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Exponentially practical stability of discrete time singular system with delay and disturbance

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Abstract

In this paper, we consider exponentially practical stability of a discrete time singular system with disturbance. By using Lyapunov–Krasovskii stability theory, some criteria for exponentially practical stability of such a system are derived. Moreover, by using a Razumikhin-type technique, the criteria for exponentially practical stability of a discrete time singular system with delay and disturbance are also obtained. Some numerical examples are given to show the success of our theoretical results.

Keywords: Discrete time system; Singular system; Delay; Exponentially practical stability; Disturbance

1 Introduction

Singular systems, which are also called descriptor systems, implicit systems, or generalized systems, have been investigated extensively in many areas [1–21]. Generally, the systems can be described using algebraic and differential equations. Such systems are natural presentations of several dynamic systems which are better than regular systems, such as economical systems, chemical systems, robotic systems, etc. [1–10, 12–14, 20–22]. Moreover, singular systems are very complicated because we have to consider the stability of the systems as well as the regularity and also impulse free (in case of continuous singular systems) or causality (in case of discrete singular systems) [2, 11, 18]. In addition, a discrete time system is often represented in the real world systems such as population models and switched systems. There are several studies on the stability of a discrete time system [2–4, 8–15, 17, 18, 22–24].

In real world systems, the variation of systems' current status often depends not only on the current state but also on the past state of the systems; such systems are called time delay systems. Examples of time delay systems are population dynamic models, mechanical transmissions, and digital control systems [4, 7–10, 12–14]. It is well known that time delay may cause instability, oscillation, and poor performance of systems. For the above-mentioned reasons, time delay systems have been extensively discussed in many literature works [2, 4, 6–10, 13–16, 18, 19, 22, 23]. As is known, most common approaches to studying stability analysis of a time delay system are Lyapunov–Krasovskii functional approach and Razumikhin-type technique. In the case of Lyapunov–Krasovskii functional method, it requires that a candidate Lyapunov–Krasovskii functional is decreasing on the whole state space. Meanwhile, the Razumikhin-type technique has an advantage that the Lyapunov–Krasovskii functional is not required to be decreasing on the whole state space. In general, disturbance inputs often occur in modeling of phenomena and engineering systems which may be due to data transformation, unknown disturbances, or measurement errors [4, 10, 24, 25]. Therefore, it is important to study the stability of a discrete time singular system with delay and disturbance.

Considering asymptotic stability, it is more desirable to consider exponential stability criterion for dynamical systems [1, 6–10, 18–21, 24, 25]. For exponential stability, it is required that all solutions starting near an equilibrium point not only stay nearby, but tend to the equilibrium point very fast with exponential decay rate. In practice, we may only need to stabilize a system into the region of a phase space where the system may oscillate near the state in which the implementation is still acceptable. This concept is called practical stability [18, 22, 26–29] which is very useful for studying the asymptotic behavior of the system in which the origin is not necessarily an equilibrium point. In this case, practical stability is an important concept to analyze the asymptotic behavior of solutions with respect to a small neighborhood of the origin. Recently, there have been several studies on practical stability of continuous time systems with delay, see [26–29]. However, there have been few studies on practical stability of discrete time systems with delay [22, 23]. In [3], the authors studied discrete time singular systems with disturbance and obtained some stability criteria by using Lyapunov stability theory. In [23], the authors used the Razumikhin-type technique to derive the exponentially practical stability condition for impulsive discrete time systems with delay. Motivated by the above discussions, we propose to study exponentially practical stability of a discrete time singular system with delay and disturbance. We shall derive a new criterion for exponentially practical stability of the system, namely the solutions tend to the origin state with exponential decay rate in the early stage (but eventually oscillate in a neighborhood of the origin), in which the performance is still acceptable.

This work is organized as follows. In Sect. 2, some notations and definitions are introduced. In Sect. 3, we present some criteria for exponentially practical stability of a discrete time system with disturbance, exponentially practical stability of a discrete time singular system with disturbance, exponentially practical stability of a discrete time system with delay and disturbance, and exponentially practical stability of a discrete time singular system with delay and disturbance; definitions and assumption will be used in the proof our result. Some numerical examples are given to show the effectiveness of our theoretical result in Sect. 4. The last section concludes the work.

2 Preliminaries

Consider the following discrete time singular system with delay and disturbance:

$$\begin{cases} Ex(k+1) = Ax(k) + Bx(k-\tau) + Gw(k), \\ x(s) = \varphi(s), \quad s \in \mathbb{N}_{k_0-\tau}, \end{cases} \tag{2.1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector, $w(k) \in \Omega = \{w(k) \in \mathbb{R}^m / \|w(k)\| \leq \overline{w}, \exists \overline{w} > 0, \forall k \geq k_0\}$ is the disturbance, A, B, G, and E are constant matrices with appropriate dimensions where E is a singular matrix with $\operatorname{rank}(E) = r < n$, τ is constant delay, $\varphi \in \mathcal{C}_\tau := \{f : \mathbb{N}_{-\tau} \longrightarrow \mathbb{R}^n, f \text{ is continuous}\}$ is the initial function with $\|\varphi\| = \max_{\theta \in \mathbb{N}_{k_0-\tau}}\{\|\varphi(\theta)\|\}$, where

$\mathbb{N}_{k_0-\tau} = \{k_0-\tau, k_0-\tau+1, \ldots, k_0-1, k_0\}$, and $\mathbb{N}_{k_0} = \{k_0, k_0+1, k_0+2, \ldots\}$, $k_0 \geq 0$. A function $\rho : \mathbb{R}_0^+ \longrightarrow \mathbb{R}$, is called a K-function if it is a nonnegative continuous function where $\mathbb{R}_0^+$ is the set of nonnegative real numbers. Let floor$(r) := \lfloor r \rfloor$ be the greatest integer that is less than or equal to r, and let ceil$(r) := \lceil r \rceil$ be the least integer that is greater than or equal to r.

Definition 2.1 ([18]) System (2.1) is said to be *regular* if $\det(zE - A) \not\equiv 0$ for some $z \in \mathbb{C}$. System (2.1) is said to be *causal* if $\deg(\det(zE - A)) = \operatorname{rank}(E)$.

3 Main results

In this section, we consider exponentially practical stability problems for the following four cases: a discrete time system with disturbance, a discrete time singular system with disturbance, a discrete time system with delay and disturbance, and a discrete time singular system with delay and disturbance.

First, we consider the discrete time system with disturbance as follows:

$$\begin{cases} x(k+1) = f(k, x(k), w(k)), \\ x(k_0) = x_0, \end{cases} \tag{3.1}$$

where disturbance $w(k) \in \Omega$, where Ω is defined as in (2.1), $f : \mathbb{R} \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ is continuous and locally Lipschitz in (x, w), uniformly in k with Lipschitz constant L which satisfies $f(k, 0, 0) = 0$. Let $x(k; k_0, x_0, w)$ denote the trajectory of system (3.1) with initial state $x(k_0) = x_0$ and disturbance signal $w(k) \in \Omega$.

Definition 3.1 System (3.1) is exponentially practically stable in the pth-moment for some $p > 0$ if for all $k \geq k_0$ there exist constants $0 < \lambda < 1, \eta > 0, r > 0$ such that

$$\left\|x(k; k_0, x_0, w)\right\|^p \leq \eta \|x_0\|^p \lambda^{k-k_0} + r, \quad \forall w(k) \in \Omega.$$

Theorem 3.1 *If there exist a Lyapunov function $V(k, x(k))$, a K-function ρ, and positive constants $c_1, c_2, c_3, a, p; c_3 < c_2$ such that the following conditions hold for all $k \geq k_0, x(k) \in \mathbb{R}^n, w(k) \in \Omega$:*

(i) $c_1\|x(k)\|^p \leq V(k, x(k)) \leq c_2\|x(k)\|^p + a$,

(ii) $\Delta V(k, x(k)) = V(k+1, x(k+1)) - V(k, x(k)) \leq -c_3\|x(k)\|^p + \rho(\|w(k)\|)$.

Then system (3.1) *is exponentially practically stable in the pth-moment with $\eta = \frac{c_2}{c_1}, \lambda = 1 - \frac{c_3}{c_2}$, and $r = \frac{a}{c_1} + \frac{a_1}{c_1(1-\sigma)}$, where $a_1 = \frac{c_3 a}{c_2} + \rho_1$ and $\rho_1 = \sup_{w(k)\in\Omega}\{\rho(\|w(k)\|)\}$.*

Proof From (i) and (ii), we obtain that

$$\begin{aligned} &V\big(k+1, x(k+1; k_0, x_0, w)\big) - V\big(k, x(k; k_0, x_0, w)\big) \\ &\quad \leq -c_3\left\|x(k; k_0, x_0, w)\right\|^p + \rho\big(\left\|w(k)\right\|\big) \\ &\quad \leq -\frac{c_3}{c_2} V\big(k, x(k; k_0, x_0, w)\big) + \frac{c_3 a}{c_2} + \rho_1 \\ &\quad = -\frac{c_3}{c_2} V\big(k, x(k; k_0, x_0, w)\big) + a_1 \end{aligned}$$

for all $k \geq k_0, w(k) \in \Omega$, where $\rho_1 = \sup_{w(k)\in\Omega}\{\rho(\|w(k)\|)\}, a_1 = \frac{c_3 a}{c_2} + \rho_1$. Without loss of generality, we may assume that $c_3 < c_2$, and let $\sigma = 1 - \frac{c_3}{c_2}$. Then $0 < \sigma < 1$, and it follows that

$$\begin{aligned}
&V\big(k+1, x(k+1;k_0,x_0,w)\big)\\
&\quad \leq \left(1 - \frac{c_3}{c_2}\right) V\big(k, x(k;k_0,x_0,w)\big) + a_1\\
&\quad = \sigma V\big(k, x(k;k_0,x_0,w)\big) + a_1\\
&\quad \leq \sigma\big[\sigma\big(V\big(k-1, x(k-1;k_0,x_0,w)\big) + a_1\big)\big] + a_1\\
&\quad \leq \sigma^2\big[\sigma\big(V\big(k-2, x(k-2;k_0,x_0,w)\big) + a_1\big)\big] + \sigma a_1 + a_1\\
&\quad \vdots\\
&\quad \leq \sigma^{k+1} V(k_0,x_0,w) + \sigma^k a_1 + \sigma^{k-1} a_1 + \cdots + a_1.
\end{aligned}$$

Thus,

$$V\big(k, x(k;k_0,x_0,w)\big) \leq \sigma^{k-k_0} V(k_0,x_0,w) + \frac{a_1}{1-\sigma}.$$

From (i), we can see that

$$\begin{aligned}
c_1 \big\|x(k;k_0,x_0,w)\big\|^p \leq V\big(k, x(k;k_0,x_0,w)\big) &\leq \sigma^{k-k_0} V(k_0,x_0,w) + \frac{a_1}{1-\sigma}\\
&\leq \sigma^{k-k_0}\big(c_2\|x_0\|^p + a\big) + \frac{a_1}{1-\sigma}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\big\|x(k;k_0,x_0,w)\big\|^p &\leq \frac{c_2}{c_1}\sigma^{k-k_0}\|x_0\|^p + \frac{\sigma^{k-k_0} a}{c_1} + \frac{a_1}{c_1(1-\sigma)}\\
&\leq \frac{c_2}{c_1}\sigma^{k-k_0}\|x_0\|^p + \frac{a}{c_1} + \frac{a_1}{c_1(1-\sigma)}\\
&= c_4 \sigma^{k-k_0}\|x_0\|^p + a_2,
\end{aligned}$$

where $c_4 = \frac{c_2}{c_1}$ and $a_2 = \frac{a}{c_1} + \frac{a_1}{c_1(1-\sigma)}$. It follows that

$$\big\|x(k;k_0,x_0,w)\big\|^p \leq c_4 \sigma^{k-k_0}\|x_0\|^p + a_2 \quad \text{for all } k \geq k_0, w(k) \in \Omega.$$

Therefore, system (3.1) is exponentially practically stable in the pth-moment with $\eta = c_4, \lambda = \sigma$, and $r = a_2$. □

Then, we will consider system (2.1) without delay, namely a discrete time singular system with disturbance, as follows:

$$\begin{cases} Ex(k+1) = Ax(k) + Gw(k), \\ x(k_0) = x_0. \end{cases} \tag{3.2}$$

Definition 3.2 The singular system (3.2) is said to be exponentially practically stable in the pth-moment for some $p > 0$ if, for all $k \geq k_0$, there exist constants $0 < \lambda < 1, \eta > 0, r > 0$ for each disturbance $w(k) \in \Omega$ such that

$$\|Ex(k;k_0,x_0,w)\|^p \leq \eta\|x_0\|^p\lambda^{k-k_0} + r,$$

where $x(k,k_0,x_0,w)$ is the state trajectory of a system with initial state x_0.

Theorem 3.2 *Assume that the singular system* (3.2) *is regular and causal. Then the singular system* (3.2) *is exponentially practically stable in the pth-moment with respect to* $w(k)$, *if there exists a Lyapunov function* $V(k,x(k))$ *such that*

(i) $c_1\|Ex(k)\|^p \leq V(k,x(k)) \leq c_2\|Ex(k)\|^p + a$,

(ii) $\Delta V(k,x(k)) = V(k+1,x(k+1)) - V(k,x(k)) \leq -c_3\|x(k)\|^p + \rho(\|w(k)\|)$ *hold for some positive constants* $a, c_1, c_2, c_3, p; c_3 < c_2$ *and a* K*-function* ρ.

Proof Assume that system (3.2) is regular and causal. Then, from [30], there exist nonsingular matrices M, N with appropriate dimensions such that

$$MEN = \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}, \qquad MAN = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{pmatrix}, \qquad MG = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}.$$

Let $y(k) = N^{-1}x(k) = \left[\begin{smallmatrix} y_1(k) \\ y_2(k) \end{smallmatrix}\right]$, then system (3.2) is transformed to the system

$$y_1(k+1) = A_1y_1(k) + G_1w(k), \tag{3.3}$$

$$y_2(k) = -G_2w(k) \tag{3.4}$$

with initial state y_0 satisfying $\left(\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right)y_0 = \left(\begin{smallmatrix} y_{10} \\ 0 \end{smallmatrix}\right)$. From the Lyapunov function $V(k,x(k))$ satisfying conditions (i) and (ii), we obtain the following estimations:

$$\begin{aligned}
&c_1\|Ex(k)\|^p \leq V(k,x(k)) \leq c_2\|Ex(k)\|^p + a, \\
&c_1\|ENy(k)\|^p \leq V(k,Ny(k)) \leq c_2\|ENy(k)\|^p + a, \\
&c_1\|M^{-1}MENy(k)\|^p \leq V(k,Ny(k)) \leq c_2\|M^{-1}MENy(k)\|^p + a, \\
&c_1\left\|M^{-1}\begin{bmatrix} y_1(k) \\ 0 \end{bmatrix}\right\|^p \leq V(k,Ny(k)) \leq c_2\left\|M^{-1}\begin{bmatrix} y_1(k) \\ 0 \end{bmatrix}\right\|^p + a, \\
&c_1\left\|M^{-1}N^{-1}N\begin{bmatrix} y_1(k) \\ 0 \end{bmatrix}\right\|^p \leq V(k,Ny(k)) \leq c_2\left\|M^{-1}N^{-1}N\begin{bmatrix} y_1(k) \\ 0 \end{bmatrix}\right\|^p + a.
\end{aligned}$$

Let $\overline{y}(k) = Ny(k), \overline{y_1}(k) = N\left[\begin{smallmatrix} y_1(k) \\ 0 \end{smallmatrix}\right]$. Then we obtain

$$c_1\|M^{-1}N^{-1}\overline{y_1}(k)\|^p \leq V(k,\overline{y}(k)) \leq c_2\|M^{-1}N^{-1}\overline{y_1}(k)\|^p + a,$$

which in turn gives

$$c_1\|M^{-1}N^{-1}\overline{y_1}(k)\|^p \leq V(k,\overline{y_1}(k)) \leq c_2\|M^{-1}N^{-1}\overline{y_1}(k)\|^p + a.$$

From which it follows that

$$\begin{aligned} V\big(k,\overline{y_1}(k)\big) &\geq c_1\big(\big\|M^{-1}N^{-1}\overline{y_1}(k)\big\|^2\big)^{\frac{p}{2}} \\ &= c_1\big(\big\|\overline{y_1}^T(k)\big(M^{-1}N^{-1}\big)^T\big(M^{-1}N^{-1}\big)\overline{y_1}(k)\big\|\big)^{\frac{p}{2}} \\ &\geq c_1\big(\lambda_{\min}\big(\big(M^{-1}N^{-1}\big)^T\big(M^{-1}N^{-1}\big)\big)\big)^{\frac{p}{2}}\big\|\overline{y_1}(k)\big\|^p. \end{aligned}$$

Similarly, we have

$$V\big(k,\overline{y_1}(k)\big) \leq c_2\big\|M^{-1}\big\|^p\big\|N^{-1}\big\|^p\big\|\overline{y_1}(k)\big\|^p + a.$$

Thus,

$$\overline{c}_1\big\|\overline{y_1}(k)\big\|^p \leq V\big(k,\overline{y_1}(k)\big) \leq \overline{c}_2\big\|\overline{y_1}(k)\big\|^p + a,$$

where $\overline{c}_1 = c_1(\lambda_{\min}((M^{-1}N^{-1})^T(M^{-1}N^{-1})))^{\frac{p}{2}}$, and $\overline{c}_2 = c_2\|M^{-1}\|^p\|N^{-1}\|^p$.

Furthermore, we have the following estimations for $\Delta V(k,x(k))$:

$$\begin{aligned} \Delta V\big(k,x(k)\big) &= V\big(k+1,x(k+1)\big) - V\big(k,x(k)\big) \leq -c_3\big\|x(k)\big\|^p + \rho\big(\big\|w(k)\big\|\big), \\ \Delta V\big(k,Ny(k)\big) &= V\big(k+1,Ny(k+1)\big) - V\big(k,Ny(k)\big) \leq -c_3\big\|Ny(k)\big\|^p + \rho\big(\big\|w(k)\big\|\big), \\ \Delta V\big(k,\overline{y}(k)\big) &= V\big(k+1,\overline{y}(k+1)\big) - V\big(k,\overline{y}(k)\big) \leq -c_3\big\|\overline{y}(k)\big\|^p + \rho\big(\big\|w(k)\big\|\big), \\ \Delta V\big(k,\overline{y_1}(k)\big) &= V\big(k+1,\overline{y_1}(k+1)\big) - V\big(k,\overline{y_1}(k)\big) \leq -c_3\big\|\overline{y_1}(k)\big\|^p + \rho\big(\big\|w(k)\big\|\big). \end{aligned}$$

From (3.3) with $\overline{y_1}(k) = N\big[\begin{smallmatrix} y_1(k) \\ 0 \end{smallmatrix}\big]$, we have

$$\overline{y_1}(k+1) = \overline{A_1}\,\overline{y_1}(k) + \overline{G_1}\,\overline{w}(k), \tag{3.5}$$

where $\overline{A_1} = N\big[\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\big]N^{-1}, \overline{G_1} = N\big[\begin{smallmatrix} G_1 & 0 \\ 0 & 0 \end{smallmatrix}\big]$, and $\overline{w}(k) = \big[\begin{smallmatrix} w(k) \\ 0 \end{smallmatrix}\big]$. Therefore, we may conclude that there exists a Lyapunov function $V(k,\overline{y_1}(k))$ for system (3.5) which satisfies the following conditions:

(i) $\overline{c}_1\|\overline{y_1}(k)\|^p \leq V(k,\overline{y_1}(k)) \leq \overline{c}_2\|\overline{y_1}(k)\|^p + a$,

(ii) $\Delta V(k,\overline{y_1}(k)) = V(k+1,\overline{y_1}(k+1)) - V(k,\overline{y_1}(k)) \leq -c_3\|\overline{y_1}(k)\|^p + \rho(\|w(k)\|)$.

Hence, from Theorem 3.1, there exist constants $0 < \lambda < 1, \eta_0 > 0, r_0 > 0$ such that

$$\big\|\overline{y_1}(k)\big\|^p \leq \eta_0\|\overline{y_{10}}\|^p\lambda^{k-k_0} + r_0, \quad \forall k \geq k_0, \tag{3.6}$$

where $\overline{y_{10}} = N\big[\begin{smallmatrix} y_{10} \\ 0 \end{smallmatrix}\big]$. Thus, we have

$$\begin{aligned} \big\|y_1(k)\big\|^p &\leq \eta_0\big\|N^{-1}\big\|^p\|N\|^p\|y_{10}\|^p\lambda^{k-k_0} + \big\|N^{-1}\big\|^p r_0 \\ &= \eta_1\|y_{10}\|^p\lambda^{k-k_0} + r_1, \quad \forall k \geq k_0, \end{aligned} \tag{3.7}$$

where $\eta_1 = \eta_0\|N^{-1}\|^p\|N\|^p, r_1 = \|N^{-1}\|^p r_0$. From (3.4), we have $y_2(k) = -G_2w(k)$, which in turn gives

$$\big\|y_2(k)\big\| \leq \|G_2\|\big\|w(k)\big\|. \tag{3.8}$$

Thus, it follows from (3.7) and (3.8) that

$$\|y(k)\| \le \|y_1(k)\| + \|y_2(k)\|.$$

Therefore,

$$\|y(k)\| \le \left(\eta_1\|y_{10}\|^p\lambda^{k-k_0} + r_1\right)^{\frac{1}{p}} + \overline{w}\|G_2\|. \tag{3.9}$$

From (3.9), there are four cases to be considered according to the values of p and r_1 as follows:

(*Case* I). $0 < p < 1, 0 < r_1 < 1$. Since $0 < \lambda < 1$, we may assume without loss of generality that

$$\eta_1\|y_{10}\|^p\lambda^{k-k_0} + r_1 < 1$$

for all $k \ge k_0$. Let $\lfloor \frac{1}{p} \rfloor = n, n \in \mathbb{I}^+$. Then, by the binomial theorem, we obtain

$$\begin{aligned}
\|y(k)\| &\le \left(\eta_1\|y_{10}\|^p\lambda^{k-k_0} + r_1\right)^{\frac{1}{p}} + \overline{w}\|G_2\| \\
&\le \left(\eta_1\|y_{10}\|^p\lambda^{k-k_0} + r_1\right)^{\lfloor \frac{1}{p} \rfloor} + \overline{w}\|G_2\| \\
&= \left(\eta_1\|y_{10}\|^p\lambda^{k-k_0} + r_1\right)^{n} + \overline{w}\|G_2\| \\
&= \left(\eta_1\|y_{10}\|^p\lambda^{k-k_0}\right)^{n} + \binom{n}{1}\left(\eta_1\|y_{10}\|^p\lambda^{k-k_0}\right)^{n-1} r_1 \\
&\quad + \binom{n}{2}\left(\eta_1\|y_{10}\|^p\lambda^{k-k_0}\right)^{n-2} r_1^2 + \cdots + r_1^n + \overline{w}\|G_2\| \\
&\le \binom{n}{\lceil \frac{n}{2} \rceil}\eta_1\|y_{10}\|^p\lambda^{k-k_0} + \binom{n}{\lceil \frac{n}{2} \rceil}\eta_1\|y_{10}\|^p\lambda^{k-k_0} r_1 \\
&\quad + \binom{n}{\lceil \frac{n}{2} \rceil}\eta_1\|y_{10}\|^p\lambda^{k-k_0} r_1 + \cdots + \binom{n}{\lceil \frac{n}{2} \rceil} r_1 + \overline{w}\|G_2\| \\
&= \binom{n}{\lceil \frac{n}{2} \rceil}\eta_1\|y_{10}\|^p\lambda^{k-k_0} + (n-1)\binom{n}{\lceil \frac{n}{2} \rceil}\eta_1\|y_{10}\|^p\lambda^{k-k_0} r_1 + \binom{n}{\lceil \frac{n}{2} \rceil} r_1 + \overline{w}\|G_2\| \\
&= \left[\binom{n}{\lceil \frac{n}{2} \rceil} + (n-1)\binom{n}{\lceil \frac{n}{2} \rceil} r_1\right]\eta_1\|y_{10}\|^p\lambda^{k-k_0} + \left[\binom{n}{\lceil \frac{n}{2} \rceil} r_1 + \overline{w}\|G_2\|\right] \\
&= \eta_2\|y_{10}\|^p\lambda^{k-k_0} + r_2,
\end{aligned}$$

where $\eta_2 = \left[\binom{n}{\lceil \frac{n}{2} \rceil} + (n-1)\binom{n}{\lceil \frac{n}{2} \rceil} r_1\right]\eta_1$ and $r_2 = \binom{n}{\lceil \frac{n}{2} \rceil} r_1 + \overline{w}\|G_2\|$.

(*Case* II) $0 < p < 1, r_1 \ge 1$. Since $0 < \lambda < 1$, we may assume without loss of generality that

$$\eta_1\|y_{10}\|^p\lambda^{k-k_0} < 1$$

for all $k \ge k_0$. Let $\lceil \frac{1}{p} \rceil = n, n \in \mathbb{I}^+$. Then, by the binomial theorem, we obtain

$$\|y(k)\| \le \left(\eta_1\|y_{10}\|^p\lambda^{k-k_0} + r_1\right)^{\frac{1}{p}} + \overline{w}\|G_2\|$$

$$
\begin{aligned}
&\leq \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0} + r_1\right)^n + \overline{w}\|G_2\| \\
&= \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0}\right)^n + \binom{n}{1} \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0}\right)^{n-1} r_1 \\
&\quad + \binom{n}{2} \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0}\right)^{n-2} r_1^2 + \cdots + r_1^n + \overline{w}\|G_2\| \\
&\leq \binom{n}{\lceil \frac{n}{2} \rceil} \eta_1 \|y_{10}\|^p \lambda^{k-k_0} r_1^n + \binom{n}{\lceil \frac{n}{2} \rceil} \eta_1 \|y_{10}\|^p \lambda^{k-k_0} r_1^n \\
&\quad + \binom{n}{\lceil \frac{n}{2} \rceil} \eta_1 \|y_{10}\|^p \lambda^{k-k_0} r_1^n + \cdots + \binom{n}{\lceil \frac{n}{2} \rceil} r_1^n + \overline{w}\|G_2\| \\
&= n \binom{n}{\lceil \frac{n}{2} \rceil} \eta_1 \|y_{10}\|^p \lambda^{k-k_0} r_1^n + \binom{n}{\lceil \frac{n}{2} \rceil} r_1^n + \overline{w}\|G_2\| \\
&= \eta_3 \|y_{10}\|^p \lambda^{k-k_0} + r_3,
\end{aligned}
$$

where $\eta_3 = n\binom{n}{\lceil \frac{n}{2} \rceil}\eta_1 r_1^n$, and $r_3 = \binom{n}{\lceil \frac{n}{2} \rceil} r_1^n + \overline{w}\|G_2\|$.

(*Case* III) $p \geq 1, 0 < r_1 < 1$. Since $0 < \lambda < 1$, we may assume without loss of generality that

$$
\eta_1 \|y_{10}\|^p \lambda^{k-k_0} + r_1 < 1
$$

for all $k \geq k_0$. Let $\lceil p \rceil = n, n \in \mathbb{I}^+$. Then, by the binomial theorem, we obtain

$$
\begin{aligned}
\|y(k)\| &\leq \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0} + r_1\right)^{\frac{1}{p}} + \overline{w}\|G_2\| \\
&\leq \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0} + r_1\right)^{\frac{1}{n}} + \overline{w}\|G_2\| \\
&\leq \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0}\right)^{\frac{1}{n}} + r_1^{\frac{1}{n}} + \overline{w}\|G_2\| \\
&\leq \eta_1^{\frac{1}{n}} \|y_{10}\|^{\frac{p}{n}} \lambda^{\frac{k-k_0}{n}} + r_1^{\frac{1}{n}} + \overline{w}\|G_2\| \\
&\leq \eta_1^{\frac{1}{n}} \|y_{10}\|^{\frac{p}{n}-p} \|y_{10}\|^p \left(\lambda^{\frac{1}{n}}\right)^{k-k_0} + r_1^{\frac{1}{n}} + \overline{w}\|G_2\| \\
&= \eta_4 \|y_{10}\|^p \lambda_1^{k-k_0} + r_4,
\end{aligned}
$$

where $\eta_4 = \eta_1^{\frac{1}{n}} \|y_{10}\|^{\frac{p}{n}-p}, r_4 = r_1^{\frac{1}{n}} + \overline{w}\|G_2\|$, and $0 < \lambda_1 = \lambda^{\frac{1}{n}} < 1$.

(*Case* IV) $p \geq 1, r_1 \geq 1$. Let $\lfloor p \rfloor = n, n \in \mathbb{I}^+$. Then, by the binomial theorem, we obtain

$$
\begin{aligned}
\|y(k)\| &\leq \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0} + r_1\right)^{\frac{1}{p}} + \overline{w}\|G_2\| \\
&\leq \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0} + r_1\right)^{\frac{1}{n}} + \overline{w}\|G_2\| \\
&\leq \left(\eta_1 \|y_{10}\|^p \lambda^{k-k_0}\right)^{\frac{1}{n}} + r_1^{\frac{1}{n}} + \overline{w}\|G_2\| \\
&\leq \eta_1^{\frac{1}{n}} \|y_{10}\|^{\frac{p}{n}} \lambda^{\frac{k-k_0}{n}} + r_1 + \overline{w}\|G_2\| \\
&\leq \eta_1^{\frac{1}{n}} \|y_{10}\|^{\frac{p}{n}-p} \|y_{10}\|^p \left(\lambda^{\frac{1}{n}}\right)^{k-k_0} + r_1 + \overline{w}\|G_2\| \\
&= \eta_4 \|y_{10}\|^p \lambda_1^{k-k_0} + r_5,
\end{aligned}
$$

where $\eta_4 = \eta_1^{\frac{1}{n}} \|y_{10}\|^{\frac{p}{n}-p}, r_5 = r_1 + \overline{w}\|G_2\|$, and $0 < \lambda_1 = \lambda^{\frac{1}{n}} < 1$. Therefore, from (*Case* I) to (*Case* IV), we obtain

$$\|y(k)\| \leq \eta_5 \|y_{10}\|^p \lambda_2^{k-k_0} + r_6, \tag{3.10}$$

where $\eta_5 = \max\{\eta_2, \eta_3, \eta_4\}, r_6 = \max\{r_2, r_3, r_4, r_5\}$, and $0 < \lambda_2 = \min\{\lambda, \lambda_1\} < 1$.

From $MEx_0 = MENy_0 = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} y_0$ and $y_0 = \begin{pmatrix} y_{10} \\ 0 \end{pmatrix}$, it follows that $\|y_{10}\| \leq \|M\|\|E\|\|x_0\|$.

Since $x(k) = Ny(k)$, it follows from (3.10) that

$$\begin{aligned} \|Ex(k)\| &\leq \|E\|\|N\|\|y(k)\| \\ &\leq \|E\|\|N\|\big(\eta_5 \|y_{10}\|^p \lambda_2^{k-k_0} + r_6\big) \\ &\leq \eta_5 \|E\|\|N\|\|M\|^p \|E\|^p \|x_0\|^p \lambda_2^{k-k_0} + \|E\|\|N\| r_6 \\ &= \eta_6 \|x_0\| \lambda_2^{k-k_0} + r_7, \end{aligned}$$

where $\eta_6 = \eta_5 \|E\|^{p+1} \|N\|\|M\|^p \|x_0\|^{p-1}$ and $r_7 = \|E\|\|N\| r_6$.

Therefore, the singular system (3.2) is exponentially practically stable with respect to $w(k)$ with $\eta = \eta_6, \lambda = \lambda_2$, and $r = r_7$. □

Remark 3.1 In Theorem 3.2, for $p = 1$, we can show that the singular system (3.2) is exponentially practically stable in the pth-moment with respect to $w(k)$ by considering the explicit form of solution of the system. System (3.2) may be reduced to system (3.3) and (3.4). Moreover, the explicit solution of (3.3) and (3.4) with $k_0 = 0$ is given as follows:

$$y_1(k) = A_1^k y_1(0) + \sum_{i=0}^{k-1} A_1^{k-1-i} G_1 w(i), \tag{3.11}$$

$$y_2(k) = -G_2 w(k). \tag{3.12}$$

Thus, from (3.11) and (3.12), we have

$$\begin{aligned} \|y(k)\| &\leq \|y_1(k)\| + \|y_2(k)\| \\ &\leq \|A_1\|^k \|y_1(0)\| + \sum_{i=0}^{k-1} \|A_1\|^{k-1-i} \|G_1\| \|w(i)\| + \|G_2\| \|w(k)\|. \end{aligned}$$

By assuming that $\|A_1\| < 1$, we obtain

$$\begin{aligned} \|y(k)\| &\leq \|A_1\|^k \|y_1(0)\| + \overline{w}\|G_1\| \frac{1}{1 - \|A_1\|} + \overline{w}\|G_2\| \\ &= \|A_1\|^k \|y_1(0)\| + r_0, \end{aligned}$$

where $r_0 = \overline{w}\|G_1\| \frac{1}{1-\|A_1\|} + \overline{w}\|G_2\|$. Since $x(k) = Ny(k)$, we have that

$$\begin{aligned} \|Ex(k)\| &\leq \|E\|\|N\|\|y(k)\| \\ &\leq \|E\|\|N\|\big(\|A_1\|^k \|y_1(0)\| + r_0\big) \end{aligned}$$

$$\begin{aligned}&\le \|E\|\|N\|\big(\|A_1\|^k\|M\|\|E\|\big\|x(0)\big\| + r_0\big)\\&= \overline{\eta}\big\|x(0)\big\|\|A_1\|^k + \overline{r},\end{aligned}$$

where $\overline{\eta} = \|E\|^2\|N\|\|M\|$ and $\overline{r} = \|E\|\|N\|r_0$. Hence, system (3.2) is exponentially practically stable.

Next, we consider the discrete time system with delay and disturbance as follows:

$$\begin{cases} x(k+1) = f(k, x_k, w(k)), \\ x(s) = \varphi(s), \quad s \in \mathbb{N}_{k_0-\tau}, \end{cases} \tag{3.13}$$

where x_k is defined by $x_k(s) = x(k+s)$ for any $s \in \mathbb{N}_{k_0-\tau}$, disturbance $w(k) \in \Omega$, where Ω is defined as in (2.1), $f : \mathbb{R} \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R}^n$ is continuous and locally Lipschitz in (x, w), uniformly in k with Lipschitz constant L which satisfies $f(k, 0, 0) = 0$. Let $x(k; k_0, \varphi, w)$ denote the trajectory of system (3.13) with initial condition φ and disturbance signal $w(k) \in \Omega$.

Definition 3.3 System (3.13) is exponentially practically stable in the pth-moment with respect to $w(k)$ for some $p > 0$ if, for any $k \ge k_0$, there exist constants $0 < \lambda < 1, \eta > 0, r > 0$ such that

$$\big\|x(k; k_0, \phi, w)\big\|^p \le \eta\|\varphi\|^p\lambda^{k-k_0} + r, \quad \forall w(k) \in \Omega.$$

Theorem 3.3 *If there exist a Lyapunov–Krasovskii functional* $V(k, x(k))$, *a* K*-function* ρ, *and positive constants* c_1, c_2, a, p, q, β, *where* $q > 1, 0 < \beta < 1, \rho(\|w(k)\|) < \beta a$ *such that the following conditions hold for all* $w(k) \in \Omega$:

(i) $c_1\|x(k)\|^p \le V(k, x(k)) \le c_2\|x(k)\|^p + a$,

(ii) *If* $V(k+s, x(k+s)) \le qV(k+1, x(k+1))$ *with* $s \in \mathbb{N}_{k_0-\tau}$, *then*

$$\Delta V\big(k, x(k)\big) = V\big(k+1, x(k+1)\big) - V\big(k, x(k)\big) \le -\beta V\big(k, x(k)\big) + \rho\big(\big\|w(k)\big\|\big).$$

Then system (3.13) *is exponentially practically stable in the pth-moment with* $\eta = \frac{c_2}{c_1}, q^{-\frac{1}{\tau+1}} \le \lambda < 1$, *and* $r = \frac{a}{c_1}$.

Proof For $q > 1$, there exists $0 < \lambda < 1$ such that $q \ge \lambda^{-(\tau+1)}$, or equivalently, $q^{-\frac{1}{\tau+1}} \le \lambda < 1$. By employing a similar approach as in the proof of Theorem (3.1) in [23], it follows from $0 < \beta < 1$ and $\rho(\|w(k)\|) < \beta a$ that

$$V\big(k, x(k)\big) \le c_2\|\varphi\|^p\lambda^{k-k_0} + a$$

for $k \ge k_0, w(k) \in \Omega$, from which it follows that

$$\big\|x(k)\big\|^p \le \frac{c_2}{c_1}\|\varphi\|^p\lambda^{k-k_0} + \frac{a}{c_1}, \quad k \ge k_0, w(k) \in \Omega.$$

Therefore, system (3.13) is exponentially practically stable in the pth-moment with $\eta = \frac{c_2}{c_1}, q^{-\frac{1}{\tau+1}} \le \lambda < 1$, and $r = \frac{a}{c_1}$. $\square$

Finally, we consider exponentially practical stability for system (2.1) with delay and disturbance.

Definition 3.4 The discrete time singular system (2.1) is said to be exponentially practically stable in the pth-moment for some $p > 0$ if there exist constants $0 < \lambda < 1, \eta > 0, r > 0$ for each disturbance $w(k) \in \Omega$ such that

$$\|Ex(k;k_0,\varphi,w)\|^p \leq \eta\|\varphi\|^p\lambda^{k-k_0} + r$$

for all $k \geq k_0$, where $x(k;k_0,\varphi,w)$ is the state trajectory of the system with initial condition φ. In particular, when $p = 2$, the system is said to be exponentially practically stable in the mean square.

In order to proceed with the main result on exponentially practical stability in the pth-moment of a singular system with delay and disturbance (2.1), we make the following assumption; an explanation for this assumption is given in Remark 3.2.

Assumption 3.1 There exist nonsingular matrices M, N with appropriate dimensions such that $MEN = \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}$, $MAN = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{pmatrix}$, $MBN = \begin{pmatrix} B_1 & 0 \\ B_3 & B_4 \end{pmatrix}$, and $MG = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix}$, where $\|B_4\| < 1$.

Remark 3.2 For a physical meaning of Assumption 3.1, it implies that there is a plant (y_1) in this situation which does not dynamically depend on the other plant (y_2). For future investigation, we propose to study a more general case in which $MBN = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$, where B_2 may not be zero; namely, all plants dynamically interact, and that $\|B_4\|$ may not be less than 1.

Theorem 3.4 *Assume that the singular system* (2.1) *is regular and causal, and that Assumption* 3.1 *holds. Then the singular system* (2.1) *is exponentially practically stable in the pth-moment with respect to* $w(k)$, *if there exists a Lyapunov–Krasovskii functional* $V(k,x(k))$ *such that*

(i) $c_1\|Ex(k)\|^p \leq V(k,x(k)) \leq c_2\|Ex(k)\|^p + a$,

(ii) *If* $V(k+s,x(k+s)) \leq qV(k+1,x(k+1))$ *with* $s \in \mathbb{N}_{k_0-\tau}$, *then*

$$\Delta V(k,x(k)) = V(k+1,x(k+1)) - V(k,x(k)) \leq -\beta V(k,x(k)) + \rho(\|w(k)\|)$$

hold for some positive constants a, c_1, c_2, p, q, β, *where* $q > 1, 0 < \beta < 1$, *and some* K*-function* ρ.

Proof Assume that system (2.1) is regular and causal. Then, from [30] and Assumption 3.1, there exist nonsingular matrices M, N such that

$$MEN = \begin{pmatrix} I_r & 0 \\ 0 & 0_{n-r} \end{pmatrix}, \qquad MAN = \begin{pmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{pmatrix},$$
$$MBN = \begin{pmatrix} B_1 & 0 \\ B_3 & B_4 \end{pmatrix}, \qquad MG = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix},$$

where $\|B_4\| < 1$. Let $y(k) = N^{-1}x(k) = \left[\begin{smallmatrix} y_1(k) \\ y_2(k) \end{smallmatrix}\right]$, then system (2.1) is transformed to the following system:

$$y_1(k+1) = A_1 y_1(k) + B_1 y_1(k-\tau) + G_1 w(k), \tag{3.14}$$

$$y_2(k) = -B_3 y_1(k-\tau) - B_4 y_2(k-\tau) - G_2 w(k), \tag{3.15}$$

$$y_1(s) := \phi_1(s) = N^{-1}\varphi_1(s), \quad s \in \mathbb{N}_{k_0-\tau},$$

$$y_2(s) := \phi_2(s) = N^{-1}\varphi_2(s), \quad s \in \mathbb{N}_{k_0-\tau}.$$

From the assumption, there exists a Lyapunov–Krasovskii functional $V(k, x(k))$ which satisfies conditions (i) and (ii). Then we obtain the following estimations:

$$\begin{aligned}
&c_1 \|Ex(k)\|^p \le V\bigl(k, x(k)\bigr) \le c_2 \|Ex(k)\|^p + a, \\
&c_1 \|ENy(k)\|^p \le V\bigl(k, Ny(k)\bigr) \le c_2 \|ENy(k)\|^p + a, \\
&c_1 \|M^{-1}MENy(k)\|^p \le V\bigl(k, Ny(k)\bigr) \le c_2 \|M^{-1}MENy(k)\|^p + a, \\
&c_1 \left\| M^{-1} \begin{bmatrix} y_1(k) \\ 0 \end{bmatrix} \right\|^p \le V\bigl(k, Ny(k)\bigr) \le c_2 \left\| M^{-1} \begin{bmatrix} y_1(k) \\ 0 \end{bmatrix} \right\|^p + a, \\
&c_1 \left\| M^{-1}N^{-1}N \begin{bmatrix} y_1(k) \\ 0 \end{bmatrix} \right\|^p \le V\bigl(k, Ny(k)\bigr) \le c_2 \left\| M^{-1}N^{-1}N \begin{bmatrix} y_1(k) \\ 0 \end{bmatrix} \right\|^p + a.
\end{aligned}$$

Let $\overline{y}(k) = Ny(k), \overline{y_1}(k) = N\left[\begin{smallmatrix} y_1(k) \\ 0 \end{smallmatrix}\right]$. Then we obtain

$$c_1 \|M^{-1}N^{-1}\overline{y_1}(k)\|^p \le V\bigl(k, \overline{y}(k)\bigr) \le c_2 \|M^{-1}N^{-1}\overline{y_1}(k)\|^p + a,$$

which gives

$$c_1 \|M^{-1}N^{-1}\overline{y_1}(k)\|^p \le V\bigl(k, \overline{y_1}(k)\bigr) \le c_2 \|M^{-1}N^{-1}\overline{y_1}(k)\|^p + a.$$

Thus, we get

$$\begin{aligned}
V\bigl(k, \overline{y_1}(k)\bigr) &\ge c_1 \bigl(\|M^{-1}N^{-1}\overline{y_1}(k)\|^2\bigr)^{\frac{p}{2}} \\
&= c_1 \bigl(\|\overline{y_1}(k)^T \bigl(M^{-1}N^{-1}\bigr)^T \bigl(M^{-1}N^{-1}\bigr)\overline{y_1}(k)\|\bigr)^{\frac{p}{2}} \\
&\ge c_1 \bigl(\lambda_{\min}\bigl(\bigl(M^{-1}N^{-1}\bigr)^T \bigl(M^{-1}N^{-1}\bigr)\bigr)\bigr)^{\frac{p}{2}} \|\overline{y_1}(k)\|^p.
\end{aligned}$$

Similarly, we have

$$V\bigl(k, \overline{y_1}(k)\bigr) \le c_2 \|M^{-1}\|^p \|N^{-1}\|^p \|\overline{y_1}(k)\|^p + a.$$

Thus, we obtain

$$\overline{c}_1 \|\overline{y_1}(k)\|^p \le V\bigl(k, \overline{y_1}(k)\bigr) \le \overline{c}_2 \|\overline{y_1}(k)\|^p + a,$$

where $\overline{c}_1 = c_1(\lambda_{\min}((M^{-1}N^{-1})^T(M^{-1}N^{-1})))^{\frac{p}{2}}$ and $\overline{c}_2 = c_2\|M^{-1}\|^p\|N^{-1}\|^p$.

Now, assume that the following inequalities hold:

$$V\big(k+s,x(k+s)\big) \le qV\big(k+1,x(k+1)\big) \quad \text{with } s \in \mathbb{N}_{k_0-\tau},$$
$$V\big(k+s,Ny(k+s)\big) \le qV\big(k+1,Ny(k+1)\big) \quad \text{with } s \in \mathbb{N}_{k_0-\tau},$$
$$V\big(k+s,\overline{y}(k+s)\big) \le qV\big(k+1,\overline{y}(k+1)\big) \quad \text{with } s \in \mathbb{N}_{k_0-\tau},$$
$$V\big(k+s,\overline{y_1}(k+s)\big) \le qV\big(k+1,\overline{y_1}(k+1)\big) \quad \text{with } s \in \mathbb{N}_{k_0-\tau}.$$

Then it follows from (ii) that

$$\Delta V\big(k,x(k)\big) = V\big(k+1,x(k+1)\big) - V\big(k,x(k)\big) \le -\beta V\big(k,x(k)\big) + \rho\big(\|w(k)\|\big),$$
$$\Delta V\big(k,Ny(k)\big) = V\big(k+1,Ny(k+1)\big) - V\big(k,Ny(k)\big) \le -\beta V\big(k,Ny(k)\big) + \rho\big(\|w(k)\|\big),$$
$$\Delta V\big(k,\overline{y}(k)\big) = V\big(k+1,\overline{y}(k+1)\big) - V\big(k,\overline{y}(k)\big) \le -\beta V\big(k,\overline{y}(k)\big) + \rho\big(\|w(k)\|\big),$$
$$\Delta V\big(k,\overline{y_1}(k)\big) = V\big(k+1,\overline{y_1}(k+1)\big) - V\big(k,\overline{y_1}(k)\big) \le -\beta V\big(k,\overline{y_1}(k)\big) + \rho\big(\|w(k)\|\big).$$

Since $\overline{y_1}(k) = N\big[\begin{smallmatrix} y_1(k) \\ 0 \end{smallmatrix}\big]$, it follows from (3.14) that

$$\overline{y_1}(k+1) = \overline{A_1}\,\overline{y_1}(k) + \overline{B_1}\,\overline{y_1}(k-\tau) + \overline{G_1}\,\overline{w}(k), \tag{3.16}$$

where $\overline{A_1} = N\big[\begin{smallmatrix} A_1 & 0 \\ 0 & 0 \end{smallmatrix}\big]N^{-1}, \overline{B_1} = N\big[\begin{smallmatrix} B_1 & 0 \\ 0 & 0 \end{smallmatrix}\big]N^{-1}, \overline{G_1} = N\big[\begin{smallmatrix} G_1 & 0 \\ 0 & 0 \end{smallmatrix}\big]$, and $\overline{w}(k) = \big[\begin{smallmatrix} w(k) \\ 0 \end{smallmatrix}\big]$. Therefore, we may conclude that there exists a Lyapunov–Krasovskii functional $V(k,\overline{y_1}(k))$ for system (3.16) which satisfies the following conditions:

(i) $\bar{c_1}\|\overline{y_1}(k)\|^p \le V(k,\overline{y_1}(k)) \le \bar{c_2}\|\overline{y_1}(k)\|^p + a$,

(ii) If $V(k+s,\overline{y_1}(k+s)) \le qV(k+1,\overline{y_1}(k+1))$ with $s \in \mathbb{N}_{k_0-\tau}$, then

$$\Delta V\big(k,\overline{y_1}(k)\big) = V\big(k+1,\overline{y_1}(k+1)\big) - V\big(k,\overline{y_1}(k)\big) \le -\beta V\big(k,\overline{y_1}(k)\big) + \rho\big(\|w(k)\|\big).$$

Thus, from Theorem 3.3, there exist constants $0 < \lambda < 1, \eta_0 \ge 0, r_0 > 0$ such that

$$\big\|\overline{y_1}(k)\big\|^p \le \eta_0\|\overline{\phi}_1\|^p\lambda^{k-k_0} + r_0, \quad \forall k \ge k_0, \tag{3.17}$$

where $\overline{\phi}_1 = N\big[\begin{smallmatrix} \phi_1 \\ 0 \end{smallmatrix}\big]$. It follows that

$$\begin{aligned}\big\|y_1(k)\big\|^p &\le \eta_0\big\|N^{-1}\big\|^p\|N\|^p\|\phi_1\|^p\lambda^{k-k_0} + \big\|N^{-1}\big\|^p r_0 \\ &= \eta_1\|\phi_1\|^p\lambda^{k-k_0} + r_1, \quad \forall k \ge k_0,\end{aligned} \tag{3.18}$$

where $\eta_1 = \eta_0\|N^{-1}\|^p\|N\|^p, r_1 = \|N^{-1}\|^p r_0$. Therefore, we obtain

$$\big\|y_1(k)\big\| \le \big(\eta_1\|\phi_1\|^p\lambda^{k-k_0} + r_1\big)^{\frac{1}{p}}. \tag{3.19}$$

From (3.19), there are four cases to be considered according to the values of p and r_1 as follows.

(*Case* I) $0 < p < 1, 0 < r_1 < 1$. Let $\lfloor\frac{1}{p}\rfloor = n, n \in \mathbb{I}^+$. Since $0 < \lambda < 1$, we may assume without loss of generality that

$$\eta_1\|\phi_1\|^p\lambda^{k-k_0} + r_1 < 1$$

for all $k \geq k_0$. Then, by the binomial theorem, we obtain

$$\begin{aligned}
\|y_1(k)\| &\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0} + r_1\right)^{\frac{1}{p}} \\
&\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0} + r_1\right)^{\lfloor\frac{1}{p}\rfloor} \\
&= \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0} + r_1\right)^{n} \\
&= \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}\right)^{n} + \binom{n}{1}\left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}\right)^{n-1}r_1 \\
&\quad + \binom{n}{2}\left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}\right)^{n-2}r_1^2 + \cdots + r_1^n \\
&\leq \binom{n}{\lceil\frac{n}{2}\rceil}\eta_1\|\phi_1\|^p\lambda^{k-k_0} + \binom{n}{\lceil\frac{n}{2}\rceil}\eta_1\|\phi_1\|^p\lambda^{k-k_0}r_1 + \binom{n}{\lceil\frac{n}{2}\rceil}\eta_1\|\phi_1\|^p\lambda^{k-k_0}r_1 \\
&\quad + \cdots + \binom{n}{\lceil\frac{n}{2}\rceil}r_1 \\
&= \binom{n}{\lceil\frac{n}{2}\rceil}\eta_1\|\phi_1\|^p\lambda^{k-k_0} + (n-1)\binom{n}{\lceil\frac{n}{2}\rceil}\eta_1\|\phi_1\|^p\lambda^{k-k_0}r_1 + \binom{n}{\lceil\frac{n}{2}\rceil}r_1 \\
&= \left[\binom{n}{\lceil\frac{n}{2}\rceil} + (n-1)\binom{n}{\lceil\frac{n}{2}\rceil}r_1\right]\eta_1\|\phi_1\|^p\lambda^{k-k_0} + \left[\binom{n}{\lceil\frac{n}{2}\rceil}r_1\right] \\
&= \eta_2\|\phi_1\|^p\lambda^{k-k_0} + r_2,
\end{aligned}$$

where $\eta_2 = \left[\binom{n}{\lceil\frac{n}{2}\rceil} + (n-1)\binom{n}{\lceil\frac{n}{2}\rceil}r_1\right]\eta_1$ and $r_2 = \binom{n}{\lceil\frac{n}{2}\rceil}r_1$.

(*Case* II) $0 < p < 1, r_1 \geq 1$. Let $\lceil\frac{1}{p}\rceil = n, n \in \mathbb{I}^+$. Since $0 < \lambda < 1$, we may assume without loss of generality that

$$\eta_1\|\phi_1\|^p\lambda^{k-k_0} < 1$$

for all $k \geq k_0$. Then, by the binomial theorem, we obtain

$$\begin{aligned}
\|y(k)\| &\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0} + r_1\right)^{\frac{1}{p}} \\
&\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0} + r_1\right)^{n} \\
&= \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}\right)^{n} + \binom{n}{1}\left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}\right)^{n-1}r_1 \\
&\quad + \binom{n}{2}\left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}\right)^{n-2}r_1^2 + \cdots + r_1^n \\
&\leq \binom{n}{\lceil\frac{n}{2}\rceil}\eta_1\|\phi_1\|^p\lambda^{k-k_0}r_1^n + \binom{n}{\lceil\frac{n}{2}\rceil}\eta_1\|\phi_1\|^p\lambda^{k-k_0}r_1^n \\
&\quad + \binom{n}{\lceil\frac{n}{2}\rceil}\eta_1\|\phi_1\|^p\lambda^{k-k_0}r_1^n + \cdots + \binom{n}{\lceil\frac{n}{2}\rceil}r_1^n
\end{aligned}$$

$$
\begin{aligned}
&= n\binom{n}{\lceil \frac{n}{2}\rceil}\eta_1\|\phi_1\|^p\lambda^{k-k_0}r_1^n+\binom{n}{\lceil \frac{n}{2}\rceil}r_1^n \\
&= \eta_3\|\phi_1\|^p\lambda^{k-k_0}+r_3,
\end{aligned}
$$

where $\eta_3 = n\binom{n}{\lceil \frac{n}{2}\rceil}\eta_1 r_1^n$ and $r_3 = \binom{n}{\lceil \frac{n}{2}\rceil}r_1^n$.

(*Case* III) $p \geq 1, 0 < r_1 < 1$. Let $\lfloor p \rfloor = n, n \in \mathbb{I}^+$. Since $0 < \lambda < 1$, we may assume without loss of generality that

$$
\eta_1\|\phi_1\|^p\lambda^{k-k_0}+r_1<1
$$

for all $k \geq k_0$. Then, by the binomial theorem, we obtain

$$
\begin{aligned}
\|y(k)\| &\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}+r_1\right)^{\frac{1}{p}} \\
&\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}+r_1\right)^{\frac{1}{n}} \\
&\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}\right)^{\frac{1}{n}}+r_1^{\frac{1}{n}} \\
&\leq \eta_1^{\frac{1}{n}}\|\phi_1\|^{\frac{p}{n}}\lambda^{\frac{k-k_0}{n}}+r_1^{\frac{1}{n}} \\
&\leq \eta_1^{\frac{1}{n}}\|\phi_1\|^{\frac{p}{n}-p}\|\phi_1\|^p\left(\lambda^{\frac{1}{n}}\right)^{k-k_0}+r_1^{\frac{1}{n}} \\
&= \eta_4\|\phi_1\|^p\lambda_1^{k-k_0}+r_4,
\end{aligned}
$$

where $\eta_4 = \eta_1^{\frac{1}{n}}\|\phi_1\|^{\frac{p}{n}-p}, r_4 = r_1^{\frac{1}{n}}$, and $\lambda_1 = \lambda^{\frac{1}{n}}$.

(*Case* IV) $p \geq 1, r_1 \geq 1$. Let $\lfloor p \rfloor = n, n \in \mathbb{I}^+$. Since $0 < \lambda < 1$, we may assume without loss of generality that

$$
\eta_1\|\phi_1\|^p\lambda^{k-k_0}<1
$$

for all $k \geq k_0$. Then, by the binomial theorem, we obtain

$$
\begin{aligned}
\|y(k)\| &\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}+r_1\right)^{\frac{1}{p}} \\
&\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}+r_1\right)^{\frac{1}{n}} \\
&\leq \left(\eta_1\|\phi_1\|^p\lambda^{k-k_0}\right)^{\frac{1}{n}}+r_1^{\frac{1}{n}} \\
&\leq \eta_1^{\frac{1}{n}}\|\phi_1\|^{\frac{p}{n}}\lambda^{\frac{k-k_0}{n}}+r_1 \\
&\leq \eta_1^{\frac{1}{n}}\|\phi_1\|^{\frac{p}{n}-p}\|\phi_1\|^p\left(\lambda^{\frac{1}{n}}\right)^{k-k_0}+r_1 \\
&= \eta_4\|\phi_1\|^p\lambda_1^{k-k_0}+r_1,
\end{aligned}
$$

where $\eta_4 = \eta_1^{\frac{1}{n}}\|\phi_1\|^{\frac{p}{n}-p}$ and $0 < \lambda_1 = \lambda^{\frac{1}{n}} < 1$.

From (*Case* I)–(*Case* IV), we obtain that

$$
\begin{aligned}
\|y_1(k)\| &\leq \eta_5\|\phi_1\|^p\lambda_2^{k-k_0}+r_5 \\
&\leq \eta_5\|\phi\|^p\lambda_2^{k-k_0}+r_5,
\end{aligned}
\tag{3.20}
$$

where $\eta_5 = \max\{\eta_2, \eta_3, \eta_4\}, r_5 = \max\{r_1, r_2, r_3, r_4\}, \lambda_2 = \min\{\lambda, \lambda_1\}, \|\phi\| = \max_{s\in\mathbb{N}_{k_0-\tau}}\{\|\phi_1(s)\|, \|\phi_2(s)\|\}$.

From (3.15), we have

$$\|y_2(k)\| \le \|B_3\|\|y_1(k-\tau)\| + \|B_4\|\|y_2(k-\tau)\| + \|G_2\|\|w(k)\|, \quad \forall k \ge k_0. \tag{3.21}$$

For $k \ge k_0$, we proceed with the proof as follows:

- For $k \in [k_0, k_0 + \tau]$, we have

$$\begin{aligned}\|y_2(k)\| &\le \|B_3\|\|\phi_1\| + \|B_4\|\|\phi_2\| + \|G_2\|\|w(k)\| \\ &\le \|B_3\|\|\phi\| + \|B_4\|\|\phi\| + \|G_2\|\|w(k)\| \\ &\le (\|B_3\| + \|B_4\|)\|\phi\| + \gamma,\end{aligned}$$

where $\gamma = \|G_2\|\|w(k)\| \le \|G_2\|\overline{w}$.

- For $k \in [k_0 + \tau, k_0 + 2\tau]$, we have

$$\begin{aligned}\|y_2(k)\| &\le \|B_3\|(\eta_5\|\phi\|^p\lambda_2^{k-k_0} + r_5) + \|B_4\|\big[(\|B_3\| + \|B_4\|)\|\phi\| + \gamma\big] + \gamma \\ &= \eta_5\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} + \|B_3\|r_5 + (\|B_4\|\|B_3\| + \|B_4\|^2)\|\phi\| + \|B_4\|\gamma + \gamma.\end{aligned}$$

- For $k \in [k_0 + 2\tau, k_0 + 3\tau]$, we have

$$\begin{aligned}\|y_2(k)\| &\le \|B_3\|(\eta_5\|\phi\|^p\lambda_2^{k-k_0} + r_5) + \eta_5\|B_4\|\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} + \|B_4\|\|B_3\|r_5 \\ &\quad + (\|B_4\|^2\|B_3\| + \|B_4\|^3)\|\phi\| + \|B_4\|^2\gamma + \|B_4\|\gamma + \gamma.\end{aligned}$$

- For $k \in [k_0 + 3\tau, k_0 + 4\tau]$, we have

$$\begin{aligned}\|y_2(k)\| &\le \eta_5\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} + \|B_3\|r_5 + \eta_5\|B_4\|\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} \\ &\quad + \|B_4\|\|B_3\|r_5 + \eta_5\|B_4\|^2\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} + \|B_4\|^2\|B_3\|r_5 \\ &\quad + (\|B_4\|^3\|B_3\| + \|B_4\|^4)\|\phi\| + \|B_4\|^3\gamma + \|B_4\|^2\gamma + \|B_4\|\gamma + \gamma.\end{aligned}$$

- For $k \in [k_0 + 4\tau, k_0 + 5\tau]$, we have

$$\begin{aligned}\|y_2(k)\| &\le \eta_5\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} + \|B_3\|r_5 + \eta_5\|B_4\|\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} \\ &\quad + \|B_4\|\|B_3\|r_5 + \eta_5\|B_4\|^2\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} + \|B_4\|^2\|B_3\|r_5 \\ &\quad + \eta_5\|B_4\|^3\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} + \|B_4\|^3\|B_3\|r_5 + (\|B_4\|^4\|B_3\| + \|B_4\|^5)\|\phi\| \\ &\quad + \|B_4\|^4\gamma + \|B_4\|^3\gamma + \|B_4\|^2\gamma + \|B_4\|\gamma + \gamma.\end{aligned}$$

By repeating the above process, for all $k \in [k_0 + (h-1)\tau, k_0 + h\tau]$, where $h \in \mathbb{I}^+$, we get that

$$\begin{aligned}\|y_2(k)\| &\le \eta_5(1 + \|B_4\| + \|B_4\|^2 + \cdots + \|B_4\|^{h-2})\|B_3\|\|\phi\|^p\lambda_2^{k-k_0} \\ &\quad + (1 + \|B_4\| + \|B_4\|^2 + \cdots + \|B_4\|^{h-2})\|B_3\|r_5 + (\|B_4\|^{h-1}\|B_3\| + \|B_4\|^h)\|\phi\| \\ &\quad + (1 + \|B_4\| + \|B_4\|^2 + \|B_4\|^3 + \cdots + \|B_4\|^{h-1})\gamma.\end{aligned} \tag{3.22}$$

Thus, from (3.21) and (3.22), for $k \in [k_0 + h\tau, k_0 + (h+1)\tau]$, we obtain

$$\begin{aligned}\|y_2(k)\| &\leq \eta_5\big(1 + \|B_4\| + \|B_4\|^2 + \cdots + \|B_4\|^{h-1}\big)\|B_3\|\|\phi\|^p \lambda_2^{k-k_0} \\ &\quad + \big(1 + \|B_4\| + \|B_4\|^2 + \cdots + \|B_4\|^{h-1}\big)\|B_3\| r_5 + \big(\|B_4\|^h \|B_3\| + \|B_4\|^{h+1}\big)\|\phi\| \\ &\quad + \big(1 + \|B_4\| + \|B_4\|^2 + \|B_4\|^3 + \cdots + \|B_4\|^h\big)\gamma.\end{aligned}$$

Hence, from $\|B_4\| < 1$, we obtain by mathematical induction that

$$\begin{aligned}\|y_2(k)\| &\leq \eta_5\big(1 + \|B_4\| + \|B_4\|^2 + \cdots\big)\|B_3\|\|\phi\|^p \lambda_2^{k-k_0} \\ &\quad + \big(1 + \|B_4\| + \|B_4\|^2 + \cdots\big)\|B_3\| r_5 + \big(\|B_3\| + \|B_4\|\big)\|\phi\| \\ &\quad + \big(1 + \|B_4\| + \|B_4\|^2 + \cdots\big)\gamma \\ &\leq \frac{\eta_5\|B_3\|}{1 - \|B_4\|}\|\phi\|^p \lambda_2^{k-k_0} + \frac{\|B_3\| r_5}{1 - \|B_4\|} + \big(\|B_3\| + \|B_4\|\big)\|\phi\| + \frac{\gamma}{1 - \|B_4\|} \\ &= \eta_6\|\phi\|^p \lambda_2^{k-k_0} + r_6, \end{aligned} \tag{3.23}$$

where $\eta_6 = \frac{\eta_5\|B_3\|}{1-\|B_4\|}$ and $r_6 = \frac{\|B_3\| r_5 + \gamma}{1-\|B_4\|} + (\|B_3\| + \|B_4\|)\|\phi\|$.

Thus, it follows from (3.20) and (3.23) that

$$\begin{aligned}\|y(k)\| &\leq \|y_1(k)\| + \|y_2(k)\| \\ &\leq \eta_5\|\phi\|^p \lambda_2^{k-k_0} + r_5 + \eta_6\|\phi\|^p \lambda_2^{k-k_0} + r_6 \\ &= (\eta_5 + \eta_6)\|\phi\|^p \lambda_2^{k-k_0} + r_5 + r_6 \\ &= \eta_7\|\phi\|^p \lambda_2^{k-k_0} + r_7, \end{aligned} \tag{3.24}$$

where $\eta_7 = \eta_5 + \eta_6$ and $r_7 = r_5 + r_6$.

From $x(k) = Ny(k)$ and (3.24), we get

$$\begin{aligned}\|Ex(k)\| &\leq \|E\|\|N\|\|y(k)\| \\ &\leq \|E\|\|N\|\big(\eta_7\|\phi\|^p \lambda_2^{k-k_0} + r_7\big) \\ &\leq \eta_7\|E\|\|N\|\|N^{-1}\|\|\varphi\|^p \lambda_2^{k-k_0} + \|E\|\|N\| r_7 \\ &= \eta_8\|\varphi\|\lambda_2^{k-k_0} + r_8,\end{aligned}$$

where $\eta_8 = \eta_7\|E\|\|N\|\|N^{-1}\|\|\varphi\|^{p-1}$ and $r_8 = \|E\|\|N\| r_7$.

Therefore, the discrete time singular system (2.1) is exponentially practically stable with respect to $w(k)$ with $\eta = \eta_8$, $\lambda = \lambda_2$, and $r = r_8$. □

Remark 3.3 From the method of proof of Theorem 3.4, it is clear that this method can be applied for a discrete time singular system with disturbance and time varying delay $\tau(k)$ with $0 \leq \tau(k) \leq \tau$, $\tau > 0$.

Remark 3.4 As is known, Razumikhin techniques only require less restrictive assumptions, namely they employ a type of Lyapunov–Krasovskii functional which is required to decrease only if a certain condition on the past state trajectory and the current state is satisfied. However, such Razumikhin-type techniques usually lead to a delay-independent criterion which is less conservative than a delay-dependent result, especially for constant delay systems. To deal with this conservativeness, several mathematical approaches have been considered in recent works, e.g., the LMI approach and the time-dependent Lyapunov functional method. Recently, in [31, 32], the Razumikhin technique was expressed by utilizing the LMI approach and the time-invariant Lyapunov functional method which avoid the conservativeness of the Razumikhin-type techniques. It is our future investigation to apply the above mentioned approaches to obtain less conservative criteria for exponentially practical stability of discrete time singular systems with delay and disturbance.

Remark 3.5 Obviously, exponential stability implies exponentially practical stability but not conversely. However, in several practical applications, it only needs to stabilize a system into the region of a phase space, namely the system may oscillate near the equilibrium point, in which the performance is still acceptable. To the best of our knowledge, the present work is the first result on exponentially practical stability of a discrete time singular system with delay and disturbance. Moreover, compared to [22] which proposed asymptotically practical stability criteria for a discrete time system with delay, we derive an exponentially practical stability condition which is more desirable.

4 Numerical examples

Remark 4.1 We provide an algorithm for the implementation and computational corresponding to Theorem 3.2 and Theorem 3.4 as follows:

1. First, we choose an appropriate Lyapunov functional or Lyapunov–Krasovskii functional candidate according to the assumptions of Theorem 3.2 or Theorem 3.4, respectively. Then, we estimate the values of c_1, c_2, and a which satisfy condition (i) of the corresponding theorems.
2. From the estimations of c_1, c_2, and a obtained in 1, we choose appropriate q, β, c_3, and ρ which satisfy condition (ii) of the corresponding theorems.

Example 4.1 Consider system (3.2) with $E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \end{bmatrix}$, $G = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $w(k) \in \mathbb{R}$, and $k_0 = 0$ with the initial condition given by $x(0) = [5, -0.04]^T$. We can see that $\det(zE - A) = -z + 0.5 \neq 0$ for some $z \in \mathbb{C}$ and $\deg(\det(zE - A)) = \operatorname{rank}(E) = 1$. Thus, system (3.2) is regular and causal. For nonsingular matrices $M = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}, N = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$, we obtain

$$MEN = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad MAN = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \qquad MG = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We choose a Lyapunov functional as $V(k, x(k)) = x^T(k)E^T Ex(k) + a$ with $a > 0$. Then we obtain that

(i) $\|Ex(k)\|^2 \leq V(k, x(k)) \leq \|Ex(k)\|^2 + a$,

(ii)

$$\begin{aligned}
&\Delta V\bigl(k, x(k)\bigr) \\
&\quad = V\bigl(k+1, x(k+1)\bigr) - V\bigl(k, x(k)\bigr) \\
&\quad = x^T(k+1)E^T Ex(k+1) + a - x^T(k)E^T Ex(k) - a
\end{aligned}$$

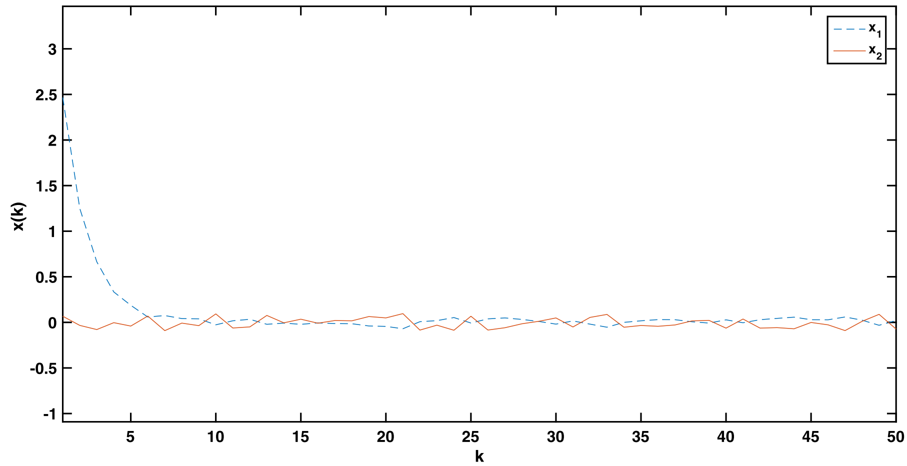

Figure 1 The trajectory of solution of system (3.2)

$$
\begin{aligned}
&= x^T(k)A^TAx(k) + 2x^T(k)A^TGw(k) + w^T(k)G^TGw(k) - x^T(k)E^TEx(k) \\
&= \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}^T \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + 2 \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}^T \begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(k) \\
&\quad + w^T(k) \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} w(k) - \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} \\
&= 0.25x_1^2(k) + 0.5x_1(k)x_2(k) + 1.25x_2^2(k) + x_1(k)w(k) + 3x_2(k)w(k) \\
&\quad + 2w(k)^2 - x_1(k)^2 \\
&= -0.75x_1^2(k) + 1.25w^2(k) + 0.5x_1(k)w(k) - w(k)^2 \\
&\leq -0.75x_1^2(k) + 0.25x_1^2(k) + 0.25w^2(k) + 0.25w(k)^2 \\
&= -0.75x_1^2(k) + 0.25x_1^2(k) + 0.25x_2^2(k) + 0.25w(k)^2 \\
&= -0.5x_1^2(k) + 0.25x_2^2(k) + 0.25w(k)^2 \\
&= -0.5x_1^2(k) - 0.5x_2^2(k) + 0.75x_2^2(k) + 0.25w(k)^2 \\
&= -0.5\big(x_1^2(k) + x_2^2(k)\big) + 0.75w_2^2(k) + 0.25w(k)^2 \\
&= -0.5\big\|x(k)\big\|^2 + w^2(k).
\end{aligned}
$$

Therefore, by Theorem 3.2, we may show that system (3.2) is exponentially practically stable in the pth-moment with $\eta = 35.6, \lambda = 0.5, r = 0.5$. For simulation purpose, we let $a = 0.3$, $\|w(k)\| \leq 0.1$. Then Theorem 3.2 is satisfied with the parameters $c_1 = 1, c_2 = 1, a = 0.3, c_3 = 0.5, p = 2, \rho(\|w(k)\|) = w(k)^2 \leq 0.01$. Figure 1 shows the trajectories of solution of Example 4.1 with disturbance. Figure 2 shows the trajectories of solution of Example 4.1 without disturbance.

Example 4.2 Consider system (2.1) with $E = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $A = \left[\begin{smallmatrix} -0.05 & 0 \\ -0.05 & -0.05 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 0.4 & 0 \\ -0.05 & -0.02 \end{smallmatrix}\right]$, $G = \left[\begin{smallmatrix} 0.5 \\ 0.5 \end{smallmatrix}\right]$, $w(k) \in \mathbb{R}, \tau = 1$, and $k_0 = 0$ with the initial conditions given by $x(-1) = [5,5]^T, x(0) = [5, 1.75]^T$. We can see that $\det(zE - A) = 0.05z + 0.0025 \not\equiv 0$ for some $z \in \mathbb{C}$ and deg(det($zE -$

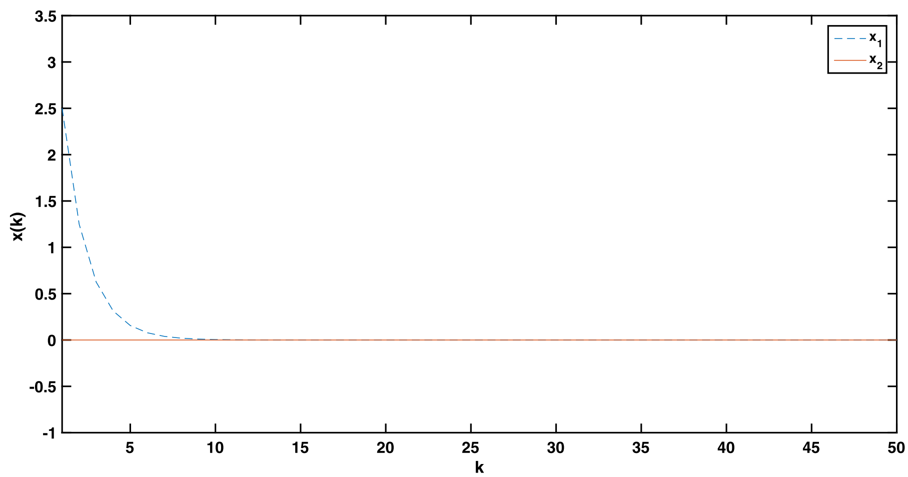

Figure 2 The trajectory of solution of system (3.2) without disturbance

$A)) = \operatorname{rank}(E) = 1$. Thus, system (2.1) is regular and causal. For nonsingular matrices $M = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right], N = \left[\begin{smallmatrix} 0.5 & 0 \\ -0.5 & -10 \end{smallmatrix}\right]$, we get

$$MEN = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad MAN = \begin{bmatrix} -0.05 & 0 \\ 0 & 1 \end{bmatrix},$$
$$MBN = \begin{bmatrix} 0.4 & 0 \\ -0.03 & 0.4 \end{bmatrix}, \qquad MG = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We choose a Lyapunov–Krasovskii functional as $V(k, x(k)) = |x_1(k)| + a$ with $a > 0$.

From $Ex(k) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} x_1(k) \\ x_2(k) \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_1(k) \\ 0 \end{smallmatrix}\right]$ and

$$\begin{aligned} Ex(k+1) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} \\ &= \begin{bmatrix} -0.05 & 0 \\ -0.05 & -0.05 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0.4 & 0 \\ -0.05 & -0.02 \end{bmatrix} \begin{bmatrix} x_1(k-\tau) \\ x_2(k-\tau) \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} w(k), \end{aligned}$$

we obtain

$$\begin{bmatrix} x_1(k+1) \\ 0 \end{bmatrix} = \begin{bmatrix} -0.05x_1(k) + 0.4x_1(k-\tau) + 0.5w(k) \\ -0.05x_1(k) - 0.05x_2(k) - 0.05x_1(k-\tau) - 0.02x_2(k-\tau) + 0.5w(k) \end{bmatrix}.$$

Thus, we obtain

(i) $\|Ex(k)\| \le V(k, x(k)) = |x_1(k)| + a \le \|Ex(k)\| + a$,

(ii) If $V(k+s, x(k+s)) \le qV(k+1, x(k+1))$ with $s \in \mathbb{N}_{-\tau}$, then we have

$$\begin{aligned} &x_1(k+1) = -0.05x_1(k) + 0.4x_1(k-\tau) + 0.5w(k), \\ &\big|x_1(k+1)\big| \le 0.05\big|x_1(k)\big| + 0.4\big|x_1(k-\tau)\big| + 0.5\big|w(k)\big|, \\ &\big|x_1(k+1)\big| + a - a \le 0.05\big|x_1(k)\big| + 0.05a - 0.05a + 0.4\big|x_1(k-\tau)\big| + 0.4a - 0.4a \end{aligned}$$

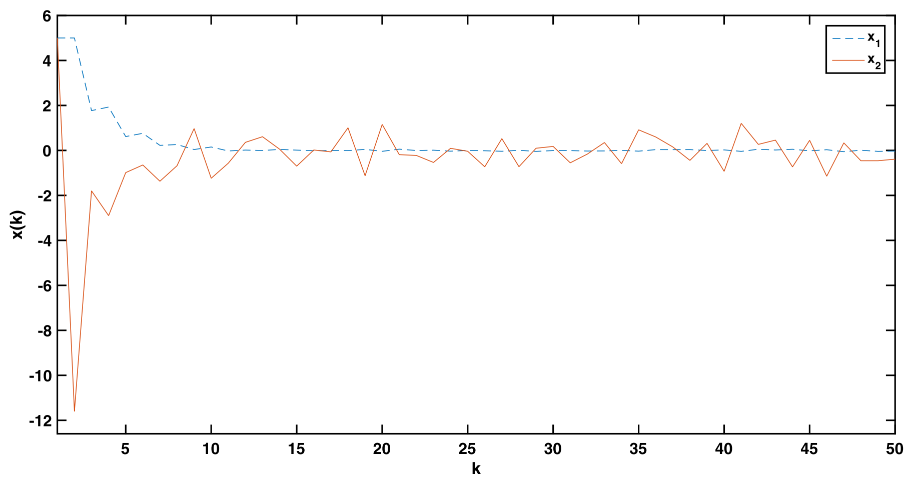

Figure 3 The trajectory of solution of system (2.1)

$$
\begin{aligned}
&+ 0.5\big|w(k)\big|, \\
&\big|x_1(k+1)\big| + a \le 0.05\big(\big|x_1(k)\big| + a\big) + 0.4\big(\big|x_1(k-\tau)\big| + a\big) + 0.55a + 0.5\big|w(k)\big|, \\
&V\big(k+1, x(k+1)\big) \le 0.05V\big(k, x(k)\big) + 0.4V\big(k-\tau, x(k-\tau)\big) + 0.55a + 0.5\big\|w(k)\big\| \\
&\qquad \le 0.05V\big(k, x(k)\big) + 0.4qV\big(k+1, x(k+1)\big) + 0.55a + 0.5\big\|w(k)\big\| \\
&\qquad \le \frac{0.05}{1-0.4q}V\big(k, x(k)\big) + \frac{0.55a + 0.5\|w(k)\|}{1-0.4q}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\Delta V\big(k, x(k)\big) &= V\big(k+1, x(k+1)\big) - V\big(k, x(k)\big) \\
&\le \frac{0.05}{1-0.4q}V\big(k, x(k)\big) + \frac{0.55a + 0.5\|w(k)\|}{1-0.4q} - V\big(k, x(k)\big) \\
&= -\left(1 - \frac{0.05}{1-0.4q}\right)V\big(k, x(k)\big) + \frac{0.55a + 0.5\|w(k)\|}{1-0.4q} \\
&= -\beta V\big(k, x(k)\big) + \rho\big(\big\|w(k)\big\|\big),
\end{aligned}
$$

where $\beta \le 1 - \frac{0.05}{1-0.4q}$ and $\rho(\|w(k)\|) = \frac{0.55a+0.5\|w(k)\|}{1-0.4q}$.

Therefore, by Theorem 3.4, system (2.1) is exponentially practically stable in the pth-moment with $\eta = 82.41, \lambda = 0.8$, and $r = 54.58$. For simulation purpose, we let $a = 0.5, \|w(k)\| \le 0.1$. Then Theorem 3.4 is satisfied with $c_1 = 1, c_2 = 1, a = 0.5, q = 1.6, \beta = 0.8, p = 1, \rho(\|w(k)\|) \le 0.9$. Figure 3 shows the trajectories of solution of Example 4.2. Figure 4 shows the trajectories of solution of system (2.1) without disturbance.

5 Conclusion

In this paper, exponentially practical stability of a discrete time singular system with delay and disturbance has been investigated. For systems with disturbance but without delay, by using Lyapunov stability theory, we obtained a criterion for exponentially practical stability of a general discrete time system and a linear discrete time singular system, respec-

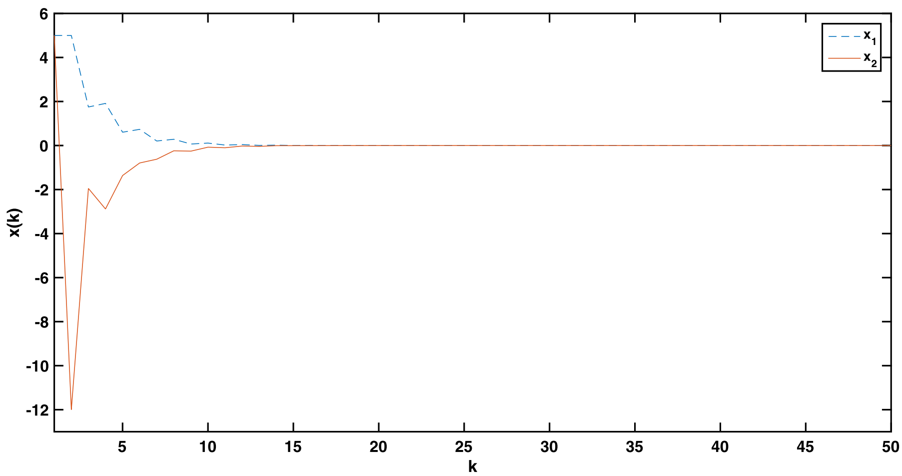

Figure 4 The trajectory of solution of system (2.1) without disturbance

tively. For systems with delay and disturbances, by using the Razumikhin-type technique, we derived exponentially practical stability criteria for a general discrete time system and a linear singular system, respectively. Numerical examples were given to show effectiveness of our theoretical results.

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The authors declare that they have no competing interests.

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