v(t - \tau_2)] \right. \\ &\quad \left. - \frac{C_2(t)(1 - \theta(t))}{\alpha_1(t)} \left[ \frac{u(t)}{v^{1-m}(t)} - \frac{\hat{u}(t)}{\hat{v}^{1-m}(t)} \right] \right. \\ &\quad \left. + \frac{C_3(t)(1 - \theta(t))}{\alpha_2(t)} [\hat{w}^m(t) - w^m(t)] \right\} \\ &\quad + \operatorname{sgn}(w(t) - \hat{w}(t)) \left\{ B_3(t) [\hat{w}(t - \tau_1) - w(t - \tau_1)] \right. \\ &\quad \left. + \frac{C_4(t)(1 - \theta(t))}{\alpha_2(t)} \left[ \frac{v(t)}{w^{1-m}(t)} - \frac{\hat{v}(t)}{\hat{w}^{1-m}(t)} \right] \right\} \\ &\leq -\kappa B_1(t) |u(t) - \hat{u}(t)| - B_2(t) |v(t) - \hat{v}(t)| - B_3(t) |w(t) - \hat{w}(t)| \\ &\quad + \kappa \frac{C_1(t)(1 - \theta(t))}{\alpha_1(t)} |\hat{v}^m(t) - v^m(t)| \\ &\quad - \frac{C_2(t)(1 - \theta(t))}{\alpha_1(t)} \operatorname{sgn}(v(t) - \hat{v}(t)) \left[ \frac{u(t)}{v^{1-m}(t)} - \frac{\hat{u}(t)}{\hat{v}^{1-m}(t)} \right] \\ &\quad + \frac{C_3(t)(1 - \theta(t))}{\alpha_2(t)} |\hat{w}^m(t) - w^m(t)| \\ &\quad + \frac{C_4(t)(1 - \theta(t))}{\alpha_2(t)} \operatorname{sgn}(w(t) - \hat{w}(t)) \left[ \frac{v(t)}{w^{1-m}(t)} - \frac{\hat{v}(t)}{\hat{w}^{1-m}(t)} \right] \\ &\quad + \kappa B_1(t) \int_{t-\tau_1}^t |u'(s) - \hat{u}'(s)| ds + B_2(t) \int_{t-\tau_2}^t |v'(s) - \hat{v}'(s)| ds \\ &\quad + B_3(t) \int_{t-\tau_3}^t |w'(s) - \hat{w}'(s)| ds \end{aligned}$$

which, together with

$$\begin{aligned} \operatorname{sgn}(v(t) - \hat{v}(t)) \left[ \frac{u(t)}{v^{1-m}(t)} - \frac{\hat{u}(t)}{\hat{v}^{1-m}(t)} \right] &\leq -g_1 |v^{m-1}(t) - \hat{v}^{m-1}(t)| + g_2^{m-1} |u(t) - \hat{u}(t)|, \\ \operatorname{sgn}(w(t) - \hat{w}(t)) \left[ \frac{v(t)}{w^{1-m}(t)} - \frac{\hat{v}(t)}{\hat{w}^{1-m}(t)} \right] &\leq -g_2 |w^{m-1}(t) - \hat{w}^{m-1}(t)| + g_3^{m-1} |v(t) - \hat{v}(t)|, \end{aligned}$$

$$\begin{aligned} \int_{t-\tau_1}^t |u'(s) - \hat{u}'(s)| ds &\leq \int_{t-\tau_1}^t \left[ a_1(s) + G_1 B_1(s) + G_2^m \frac{C_1(s)(1-\theta(s))}{\alpha_1(s)} \right] |u(s) - \hat{u}(s)| ds \\ &\quad + G_1 \int_{t-\tau_1}^t B_1(s) |u(s - \tau_1) - \hat{u}(s - \tau_1)| ds \\ &\quad + G_1 \int_{t-\tau_1}^t \frac{C_1(s)(1-\theta(s))}{\alpha_1(s)} |v^m(s) - \hat{v}^m(s)| ds \end{aligned}$$

and

$$\begin{aligned} \int_{t-\tau_2}^t |v'(s) - \hat{v}'(s)| ds &\leq \int_{t-\tau_2}^t [a_2(s) + G_2 B_2(s)] |v(s) - \hat{v}(s)| ds \\ &\quad + G_2 \int_{t-\tau_2}^t B_2(s) |v(s - \tau_2) - \hat{v}(s - \tau_2)| ds \\ &\quad + G_2^m \int_{t-\tau_2}^t \frac{C_2(s)(1-\theta(s))}{\alpha_1(s)} |u(s) - \hat{u}(s)| ds \\ &\quad + G_3^m \int_{t-\tau_2}^t \frac{C_3(s)(1-\theta(s))}{\alpha_2(s)} |v(s) - \hat{v}(s)| ds \\ &\quad + G_1 \int_{t-\tau_2}^t \frac{C_2(s)(1-\theta(s))}{\alpha_1(s)} |v^m(s) - \hat{v}^m(s)| ds \\ &\quad + G_2 \int_{t-\tau_2}^t \frac{C_3(s)(1-\theta(s))}{\alpha_2(s)} |w^m(s) - \hat{w}^m(s)| ds \end{aligned}$$

and

$$\begin{aligned} \int_{t-\tau_1}^t |w'(s) - \hat{w}'(s)| ds &\leq \int_{t-\tau_3}^t [a_3(s) + G_3 B_3(s)] |w(s) - \hat{w}(s)| ds \\ &\quad + G_3^m \int_{t-\tau_2}^t \frac{C_4(s)(1-\theta(s))}{\alpha_2(s)} |v(s) - \hat{v}(s)| ds \\ &\quad + G_3 \int_{t-\tau_3}^t B_3(s) |w(s - \tau_3) - \hat{w}(s - \tau_3)| ds \\ &\quad + G_2 \int_{t-\tau_1}^t \frac{C_4(s)(1-\theta(s))}{\alpha_2(s)} |w^m(s) - \hat{w}^m(s)| ds, \end{aligned}$$

yields

$$\begin{aligned} D^+ V_1(t) &\leq \left( -\kappa B_1(t) + \frac{C_2(t)(1-\theta(s))}{\alpha_1(t)} g_2^{m-1} \right) |u(t) - \hat{u}(t)| - B_2(t) |v(t) - \hat{v}(t)| \\ &\quad - B_3(t) |w(t) - \hat{w}(t)| + \alpha \frac{C_1(t)(1-\theta(s))}{\alpha_1(t)} |\hat{v}^m(t) - v^m(t)| \\ &\quad - \frac{C_2(t)(1-\theta(s))}{\alpha_1(t)} g_1 |v^{m-1}(t) - \hat{v}^{m-1}(t)| + \frac{C_3(t)(1-\theta(s))}{\alpha_2(t)} |\hat{w}^m(t) - w^m(t)| \\ &\quad - \frac{C_4(t)(1-\theta(s))}{\alpha_2(t)} g_2 |w^{m-1}(t) - \hat{w}^{m-1}(t)| + \frac{C_4(t)(1-\theta(s))}{\alpha_2(t)} g_3^{m-1} |v(t) - \hat{v}(t)| \\ &\quad + \kappa B_1(t) \left( \int_{t-\tau_1}^t \left[ a_1(s) + G_1 B_1(s) + G_2^m \frac{C_1(s)(1-\theta(s))}{\alpha_1(s)} \right] |u(s) - \hat{u}(s)| ds \right) \end{aligned}$$

$$\begin{aligned}
 &+ G_1 \int_{t-\tau_1}^t B_1(s) |u(s - \tau_1) - \hat{u}(s - \tau_1)| ds \\
 &+ G_1 \int_{t-\tau_1}^t \frac{C_1(s)(1 - \theta(s))}{\alpha_1(s)} |v^m(s) - \hat{v}^m(s)| ds \\
 &+ B_2(t) \left( \int_{t-\tau_2}^t [a_2(s) + G_2 B_2(s)] |v(s) - \hat{v}(s)| ds \right. \\
 &+ G_2 \int_{t-\tau_2}^t B_2(s) |v(s - \tau_2) - \hat{v}(s - \tau_2)| ds \\
 &+ G_2^m \int_{t-\tau_2}^t \frac{C_2(s)(1 - \theta(s))}{\alpha_1(s)} |u(s) - \hat{u}(s)| ds \\
 &+ G_3^m \int_{t-\tau_2}^t \frac{C_3(s)(1 - \theta(s))}{\alpha_2(s)} |v(s) - \hat{v}(s)| ds \\
 &+ G_1 \int_{t-\tau_2}^t \frac{C_2(s)(1 - \theta(s))}{\alpha_1(s)} |v^m(s) - \hat{v}^m(s)| ds \\
 &+ G_2 \int_{t-\tau_2}^t \frac{C_3(s)(1 - \theta(s))}{\alpha_2(s)} |w^m(s) - \hat{w}^m(s)| ds \left. \right) \\
 &+ B_3(t) \left( \int_{t-\tau_3}^t [a_3(s) + G_3 B_3(s)] |w(s) - \hat{w}(s)| ds \right. \\
 &+ G_3^m \int_{t-\tau_2}^t \frac{C_4(s)(1 - \theta(s))}{\alpha_2(s)} |v(s) - \hat{v}(s)| ds \\
 &+ G_3 \int_{t-\tau_3}^t B_3(s) |w(s - \tau_3) - \hat{w}(s - \tau_3)| ds \\
 &+ G_2 \int_{t-\tau_1}^t \frac{C_4(s)(1 - \theta(s))}{\alpha_2(s)} |w^m(s) - \hat{w}^m(s)| ds \left. \right).
 \end{aligned}$$

Define further

$$\begin{aligned}
 V_2(t) &= \int_t^{t+\tau_1} \int_{\sigma-\tau_1}^t B_1(\sigma) \left[ a_1(s) + G_1 B_1(s) + G_2^m \frac{C_1(s)(1 - \theta(s))}{\alpha_1(s)} \right] |u(s) - \hat{u}(s)| ds d\sigma \\
 &+ G_1 \int_t^{t+\tau_1} \int_{\sigma-\tau_1}^t B_1(\sigma) B_1(s) |u(s - \tau_1) - \hat{u}(s - \tau_1)| ds d\sigma \\
 &+ G_1 \int_t^{t+\tau_1} \int_{\sigma-\tau_1}^t B_1(\sigma) \frac{C_1(s)(1 - \theta(s))}{\alpha_1(s)} |v^m(s) - \hat{v}^m(s)| ds d\sigma, \\
 V_3(t) &= \int_t^{t+\tau_2} \int_{\sigma-\tau_2}^t B_2(\sigma) [a_2(s) + G_2 B_2(s)] |v(s) - \hat{v}(s)| ds d\sigma \\
 &+ G_2 \int_t^{t+\tau_2} \int_{\sigma-\tau_2}^t B_2(\sigma) B_2(s) |v(s - \tau_2) - \hat{v}(s - \tau_2)| ds d\sigma \\
 &+ G_2^m \int_t^{t+\tau_2} \int_{\sigma-\tau_2}^t B_2(\sigma) \frac{C_2(s)(1 - \theta(s))}{\alpha_1(s)} |u(s) - \hat{u}(s)| ds d\sigma \\
 &+ G_3^m \int_t^{t+\tau_2} \int_{\sigma-\tau_2}^t B_2(\sigma) \frac{C_3(s)(1 - \theta(s))}{\alpha_2(s)} |v(s) - \hat{v}(s)| ds d\sigma
 \end{aligned}$$

$$\begin{aligned}
 &+ G_1 \int_t^{t+\tau_2} \int_{\sigma-\tau_2}^t B_2(\sigma) \frac{C_2(s)(1-\theta(s))}{\alpha_1(s)} |v^m(s) - \hat{v}^m(s)| ds d\sigma \\
 &+ G_2 \int_t^{t+\tau_2} \int_{\sigma-\tau_2}^t B_2(\sigma) \frac{C_3(s)(1-\theta(s))}{\alpha_2(s)} |w^m(s) - \hat{w}^m(s)| ds d\sigma, \\
 V_4 = &\int_t^{t+\tau_3} \int_{\sigma-\tau_3}^t [a_3(s) + G_3 B_3(s)] |w(s) - \hat{w}(s)| ds d\sigma \\
 &+ G_3^m \int_t^{t+\tau_3} \int_{\sigma-\tau_2}^t \frac{C_4(s)(1-\theta(s))}{\alpha_2(s)} |v(s) - \hat{v}(s)| ds d\sigma \\
 &+ G_3 \int_t^{t+\tau_3} \int_{\sigma-\tau_3}^t B_3(s) |w(s - \tau_3) - \hat{w}(s - \tau_3)| ds d\sigma \\
 &+ G_2 \int_t^{t+\tau_3} \int_{\sigma-\tau_1}^t \frac{C_4(s)(1-\theta(s))}{\alpha_2(s)} |w^m(s) - \hat{w}^m(s)| ds d\sigma
 \end{aligned}$$

and

$$\begin{aligned}
 V_5 = &\kappa G_1 \int_t^{t+\tau_1} \int_s^{s+\tau_1} B_1(\sigma) B_1(s) |u(s - \tau_1) - \hat{u}(s - \tau_1)| d\sigma ds \\
 &+ G_2 \int_t^{t+\tau_2} \int_s^{s+\tau_2} B_2(\sigma) B_2(s) |v(s - \tau_2) - \hat{v}(s - \tau_2)| d\sigma ds \\
 &+ G_3 \int_t^{t+\tau_3} \int_s^{s+\tau_3} B_3(\sigma) B_3(s) |w(s - \tau_3) - \hat{w}(s - \tau_3)| d\sigma ds.
 \end{aligned}$$

We choose the Lyapunov functional as follows:  $V(t) = V_1(t) + \kappa V_2(t) + V_3(t) + V_4(t) + V_5(t)$ .

Thus, for  $t \geq T^*$ ,

$$\begin{aligned}
 D^+ V(t) = &D^+ V_1(t) + \kappa V_2'(t) + V_3'(t) + V_4'(t) + V_5'(t) \\
 \leq &\left( \kappa B_1(t) - \frac{C_2(t)(1-\theta(s))}{\alpha_1(t)} g_2^{m-1} \right) |u(t) - \hat{u}(t)| - B_2(t) |v(t) - \hat{v}(t)| \\
 &- B_3(t) |w(t) - \hat{w}(t)| + \alpha \frac{C_1(t)(1-\theta(s))}{\alpha_1(t)} |\hat{v}^m(t) - v^m(t)| \\
 &- \frac{C_2(t)(1-\theta(s))}{\alpha_1(t)} g_1 |v^{m-1}(t) - \hat{v}^{m-1}(t)| + \frac{C_3(t)(1-\theta(s))}{\alpha_2(t)} |\hat{w}^m(t) - w^m(t)| \\
 &- \frac{C_4(t)(1-\theta(s))}{\alpha_2(t)} g_2 |w^{m-1}(t) - \hat{w}^{m-1}(t)| + \frac{C_4(t)(1-\theta(s))}{\alpha_2(t)} g_3^{m-1} |v(t) - \hat{v}(t)| \\
 &+ \kappa \tilde{B}_{\tau_{11}}(t) \left[ a_1(t) + G_1 B_1(t) + G_2^m \frac{C_1(t)(1-\theta(s))}{\alpha_1(t)} \right] |u(t) - \hat{u}(t)| \\
 &+ \kappa \tilde{B}_{\tau_{11}}(t) G_1 \frac{C_1(t)(1-\theta(s))}{\alpha_1(t)} |v^m(t) - \hat{v}^m(t)| \\
 &+ \kappa \tilde{B}_{\tau_{12}}(t) G_1 B_1(t + \tau_1) |u(t) - \hat{u}(t)| \\
 &+ \tilde{B}_{\tau_{21}}(t) \left[ a_2(t) + M_2 B_2(t) + G_3^m \frac{C_3(t)(1-\theta(s))}{\alpha_2(t)} \right] |v(t) - \hat{v}(t)| \\
 &+ \tilde{B}_{\tau_{21}}(t) G_2^m \frac{C_2(t)(1-\theta(s))}{\alpha_1(t)} |u(t) - \hat{u}(t)| + \tilde{B}_{\tau_{22}}(t) G_2 B_2(t + \tau_2) |v(t) - \hat{v}(t)| \\
 &+ \tilde{B}_{\tau_{21}}(t) G_1 \frac{C_2(t)(1-\theta(s))}{\alpha_1(t)} |v^m(t) - \hat{v}^m(t)|
 \end{aligned}$$

$$\begin{aligned}
 & + \tilde{B}_{\tau_{21}}(t)G_2 \frac{C_3(t)(1-\theta(s))}{\alpha_2(t)} |w^m(t) - \hat{w}^m(t)| \\
 & + \tilde{B}_{\tau_{31}}(t)[a_3(t) + G_3B_3(t)] |w(t) - \hat{w}(t)| \\
 & + \tilde{B}_{\tau_{31}}(t)G_3^m \frac{C_4(t)(1-\theta(s))}{\alpha_2(t)} |v(t) - \hat{v}(t)| \\
 & + \tilde{B}_{\tau_{32}}(t)G_3B_3(t + \tau_3) |w(t) - \hat{w}(t)| \\
 & + \tilde{B}_{\tau_{31}}(t)G_2 \frac{C_4(t)(1-\theta(s))}{\alpha_2(t)} |w^m(t) - \hat{w}^m(t)| \\
 & \leq -\Phi(t)|u(t) - \hat{u}(t)| - \Psi(t)|v(t) - \hat{v}(t)| - E(t)|w(t) - \hat{w}(t)|.
 \end{aligned}$$

It follows from  $\Phi^L > 0$ ,  $\Psi^L > 0$ , and  $E^L > 0$  that

$$D^+V(t) \leq -\Phi^L(t)|u(t) - \hat{u}(t)| - \Psi^L(t)|v(t) - \hat{v}(t)| - E^L(t)|w(t) - \hat{w}(t)| \quad \text{for } t \geq T^*.$$

Integrating both sides of the above inequality gives

$$\begin{aligned}
 V(t) + \Phi^L \int_{T^*}^t |u(s) - \hat{u}(s)| ds + \Psi^L \int_{T^*}^t |v(s) - \hat{v}(s)| ds - E^L \int_{T^*}^t |w(s) - \hat{w}(s)| ds \\
 \leq V(T^*) < +\infty
 \end{aligned}$$

which, together with the boundedness of  $u'(t)$ ,  $v'(t)$ , and  $w'(t)$ , yields that  $|u(s) - \hat{u}(s)|$ ,  $|v(s) - \hat{v}(s)|$ , and  $|w(s) - \hat{w}(s)|$  are uniformly continuous on  $[T^*, +\infty)$ . By Barbalat's lemma [22], we have

$$\begin{aligned}
 \lim_{t \rightarrow +\infty} |u(s) - \hat{u}(s)| &= \lim_{t \rightarrow +\infty} \Delta_{d_1}^{-1} |x(s) - \hat{x}(s)| = 0, \\
 \lim_{t \rightarrow +\infty} |v(s) - \hat{v}(s)| &= \lim_{t \rightarrow +\infty} \Delta_{d_2}^{-1} |y(s) - \hat{y}(s)| = 0, \\
 \lim_{t \rightarrow +\infty} |w(s) - \hat{w}(s)| &= \lim_{t \rightarrow +\infty} \Delta_{d_3}^{-1} |z(s) - \hat{z}(s)| = 0,
 \end{aligned}$$

which yields  $\lim_{t \rightarrow +\infty} (|x(s) - \hat{x}(s)| + |y(s) - \hat{y}(s)| + |z(s) - \hat{z}(s)|) = 0$ . The proof is now finished. □

From Theorem 3.1, if we let  $\theta = 0$ ,  $\alpha_i = 1$  ( $i = 1, 2$ ),  $z(t) = 0$ , then we can get the following corollary.

**Corollary 4.1** *If  $\Delta_1 > 0$  and  $\Delta_2 > 0$ ,*

$$\liminf_{t \rightarrow +\infty} \{ \Phi(t), \Psi(t) \} > 0,$$

*then system (1.2) is globally attractive, where*

$$\begin{aligned}
 \Psi(t) &= \Psi_1(t) - \Psi_2(t)mg_2^{m-1} + C_2(t)(1-m)g_1G_2^{m-2}, \\
 \Phi(t) &= (\kappa B_1(t) - C_2(t)g_2^{m-1}) - \kappa \tilde{B}_{\tau_{11}}(t)[a_1(t) + G_1B_1(t) + G_2^m C_1(t)] \\
 &\quad - \kappa \tilde{B}_{\tau_{12}}(t)G_1B_1(t + \tau_1) - \tilde{B}_{\tau_{21}}(t)G_2^m C_2(t),
 \end{aligned}$$

$$\begin{aligned} \Psi_1(t) &= B_2(t) - \tilde{B}_{\tau_{22}}(t)G_2B_2(t + \tau_2) - \tilde{B}_{\tau_{21}}(t)[a_2(t) + G_2B_2(t)], \\ \Psi_2(t) &= \kappa C_1(t)(1 + \tilde{B}_{\tau_{11}}(t)G_1) + \tilde{B}_{\tau_{21}}(t)G_1C_2(t), \\ \tilde{B}_{\tau_{11}}(t) &= \int_t^{t+\tau_1} B_1(\sigma) d\sigma, \quad \tilde{B}_{\tau_{21}}(t) = \int_t^{t+\tau_2} B_2(\sigma) d\sigma, \\ \tilde{B}_{\tau_{12}}(t) &= \int_{t+\tau_1}^{t+2\tau_1} B_1(\sigma) d\sigma, \quad \tilde{B}_{\tau_{22}}(t) = \int_{t+\tau_1}^{t+2\tau_2} B_2(\sigma) d\sigma. \end{aligned}$$

*Remark 4.1* Corollary 4.1 is the same as Theorem 4.1 of [4]. Their model is the special case of (1.2). In this sense, we generalize or improve their results.

### 5 Examples and simulation

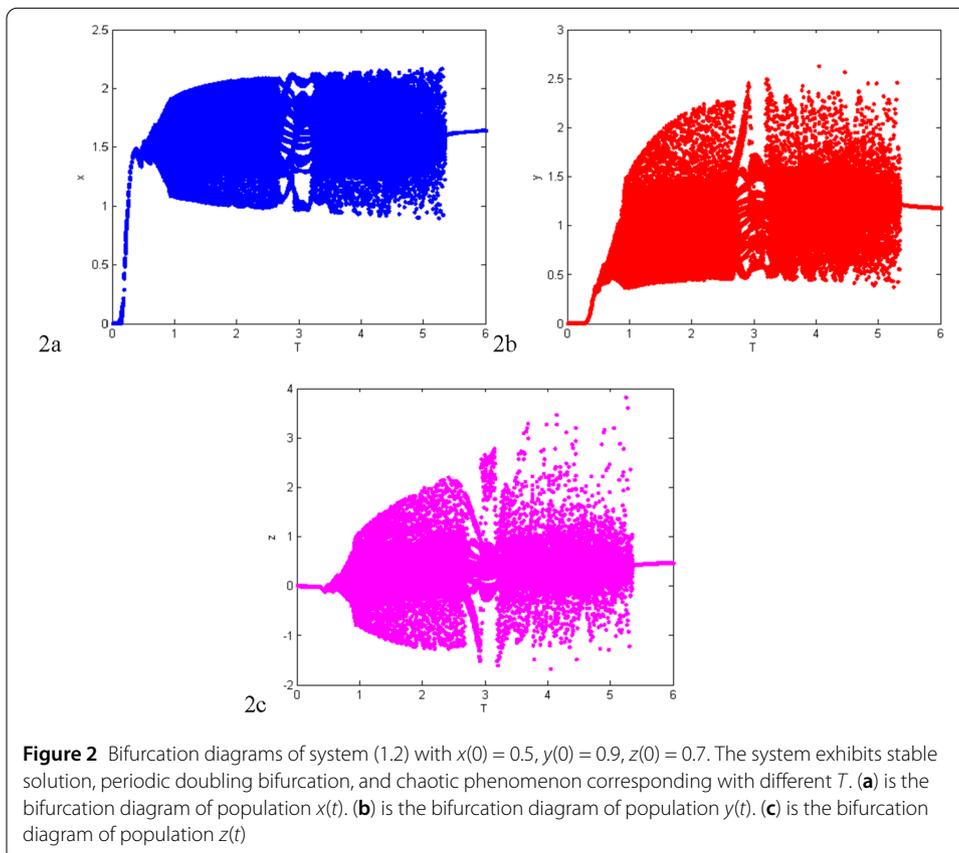
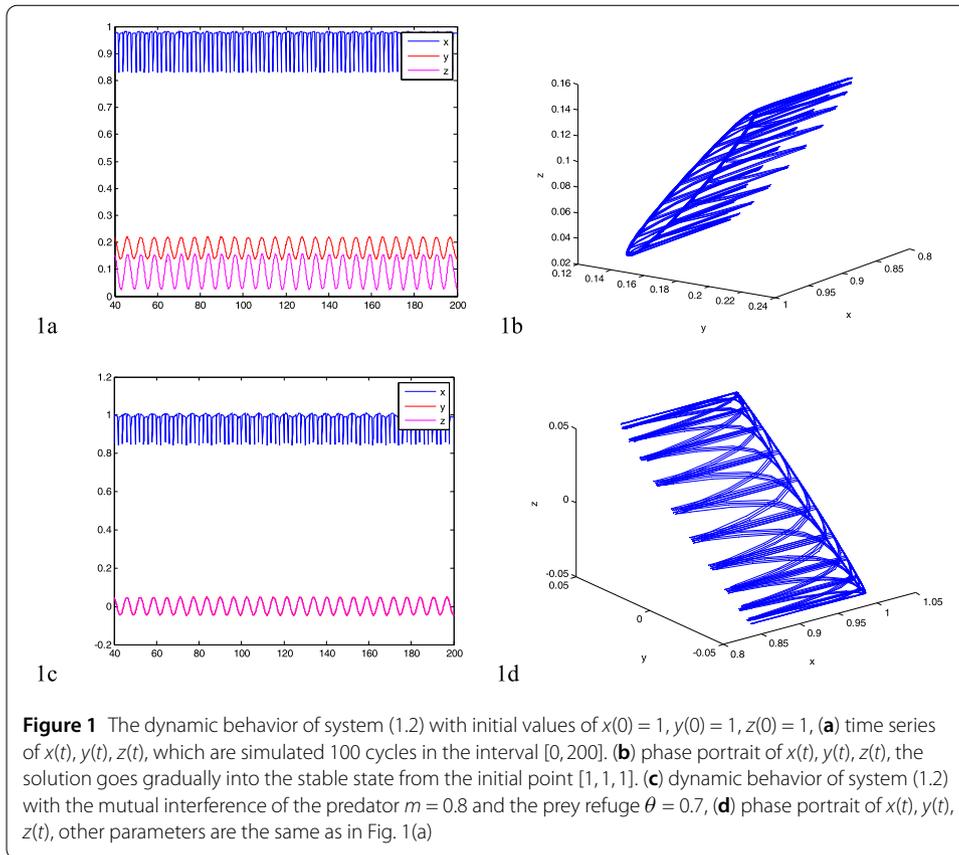
In this section, we give some examples and numerical simulations to verify the feasibility of our theoretical results, and further study the complexity and variety of system (1.2).

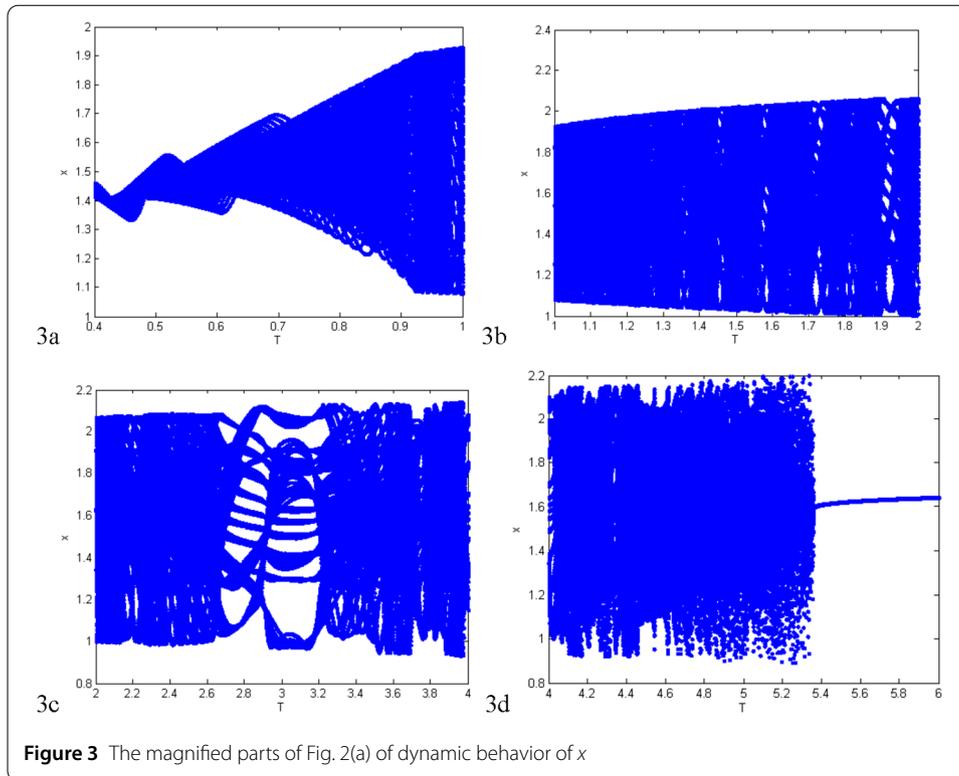
In order to verify the feasibility of our results, we consider the following system:

$$\begin{cases} x'(t) = x(t)(10 + 0.1 \sin(t) - 10x(t - \tau_1) - \frac{(1+0.29 \sin(t))(1-\theta(t))y^{0.1}(t)}{1.1+x(t)+y(t)}), \\ y'(t) = y(t)(-2.3 + 0.1 \sin(t) - 0.4(t)y(t - \tau_2) + \frac{(2+0.1 \sin(t))(1-\theta(t))x(t)y^{-0.9}(t)}{1.1+x(t)+y(t)} \\ \quad - \frac{(0.1+0.29 \sin(t))(1-\theta(t))z^{0.1}(t)}{1.1+y(t)+z(t)}), \\ z'(t) = z(t)(-2 + 0.1 \sin(t) - 0.1z(t - \tau_3) + \frac{(3+0.1 \sin(t))(1-\theta(t))y(t)z^{-0.9}(t)}{1.1+y(t)+z(t)}). \end{cases} \tag{5.1}$$

We take, corresponding to system (3.1),  $a_1 = 10 + 0.1 \sin(t)$ ,  $a_2 = 2.3 - 0.1 \sin(t)$ ,  $a_3 = 2 - 0.1 \sin(t)$ ,  $B_1(t) = 10$ ,  $B_2(t) = 0.4$ ,  $B_3(t) = 0.1$ ,  $C_1(t) = 1 + 0.29 \sin(t)$ ,  $C_2(t) = 2 + 0.1 \sin(t)$ ,  $C_3(t) = 0.1 + 0.29 \sin(t)$ ,  $C_4(t) = 3 + 0.1 \sin(t)$ ,  $\alpha_1(t) = \alpha_2(t) = 1.1$ ,  $\beta_1(t) = \beta_2(t) = 1$ ,  $\gamma_1(t) = \gamma_2(t) = 1$ ,  $\tau_1 = \tau_2 = \tau_3 = 0.002$ ,  $T = 1$ ,  $d_{1k} = 0.15$ ,  $d_{2k} = 0.005$ ,  $d_{3k} = 0.001$ ,  $m = 0.1$ ,  $\theta = 0.4$ , and let  $\sum_{k=1}^q \ln(1 + d_{1k}) \geq -10$ . By direct computation we have  $\int_0^T a_1(t) dt \approx 10.045969769$ ,  $\Delta_1 = 9.2434 > 0$ ,  $\Delta_2 = 0.2053 > 0$ ,  $\Delta_3 = 0.0748 > 0$ ,  $\lim_{t \rightarrow +\infty} \inf\{\Phi(t), \Psi(t), E(t)\} \approx 0.1936 > 0$ . By verification, these parameters satisfy the conditions of Theorems 2.1, 3.1, 4.1 that system (1.2) is permanent and has positive  $T$ -periodic solution, which is globally attractive. In order to see these dynamic properties clearly, we drew the figures for the evolution of the solutions of system (1.2) by using the function dde23 in Matlab; see Figs. 1(a)–1(b). If we change the mutual interference of the predator  $m$  and the prey refuge constant  $\theta$ , the other parameters are the same as in Fig. 1, then we get another dynamic portrait that is very different from Fig. 1. We give examples for  $m = 0.8$  and  $\theta = 0.7$ , see Fig. 1(c) and Fig. 1(d), respectively. Figure 1(c) and Fig. 1 (d) show that the mutual interference of the predator  $m$  and the prey refuge constant  $\theta$  affects the dynamic behaviors of system (1.2).

In order to show the complex dynamic behaviors of system (1.2), we choose another set of parameters as  $a_1 = 1$ ,  $a_2 = 0.1$ ,  $a_3 = 0.1$ ,  $b_1(t) = 0.35$ ,  $b_2(t) = 0.1$ ,  $b_3(t) = 0.45$ ,  $c_1(t) = 1$ ,  $c_2(t) = 0.8$ ,  $c_3(t) = 1$ ,  $c_4(t) = 0.8$ ,  $\alpha_1(t) = \alpha_2(t) = 1$ ,  $\beta_1(t) = \beta_2(t) = 0.5$ ,  $\gamma_1(t) = \gamma_2(t) = 0.5$ ,  $\tau_1 = \tau_2 = \tau_3 = 0.01$ ,  $d_{1k} = 0.05$ ,  $d_{2k} = 0.05$ ,  $d_{3k} = 0.05$ ,  $m = 0.9$ ,  $\theta = 0.1$ . The bifurcation diagrams of  $x(t)$ ,  $y(t)$ , and  $z(t)$  with respect to parameter  $T$  in range  $[0, 6]$  are shown in Fig. 2. Figure 2(a)–(c) are bifurcation diagrams of  $x(t)$ ,  $y(t)$ , and  $z(t)$ , respectively. From these diagrams, we can see that impulsive period  $T$  heavily affects the dynamic behaviors. For example, Fig. 2(a) shows the complex dynamic behaviors of  $x$ . Figure 3 shows the magnified parts of Fig. 2(a). From Fig. 3(a) we can see, when impulsive period  $T < 0.43$ , it is





stable; if  $T \in [2.62, 3.31]$ , we found that system (1.2) exhibits stable solutions, chaos, bifurcations, stable periodic solutions, and so on (see Fig. 3(c)); it seems more chaotic on the interval (see Figs. 3(a), 3(b), and 3(d)) when  $T \in [0.43, 2.62] \cup [3.31, 6]$ ; when  $T > 5.35$ , it is stable. These phenomena support strongly that system (1.2) has rich and complicated behavior.

### 6 Conclusion

In this paper, the dynamic behaviors of a delayed impulsive food chain system with prey refuge and mutual interference of predator are investigated and numerical simulations are given. Conditions for the existence and globally asymptotic stability of periodic solution of system (1.2) are obtained in Theorem 2.1 and Theorem 3.1, respectively. Theorem 4.1 gives the sufficient conditions of the permanence of this system (1.2). By computer simulation, the abundant dynamic behaviors of system (1.2) are exhibited.

Theorem 2.1, Theorem 3.1, Theorem 4.1, and simulations show that dynamic properties of (1.2) depend on the mutual interference of the predator  $m$ , the prey refuge  $\theta$ , delays  $\tau_i$  ( $i = 1, 2, 3$ ), and the parameter  $d_i$  ( $i = 1, 2, 3$ ) of pests or predator which represent the regular harvest or deaths from spraying pesticide. Figure 1(c)–(d) imply that different values of mutual interference of predator  $m$  and the prey refuge  $\theta$  bring different dynamic properties for system (1.2). Figure 2 and Fig. 3 show that impulsive period  $T$  heavily affects the dynamic behavior of system (1.2). As  $T$  changes, periodic behaviors, bifurcations, and chaotic phenomenon appear, respectively. Hence, we can choose a moderate value of  $T$  for some different control strategies.

In addition, model (1.2) generalizes the model of reference [4]. In reference [4], Wang did not consider the prey refuge  $\theta$ , the Beddington–DeAngelis functional response, and

the case that a predator ( $y(t)$ ) also has a natural enemy ( $z(t)$ ) for understanding and investigating more complex food-chain system. In reference [9], Do studied a three-species food chain system with impulsive control and Beddington–DeAngelis functional response, but he did not consider the prey refuge  $\theta$ , mutual interference of predator  $m$ , and he considered the periodic pulse system. In reference [10], the author studied the existence and global asymptotic stability of positive periodic solutions of periodic  $n$ -species Lotka–Volterra impulsive systems with several deviating arguments. In part 4.4 of reference [1], the authors investigated the existence of almost periodic solutions of the impulsive  $n$ -species Lotka–Volterra type system. But they did not study the persistence and numerical simulation of the system. However, this paper makes a comprehensive study of the above factors. Our main results generalize or improve their corresponding results in this sense. All above results show that the dynamic behaviors of system (1.2) become more complex than those of system (1.1).

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#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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