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Optimal error estimate of the Legendre spectral approximation for space-fractional reaction–advection–diffusion equation

Wenping Chen[1*], Shujuan Lü[1], Hu Chen[2] and Haiyu Liu[1]

[*]Correspondence:
anhuicwp@163.com
[1]School of Mathematics and Systems Science & LMIB, Beihang University, Beijing, China
Full list of author information is available at the end of the article

Abstract

In this paper, we consider the space-fractional reaction–advection–diffusion equation with fractional diffusion and integer advection terms. By treating the first-order integer derivative as the composition of two Riemann–Liouville fractional derivative operators, we construct a fully discrete scheme by Legendre spectral method in a spatial and Crank–Nicolson scheme in temporal discretizations. Using thee right Riemann–Liouville fractional derivative, a novel duality argument is established, the optimal error estimate is proved to be $O(\tau^2 + N^{-m})$ in L^2-norm. Numerical tests are carried out to support the theoretical results, and the coefficient matrix with respect to first-order derivative obtained here is compared with that of traditional Legendre spectral method.

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Keywords: Space-fractional reaction–advection–diffusion equation; Legendre spectral method; Crank-Nicolson scheme; Stability; Optimal error estimate

1 Introduction

Fractional differential equations (FDEs), in which standard temporal and/or spatial derivative are replaced by fractional derivative operators, are widely used as a modeling tool and have a long history in many scientific and engineering fields, such as physics [1–3], finance [4–7], bioengineering [8–10], hydrology [11–14], and so on, for their cumulative memory effect.

There are several analytical methods to solve the FDEs, such as homotopy analysis method [15]. But in most cases, analytical methods do not work well on most of FDEs, so it is natural to resort to numerical methods. Up to now, there have been several numerical techniques to solve FDEs, such as finite difference method (FDM) [16–21], finite element method (FEM) [22–25], boundary elements method [26], spectral methods (SM) [27–36], etc. As far as we know, Lubich [20, 21] first introduced the idea of FDM to discretized fractional calculus. In [24], Roop introduced several equivalent fractional Sobolev spaces, and developed a theoretical framework for the Galerkin finite element approximation to steady-state and time-dependent fractional advection dispersion equations with Dirichlet boundary conditions.

The fractional derivative is essentially a global differential operator, whereas FDM and FEM are inherently local methods that lack the capability to deal with the fractional derivative effectively. Therefore, it is natural to consider a global method. Spectral method is well known for its global feature and high order accuracy. To compute the fractional differential equation accurately, spectral method is a suitable choice for its nature. Lin and Xu [35] proposed a finite difference scheme in time and Legendre collocation spectral method in space for the time-fractional diffusion equation in Caputo sense. In [33, 34], Li and Xu developed a time-space spectral method for the time-fractional diffusion equation, and spectral accuracy was received both in time and space. Li et al. [32] proposed the spectral collocation method to solve the fractional initial value problems and boundary value problems. Bueno-Orovio et al. [27] introduced Fourier spectral methods for the space fractional reaction-diffusion equations described by the fractional Laplacian in bounded domain. In [36], Nie et al. considered the backward Euler scheme in time and Galerkin-Legendre spectral method in space for spatial-fractional diffusion equations on a bounded interval, the convergence order was $O(\tau + N^{\alpha-m})$ in L^2-norm. Huang et al. [31] combined the second order finite difference method in time and the spectral Galerkin method in space for space-fractional diffusion equations, the convergence order of the proposed method was proved to be $O(\tau^2 + N^{\alpha-m})$ in L^2-norm.

The model problem considered in this paper is the space-fractional reaction–advection–diffusion equation (SFRADE):

$$u_t - K_1 {}_{-1}D_x^{2\alpha} u + K_2 u_x + K_3 u = f(x,t), \quad (x,t) \in \mathbb{D} = \Omega \times I, \tag{1}$$

subject to the following initial and boundary conditions:

$$\begin{cases} u(x,0) = u_0(x), \quad x \in \Omega, & (2) \\ u(\pm 1, t) = 0, \quad t \in I, & (3) \end{cases}$$

where $\Omega = (-1,1)$, $I = (0,T]$, $\alpha \in (1/2,1]$. $K_1 > 0$, $K_i \geq 0$ $(i = 2,3)$ are diffusion, advection, and reaction coefficients, respectively. $K_1 - C_p(K_2 + K_3) > 0$, C_p is a positive constant (see below). The fractional order operator ${}_{-1}D_x^{2\alpha}$ is the left Riemann–Liouville fractional derivative of order 2α with respect to x, commonly referred to as an anomalous diffusion operator. If we take $\alpha = 1$ in (1), it is the integer diffusion operator, and the classical reaction–advection–diffusion equation is obtained. The key difference between the fractional operator and the usual one is that the former is nonlocal.

Definition 1 (Riemann–Liouville (R–L) fractional integral [37, 38]) Let v be a function defined on (a,b), and $\sigma > 0$.

Then the left Riemann–Liouville fractional integral of order σ is defined to be

$${}_aD_x^{-\sigma} v(x) = \frac{1}{\Gamma(\sigma)} \int_a^x (x-\xi)^{\sigma-1} v(\xi)\,\mathrm{d}\xi, \quad x > a.$$

The right Riemann–Liouville fractional integral of order σ is defined to be

$${}_xD_b^{-\sigma} v(x) = \frac{1}{\Gamma(\sigma)} \int_x^b (\xi - x)^{\sigma-1} v(\xi)\,\mathrm{d}\xi, \quad x < b.$$

Definition 2 (Riemann–Liouville (R–L) fractional derivative [37, 38]) Let v be a function defined on (a,b), $\beta > 0$, n be the smallest integer greater than β ($n-1 \le \beta < n$), and $\sigma = n-\beta$.

Then the left fractional derivative of order β is defined to be

$$ {}_aD_x^{\beta} v(x) = D^n {}_aD_x^{-\sigma} v(x) = \frac{1}{\Gamma(\sigma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_a^x (x-\xi)^{\sigma-1} v(\xi)\, \mathrm{d}\xi, \quad x \in (a,b). $$

The right fractional derivative of order β is defined to be

$$ {}_xD_b^{\beta} v(x) = (-D)^n {}_xD_b^{-\sigma} v(x) = \frac{(-1)^n}{\Gamma(\sigma)} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \int_x^b (\xi - x)^{\sigma-1} v(\xi)\, \mathrm{d}\xi, \quad x \in (a,b). $$

In this paper, we consider the first order derivative as composition of two 1/2-order left R–L derivatives for the advection term of (1)–(3) according to the property of R–L fractional derivative as that in [39]. We also establish a duality argument by using the right R–L fractional derivative, the purpose is to construct an efficient approach by applying the Legendre spectral method in spatial and Crank–Nicolson scheme in temporal discretizations such that it can be implemented efficiently and has an optimal convergence rate.

The organization of the paper is as follows. We commence by reviewing some preliminaries of fractional order functional spaces endowed with inner products and norms, and the weak formulation for the space-fractional reaction–advection–diffusion equation is given in the next section. The fully discrete spectral scheme by applying Legendre spectral method to the spatial component and Crank–Nicolson difference scheme to time derivative is constructed, and the existence and uniqueness of the fully discrete scheme are proved by the Lax–Milgram theorem in Sect. 3. In Sect. 4, by constructing a fractional duality argument, we carry out the stability and optimal convergence analysis of the fully discrete scheme, respectively. In Sect. 5, we present some numerical experiments, which support the theoretical estimates. We conclude by summary and discussion of our method for fractional differential equation in the last section.

2 Preliminaries

In this section, we introduce some definitions and notations of fractional derivative spaces endowed with inner products and norms, then give some basic properties of fractional derivative, which will be used in the context.

The $L^2(\Omega)$ inner product is denoted by $(\cdot,\cdot)$ and the $L^p(\Omega)$ norm by $\|\cdot\|_{L^p}$ with the special case of $L^2(\Omega)$ and $L^\infty(\Omega)$ norms being written as $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively. For $k \in \mathbb{N}$, we denote the semi-norm and the norm associated with the Sobolev space $H^k(\Omega)$ by $|\cdot|_k$ and $\|\cdot\|_k$, respectively. For nonnegative real number $r \in \mathbb{R}^+ \backslash \mathbb{Z}^+$, we use $H^r(\Omega)$ to denote the fractional Sobolev spaces, the semi-norm $|\cdot|_r$ and the norm $\|\cdot\|_r$ as defined below.

Definition 3 (see [24, 40]) Let $r > 0$. Define the semi-norm

$$ |u|_r = \left\| |\omega|^r \hat{u} \right\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |\omega|^{2r} |\hat{u}|^2 \,\mathrm{d}\omega \right)^{\frac{1}{2}}, $$

and the norm

$$\|u\|_r = \left(\|u\|^2 + |u|_r^2\right)^{\frac{1}{2}},$$

where $\hat{u} = \int_{\mathbb{R}} u(x)e^{-\mathrm{i}\omega x}\,\mathrm{d}x$ denotes the Fourier transform of u, the transform variable ω is a real number. Define $H_0^r(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $H^r(\Omega)$ with respect to the norm $\|\cdot\|_r$, and use $H^{-r}(\Omega)$ to denote the dual space of $H_0^r(\Omega)$, with the norm denoted by $\|\cdot\|_{-r}$.

Remark 1 (see [24]) Let $\tilde{u}$ be the expansion of u by zero outside of Ω, then $|u|_r = |\tilde{u}|_{H^r(\mathbb{R})}$.

Throughout the paper, we use C to denote a generic nonnegative constant whose actual value may change from line to line.

Next, we introduce some useful fractional derivative spaces and related properties, which are used in the formulation of the numerical analysis, one can refer to [24, 40] for more details.

Definition 4 (see [24, 40]) Let $\mu > 0$. Define the semi-norm

$$|u|_{J_L^\mu(\Omega)} = \left\|{}_{-1}D_x^\mu u\right\|,$$

and the norm

$$\|u\|_{J_L^\mu(\Omega)} = \left(\|u\|^2 + |u|^2_{J_L^\mu(\Omega)}\right)^{\frac{1}{2}}.$$

Denote $J_{L,0}^\mu(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{J_L^\mu(\Omega)}$.

Definition 5 (see [24, 40]) Let $\mu > 0$. Define the semi-norm

$$|u|_{J_R^\mu(\Omega)} = \left\|{}_xD_1^\mu u\right\|,$$

and the norm

$$\|u\|_{J_R^\mu(\Omega)} = \left(\|u\|^2 + |u|^2_{J_R^\mu(\Omega)}\right)^{\frac{1}{2}}.$$

Denote $J_{R,0}^\mu(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{J_R^\mu(\Omega)}$.

Definition 6 (see [24, 40]) Let $\mu > 0$, $\mu \neq n - \frac{1}{2}$, $n \in \mathbb{N}$. Define the semi-norm

$$|u|_{J_S^\mu(\Omega)} = \left|\left({}_{-1}D_x^\mu u, {}_xD_1^\mu u\right)\right|^{\frac{1}{2}},$$

and the norm

$$\|u\|_{J_S^\mu(\Omega)} = \left(\|u\|^2 + |u|^2_{J_S^\mu(\Omega)}\right)^{\frac{1}{2}}.$$

Define $J_{S,0}^\mu(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{J_S^\mu(\Omega)}$.

Lemma 1 (see [24, 40]) *Let $\mu > 0$, $\mu \neq n - 1/2$, $n \in \mathbb{N}$. Then spaces $J^{\mu}_{L,0}(\Omega)$, $J^{\mu}_{R,0}(\Omega)$, $J^{\mu}_{S,0}(\Omega)$, and $H^{\mu}_0(\Omega)$ are equal, with equivalent semi-norms and norms.*

Remark 2 If the domain Ω in Definitions 4–6 is replaced by the entire line $\mathbb{R}$, the corresponding semi-norms should be denoted, respectively, by

$$\begin{cases} |u|_{J^{\mu}_L(\mathbb{R})} = \|{}_{-\infty}D^{\mu}_x u\|_{L^2(\mathbb{R})}, \\ |u|_{J^{\mu}_R(\mathbb{R})} = \|{}_x D^{\mu}_{\infty} u\|_{L^2(\mathbb{R})}, \\ |u|_{J^{\mu}_S(\mathbb{R})} = (|({}_{-\infty}D^{\mu}_x u, {}_x D^{\mu}_{\infty} u)|)^{\frac{1}{2}}. \end{cases}$$

Let $J^{\mu}_L(\mathbb{R})$, $J^{\mu}_R(\mathbb{R})$, $J^{\mu}_S(\mathbb{R})$, and $H^{\mu}(\mathbb{R})$ denote the closure of $C^{\infty}_0(\mathbb{R})$ with respect to $\|u\|_{J^{\mu}_L(\mathbb{R})}$, $\|u\|_{J^{\mu}_R(\mathbb{R})}$, $\|u\|_{J^{\mu}_S(\mathbb{R})}$, and $\|u\|_{H^{\mu}(\mathbb{R})}$, respectively.

Lemma 2 (see [24, 40]) *Let $\mu > 0$, $\mu \neq n - 1/2$, $n \in \mathbb{N}$. Then spaces $J^{\mu}_L(\mathbb{R})$, $J^{\mu}_R(\mathbb{R})$, $J^{\mu}_S(\mathbb{R})$, and $H^{\mu}(\mathbb{R})$ are equal, with equivalent semi-norms and norms.*

Lemma 3 (see [24, 40]) *Let $\mu > 0$. A function $u \in L^2(\mathbb{R})$ belongs to $J^{\mu}_L(\mathbb{R})$ if and only if $|\omega|^{\mu}\hat{u} \in L^2(\mathbb{R})$, specifically $|u|_{J^{\mu}_L(\mathbb{R})} = \||\omega|^{\mu}\hat{u}\|_{L^2(\mathbb{R})} = |u|_{H^{\mu}(\mathbb{R})}$. Similarly, $|u|_{J^{\mu}_R(\mathbb{R})} = |u|_{H^{\mu}(\mathbb{R})}$.*

Lemma 4 (see [24, 40]) *Let $\mu > 0$ be given. Then*

$$\left({}_{-1}D^{\mu}_x u, {}_x D^{\mu}_1 u\right) = \left({}_{-\infty}D^{\mu}_x \tilde{u}, {}_x D^{\mu}_{\infty} \tilde{u}\right) = \cos(\pi\mu)\left\|{}_{-\infty}D^{\mu}_x \tilde{u}\right\|^2_{L^2(\mathbb{R})} = \cos(\pi\mu)\left\|{}_x D^{\mu}_{\infty} \tilde{u}\right\|^2_{L^2(\mathbb{R})}.$$

Hence we have the following relations.

Lemma 5 (see [24, 40]) *Let $\mu > 0$, $\Omega = (-1, 1)$, $u \in J^{\mu}_{L,0}(\Omega) \cap J^{\mu}_{R,0}(\Omega)$. Then*

$$\left({}_{-1}D^{\mu}_x u, {}_x D^{\mu}_1 u\right) = \cos(\pi\mu)|u|_{\mu}.$$

Proof We can get the result by Remark 1 and Lemmas 3, 4, immediately. □

Lemma 6 (Fractional Poincaré–Friedrichs inequality [40]) *For $u \in J^{\mu}_{L,0}(\Omega)$,*

$$\|u\| \leq C|u|_{J^{\mu}_L(\Omega)},$$

and for $0 < s < \mu$,

$$|u|_{J^s_L(\Omega)} \leq C|u|_{J^{\mu}_L(\Omega)}.$$

The results for $J^{\mu}_{R,0}(\Omega)$ follow analogously. For $u \in H^{\mu}_0(\Omega)$,

$$\|u\| \leq C_p|u|_{\mu},$$

and for $0 < s < \mu$, $s \neq n - 1/2$, $n \in \mathbb{N}$,

$$|u|_s \leq C_p|u|_{\mu}.$$

Via integration by parts, one can verify the following readily.

Lemma 7 (see [38]) *Let* $0 < s < 1$, $u \in H_0^{2s}(\Omega)$, $v \in H_0^s(\Omega)$. *Then we have*

$$\big({}_{-1}D_x^{2s}u, v\big) = \big({}_{-1}D_x^{s}u, {}_xD_1^{s}v\big), \qquad \big({}_xD_1^{2s}u, v\big) = \big({}_xD_1^{s}u, {}_{-1}D_x^{s}v\big).$$

Remark 3 Since u_x can be expressed as $u_x = {}_{-1}D_x^{\frac{1}{2}}\,{}_{-1}D_x^{\frac{1}{2}}u$, we can get from Lemma 7 that, for $u \in H_0^1(\Omega)$, $v \in H_0^{\frac{1}{2}}(\Omega)$,

$$\begin{aligned}
(u_x, v) &= \big({}_{-1}D_x^{\frac{1}{2}}\,{}_{-1}D_x^{\frac{1}{2}}u, v\big) = \big({}_{-1}D_x^{\frac{1}{2}}u, {}_xD_1^{\frac{1}{2}}v\big),\\
(u_x, v) &= -\big({}_xD_1^{\frac{1}{2}}\,{}_xD_1^{\frac{1}{2}}u, v\big) = -\big({}_xD_1^{\frac{1}{2}}u, {}_{-1}D_x^{\frac{1}{2}}v\big).
\end{aligned}$$

Moreover, due to $(u_x, u) = 0$, we can also deduce that

$$\big({}_{-1}D_x^{\frac{1}{2}}u, {}_xD_1^{\frac{1}{2}}u\big) = 0, \quad u \in H_0^1(\Omega).$$

Now, we introduce some space-time functional spaces. Let E be a Hilbert space, we define the space $L^2(0, T; E)$ as

$$L^2(0, T; E) := \left\{u : (0, T) \mapsto E \,\middle|\, \int_0^T \|u\|_E^2 \,\mathrm{d}t < \infty, u \text{ is measurable}\right\},$$

and similarly we can define some other spaces for space-time functions.

We define $a(u, v) = -K_1({}_{-1}D_x^{\alpha}u, {}_xD_1^{\alpha}v) + K_2({}_{-1}D_x^{\frac{1}{2}}u, {}_xD_1^{\frac{1}{2}}v) + K_3(u, v)$ for convenience. By the linearity of the left and right R–L derivatives, we can verify readily that $a(\cdot, \cdot)$ is a bilinear form. The variational formulation of equation (1) with the homogeneous boundary condition is as follows: Let $f \in L^2(\mathbb{D})$ and $u_0 \in L^2(\Omega)$, find $u \in L^2(0, T; H_0^{\alpha}(\Omega)) \cap C([0, T]; L^2(\Omega))$ such that

$$\begin{cases}
(u_t, v) + a(u, v) = (f, v), \quad \forall v \in H_0^{\alpha}(\Omega), & (4)\\
u(0) = u_0. & (5)
\end{cases}$$

Lemma 8 *The bilinear form* $a(\cdot, \cdot)$ *is continuous and coercive on* $H_0^{\alpha}(\Omega) \times H_0^{\alpha}(\Omega)$.

Proof Hölder's inequality, Lemmas 1 and 6 yield

$$\begin{aligned}
\big|a(u, v)\big| &\le K_1 \big\|{}_{-1}D_x^{\alpha}u\big\| \big\|{}_xD_1^{\alpha}v\big\| + K_2 \big\|{}_{-1}D_x^{\frac{1}{2}}u\big\| \big\|{}_xD_1^{\frac{1}{2}}v\big\| + K_3\|u\|\|v\|\\
&\le C_1\|u\|_{\alpha}\|v\|_{\alpha}, \quad \forall u, v \in H_0^{\alpha}(\Omega),
\end{aligned}$$

i.e., $a(\cdot, \cdot)$ is continuous on $H_0^{\alpha}(\Omega) \times H_0^{\alpha}(\Omega)$.

On the other hand, by Remark 3, Lemmas 1-2, 5-6, we have

$$\begin{aligned}
a(u, u) &= -K_1\big({}_{-1}D_x^{\alpha}u, {}_xD_1^{\alpha}u\big) + K_2\big({}_{-1}D_x^{\frac{1}{2}}u, {}_xD_1^{\frac{1}{2}}u\big) + K_3(u, u)\\
&= -K_1 \cos\pi\alpha |u|_{\alpha}^2 + K_3\|u\|^2
\end{aligned}$$

$$\geq C\min\bigl(K_1|\cos\pi\alpha|,K_3\bigr)\|u\|_\alpha^2$$
$$\triangleq C_2\|u\|_\alpha^2,\quad \forall u\in H_0^\alpha(\Omega),$$

viz., $a(\cdot,\cdot)$ is coercive on $H_0^\alpha(\Omega)$.

The proof of the lemma is completed. □

According to the above lemma on the continuity and coercivity of the bilinear form $a(\cdot,\cdot)$, the existence and uniqueness of solution for the weak form above (4)–(5) could be proved.

Theorem 1 *Let $u_0\in L^2(\Omega)$, $f\in L^2(\mathbb{D})$. Then* (4)–(5) *has a unique solution $u\in L^2(0,T;H_0^\alpha(\Omega))\cap C^0([0,T];L^2(\Omega))$. Furthermore, $\frac{\partial u}{\partial t}\in L^2(0,T;H^{-\alpha}(\Omega))$ and*

$$\|u\|^2+C\int_0^t\|u\|_\alpha^2\,\mathrm{d}s\leq\|u_0\|^2+C\int_0^t\|f\|^2\,\mathrm{d}s,\quad\forall t\in[0,T].\tag{6}$$

Proof Assume that $\{\phi_k(x),k=0,1,\dots,N\}$ is the complete orthogonal basis of $V=H_0^\alpha(\Omega)$. Define $V_N=\operatorname{span}\{\phi_0,\dots,\phi_N\}$, then we get the approximation problem: for each $t\in[0,T]$, find $U_N(x,t)\in V_N$ such that

$$\begin{cases}\frac{\mathrm{d}}{\mathrm{d}t}(U_N(t),v)+a(U_N(t),v)=(f,v),\quad\forall v\in V, & (7)\\ U_N(0)=u_{0,N}=\sum_{j=0}^N\rho_j\phi_j. & (8)\end{cases}$$

Let $v=\phi_j$, $U_N=\sum_{k=0}^N c_k(t)\phi_k$, we have the following linear ordinary differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}c_j(t)(\phi_j,\phi_j)+\sum_{k=0}^N a(\phi_k,\phi_j)c_k(t)=(f,\phi_j),\quad j=0,1,\dots,N.\tag{9}$$

Since $(\phi_j,\phi_j)>0$, $\forall j=0,1,\dots,N$, there exists a unique solution $c^N=(c_0^N,c_1^N,\dots,c_N^N)^T$ to (9) and $U_N\in H^1(0,T;V)$.

Choosing $U_N(t)$ as a test function, we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}U_N(t),U_N(t)\right)+a\bigl(U_N(t),U_N(t)\bigr)=\bigl(f(t),U_N(t)\bigr).$$

Then, by the coercivity of $a(\cdot,\cdot)$, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\bigl\|U_N(t)\bigr\|^2+C_2\bigl\|U_N(t)\bigr\|_\alpha^2\leq\|f\|\bigl\|U_N(t)\bigr\|\leq\frac{1}{2C_2}\|f\|^2+\frac{C_2}{2}\bigl\|U_N(t)\bigr\|_\alpha^2.$$

Integrating the inequality over $(0,t)$, $t\in(0,T]$, we have

$$\bigl\|U_N(t)\bigr\|^2+C_2\int_0^t\bigl\|U_N(t)\bigr\|_\alpha^2\,\mathrm{d}s\leq\|u_0\|^2+C\int_0^t\|f\|^2\,\mathrm{d}s.\tag{10}$$

Thus $U_N(x,t)\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;V)$. There exists a subsequence (still denoted by U_N), which weakly star converges in $L^\infty(0,T;L^2(\Omega))$ and weakly converges in $L^2(0,T;V)$, i.e., there exists $\tilde{u} \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V)$ such that

$$\lim_{N\to\infty}\int_0^T \big(U_N(t),\varphi(t)\big)\,\mathrm{d}t = \int_0^T \big(\tilde{u}(t),\varphi(t)\big)\,\mathrm{d}t, \quad \varphi \in L^1\big(0,T;L^2(\Omega)\big),$$
$$\lim_{N\to\infty}\int_0^T \big(D^\alpha U_N(t),\psi(t)\big)\,\mathrm{d}t = \int_0^T \big(D^\alpha \tilde{u}(t),\psi(t)\big)\,\mathrm{d}t, \quad \psi \in L^2\big(0,T;L^2(\Omega)\big),$$
$$\lim_{N\to\infty}\int_0^T \big(D^{\frac{1}{2}} U_N(t),\varphi(t)\big)\,\mathrm{d}t = \int_0^T \big(D^{\frac{1}{2}} \tilde{u}(t),\varphi(t)\big)\,\mathrm{d}t, \quad \varphi \in L^1\big(0,T;L^2(\Omega)\big),$$

where D^α and $D^{\frac{1}{2}}$ are left/right Riemann–Liouville fractional derivatives, respectively. Let $\phi(t) \in C^1[0,T]$ with $\phi(T) = 0$, multiplying (7) by $\phi(t)$, integrating with respect to t, and using integration by parts, we get

$$-\int_0^T \big(U_N(t),v\big)\frac{\mathrm{d}\phi}{\mathrm{d}t}\,\mathrm{d}t - \big(U_N(0),v\big)\phi(0) + \int_0^T a\big(U_N(t),v\big)\,\mathrm{d}t = \int_0^T \big(f(t),v\big)\,\mathrm{d}t.$$

Since $U_N(0)$ converges to u_0 in $L^2(\Omega)$, we have

$$-\int_0^T \big(\tilde{u}(t),v\big)\frac{\mathrm{d}\phi}{\mathrm{d}t}\,\mathrm{d}t - (u_0,v)\phi(0) + \int_0^T a\big(\tilde{u}(t),v\big)\,\mathrm{d}t = \int_0^T \big(f(t),v\big)\,\mathrm{d}t, \quad \forall v \in V. \tag{11}$$

Taking $\phi \in C_0^\infty(0,T)$, then we get by the variational principle

$$\left(\frac{\mathrm{d}\tilde{u}}{\mathrm{d}t},v\right) + a\big(\tilde{u}(t),v\big) = \big(f(t),v\big)\,\mathrm{d}t, \quad \forall v \in V. \tag{12}$$

Multiplying (12) by $\phi \in H^1(0,T;V)$, $\phi(T) = 0$, integrating it with respect to t, and using integration by parts, we obtain

$$\begin{aligned}
&-\int_0^T (\tilde{u},v)\frac{\mathrm{d}\phi}{\mathrm{d}t}\,\mathrm{d}t - \big(\tilde{u}(0),v\big)\phi(0) + \int_0^T a\big(\tilde{u}(t),v\big)\phi(t)\,\mathrm{d}t \\
&\quad = \int_0^T \big(f(t),v\big)\phi(t)\,\mathrm{d}t, \quad \forall v \in V.
\end{aligned} \tag{13}$$

Comparing (11) with (13), we get

$$\big(\tilde{u}(0),v\big) = (u_0,v), \quad \forall v \in V.$$

Thus $\tilde{u}(0) = u_0$, the existence of the solution of (4)–(5) is established. Let $N \to \infty$ in (10), we obtain estimate (6).

$u \in L^2(0,T;V)$ leads to $-K_{1-1}D_x^{2\alpha}u + K_2u_x + K_3u \in L^2(0,T;V')$, where V' is the dual space of V. By $f \in L^2(\mathbb{D}) \subset L^2(0,T;V')$, we get $\frac{du}{dt} \in L^2(0,T;V')$ and $u \in H^1(0,T;V')$, which means that $u \in C^0([0,T];L^2(\Omega))$.

Next we prove the uniqueness of solution of (4)–(5).

Let u_1 be another solution of (4)–(5), $w = u - u_1$ satisfies

$$\begin{cases} (w_t,v) + a(w,v) = 0, \quad \forall v \in V, \\ w(0) = 0. \end{cases}$$

From (6) we have

$$\|w\|^2 + C_2 \int_0^t \|w\|_\alpha^2 \,\mathrm{d}t \le 0,$$

which implies $w = 0$, i.e., $u_1 = u$. The proof of the theorem is completed. □

3 Existence and uniqueness of the fully discrete scheme

In this section, we study the existence and uniqueness of the fully spectral discrete scheme of (1)–(3). By introducing some lemmas and tools, we prove that the fully discrete scheme has a unique solution.

Let $P_N(\Omega)$ be the set of all algebraic polynomials defined on domain Ω with the degree less than or equal to $N \in \mathbb{Z}^+$. $V_N^0 = P_N(\Omega) \cap H_0^1(\Omega)$. Let τ be the step size for time t, $t_k = k\tau$, $k = 0, 1, \dots, n_T$, and $T = n_T\tau$, $t_{k-\frac{1}{2}} = (t_k + t_{k-1})/2$. For convenience, we introduce the following notations for the function $u(x,t)$:

$$u^k = u^k(\cdot) = u(\cdot, t_k), \qquad u^{k-\frac{1}{2}} = u(t_{k-\frac{1}{2}}),$$
$$\bar{\partial}_t u^k = \frac{u^k - u^{k-1}}{\tau}, \qquad u^{\hat{k}} = \frac{u^k + u^{k-1}}{2}.$$

Due to homogeneous boundary conditions, we adopt Legendre–Gauss–Lobatto (LGL) points here. Let $I_N : C(\bar{\Omega}) \to P_N$ be the interpolating operator associated with LGL points.

$$I_N u(x_i) = u(x_i), \quad i = 0, 1, \dots, N,$$

where $x_i \in \bar{\Omega}$ are LGL points.

For the time advance, the Crank–Nicolson scheme will be used to discrete the temporal derivative of (1). We can obtain the fully discrete Crank–Nicolson–Legendre spectral method for (1)–(3): find $u_N^k \in V_N^0$, $1 \le k \le n_T$, such that

$$\begin{cases} (\bar{\partial}_t u_N^k, v) + a(u_N^{\hat{k}}, v) = (f^{k-\frac{1}{2}}, v), & \forall v \in V_N^0, \\ u_N^0 = u_{0N}, \end{cases} \tag{14}$$

where $u_{0N} \in V_N^0$ is an approximation of u_0 in the space V_N^0.

Now, we consider the existence and uniqueness of the Crank–Nicolson fully discrete scheme (14), we have the following theorem.

Theorem 2 *Let $u_0 \in L^2(\Omega), f \in L^2(\mathbb{D})$. The Crank–Nicolson fully discrete scheme* (14) *has a unique solution $u_N^k \in V_N^0$.*

Proof Rewrite scheme (14) as the following equivalent form:

$$\left(u_N^k, v\right) + \frac{\tau}{2} a\left(u_N^k, v\right) = \left(u_N^{k-1}, v\right) - \frac{\tau}{2} a\left(u_N^{k-1}, v\right) + \tau\left(f^{k-\frac{1}{2}}, v\right).$$

For simplicity, we denote $A(u,v) = (u,v) + \frac{\tau}{2}a(u,v)$, $F(v) = (u_N^{k-1}, v) - \frac{\tau}{2}a(u_N^{k-1}, v) + \tau(f^{k-\frac{1}{2}}, v)$. On the one hand, via Lemma 8, we have

$$\left|A\left(u_N^k, v\right)\right| \le \left\|u_N^k\right\| \|v\| + C\tau \left\|u_N^k\right\|_\alpha \|v\|_\alpha \le (1 + C\tau)\left\|u_N^k\right\|_\alpha \|v\|_\alpha$$

and

$$A\big(u_N^k,u_N^k\big) \geq \|u_N^k\|^2 + \frac{C_2\tau}{2}\|u_N^k\|_\alpha^2 \geq C\tau\|u_N^k\|_\alpha^2.$$

Therefore $A(u,v)$ is continuous on $V_N^0 \times V_N^0$ and coercive on V_N^0.

On the other hand, by virtue of Hölder's inequality, we deduce that

$$\begin{aligned}\big|F(v)\big| &= \left|\big(u_N^{k-1},v\big) - \frac{\tau}{2}a\big(u_N^{k-1},v\big) + \tau\big(f^{k-\frac{1}{2}},v\big)\right| \\ &\leq \|u_N^{k-1}\|\|v\| + C\tau\|u_N^{k-1}\|_\alpha\|v\|_\alpha + \tau\|f^{k-\frac{1}{2}}\|\|v\|_\alpha \\ &\leq \big(\|u_N^{k-1}\| + C\tau\|u_N^{k-1}\|_\alpha + \tau\|f^{k-\frac{1}{2}}\|\big)\|v\|_\alpha,\end{aligned}$$

viz., $F(v)$ is continuous on V_N^0.

Thus the existence and uniqueness of (14) are ensured by the Lax–Milgram theorem. The proof is completed. □

4 Stability and convergence of the fully discrete scheme

In this section, we consider the stability and convergence analysis for the fully discrete scheme (14). Let us first consider the stability of scheme (14), we have the following result.

Theorem 3 *The fully discrete scheme* (14) *is unconditionally stable.*

Proof Taking $v = u_N^{\hat{k}}$ in (14), we obtain

$$\big(\bar{\partial}_t u_N^k, u_N^{\hat{k}}\big) + a\big(u_N^{\hat{k}}, u_N^{\hat{k}}\big) = \big(f^{k-\frac{1}{2}}, u_N^{\hat{k}}\big). \tag{15}$$

For the left-hand side of (15), we infer that

$$\big(\bar{\partial}_t u_N^k, u_N^{\hat{k}}\big) + a\big(u_N^{\hat{k}}, u_N^{\hat{k}}\big) \geq \frac{1}{2\tau}\big(\|u_N^k\|^2 - \|u_N^{k-1}\|^2\big) + C_2\left\|\frac{u_N^k + u_N^{k-1}}{2}\right\|_\alpha^2.$$

On the other hand, for the right-hand side of (15), we have

$$\big(f^{k-\frac{1}{2}}, u_N^{\hat{k}}\big) \leq \frac{1}{2C_2}\|f^{k-\frac{1}{2}}\|^2 + \frac{C_2}{2}\left\|\frac{u_N^k + u_N^{k-1}}{2}\right\|_\alpha^2.$$

Bringing the above two inequalities into (15), and summing for k from 1 to n, we deduce that

$$\begin{aligned}\|u_N^n\|^2 + C_2\tau\sum_{k=1}^n\left\|\frac{u_N^k + u_N^{k-1}}{2}\right\|_\alpha^2 &\leq \|u_N^0\|^2 + \frac{1}{C_2}\sum_{k=1}^n\big(\tau\|f^{k-\frac{1}{2}}\|^2\big) \\ &\leq \|u_N^0\|^2 + C\|f\|^2_{L^\infty(0,T;L^2(\Omega))}.\end{aligned}$$

The theorem is proved. □

Next, we analyze the convergence of the fully discrete scheme (14). Firstly, we introduce some notations and lemmas.

For the following projector $\Pi_N^{1,0}$, one can be referred to [41] for more details.

Let $\Pi_N^{1,0} : H_0^1(\Omega) \mapsto V_N^0$ be the orthogonal projection operator such that

$$\left(\partial_x\left(u - \Pi_N^{1,0}u\right), \partial_x\varphi_N\right) = 0, \quad \forall \varphi_N \in V_N^0.$$

For the operator $\Pi_N^{1,0}$, we have the following estimate.

Lemma 9 (see [41]) *Let s be a real number. For any nonnegative real number r, $0 \le s \le r$, there exists a positive constant C depending only on r such that, for any function u in $H_0^s(\Omega) \cap H^r(\Omega)$, the following estimate holds*:

$$\left\| u - \Pi_N^{1,0}u \right\|_s \le CN^{s-r}\|u\|_r.$$

Define the projector $\Pi_N^{\alpha,0} : H_0^\alpha(\Omega) \mapsto V_N^0$ such that

$$a\left(u - \Pi_N^{\alpha,0}u, v\right) = 0, \quad \forall v \in V_N^0. \tag{16}$$

By virtue of (16) and the continuity and coercivity of the bilinear form $a(\cdot,\cdot)$, we have

$$\begin{aligned} C\left\| u - \Pi_N^{\alpha,0}u \right\|_\alpha^2 &\le a\left(u - \Pi_N^{\alpha,0}u, u - \Pi_N^{\alpha,0}u\right) \\ &= a\left(u - \Pi_N^{\alpha,0}u, u - \Pi_N^{1,0}u\right) \\ &\le C\left\| u - \Pi_N^{\alpha,0}u \right\|_\alpha \left\| u - \Pi_N^{1,0}u \right\|_\alpha, \quad \forall u \in V_N^0. \end{aligned}$$

Therefore, by Lemma 9, we get

$$\left\| u - \Pi_N^{\alpha,0}u \right\|_\alpha \le C\left\| u - \Pi_N^{1,0}u \right\|_\alpha \le CN^{\alpha-r}\|u\|_r, \quad \alpha \le r.$$

We next estimate the error $\|u - \Pi_N^{\alpha,0}u\|$ using a duality argument. For any $g \in L^2(\Omega)$, we consider the auxiliary problem

$$\begin{cases} -K_{1x}D_1^{2\alpha}w - K_2w_x + K_3w = g, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{17}$$

We can get that

$$\|w\|_{2\alpha} \le C\|g\|. \tag{18}$$

The weak form of (17) is as follows:

$$a(\varphi, w) = (g, \varphi), \quad \forall \varphi \in H_0^\alpha(\Omega).$$

Taking $\varphi = u - \Pi_N^{\alpha,0}u$, we obtain

$$\begin{aligned} \left(g, u - \Pi_N^{\alpha,0}u\right) &= a\left(u - \Pi_N^{\alpha,0}u, w\right) \\ &\le \left\| u - \Pi_N^{\alpha,0}u \right\|_\alpha \left\| w - \Pi_N^{1,0}w \right\|_\alpha \\ &\le CN^{-r}\|u\|_r\|w\|_{2\alpha}. \end{aligned} \tag{19}$$

Using (18) and (19), we have

$$\left\| u - \Pi_N^{\alpha,0} u \right\| = \sup_{g \in L^2(\Omega), g \neq 0} \frac{|(g, u - \Pi_N^{\alpha,0} u)|}{\|g\|} \leq CN^{-r} \|u\|_r. \tag{20}$$

Discrete Gronwall's inequality is a useful tool in the convergence analysis.

Lemma 10 (Discrete Gronwall's inequality[42]) *Let τ, B, a_i, b_i, c_i, γ_i (integers $i \geq 0$) be nonnegative numbers such that*

$$a_n + \tau \sum_{i=0}^{n} b_i \leq \tau \sum_{i=0}^{n} \gamma_i a_i + \tau \sum_{i=0}^{n} c_i + B \quad \textit{for } n \geq 0.$$

Suppose that $\tau \gamma_i < 1$ for all i, and set $\sigma_i \equiv (1 - \tau \gamma_i)^{-1}$. Then

$$a_n + \tau \sum_{i=0}^{n} b_i \leq \left(\tau \sum_{i=0}^{n} c_i + B \right) \exp\left(\tau \sum_{i=0}^{n} \sigma_i \gamma_i \right) \quad \textit{for } n \geq 0. \tag{21}$$

Now, we consider the convergence of the fully discrete scheme.

Theorem 4 *Let $1/2 < \alpha < 1$, u and u_N^k be the solutions of (1)–(3) and (14), respectively. Assume that $u_t \in L^2(0,T;H^m(\Omega))$, $u_{tt} \in L^2(0,T;H^\alpha(\Omega))$, $u_{ttt} \in L^2(0,T;L^2(\Omega))$, and $u_0 \in H^m(\Omega)$, $m > \alpha$. Then there exists a positive constant C independent of k, τ, and N such that*

$$\left\| u^k - u_N^k \right\| \leq C\left(\tau^2 + N^{-m}\right), \quad 0 \leq k \leq n_T.$$

Proof Setting $e = u - \Pi_N^{\alpha,0} u$ and $\eta = \Pi_N^{\alpha,0} u - u_N$, by (1)–(3), (14) and the definition of projector $\Pi_N^{\alpha,0}$, we get the error equation below: for all

$$\begin{cases} (\bar{\partial}_t \eta^k, v) + a(\eta^{\hat{k}}, v) = -(\bar{\partial}_t e^k, v) + (\bar{\partial}_t u^k - u_t^{k-\frac{1}{2}}, v) \\ \qquad\qquad + a(u^{\hat{k}} - u^{k-\frac{1}{2}}, v), \quad \forall v \in V_N^0, \\ \eta^0 = 0. \end{cases} \tag{22}$$

It is easy to get $u \in L^2(0,T;H^m(\Omega)) \cap L^\infty(0,T;H^m(\Omega))$ by $u_t \in L^2(0,T;H^m(\Omega))$ and $u_0 \in H^m(\Omega)$, this means that $\|u\|_\beta$ $(\forall \beta \leq m)$ is bounded. Taking $v = \eta^{\hat{k}}$ in (22), we have

$$\left(\bar{\partial}_t \eta^k, \eta^{\hat{k}}\right) + a\left(\eta^{\hat{k}}, \eta^{\hat{k}}\right) = -\left(\bar{\partial}_t e^k, \eta^{\hat{k}}\right) + \left(\bar{\partial}_t u^k - u_t^{k-\frac{1}{2}}, \eta^{\hat{k}}\right) + a\left(u^{\hat{k}} - u^{k-\frac{1}{2}}, \eta^{\hat{k}}\right). \tag{23}$$

For the left-hand side of (23), we note that

$$\left(\bar{\partial}_t \eta^k, \eta^{\hat{k}}\right) = \left(\frac{\eta^k - \eta^{k-1}}{\tau}, \frac{\eta^k + \eta^{k-1}}{2} \right) = \frac{1}{2\tau} \left(\left\| \eta^k \right\|^2 - \left\| \eta^{k-1} \right\|^2 \right).$$

By the coercivity of $a(\cdot,\cdot)$, we get

$$a\left(\eta^{\hat{k}}, \eta^{\hat{k}}\right) \geq C_2 \left\| \eta^{\hat{k}} \right\|_\alpha^2.$$

Thus for the left-hand side of (23), we obtain

$$\left(\bar{\partial}_t \eta^k, \eta^{\hat{k}}\right) + a\left(\eta^{\hat{k}}, \eta^{\hat{k}}\right) \geq \frac{1}{2\tau}\left(\|\eta^k\|^2 - \|\eta^{k-1}\|^2\right) + C_2\|\eta^{\hat{k}}\|_\alpha^2. \tag{24}$$

Next we estimate terms on the right-hand side of (23). By Hölder's inequality and Young's inequality, we have

$$\left|\left(\bar{\partial}_t e^k, \eta^{\hat{k}}\right)\right| \leq \|\bar{\partial}_t e^k\|\|\eta^{\hat{k}}\| \leq \|\bar{\partial}_t e^k\|^2 + \frac{1}{4}\|\eta^{\hat{k}}\|^2. \tag{25}$$

According to Hölder's inequality, we deduce that

$$\begin{aligned}\|\bar{\partial}_t e^k\|^2 &= \frac{1}{\tau^2}\left\|\int_{t_{k-1}}^{t_k} e_t \mathrm{d}t\right\|^2 \leq \frac{1}{\tau^2}\int_\Omega\left(\int_{t_{k-1}}^{t_k} \mathrm{d}t\right)\left(\int_{t_{k-1}}^{t_k} e_t^2 \mathrm{d}t\right)\mathrm{d}x \\ &= \frac{1}{\tau}\int_{t_{k-1}}^{t_k}\|e_t\|^2\,\mathrm{d}t \leq \frac{1}{\tau}N^{-2m}\int_{t_{k-1}}^{t_k}\|u_t\|_m^2\,\mathrm{d}t. \end{aligned}\tag{26}$$

Substituting (26) into (25), we get

$$\left|\left(\bar{\partial}_t e^k, \eta^{\hat{k}}\right)\right| \leq \frac{1}{\tau}N^{-2m}\int_{t_{k-1}}^{t_k}\|u_t\|_m^2\,\mathrm{d}t + \frac{1}{4}\|\eta^{\hat{k}}\|^2. \tag{27}$$

Via Taylor's theorem with integral remainder and Young's inequality, we infer that

$$\begin{aligned}&\left|\left(\bar{\partial}_t u^k - u_t^{k-\frac{1}{2}}, \eta^{\hat{k}}\right)\right| \\ &\quad= \frac{1}{2\tau}\left|\left(\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t_{k-1}-t)^2 u_{ttt}\,\mathrm{d}t + \int_{t_{k-\frac{1}{2}}}^{t_k}(t_k-t)^2 u_{ttt}\,\mathrm{d}t, \eta^{\hat{k}}\right)\right| \\ &\quad\leq \frac{1}{2\tau}\left(\left\|\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t_{k-1}-t)^2 u_{ttt}\,\mathrm{d}t\right\| + \left\|\int_{t_{k-\frac{1}{2}}}^{t_k}(t_k-t)^2 u_{ttt}\,\mathrm{d}t\right\|\right)\|\eta^{\hat{k}}\| \\ &\quad\leq \frac{1}{4\tau^2}\left(\left\|\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t_{k-1}-t)^2 u_{ttt}\,\mathrm{d}t\right\|^2 + \left\|\int_{t_{k-\frac{1}{2}}}^{t_k}(t_k-t)^2 u_{ttt}\,\mathrm{d}t\right\|^2\right) + \frac{1}{4}\|\eta^{\hat{k}}\|^2. \end{aligned}\tag{28}$$

By virtue of Hölder's inequality, we obtain

$$\begin{aligned}\left\|\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t_{k-1}-t)^2 u_{ttt}\,\mathrm{d}t\right\|^2 &\leq \int_{-1}^{1}\left(\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}(t_{k-1}-t)^4\,\mathrm{d}t\right)\left(\int_{t_{k-1}}^{t_{k-\frac{1}{2}}} u_{ttt}^2\,\mathrm{d}t\right)\mathrm{d}x \\ &\leq \frac{\tau^5}{5\cdot 2^5}\int_{t_{k-1}}^{t_{k-\frac{1}{2}}}\|u_{ttt}\|^2\,\mathrm{d}t, \end{aligned}\tag{29}$$

analogously, we have

$$\left\|\int_{t_{k-\frac{1}{2}}}^{t_k}(t_k-t)^2 u_{ttt}\,\mathrm{d}t\right\|^2 \leq \frac{\tau^5}{5\cdot 2^5}\int_{t_{k-\frac{1}{2}}}^{t_k}\|u_{ttt}\|^2\,\mathrm{d}t. \tag{30}$$

Substituting (29)–(30) into (28), we get

$$\left(\bar{\partial}_t u^k - u_t^{k-\frac{1}{2}}, \eta^{\hat{k}}\right) \leq \frac{1}{4}\|\eta^{\hat{k}}\|^2 + \frac{\tau^3}{5\cdot 2^7}\int_{t_{k-1}}^{t_k}\|u_{ttt}\|^2\,\mathrm{d}t. \tag{31}$$

For the last term on the right-hand side of (23), we have

$$a\big(u^{\hat{k}} - u^{k-\frac{1}{2}}, \eta^{\hat{k}}\big) \le C_1 \big\| u^{\hat{k}} - u^{k-\frac{1}{2}} \big\|_\alpha \big\| \eta^{\hat{k}} \big\|_\alpha \le \frac{C_2}{2} \big\| \eta^{\hat{k}} \big\|_\alpha^2 + \frac{C_1^2}{2C_2} \big\| u^{\hat{k}} - u^{k-\frac{1}{2}} \big\|_\alpha^2. \tag{32}$$

Via Taylor's formula and Hölder's inequality, we deduce that

$$\begin{aligned} \big\| u^{\hat{k}} - u^{k-\frac{1}{2}} \big\|_\alpha^2 &= \frac{1}{4} \big\| u^k + u^{k-1} - 2u^{k-\frac{1}{2}} \big\|_\alpha^2 \\ &= \frac{1}{4} \bigg\| \int_{t_{k-1}}^{t_{k-\frac{1}{2}}} (t - t_{k-1}) u_{tt} \,\mathrm{d}t + \int_{t_{k-\frac{1}{2}}}^{t_k} (t_k - t) u_{tt} \,\mathrm{d}t \bigg\|_\alpha^2 \\ &\le \frac{1}{2} \bigg(\bigg\| \int_{t_{k-1}}^{t_{k-\frac{1}{2}}} (t - t_{k-1}) u_{tt} \,\mathrm{d}t \bigg\|_\alpha^2 + \bigg\| \int_{t_{k-\frac{1}{2}}}^{t_k} (t_k - t) u_{tt} \,\mathrm{d}t \bigg\|_\alpha^2 \bigg) \\ &\le C\tau^3 \int_{t_{k-1}}^{t_k} \|u_{tt}\|_\alpha^2 \,\mathrm{d}t. \end{aligned} \tag{33}$$

Inserting (33) into (32), we obtain

$$a\big(u^{\hat{k}} - u^{k-\frac{1}{2}}, \eta^{\hat{k}}\big) \le \frac{C_2}{2} \big\| \eta^{\hat{k}} \big\|_\alpha^2 + C\tau^3 \int_{t_{k-1}}^{t_k} \|u_{tt}\|_\alpha^2 \,\mathrm{d}t. \tag{34}$$

Substituting (24), (27), (31), (34) into (23) and summing for k from 1 to n ($n \le n_T$), we have

$$\begin{aligned} &\big\| \eta^n \big\|^2 + C_2 \tau \sum_{k=1}^n \big\| \eta^{\hat{k}} \big\|_\alpha^2 \\ &\quad \le 2\tau \sum_{k=0}^n \big\| \eta^k \big\|^2 + 2N^{-2m} \int_0^{t_n} \|u_t\|_m^2 \,\mathrm{d}t + C\tau^4 \int_0^{t_n} \big(\|u_{tt}\|_\alpha^2 + \|u_{ttt}\|^2 \big) \,\mathrm{d}t. \end{aligned} \tag{35}$$

Let $a_k = \|\eta^k\|^2$, $b_k = C_2 \|\eta^{\hat{k}}\|_\alpha^2$, $c_k = 0$, $\gamma_k = 2$, and

$$B = 2N^{-2m} \|u_t\|_{L^2(0,T;H^m(\Omega))}^2 + C\tau^4 \big(\|u_{tt}\|_{L^2(0,T;H^\alpha(\Omega))}^2 + \|u_{ttt}\|_{L^2(0,T;L^2(\Omega))}^2 \big).$$

Then $a_k, b_k, c_k, \gamma_k \ge 0$. In view of Lemma 10 and $\tau < 1/2$, we get

$$\big\| \eta^n \big\|^2 + C_2 \tau \sum_{k=1}^n \big\| \eta^{\hat{k}} \big\|_\alpha^2 \le \exp\bigg(\frac{2T}{1 - 2\tau} \bigg) B. \tag{36}$$

Moreover, by virtue of (20), we obtain

$$\begin{aligned} \big\| e^n \big\|^2 &= \big\| U_N - \Pi_N^{\alpha,0} U_N \big\|^2 \le CN^{-2m} \|U_N\|_m^2 = CN^{-2m} \bigg\| u_0 + \int_0^{t_n} u_t(s) \mathrm{d}s \bigg\|_m^2 \\ &\le CN^{-2m} \big(\|u_0\|_m^2 + \|u_t\|_{L^2(0,T;H^m(\Omega))}^2 \big). \end{aligned} \tag{37}$$

Finally, combining the formulae (36) and (37), we get the error estimate.

The proof of the theorem is completed. □

5 Numerical experiments

In this section, we present numerical results obtained by the proposed method.

5.1 Implementation

Now, we give the implementation of the fully discrete system (14).

Let $L_k(x)$ be the Legendre polynomial of degree k with the following recursive relations (see [43]):

$$L_0(x) = 1, \qquad L_1(x) = x, \qquad (k+1)L_{k+1}(x) = (2k+1)xL_k(x) + kL_{k-1}(x), \tag{38}$$

$$L'_{k+1}(x) - L'_{k-1}(x) = (2k+1)L_k(x), \quad k \geq 1. \tag{39}$$

A detailed discussion of the properties of Legendre polynomials is furnished in [44].

Lemma 11 ([31]) *If $0 < \mu < 1$, we have*

$$\begin{aligned}
{}_{-1}D_x^{\mu} L_n(x) &= \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1+x)^{-\mu} J_n^{\mu,-\mu}(x), \\
{}_xD_1^{\mu} L_n(x) &= \frac{\Gamma(n+1)}{\Gamma(n-\mu+1)}(1-x)^{-\mu} J_n^{-\mu,\mu}(x),
\end{aligned}$$

where $J_n^{a,b}(x)$ $(a, b > -1)$ is a Jacobi polynomial.

The function space V_N^0 can be expressed as $V_N^0 = \operatorname{span}\{\phi_i(x) : i = 1, 2, \ldots, N-1\}$, in which $\phi_i(x) = L_{i+1}(x) - L_{i-1}(x)$ $(-1 < x < 1)$, and the unknown function $u_N^k \in V_N^0$ has the following form:

$$u_N^k = \sum_{i=1}^{N-1} c_i^k \phi_i(x). \tag{40}$$

In general, the term (f, ϕ_j) cannot be computed exactly and is usually approximated by $(I_N f, \phi_j)$, where I_N is an interpolation operator upon P_N relative to the Gauss–Lobatto points. Here, we approximate $(f^{k-\frac{1}{2}}, \phi_j)$ by $(I_N f^{\hat{k}}, \phi_j)$. Thus, substituting (40) into (14), we can write

$$\left(u_N^k, v\right) + \frac{\tau}{2} a\left(u_N^k, v\right) = \left(u_N^{k-1}, v\right) - \frac{\tau}{2} a\left(u_N^{k-1}, v\right) + \tau\left(I_N f^{\hat{k}}, v\right), \quad \forall v \in V_N^0. \tag{41}$$

By the definition of $a(\cdot, \cdot)$, we deduce that

$$\begin{aligned}
&\text{LHS of (41)} \\
&\quad= \sum_{i=1}^{N-1} c_i^k (\phi_i, v) - \frac{K_1\tau}{2} \sum_{i=1}^{N-1} c_i^k \left({}_{-1}D_x^{\alpha}\phi_i, {}_xD_1^{\alpha} v\right) + \frac{K_2\tau}{2} \sum_{i=1}^{N-1} c_i^k \left({}_{-1}D_x^{\frac{1}{2}}\phi_i, {}_xD_1^{\frac{1}{2}} v\right) \\
&\qquad + \frac{K_3\tau}{2} \sum_{i=1}^{N-1} c_i^k (\phi_i, v) \\
&\quad= \sum_{i=1}^{N-1} \left[-\frac{K_1\tau}{2} \left({}_{-1}D_x^{\alpha}\phi_i, {}_xD_1^{\alpha} v\right) + \frac{K_2\tau}{2} \left({}_{-1}D_x^{\frac{1}{2}}\phi_i, {}_xD_1^{\frac{1}{2}} v\right) + \left(1 + \frac{K_3\tau}{2}\right)(\phi_i, v) \right] c_i^k,
\end{aligned} \tag{42}$$

and

$$\begin{aligned}
&\text{RHS of (41)}\\
&= \sum_{i=1}^{N-1} c_i^{k-1}(\phi_i, v) + \frac{K_1\tau}{2}\sum_{i=1}^{N-1} c_i^{k-1}\big({}_{-1}D_x^\alpha \phi_i, {}_xD_1^\alpha v\big)\\
&\quad - \frac{K_2\tau}{2}\sum_{i=1}^{N-1} c_i^{k-1}\big({}_{-1}D_x^{\frac{1}{2}} \phi_i, {}_xD_1^{\frac{1}{2}} v\big) - \frac{K_3\tau}{2}\sum_{i=1}^{N-1} c_i^{k-1}(\phi_i, v) + \tau\big(I_N f^{\hat{k}}, v\big)\\
&= \sum_{i=1}^{N-1}\left[\frac{K_1\tau}{2}\big({}_{-1}D_x^\alpha \phi_i, {}_xD_1^\alpha v\big) - \frac{K_2\tau}{2}\big({}_{-1}D_x^{\frac{1}{2}} \phi_i, {}_xD_1^{\frac{1}{2}} v\big) + \left(1 - \frac{K_3\tau}{2}\right)(\phi_i, v)\right]c_i^{k-1}\\
&\quad + \tau\big(I_N f^{\hat{k}}, v\big). \qquad (43)
\end{aligned}$$

Set the matrices $M, S^\alpha \in \mathbb{R}^{(N-1)\times(N-1)}$ that satisfy

$$M_{ij} = (\phi_j, \phi_i), \qquad S_{ij}^\alpha = \big({}_xD_1^\alpha \phi_j, {}_{-1}D_x^\alpha \phi_i\big), \quad 1 \le i, j \le N-1.$$

Substituting (42), (43) into (41) and letting $v = \phi_j$, we obtain the matrix form of the discrete scheme as follows:

$$\begin{aligned}
&\left(-\frac{\tau}{2}K_1 S^\alpha + \frac{\tau}{2}K_2 S^{\frac{1}{2}} + \left(1 + \frac{\tau}{2}K_3\right)M\right)C^k\\
&\quad = \left(\frac{\tau}{2}K_1 S^\alpha - \frac{\tau}{2}K_2 S^{\frac{1}{2}} + \left(1 - \frac{\tau}{2}K_3\right)M\right)C^{k-1} + \tau F^k, \qquad (44)
\end{aligned}$$

where $C^k = (c_1^k, c_2^k, \dots, c_{N-1}^k)^T$, $F^k = (\tilde{f}_1^k, \tilde{f}_2^k, \dots, \tilde{f}_{N-1}^k)^T$, satisfying

$$\tilde{f}_i^k = \big(I_N f^{\hat{k}}, \phi_i\big) = \frac{1}{2}\big(I_N\big(f^k + f^{k-1}\big), \phi_i\big).$$

In the numerical experiments, we use Jacobi–Gauss–Lobatto quadrature to calculate the element S_{ij}^α for convenience.

$$\begin{aligned}
S_{ij}^\alpha &= \big({}_xD_1^\alpha \phi_j(x), {}_{-1}D_x^\alpha \phi_i(x)\big)\\
&= \big({}_xD_1^\alpha L_{j+1}(x), {}_{-1}D_x^\alpha L_{i+1}(x)\big) - \big({}_xD_1^\alpha L_{j-1}(x), {}_{-1}D_x^\alpha L_{i+1}(x)\big)\\
&\quad - \big({}_xD_1^\alpha L_{j+1}(x), {}_{-1}D_x^\alpha L_{i-1}(x)\big) + \big({}_xD_1^\alpha L_{j-1}(x), {}_{-1}D_x^\alpha L_{i-1}(x)\big)\\
&\triangleq D(j+1, i+1) - D(j+1, i-1) - D(j-1, i+1) + D(j-1, i-1).
\end{aligned}$$

By virtue of Lemma 11, we obtain

$$\begin{aligned}
D(j, i) &= \big({}_xD_1^\alpha L_j(x), {}_{-1}D_x^\alpha L_i(x)\big)\\
&= \frac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}\frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)}\int_{-1}^{1}(1-x)^{-\alpha}(1+x)^{-\alpha} J_i^{\alpha,-\alpha}(x) J_j^{-\alpha,\alpha}(x)\,\mathrm{d}x\\
&\approx \frac{\Gamma(i+1)}{\Gamma(i-\alpha+1)}\frac{\Gamma(j+1)}{\Gamma(j-\alpha+1)}\sum_{k=0}^{M} J_i^{\alpha,-\alpha}(x_k) J_j^{-\alpha,\alpha}(x_k)\omega_k, \qquad (45)
\end{aligned}$$

Table 1 $\max|a_{ij}^{\frac{1}{2}}|$ and L^1, L^2, L^∞ norms of $A^{\frac{1}{2}} = S^{\frac{1}{2}} - S$ for different N

N	$\max\|a_{ij}^{\frac{1}{2}}\|$	L^1 norm	L^2 norm	L^∞ norm
10	8.8818e–15	2.9421e–14	1.8203e–14	2.6201e–14
50	9.1038e–14	4.1622e–13	1.5942e–13	4.0790e–13
90	5.9908e–13	1.7258e–12	9.2418e–13	1.8320e–12

Table 2 $\max|a_{ij}^{0}|$ and L^1, L^2, L^∞ norms of $A^0 = S^0 - M$ for different N

N	$\max\|a_{ij}^{0}\|$	L^1 norm	L^2 norm	L^∞ norm
10	7.5495e–15	1.1657e–14	8.1202e–15	1.1657e–14
50	7.9936e–15	2.5934e–14	9.6391e–15	2.5879e–14
90	1.7186e–13	2.3348e–13	1.7794e–13	2.3354e–13

where $\{x_k\}$, $k = 0, 1, \ldots, M$, are the Jacobi–Gauss–Lobatto points with respect to the weight function $\omega^{-\alpha,-\alpha}(x) = (1+x)^{-\alpha}(1-x)^{-\alpha}$. If $M \geq N+1$, the numerical integration (45) is exact for all $0 \leq i, j \leq N$. We can also use Jacobi–Gauss or Jacobi–Guass–Radau quadrature to calculate S_{ij}^{α} [45].

Remark 4 The element of coefficient matrix $S \in \mathbb{R}^{(N-1)\times(N-1)}$, corresponding to the advection term u_x, is calculated by $S_{ij} = (\partial_x \phi_j, \phi_i)$ $(1 \leq i, j \leq N-1)$ in the traditional fully discrete Legendre spectral scheme case. In our numerical scheme, we calculate it in the same way as that of fractional diffusion term with $\alpha = \frac{1}{2}$. Let $A^{\frac{1}{2}} = (a_{ij}^{\frac{1}{2}}) = S^{\frac{1}{2}} - S$, Table 1 lists the maximum absolute value $|a_{ij}^{\frac{1}{2}}|$ as well as L^1, L^2, and L^∞ norms of the matrix $A^{\frac{1}{2}}$ for different N, respectively. It indicates that matrix $S^{\frac{1}{2}}$ obtained by our scheme and matrix S are considerably similar, so the approach we adopt here is suitable.

Remark 5 In fact, our numerical scheme is also right to calculate the coefficient matrix M corresponding to the reaction term u with $\alpha = 0$. Let $A^0 = (a_{ij}^0) = S^0 - M$. Table 2 lists the maximum absolute value $|a_{ij}^0|$ and L^1, L^2, and L^∞ norms of the matrix A^0 for different values of N.

5.2 Numerical example

Example 1 Consider the following SFRADE:

$$\begin{cases} u_t - {}_{-1}D_x^{2\alpha} u + u_x + u = f(x,t), & (x,t) \in \mathbb{D} = \Omega \times I, \\ u(x,0) = e^{\beta}(x^2-1)^2, & x \in \Omega, \\ u(\pm 1, t) = 0, & t \in I, \end{cases} \tag{46}$$

where $\Omega = (-1, 1)$, $I = (0, 1]$, and

$$\begin{aligned} &f(x,t) \\ &\quad = (\alpha+1)e^{\alpha t+\beta}\bigl(x^2-1\bigr)^2 + 4e^{\alpha t+\beta}x\bigl(x^2-1\bigr) \\ &\qquad - \frac{8e^{\alpha t+\beta}}{\Gamma(5-2\alpha)}(x+1)^{2-2\alpha}\bigl[3(x+1)^2 - 3(4-2\alpha)(x+1) + (4-2\alpha)(3-2\alpha)\bigr]. \end{aligned}$$

The exact solution of (46) is $u(x,t) = e^{\alpha t+\beta}(x^2-1)^2$. Here, we select $\beta = -3$.

Table 3 The L^2 errors and temporal convergence order with different α and τ for Example 1, $N = 50$

τ	$\alpha = 0.55$		$\alpha = 0.75$		$\alpha = 0.95$	
	Error	Order	Error	Order	Error	Order
1/2	1.2653e–04	–	2.6849e–04	–	4.7402e–04	–
1/4	3.1575e–05	2.0027	6.7038e–05	2.0018	1.1908e–04	1.9930
1/8	7.8908e–06	2.0005	1.6764e–05	1.9996	2.9811e–05	1.9981
1/16	1.9730e–06	1.9998	4.2004e–06	1.9968	7.4586e–06	1.9989
1/32	4.9393e–07	1.9980	1.0635e–06	1.9818	1.8967e–06	1.9754

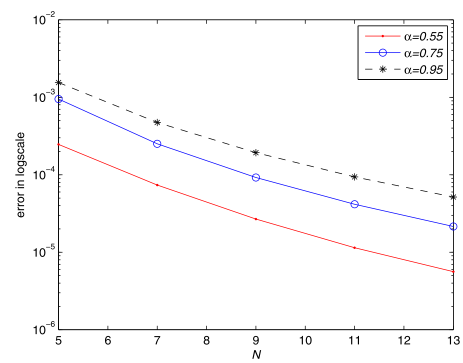


Figure 1 Numerical errors versus polynomial degree N for Example 1 with different α

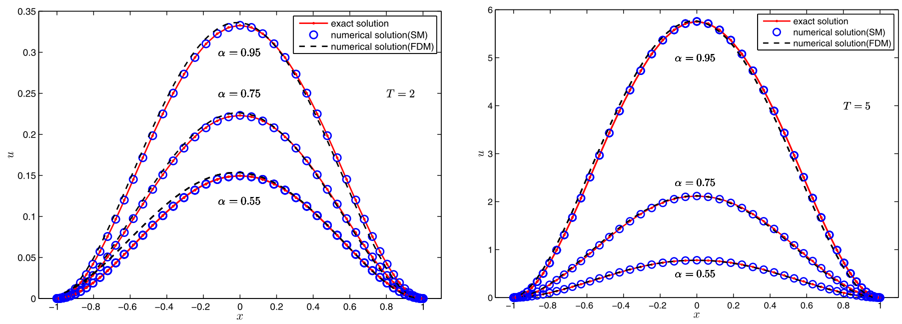
To confirm the temporal accuracy, we choose $N = 50$, which is large enough such that the spatial error is negligible compared with the temporal error. Table 3 lists the errors $\|u - u_N\|$ and temporal convergence rates. From the table, we can check that temporal convergence order, almost second-order, is in accordance with the theoretical result in Theorem 4.

Next, we investigate the spatial discretization error. We take $\tau = 0.001$ so that the temporal discretization error is negligible compared with the spatial discretization error. As shown in Fig. 1, the L^2 errors of the numerical solution with different values of α decay exponentially in an approximate line as the polynomial degree N increases. Solution u is sufficiently smooth with respect to spatial variable x, thus the numerical result coincides with Theorem 4.

In Fig. 2, comparisons between the exact solution and the numerical solutions of the present method and the finite difference method are presented at various values of the final time T and α, respectively, where $N = 50$. It can be seen that numerical results of the spectral method much better coincide with the exact solution than those of the finite difference method. This illustrates that the spectral method is a high accuracy method.

Example 2 Consider the following SFRADE:

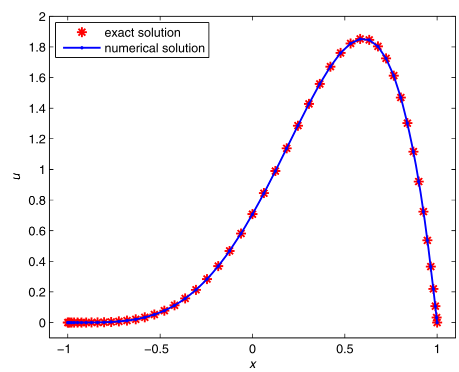
$$\begin{cases} u_t - 3\,{}_{-1}D_x^{2\alpha} u + 2u_x = f(x,t), & (x,t) \in \mathbb{D} = \varOmega \times I, \\ u(x,0) = (1-x)(x+1)^4, & x \in \varOmega, \\ u(\pm 1, t) = 0, & t \in I, \end{cases} \tag{47}$$



(a) $T = 2, \tau = 0.001$ (b) $T = 5, \tau = 0.001$

Figure 2 Comparisons of the exact solution with the numerical solutions of spectral method and finite difference method for Example 1 with different α and T, where $N = 50$

Figure 3 The numerical and the exact solution at $T = 1$ for Example 2, taking $N = 50$



where $\Omega = (-1, 1)$, $I = (0, 1]$, and

$$f(x,t) = \left(\frac{\pi}{4}\cos\frac{\pi t}{4} + K_3 \sin\frac{\pi t}{4}\right)(1-x)(x+1)^4 + K_2 \sin\frac{\pi t}{4}(x+1)^3(3-5x) - \frac{4!K_1}{\Gamma(6-2\alpha)}\sin\frac{\pi t}{4}(x+1)^{4-2\alpha}(5-5x-4\alpha).$$

It is easy to verify that the exact solution of (47) is $u(x,t) = \sin\frac{\pi t}{4}(1-x)(x+1)^4$. The comparisons of the numerical and the exact solution of problem (47) at $t = 1$ are shown in Fig. 3. In this case, we take the time step size $\tau = 0.0001$ and the degree of interpolation polynomial in spatial direction $N = 50$. It can be seen that our numerical results are in excellent agreement with the exact solution.

We take $N = 50$, a value large enough such that the spatial discretization errors are negligible compared with the temporal errors, we choose different time step size τ to obtain the numerical convergence order in time list in Table 4. We consider the errors as a function of τ, and plot the log-log graph with $T = 1$ and $\alpha = 0.9$ in Fig. 4, it coincides with τ^2. We can check that these numerical convergence orders, almost approaching 2, are consistent with the theoretical analysis in Theorem 4. This demonstrates that our method has second order convergence in time.

Table 4 The L^2 errors and temporal convergence order with different τ for Example 2, $\alpha = 0.9$, $N = 50$

τ	Error	Order	τ	Error	Order
1/4	8.9062e–04	–	1/20	3.9736e–05	–
1/8	2.2084e–04	2.0118	1/40	9.9305e–06	2.0005
1/16	5.5094e–05	2.0031	1/80	2.4824e–06	2.0001
1/32	1.3766e–05	2.0008	1/160	6.2059e–07	2.0000
1/64	3.4411e–06	2.0002	1/320	1.5515e–07	2.0000

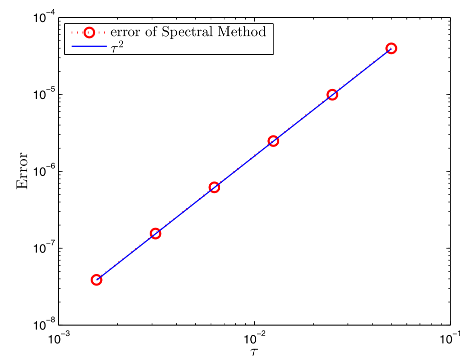


Figure 4 Numerical errors versus τ for Example 2, where $\alpha = 0.9$

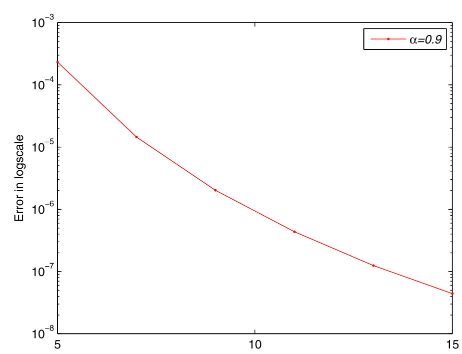


Figure 5 Numerical errors versus polynomial degree N for Example 2, where $\alpha = 0.9$

In Fig. 5, we plot the logarithm of errors $\|u - u_N\|$ as a function of the polynomial degree N at $T = 1$ with $\alpha = 0.9$. Figure 5 demonstrates that our method has spectral accuracy in space for this problem. Obviously, the exponential convergence in spatial discretization can be seen from the figure which shows almost linear curves. The above numerical experiments state that it is effective and feasible to use spectral method to solve space-fractional reaction–advection–diffusion equations.

6 Conclusions

In this paper, we have considered the reaction–advection–diffusion equation in the case of Riemann–Liouville fractional diffusion and integer advection. By treating first order integer derivative as the composition of two $\frac{1}{2}$-order fractional operators, we construct a new Crank–Nicolson fully discrete Legendre spectral scheme and a dual auxiliary problem. The scheme proved to be unconditionally stable and to converge with $O(\tau^2 + N^{-m})$ in L^2-norm. The model problems are given in the case of the left Riemann–Liouville fractional derivative, but the results are also valid for the right Riemann–Liouville fractional derivative case as well as the Riesz fractional derivative. The methodology we present here can also be suitable for problems in the 2-D and 3-D cases, and a discrete scheme with various spectral methods and different temporal discretizations will be considered in our future work.

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Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WC and SL contributed equally to this work. All authors read and approved the final manuscript.

Author details
[1]School of Mathematics and Systems Science & LMIB, Beihang University, Beijing, China. [2]Beijing Computational Science Research Center, Beijing, China.

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