Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Approximation of state variables for discrete-time stochastic genetic regulatory networks with leakage, distributed, and probabilistic measurement delays: a robust stability problem

S. Pandiselvi[1], R. Raja[2], Jinde Cao[3*], G. Rajchakit[4] and Bashir Ahmad[5]

*Correspondence:
jdcao@seu.edu.cn
[3]School of Mathematics, Southeast University, Nanjing, China
Full list of author information is available at the end of the article

Abstract

This work predominantly labels the problem of approximation of state variables for discrete-time stochastic genetic regulatory networks with leakage, distributed, and probabilistic measurement delays. Here we design a linear estimator in such a way that the absorption of mRNA and protein can be approximated via known measurement outputs. By utilizing a Lyapunov–Krasovskii functional and some stochastic analysis execution, we obtain the stability formula of the estimation error systems in the structure of linear matrix inequalities under which the estimation error dynamics is robustly exponentially stable. Further, the obtained conditions (in the form of LMIs) can be effortlessly solved by some available software packages. Moreover, the specific expression of the desired estimator is also shown in the main section. Finally, two mathematical illustrative examples are accorded to show the advantage of the proposed conceptual results.

Keywords: Genetic regulatory networks (GRNs); Time-varying delays; Distributed delays; Leakage delays; Probabilistic measurement delays

1 Introduction and system formulation

A gene is a physical structure made up of DNA, and most of the genes hold the data which is required to make molecules called as proteins. In the modern years, research in genetic regulatory networks (GRNs) has gained significance in both biological and bio-medical sciences, and a huge number of tremendous results have been issued. Distinct kinds of computational models have been applied to propagate the behaviors of GRNs; see, for instance, the Bayesian network models, the Petri net models, the Boolean models, and the differential equation models. Surrounded by the indicated models, the differential equation models describe the rate of change in the concentration of gene production, such as mRNAs and proteins, as constant values, whereas the other models do not have such a basis.

As one of the mostly investigated dynamical behaviors, the state estimation for GRNs has newly stirred increasing research interest (see [1, 2] and the references cited therein [1, 3–10]). In fact, this is an immense concern since GRNs are complex nonlinear systems. Due to the complication, it is frequently the case that only partial facts around the states of the nodes are accessible in the network outputs. In consideration of realizing the GRNs better, there has been a necessity to estimate the state of the nodes through securable measurements. In [1], the robust H_∞ problem was considered for a discrete-time stochastic GRNs with probabilistic measurement delays. In [2], the robust H_∞ state estimation problem was investigated for a general class of uncertain discrete-time stochastic neural networks with probabilistic measurement delays. By designing an adaptive controller, the authors investigated the problem of delayed GRNs stabilization in [7]. Xiao et al. discussed the stability, periodic oscillation, and bifurcation of two-gene regulatory networks with time delays [8]. The stability of continuous GRNs and discrete-time GRNs was discussed, respectively, in [11]. Huang et al. considered the bifurcation of delayed fractional GRNs by hybrid control [12].

Due to the limited signal communication speed, the measurement among the networks is always assumed to be a delayed one. So, the network measurement could not include instruction about the present gene states, while the delayed network measurement could. The most fashionable mechanism to relate the probabilistic measurement delay or some other kind of lacking measurement is to grab it as a Bernoulli distributed white classification [13–20]. The robust stochastic stability of stochastic genetic GRNs was considered, and some delay-dependent criteria were presented in the form of LMIs [18]. And the asymptotic stability of delayed stochastic GRNs with impulsive effect was discussed in [19]. The synchronization problem of dynamical system was also discussed in [21, 22]. The challenging task is how to draft the robust estimators when both uncertainties and probabilistic appeared in discrete-time GRN models.

More recently, in [23], Liu et al. developed a state estimation problem for a genetic regulatory network with Markovian jumping parameters and time delays:

$$\begin{aligned}\dot{m}(t) &= -A\bigl(r(t)\bigr)m(t) + W\bigl(r(t)\bigr)g\bigl(p\bigl(t-\sigma(t)\bigr)\bigr),\\ \dot{p}(t) &= -C\bigl(r(t)\bigr)p(t) + D\bigl(r(t)\bigr)m\bigl(t-\tau(t)\bigr).\end{aligned}$$

Also in [24], Wan et al. proposed the state estimation of discrete-time GRN with random delays governed by the following equation:

$$\begin{aligned}M(k+1) &= AM(k) + Bf\bigl(P\bigl(k-d(k)\bigr)\bigr) + V,\\ P(k+1) &= CP(k) + DM\bigl(k-\tau(k)\bigr).\end{aligned}$$

Considering the above referenced papers, the robustness of approximation of the stochastic GRNs with leakage delays, distributed delays, and probabilistic measurement delays has not been tackled. The main contributions of this paper are summarized as follows:

1. We examine the approximation concern for the discrete-time stochastic GRNs with the leakage delays, distributed delays, and probabilistic measurement delays into the problem and model the robust H_∞ state estimator for a class of discrete-time stochastic GRNs. Here, the probabilistic measurement delays, which narrate the

binary shifting sequence, are satisfied by the conditional probability distribution. So, the crisis of parameter uncertainties, including errors, stochastic disturbance, leakage delays, distributed delays, and the activation function of the addressed GRNs, is identified by sector-bounded nonlinearities.
2. By applying the Lyapunov stability theory and stochastic analysis techniques, sufficient conditions are first entrenched to assure the presence of the desired estimators in terms of a linear matrix inequality (LMI). These circumstances are reliant on both the lower and upper bounds of time-varying delays. Again, the absolute expression of the desired estimator is demonstrated to assure the estimation error dynamics to be robustly exponentially stable in the mean square for the consigned system.
3. Finally, twin mathematical examples beside with simulations are given to view the capability of the advanced criteria.

In this note, we consider the GRNs with leakage, discrete, and distributed delays described as follows:

$$\begin{aligned} x(k+1) &= -\big(\mathbb{A}+\Delta\mathbb{A}(k)\big)x(k-\rho_1)+\big(\mathbb{B}+\Delta\mathbb{B}(k)\big)\hat{g}\big(y\big(k-\delta(k)\big)\big) \\ &\quad+\big(E+\Delta E(k)\big)\sum_{s=1}^{\infty}\mu_s h\big(y(k-s)\big)+\sigma\big(k,x(k-\rho_1)\big)\omega(k)+L_x v_x(k), \\ y(k+1) &= -\big(\mathbb{C}+\Delta\mathbb{C}(k)\big)y(k-\rho_2)+\big(\mathbb{D}+\Delta\mathbb{D}(k)\big)x\big(k-\tau(k)\big) \\ &\quad+\big(F+\Delta F(k)\big)\sum_{n=1}^{\infty}\xi_n x(k-n)+L_y v_y(k), \end{aligned} \tag{1}$$

where $x(k-\rho_1)=[x_1(k-\rho_1),\dots,x_n(k-\rho_2)]^T\in\mathbb{R}^n$, $y(k-\rho_2)=[y_1(k-\rho_2),\dots,y_n(k-\rho_2)]^T\in\mathbb{R}^n$, $x_i(k-\rho_1)$, and $y_i(k-\rho_2)$ $(i=1,2,\dots,n)$ denote the concentrations of mRNA and protein of the ith node at time t, respectively; $\mathbb{A}=\operatorname{diag}\{a_1,a_2,\dots,a_n\}$, $\mathbb{C}=\operatorname{diag}\{c_1,c_2,\dots,c_n\}$, and $\mathbb{D}=\operatorname{diag}\{d_1,d_2,\dots,d_n\}$ are constant matrices; $a_i>0$, $c_i>0$, and $d_i>0$ are the degradation rates of mRNAs, protein, and the translation rate of the ith gene, respectively; the coupling matrix of the genetic regulatory network is defined as $\mathbb{B}=(b_{ij})\in\mathbb{R}^{n\times n}$; $E=\operatorname{diag}\{e_1,e_2,\dots,e_n\}$, and $F=\operatorname{diag}\{f_1,f_2,\dots,f_n\}$ are the weight matrices. $\Delta\mathbb{A}(k)$, $\Delta\mathbb{B}(k)$, $\Delta\mathbb{C}(k)$, $\Delta\mathbb{D}(k)$, $\Delta\mathbb{E}(k)$, and $\Delta\mathbb{F}(k)$ represent the parameter uncertainties; $h(y(k))=[h_1(y(k)),\dots,h_n(y(k))]^T\in\mathbb{R}^n$ denotes the activation function; the exogenous disturbance signals $v_x(k),v_y(k)\in\mathbb{R}^n$ satisfy $v_i(\cdot)\in L_2[0,\infty)$. L_x and L_y are the known real constant matrices. $\delta(k)$ denotes the feedback regulation delay and $\tau(k)$ denotes the translation delay, which satisfy

$$0\le\delta_m\le\delta(k)\le\delta_M,\qquad 0\le\tau_m\le\tau(k)\le\tau_M, \tag{2}$$

where the lower bound δ_m, τ_m and the upper bound δ_M, τ_M are known positive integers.

Furthermore, the nonlinear activation function $\hat{g}(y(k-\delta(k)))=[\hat{g}_1(y_1(k-\delta(k))),\dots,\hat{g}_n(y_n(k-\delta(k)))]^T\in\mathbb{R}^n$ represents the feedback regulation of the protein on the transcription. It is a monotonic function in the Hill form, that is, $\hat{g}_i(f)=\frac{f^{h_j}}{1+f^{h_j}}$ $(j=1,2,\dots,n)$, where h_j is the Hill co-efficient and f is a positive constant. The noise intensity function vector

$\sigma(k, x(k)) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\sigma^T\big(k, x(k-\rho_1)\big)\sigma\big(k, x(k-\rho_1)\big) \leq x^T(k-\rho_1)Hx(k-\rho_1), \tag{3}$$

where $H > 0$ is a known matrix. $\omega(k)$ is a Brownian motion with $\mathbb{E}\{\omega(k)\} = 0$, $\mathbb{E}\{\omega^2(k)\} = 1$ and $\mathbb{E}\{\omega(i)\omega(j)\} = 0$ $(i \neq j)$.

For large-scale complex networks, information around the network nodes is not often fully attainable from the network outputs (see [25, 26]). We can assume that network measurements are described as follows:

$$\begin{aligned} Z_x(k) &= Mx(k), \\ Z_y(k) &= Ny(k), \end{aligned} \tag{4}$$

where M and N are known constant matrices. $Z_x(k), Z_y(k) \in \mathbb{R}^l$ are the complete outputs of the network. The network outputs are subjected to probabilistic delays that can be described by

$$\begin{aligned} \tilde{Z}_x(k) &= \alpha_k Z_x(k) + (1-\alpha_k)Z_x(k-1), \\ \tilde{Z}_y(k) &= \beta_k Z_y(k) + (1-\beta_k)Z_y(k-1), \end{aligned} \tag{5}$$

where the stochastic variables $\alpha_k, \beta_k \in \mathbb{R}$ are Bernoulli allocated with sequences directed by

$$\begin{aligned} &\mathrm{Prob}\{\alpha_k = 1\} = \mathbb{E}\{\alpha_k\} = \alpha_0, \qquad \mathrm{Prob}\{\alpha_k = 0\} = 1 - \mathbb{E}\{\alpha_k\} = 1 - \alpha_0, \\ &\mathrm{Prob}\{\beta_k = 1\} = \mathbb{E}\{\beta_k\} = \beta_0, \qquad \mathrm{Prob}\{\beta_k = 0\} = 1 - \mathbb{E}\{\beta_k\} = 1 - \beta_0. \end{aligned} \tag{6}$$

Here $\alpha_0, \beta_0 > 0$ are known constants. Obviously, for α_k, β_k, the variance $\sigma_\alpha = \alpha_0(1-\alpha_0)$, $\sigma_\beta = \beta_0(1-\beta_0)$.

The GRN state estimator to be designed is given as follows:

$$\begin{cases} \hat{x}(k+1) = -\mathbb{A}_x\hat{x}(k) + \mathbb{B}_x\tilde{Z}_x(k), \\ \hat{y}(k+1) = -\mathbb{A}_y\hat{y}(k) + \mathbb{B}_y\tilde{Z}_y(k), \end{cases} \tag{7}$$

where $\hat{x}(k), \hat{y}(k) \in \mathbb{R}^n$ are the estimations of $x(k)$ and $y(k)$, and $\mathbb{A}_x$, $\mathbb{A}_y$, $\mathbb{B}_x$, $\mathbb{B}_y$ are the estimator gain matrices to be determined.

Assume that the estimation error vectors are $\tilde{x}(k) = x(k) - \hat{x}(k)$ and $\tilde{y}(k) = y(k) - \hat{y}(k)$; the estimation error dynamics can be defined as follows from equations (1), (5), and (7):

$$\begin{aligned} \tilde{x}(k+1) = {} & -\big(\mathbb{A} + \Delta\mathbb{A}(k)\big)x(k-\rho_1) + (\mathbb{A}_x - \alpha_k\mathbb{B}_xM)x(k) + \big(\mathbb{B} + \Delta\mathbb{B}(k)\big)\hat{g}\big(y\big(k-\delta(k)\big)\big) \\ & + \big(E + \Delta E(k)\big)\sum_{s=1}^{\infty}\mu_s h\big(y(k-s)\big) + \sigma\big(k, x(k-\rho_1)\big)\omega(k) - \mathbb{A}_x\tilde{x}(k) \\ & - (1-\alpha_k)\mathbb{B}_xMx(k-1) + L_x v_x(k), \end{aligned}$$

$$\begin{aligned}\tilde{y}(k+1) &= -\big(\mathbb{C}+\Delta\mathbb{C}(k)\big)y(k-\rho_2) + (\mathbb{A}_y - \beta_k\mathbb{B}_yN)y(k) + \big(\mathbb{D}+\Delta\mathbb{D}(k)\big)x\big(k-\tau(k)\big) \\ &\quad + \big(F+\Delta F(k)\big)\sum_{n=1}^{\infty}\xi_n x\big((k-n)\big) - \mathbb{A}_y\tilde{y}(k) - (1-\beta_k)\mathbb{B}_yNy(k-1) \\ &\quad + L_y\nu_y(k). \end{aligned} \tag{8}$$

For suitability, we denote

$$\begin{aligned} &\bar{x}(k) = \begin{bmatrix} x(k) \\ \tilde{x}(k) \end{bmatrix}, \qquad \bar{y}(k) = \begin{bmatrix} y(k) \\ \tilde{y}(k) \end{bmatrix}, \\ &\bar{x}(j) = \psi(j), \quad j = -\tau_M, -\tau_{M+1}, \dots, -1, 0, \\ &\bar{y}(j) = \varphi(j), \quad j = -\delta_M, -\delta_{M+1}, \dots, -1, 0, \end{aligned}$$

where $\psi(j)$, $j = -\tau_M, -\tau_M + 1, \dots, -1, 0$ and $\varphi(j)$, $j = -\delta_M, -\delta_M + 1, \dots, -1, 0$ are the initial conditions.

2 Preliminaries

Notations: Throughout the paper, *naturals*$^+$ refers to the position for the set of nonnegative integers; $\mathbb{R}^n$ indicates the n-dimensional Euclidean space. The superscript "T" acts as the matrix transposition. The code $X \geq Y$ (each $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semi-definitive (respectively positive definite). I means the identity matrix with consistent dimension. The symbol "$*$" denotes the term symmetry. In addition, $E\{\cdot\}$ denotes the expectation operator. $L_2[0,\infty)$ is the amplitude of square-integrable vector functions over $[0,\infty)$. $|\cdot|$ denotes the Euclidean vector norm. Matrices, if not absolutely specified, are affected to have compatible dimensions.

Assumption 1 The parameter uncertainties $\Delta\mathbb{A}(k)$, $\Delta\mathbb{B}(k)$, $\Delta\mathbb{C}(k)$, $\Delta\mathbb{D}(k)$, $\Delta E(k)$, $\Delta F(k)$ are of the following form.

The admissible parameter uncertainties are assumed to be of the form:

$$\begin{aligned} &\begin{bmatrix} \Delta\mathbb{A}(k) & \Delta\mathbb{B}(k) & \Delta\mathbb{C}(k) & \Delta\mathbb{D}(k) & \Delta E(k) & \Delta F(k) \end{bmatrix} \\ &\quad = RN(k)[W_1 \quad W_2 \quad W_3 \quad W_4 \quad W_5 \quad W_6], \end{aligned}$$

where R, $\mathbb{W}_i$ $(i = 1, 2, \dots, 6)$ are the known constant matrices with appropriate dimensions. The uncertain matrix $N(k)$ satisfies $N^T(k)N(k) \leq I$, $\forall k \in$ *naturals*$^+$.

Assumption 2 The vector-valued function $\hat{g}_i(\cdot)$ is assumed to satisfy the following sector-bounded condition, namely for $\forall x, y \in \mathbb{R}^n$:

$$\big[\hat{g}(x) - \hat{g}(y) - N_1(x-y)\big]^T\big[\hat{g}(x) - \hat{g}(y) - N_2(x-y)\big] \leq 0,$$

where N_1, N_2 are known real constant matrices, and $\tilde{N} = N_1 - N_2$ is a symmetric positive definite matrix.

Definition 2.1 If there exist constants $\alpha > 0$ and $0 < \mu < 1$, system (8) with $\nu_x(k) = 0$ and $\nu_y(k) = 0$ is global robust exponential state estimator of GRNs (1) with measurements (5) in the mean square sense such that

$$\mathbb{E}\{|\bar{x}(k)|^2 + |\bar{y}(k)|^2\} \leq \alpha\mu^k \Big(\max_{-\tau_M\leq k\leq 0} |\bar{x}(k)|^2 + \max_{-\delta_M\leq k\leq 0} |\bar{y}(k)|^2\Big).$$

Definition 2.2 If there exists a scalar $\gamma > 0$, system (8) is a robust H_∞ state estimator of GRNs (1) with measurements (5) in the mean square sense with zero initial conditions such that

$$\mathbb{E}\sum_{k=0}^{\infty}\{|\bar{x}(k)|^2 + |\bar{y}(k)|^2\} \leq \gamma^2\mathbb{E}\sum_{k=0}^{\infty}(|\nu_x(k)|^2 + |\nu_y(k)|^2)$$

for all non-zero $\nu_x(k), \nu_y(k) \in L_2[0,\infty)$.

The following lemmas are crucial in implementing our main results.

Lemma 2.3 (see [2, 26]) *Let N and S be real constant matrices; matrix $F(k)$ satisfies $F^T(k)F(k) \leq 1$. Then we have*:

(i) *For any $\epsilon > 0$, $NF(k)S + S^TF^T(k)N^T \leq \epsilon^{-1}NN^T + \epsilon S^TS$.*

(ii) *For any $P > 0$, $\pm 2x^Ty \leq x^TP^{-1}x + y^TPy$.*

Lemma 2.4 *Given the constant matrices $\hat{\Omega}_1$, $\hat{\Omega}_2$, and $\hat{\Omega}_3$, where $\hat{\Omega}_1^T = \hat{\Omega}_1$ and $\hat{\Omega}_2^T = \hat{\Omega}_2 > 0$, then $\hat{\Omega}_1 + \hat{\Omega}_3^T\hat{\Omega}_2^{-1}\hat{\Omega}_3 < 0$, if and only if*

$$\begin{bmatrix} \hat{\Omega}_1 & \hat{\Omega}_3^T \\ \hat{\Omega}_3 & -\hat{\Omega}_2 \end{bmatrix} < 0 \quad or \quad \begin{bmatrix} -\hat{\Omega}_2 & \hat{\Omega}_3 \\ \hat{\Omega}_3^T & \hat{\Omega}_1 \end{bmatrix} < 0.$$

Lemma 2.5 *Let $\mathbb{M} \in \mathbb{R}^{n\times n}$ be a positive semi-definite matrix, $x_i \in \mathbb{R}^n$, and $a_i \geq 0$ $(i = 1,2,\ldots)$. If the series distressed are convergent, the following inequality holds*:

$$\left(\sum_{i=1}^{+\infty} a_ix_i\right)^T \mathbb{M}\left(\sum_{i=1}^{+\infty} a_ix_i\right) \leq \left(\sum_{i=1}^{+\infty} a_i\right)\sum_{i=1}^{+\infty} a_ix_i^T\mathbb{M}x_i.$$

Remark 2.1 In [1] Wang et al. investigated the robust state estimation for stochastic genetic regulatory networks with probabilistic delays in discrete sense, and Lv et al. [4] developed the robust distributed state estimation for genetic regulatory networks with Markovian jumping parameters. However, the inclusion of discrete-interval GRNs with leakage delays, probabilistic measurement delays, noise, and distributed delays has not been taken into account. So, the prime intention of this work is to elucidate that the state estimation problem for the improved system (8) with leakage delays is robustly exponentially stable.

3 Exponential stability criterion

In this part, we first introduce a sufficient condition under which the augmented system (8) is robustly mean-square exponentially stable with the exogenous disturbance signals $\nu_x(k) = 0$ and $\nu_y(k) = 0$.

Theorem 3.1 *Suppose that Assumptions* 1 *and* 2 *hold. Let the leakage delays ρ_1, ρ_2 and the estimation parameters $\mathbb{A}_x$, $\mathbb{B}_x$, $\mathbb{A}_y$, and $\mathbb{B}_y$ be given and also the acceptable conditions hold.*

Then the estimation error system (8) *with* $v_x(k) = 0$ *and* $v_y(k) = 0$ *is robustly exponentially stable in the mean square if there exist positive definite matrices* R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}, R_{51}, R_{52} *and three positive constant scalars* λ, ε_1, *and* ε_2 *such that the following LMI holds*:

$$\Lambda_1 = \begin{bmatrix} \Lambda'_{11} & * & * \\ S_1 & J_1 & * \\ 0 & \bar{T}_1^T & -\varepsilon_1 I \end{bmatrix} < 0, \qquad \Lambda_2 = \begin{bmatrix} \Lambda'_{22} & * & * \\ S_2 & J_2 & * \\ 0 & \bar{T}_2^T & -\varepsilon_2 I \end{bmatrix} < 0, \tag{9}$$

where

$$\Lambda'_{11} = \begin{bmatrix} \psi_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{41} + \varepsilon_1 W_4^T W_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & HR_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I(R_{12} + R_{22}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\xi} R_{52} \end{bmatrix},$$

$$\Lambda'_{22} = \begin{bmatrix} \psi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{42} - \lambda\tilde{N}_1 + \varepsilon_2 W_2^T W_2 & -\lambda\tilde{N}_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda\tilde{N}_2 & -\lambda I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I(R_{11} + R_{21}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mu} R_{51} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\mu} R_{51} \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \bar{\Xi}_{15} & 0 & 0 \\ \Xi_{21} & -\sqrt{2}R_{21}\mathbb{A}_x & \Xi_{23} & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_x M & 0 & \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_x M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\Xi}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\Xi}_{55} & 0 \end{bmatrix},$$

where

$$\bar{\Xi}_{15} = -\sqrt{2}(R_{11} + R_{21})\mathbb{A}; \qquad \bar{\Xi}_{44} = \sqrt{2}(R_{12} + R_{22})\mathbb{D}; \qquad \bar{\Xi}_{55} = \sqrt{2}(R_{12} + R_{22})F;$$

$$\Xi_{21} = \sqrt{2}R_{21}(\mathbb{A}_x - \alpha_0\mathbb{B}_x M); \qquad \Xi_{23} = -\sqrt{2}R_{21}(1 - \alpha_0)\mathbb{B}_x M;$$

$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{16} & 0 & 0 & 0 \\ \Theta_{21} & -\sqrt{2}R_{22}\mathbb{A}_y & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta}R_{22}\mathbb{B}_y N & 0 & \sqrt{\sigma_\beta}R_{22}\mathbb{B}_y N & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\Theta}_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{57} & 0 & 0 \end{bmatrix},$$

where

$$\bar{\Theta}_{16} = -\sqrt{2}(R_{12} + R_{22})\mathbb{C}; \qquad \bar{\Theta}_{45} = \sqrt{2}(R_{11} + R_{21})\mathbb{B}; \qquad \bar{\Theta}_{57} = \sqrt{2}(R_{11} + R_{21})E;$$

$$\Theta_{21} = \sqrt{2}R_{22}(\mathbb{A}_y - \beta_0\mathbb{B}_y N); \qquad \Theta_{23} = -\sqrt{2}R_{22}(1 - \beta_0)\mathbb{B}_y N,$$

$$J_1 = \operatorname{diag}\bigl\{-(R_{11} + R_{21}), -R_{21}, -R_{21}, -(R_{12} + R_{22}), -(R_{12} + R_{22})\bigr\},$$

$$J_2 = \operatorname{diag}\bigl\{-(R_{12} + R_{22}), -R_{22}, -R_{22}, -(R_{11} + R_{21}), -(R_{11} + R_{21})\bigr\},$$

$$\bar{T}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -\sqrt{2}(R_{11}+R_{21})T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}(R_{12}+R_{22})T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{12}+R_{22})T & 0 \end{bmatrix},$$

$$\bar{T}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{2}(R_{12}+R_{22})T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}(R_{11}+R_{21})T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{11}+R_{21})T & 0 & 0 \end{bmatrix},$$

$$\bar{\mu} = \sum_{s=1}^{\infty} \mu_s, \qquad \bar{\xi} = \sum_{n=1}^{\infty} \xi_n,$$

$$\psi_{11} = -R_{11} + R_{31} + (\tau_M - \tau_m + 1)R_{41} + \bar{\xi} R_{52}; \qquad \psi_{12} = -R_{12} + R_{32} + (\delta_M - \delta_m + 1)R_{42},$$

$$\tilde{N}_1 = \frac{(N_1^T N_2 + N_2^T N_1)}{2}; \qquad \tilde{N}_2 = -\frac{(N_1^T + N_2^T)}{2}.$$

Proof Choose a Lyapunov–Krasovskii functional for the augmented system (8):

$$\mathbb{V}(k) = \mathbb{V}_1(k) + \mathbb{V}_2(k) + \mathbb{V}_3(k) + \mathbb{V}_4(k) + \mathbb{V}_5(k) + \mathbb{V}_6(k), \tag{10}$$

where

$$\begin{aligned}
\mathbb{V}_1(k) &= x^T(k)R_{11}x(k) + y^T(k)R_{12}y(k), \\
\mathbb{V}_2(k) &= \tilde{x}^T(k)R_{21}\tilde{x}(k) + \tilde{y}^T(k)R_{22}\tilde{y}(k), \\
\mathbb{V}_3(k) &= x^T(k-1)R_{31}x(k-1) + y^T(k-1)R_{32}y(k-1), \\
\mathbb{V}_4(k) &= \sum_{i=k-\tau(k)}^{k-1} x^T(i)R_{41}x(i) + \sum_{i=k-\delta(k)}^{k-1} y^T(i)R_{42}y(i), \\
\mathbb{V}_5(k) &= \sum_{j=-\tau_M+1}^{-\tau_m} \sum_{i=k+j}^{k-1} x^T(i)R_{41}x(i) + \sum_{j=-\delta_M+1}^{-\delta_m} \sum_{i=k+j}^{k-1} y^T(i)R_{42}y(i), \\
\mathbb{V}_6(k) &= \sum_{i=1}^{\infty} \mu_i \sum_{j=k-i}^{k-1} h^T\big(y(j)\big)R_{51}h\big(y(j)\big) + \sum_{i=1}^{\infty} \xi_i \sum_{j=k-i}^{k-1} x^T(i)R_{52}x(i).
\end{aligned}$$

Calculate the difference of $\mathbb{V}_i(k)$ $(i = 1, 2, \ldots, 6)$ along the trajectories of model (8) with $v_x(k) = 0$, $v_y(k) = 0$ and

$$\mathbb{E}\big\{\Delta\mathbb{V}(k)\big\} = \sum_{i=1}^{6} \mathbb{E}\big\{\mathbb{V}_i(k)\big\}. \tag{11}$$

Now, we have

$$\begin{aligned}
\mathbb{E}\big\{\Delta\mathbb{V}_1(k)\big\} &= \mathbb{E}\big\{\mathbb{V}_1(k+1) - \mathbb{V}_1(k)\big\} \\
&= \mathbb{E}\Big\{\Big[-\big(\mathbb{A} + \Delta\mathbb{A}(k)\big)x(k-\rho_1) + \big(\mathbb{B} + \Delta\mathbb{B}(k)\big)\hat{g}\big(y\big(k-\delta(k)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \big(E + \Delta E(k)\big) \sum_{s=1}^{\infty} \mu_s h\big(y(k-s)\big)\Bigg]^T \\
&\times R_{11}\Bigg[-\big(\mathbb{A} + \Delta\mathbb{A}(k)\big)x(k-\rho_1) + \big(\mathbb{B} + \Delta\mathbb{B}(k)\big)\hat{g}\big(y\big(k-\delta(k)\big) \\
&+ \big(E + \Delta E(k)\big) \sum_{s=1}^{\infty} \mu_s h\big(y(k-s)\big)\Bigg] \\
&+ \sigma^T\big(k, x(k-\rho_1)\big)R_{11}\sigma\big(k, x(k-\rho_1)\big) - x^T(k)R_{11}x(k) - y^T(k)R_{12}y(k) \\
&+ \Bigg[-\big(\mathbb{C} + \Delta\mathbb{C}(k)\big)y(k-\rho_2) + \big(\mathbb{D} + \Delta\mathbb{D}(k)\big)x\big(k-\tau(k)\big) \\
&+ \big(F + \Delta F(k)\big) \sum_{n=1}^{\infty} \xi_n x(k-n)\Bigg]^T \\
&\times R_{12}\Bigg[-\big(\mathbb{C} + \Delta\mathbb{C}(k)\big)y(k-\rho_2) + \big(\mathbb{D} + \Delta\mathbb{D}(k)\big)x\big(k-\tau(k)\big) \\
&+ \big(F + \Delta F(k)\big) \sum_{n=1}^{\infty} \xi_n x(k-n)\Bigg]\Bigg\},
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\mathbb{E}\big\{\Delta\mathbb{V}_2(k)\big\} &= \mathbb{E}\big\{\mathbb{V}_2(k+1) - \mathbb{V}_2(k)\big\} \\
&= \mathbb{E}\Bigg\{\Bigg[-\big(\mathbb{A} + \Delta\mathbb{A}(k)\big)x(k-\rho_1) + (\mathbb{A}_x - \alpha_k\mathbb{B}_x M)x(k) \\
&\quad + \big(\mathbb{B} + \Delta\mathbb{B}(k)\big)\hat{g}\big(y\big(k-\delta(k)\big)\big) + \big(E + \Delta E(k)\big) \sum_{s=1}^{\infty} \mu_s h\big(y(k-s)\big) \\
&\quad - \mathbb{A}_x\tilde{x}(k) - (1-\alpha_k)\mathbb{B}_x Mx(k-1)\Bigg]^T \\
&\quad \times R_{21}\Bigg[-\big(\mathbb{A} + \Delta\mathbb{A}(k)\big)x(k-\rho_1) + (\mathbb{A}_x - \alpha_k\mathbb{B}_x M)x(k) \\
&\quad + \big(\mathbb{B} + \Delta\mathbb{B}(k)\big)\hat{g}\big(y\big(k-\delta(k)\big)\big) + \big(E + \Delta E(k)\big) \sum_{s=1}^{\infty} \mu_s h\big(y(k-s)\big) \\
&\quad - \mathbb{A}_x\tilde{x}(k) - (1-\alpha_k)\mathbb{B}_x Mx(k-1)\Bigg] \\
&\quad + \sigma_\alpha\big[\mathbb{B}_x Mx(k) + \mathbb{B}_x Mx(k-1)\big]^T R_{21}\big[\mathbb{B}_x Mx(k) + \mathbb{B}_x Mx(k-1)\big] \\
&\quad - \tilde{x}^T(k)R_{21}\tilde{x}(k) + \Bigg[-\big(\mathbb{C} + \Delta\mathbb{C}(k)\big)y(k-\rho_2) + (\mathbb{A}_y - \beta_k\mathbb{B}_y N)y(k) \\
&\quad + \big(\mathbb{D} + \Delta\mathbb{D}(k)\big)x\big(k-\tau(k)\big) + \big(F + \Delta F(k)\big) \sum_{n=1}^{\infty} \xi_n x\big((k-n)\big) \\
&\quad - \mathbb{A}_y\tilde{y}(k) - (1-\beta_k)\mathbb{B}_y Ny(k-1)\Bigg]^T
\end{aligned}
$$

$$\begin{aligned}
&\times R_{22}\Bigg[-\big(\mathbb{C}+\Delta\mathbb{C}(k)\big)y(k-\rho_2)+(\mathbb{A}_y-\beta_k\mathbb{B}_yN)y(k) \\
&+\big(\mathbb{D}+\Delta\mathbb{D}(k)\big)x\big(k-\tau(k)\big)+\big(F+\Delta F(k)\big)\sum_{n=1}^{\infty}\xi_n x\big((k-n)\big) \\
&-\mathbb{A}_y\tilde{y}(k)-(1-\beta_k)\mathbb{B}_yNy(k-1)\Bigg] \\
&+\sigma_\beta\big[\mathbb{B}_yNy(k)+\mathbb{B}_yNy(k-1)\big]^TR_{22}\big[\mathbb{B}_yNy(k)+\mathbb{B}_yNy(k-1)\big] \\
&-\tilde{y}^T(k)R_{22}\tilde{y}(k)\Bigg\},
\end{aligned} \tag{13}$$

$$\begin{aligned}
\mathbb{E}\big\{\Delta\mathbb{V}_3(k)\big\}&=\mathbb{E}\big\{\mathbb{V}_3(k+1)-\mathbb{V}_3(k)\big\} \\
&=\mathbb{E}\big\{x^T(k)R_{31}x(k)-x^T(k-1)R_{31}x(k-1) \\
&\quad+y^T(k)R_{32}y(k)-y^T(k-1)R_{32}y(k-1)\big\},
\end{aligned} \tag{14}$$

$$\begin{aligned}
\mathbb{E}\big\{\Delta\mathbb{V}_4(k)\big\}&=\mathbb{E}\big\{\mathbb{V}_4(k+1)-\mathbb{V}_4(k)\big\} \\
&\leq\mathbb{E}\Bigg\{x^T(k)R_{41}x(k)-x^T\big(k-\tau(k)\big)R_{41}x\big(k-\tau(k)\big) \\
&\quad+\sum_{i=k-\tau_M+1}^{k-\tau_m}x^T(i)R_{41}x(i) \\
&\quad+y^T(k)R_{42}y(k)-y^T\big(k-\delta(k)\big)R_{42}y\big(k-\delta(k)\big) \\
&\quad+\sum_{i=k-\delta_M+1}^{k-\delta m}y^T(i)R_{42}y(i)\Bigg\},
\end{aligned} \tag{15}$$

$$\begin{aligned}
\mathbb{E}\big\{\Delta\mathbb{V}_5(k)\big\}&=\mathbb{E}\big\{\mathbb{V}_5(k+1)-\mathbb{V}_5(k)\big\} \\
&=\mathbb{E}\Bigg\{(\tau_M-\tau_m)x^T(k)R_{41}x(k)-\sum_{i=k-\tau_M+1}^{k-\tau_m}x^T(i)R_{41}x(i) \\
&\quad+(\delta_M-\delta_m)y^T(k)R_{42}y(k)-\sum_{i=k-\delta_M+1}^{k-\delta m}y^T(i)R_{42}y(i)\Bigg\},
\end{aligned} \tag{16}$$

$$\begin{aligned}
\mathbb{E}\big\{\Delta\mathbb{V}_6(k)\big\}&=\mathbb{E}\big\{\mathbb{V}_6(k+1)-\mathbb{V}_6(k)\big\} \\
&=\sum_{i=1}^{\infty}\mu_i\sum_{j=k+1-i}^{k+1-1}h^T\big(y(j)\big)R_{51}h\big(y(j)\big)+\sum_{i=1}^{\infty}\xi_i\sum_{j=k+1-i}^{k+1-1}x^T(i)R_{52}x(i) \\
&\quad-\sum_{i=1}^{\infty}\mu_i\sum_{j=k-i}^{k-1}h^T\big(y(j)\big)R_{51}h\big(y(j)\big)-\sum_{i=1}^{\infty}\xi_i\sum_{j=k-i}^{k-1}x^T(i)R_{52}x(i) \\
&=\sum_{i=1}^{\infty}\mu_i\big[h^T\big(y(k)\big)R_{51}h\big(y(k)\big)-h^T\big(y(k-i)\big)R_{51}h\big(y(k-i)\big)\big] \\
&\quad+\sum_{i=1}^{\infty}\xi_i\big[x^T(k)R_{52}x(k)-x^T(k-i)R_{52}x(k-i)\big].
\end{aligned}$$

Using Lemma 2.5, we get

$$\begin{aligned}\mathbb{E}\{\Delta\mathbb{V}_6(k)\} &\le \bar{\mu}h^T\big(y(k)\big)R_{51}h\big(y(k)\big) - \bar{\mu}\big[\bar{\mu}h\big(y(k-s)\big)\big]^T R_{51}\big[\bar{\mu}h\big(y(k-s)\big)\big] \\ &\quad + \bar{\xi}x^T(k)R_{52}x(k) - \bar{\xi}\big[\bar{\xi}x(k-n)\big]^T R_{52}\big[\bar{\xi}x(k-n)\big]. \end{aligned} \tag{17}$$

Substituting equations (12)–(17) into equation (11) results in

$$\begin{aligned}\mathbb{E}\{\Delta\mathbb{V}(k)\} \le \mathbb{E}\big\{&\varpi_0^T(k)\big[\Lambda_{11} + \sigma_\alpha\hat{W}_{01}^T R_{21}\hat{W}_{01} + 2\hat{G}_{01}^T(k)(R_{11}+R_{21})\hat{G}_{01}(k) \\ &+ 2\hat{F}_{01}^T(k)R_{21}\hat{F}_{01}(k) + 2\hat{G}_{11}^T(k)(R_{12}+R_{22})\hat{G}_{11}(k) \\ &+ 2\hat{S}_{01}^T(k)(R_{12}+R_{22})\hat{S}_{01}(k)\big]\varpi_0(k) \\ &+ \Gamma_0^T(k)\big[\Lambda_{12} + \sigma_\beta\hat{W}_{02}^T R_{22}\hat{W}_{02} + 2\hat{G}_{02}^T(k)(R_{12}+R_{22})\hat{G}_{02}(k) \\ &+ 2\hat{F}_{02}^T(k)R_{22}\hat{F}_{02}(k) + 2\hat{G}_{12}^T(k)(R_{11}+R_{21})\hat{G}_{12}(k) \\ &+ 2\hat{S}_{02}^T(k)(R_{11}+R_{21})\hat{S}_{02}(k)\big]\Gamma_0(k)\big\}, \end{aligned} \tag{18}$$

where

$$\begin{aligned}\varpi_0(k) &= \big[x^T(k), \tilde{x}^T(k), x^T(k-1), x^T\big(k-\tau(k)\big), x^T(k-\rho_1), x^T(k-n), \big[\bar{\xi}x(k-n)\big]^T\big], \\ \Gamma_0(k) &= \big[y^T(k), \tilde{y}^T(k), y^T(k-1), y^T\big(k-\delta(k)\big), g^T\big(y\big(k-\delta(k)\big)\big), y^T(k-\rho_2), h^T\big(y(k-s)\big), \\ &\quad h^T(k), \big[\bar{\mu}h\big(y(k-s)\big)\big]^T\big],\end{aligned}$$

where

$$\Lambda_{11} = \begin{bmatrix} \psi_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{41} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & HR_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I(R_{12}+R_{22}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\xi}R_{52} \end{bmatrix},$$

$$\Lambda_{12} = \begin{bmatrix} \psi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{42} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I(R_{11}+R_{21}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mu}R_{51} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\mu}R_{51} \end{bmatrix},$$

$$\psi_{11} = -R_{11} + R_{31} + (\tau_M - \tau_m + 1)R_{41} + \bar{\xi}R_{52}; \qquad \psi_{12} = -R_{12} + R_{32} + (\delta_M - \delta_m + 1)R_{42},$$

$$\hat{W}_{01} = [\mathbb{B}_x M, 0, \mathbb{B}_x M, 0, 0, 0, 0]; \qquad \hat{W}_{02} = [\mathbb{B}_y N, 0, \mathbb{B}_y N, 0, 0, 0, 0, 0, 0],$$

$$\hat{G}_{01}(k) = \big[0, 0, 0, 0, -\big(\mathbb{A} + \Delta\mathbb{A}(k)\big), 0, 0\big];$$

$$\hat{G}_{02}(k) = \big[0,0,0,0,0,-\big(\mathbb{C}+\Delta\mathbb{C}(k)\big),0,0,0\big],$$
$$\hat{F}_{01}(k) = \big[\mathbb{A}_x - \alpha_0\mathbb{B}_x M, -\mathbb{A}_x, -(1-\alpha_0)\mathbb{B}_x M, 0,0,0,0\big];$$
$$\hat{F}_{02}(k) = \big[\mathbb{A}_y - \beta_0\mathbb{B}_y N, -\mathbb{A}_y, -(1-\beta_0)\mathbb{B}_y N, 0,0,0,0,0,0\big],$$
$$\hat{G}_{11}(k) = \big[0,0,0,\big(\mathbb{D}+\Delta\mathbb{D}(k)\big),0,0,0\big]; \qquad \hat{G}_{12}(k) = \big[0,0,0,0,\big(\mathbb{B}+\Delta\mathbb{B}(k)\big),0,0,0,0\big],$$
$$\hat{S}_{01}(k) = \big[0,0,0,0,0,\big(F+\Delta F(k)\big),0\big]; \qquad \hat{S}_{02}(k) = \big[0,0,0,0,0,0,\big(E+\Delta E(k)\big),0,0\big].$$

From Assumption 2, we have

$$\begin{bmatrix} y(k-\delta(k)) \\ \hat{g}(y(k-\delta(k))) \end{bmatrix}^T \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_2^T & I \end{bmatrix} \begin{bmatrix} y(k-\delta(k)) \\ \hat{g}(y(k-\delta(k))) \end{bmatrix} \leq 0, \tag{19}$$

where

$$\tilde{N}_1 = \frac{(N_1^T N_2 + N_2^T N_1)}{2}; \qquad \tilde{N}_2 = -\frac{(N_1^T + N_2^T)}{2}.$$

Then, from equations (18) and (19), we have

$$\begin{aligned}
\mathbb{E}\big\{\Delta V(k)\big\} &\leq \mathbb{E}\big\{\Delta V(k)\big\} - \mathbb{E}\left\{\lambda \begin{bmatrix} y(k-\delta(k)) \\ g(y(k-\delta(k))) \end{bmatrix}^T \begin{bmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_2^T & I \end{bmatrix} \begin{bmatrix} y(k-\delta(k)) \\ g(y(k-\delta(k))) \end{bmatrix}\right\} \\
&= \mathbb{E}\big\{\varpi_0^T(k)\big[\Lambda_{11} + \sigma_\alpha \hat{W}_{01}^T R_{21}\hat{W}_{01} + 2\hat{G}_{01}^T(k)(R_{11}+R_{21})\hat{G}_{01}(k) \\
&\quad + 2\hat{F}_{01}^T(k)R_{21}\hat{F}_{01}(k) + 2\hat{G}_{11}^T(k)(R_{12}+R_{22})\hat{G}_{11}(k) \\
&\quad + 2\hat{S}_{01}^T(k)(R_{12}+R_{22})\hat{S}_{01}(k)\big]\varpi_0(k) \\
&\quad + \Gamma_0^T(k)\big[\Lambda_{22} + \sigma_\beta \hat{W}_{02}^T R_{22}\hat{W}_{02} + 2\hat{G}_{02}^T(k)(R_{12}+R_{22})\hat{G}_{02}(k) \\
&\quad + 2\hat{F}_{02}^T(k)R_{22}\hat{F}_{02}(k) + 2\hat{G}_{12}^T(k)(R_{11}+R_{21})\hat{G}_{12}(k) \\
&\quad + 2\hat{S}_{02}^T(k)(R_{11}+P_{21})\hat{S}_{02}(k)\big]\Gamma_0(k)\big\},
\end{aligned} \tag{20}$$

where

$$\Lambda_{22} = \begin{bmatrix}
\psi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -R_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -R_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -R_{42}-\lambda\tilde{N}_1 & -\lambda\tilde{N}_2^T & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\lambda\tilde{N}_2 & -\lambda I & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & I(R_{11}+R_{21}) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mu}R_{51} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\mu}R_{51}
\end{bmatrix}.$$

Notice that, since $\Lambda_1 < 0$ and $\Lambda_2 < 0$, there are two scalars $\mu_1 > 0$ and $\mu_2 > 0$ such that

$$\begin{aligned}
\hat{\Lambda}_1 &= \Lambda_1 + \mu_1 \begin{bmatrix} I_{2n\times 2n} & 0 \\ 0 & 0 \end{bmatrix} < 0, \\
\hat{\Lambda}_2 &= \Lambda_2 + \mu_2 \begin{bmatrix} I_{2n\times 2n} & 0 \\ 0 & 0 \end{bmatrix} < 0.
\end{aligned} \tag{21}$$

Equation (21) implies

$$\begin{aligned}
&\Lambda_{11} + \mu_1 \begin{bmatrix} I_{2n\times 2n} & 0 \\ 0 & 0 \end{bmatrix} + \sigma_\alpha \hat{W}_{01}^T R_{21} \hat{W}_{01} + 2\hat{G}_{01}^T(k)(R_{11} + R_{21})\hat{G}_{01}(k) + 2\hat{F}_{01}^T(k) R_{21} \hat{F}_{01}(k) \\
&\quad + 2\hat{G}_{11}^T(k)(R_{12} + R_{22})\hat{G}_{11}(k) + 2\hat{S}_{01}^T(k)(R_{12} + R_{22})\hat{S}_{01}(k) < 0, \\
&\Lambda_{22} + \mu_2 \begin{bmatrix} I_{2n\times 2n} & 0 \\ 0 & 0 \end{bmatrix} + \sigma_\beta \hat{W}_{02}^T R_{22} \hat{W}_{02} + 2\hat{G}_{02}^T(k)(R_{12} + R_{22})\hat{G}_{02}(k) + 2\hat{F}_{02}^T(k) R_{22} \hat{F}_{02}(k) \\
&\quad + 2\hat{G}_{12}^T(k)(R_{11} + R_{21})\hat{G}_{12}(k) + 2\hat{S}_{02}^T(k)(R_{11} + R_{21})\hat{S}_{02}(k) < 0.
\end{aligned} \tag{22}$$

First we satisfy (21) before proving the exponential stability. Using Lemma 2.4, the above equalities are equivalent to

$$\Lambda_3(k) = \begin{bmatrix} \hat{\Lambda}_{11} & * \\ S_1(k) & J_1 \end{bmatrix} < 0, \qquad \Lambda_4(k) = \begin{bmatrix} \hat{\Lambda}_{22} & * \\ S_2(k) & J_2 \end{bmatrix} < 0, \tag{23}$$

where

$$\begin{aligned}
\hat{\Lambda}_{11} &= \Lambda_{11} + \mu_1 \begin{bmatrix} I_{2n\times 2n} & 0 \\ 0 & 0 \end{bmatrix}, \\
\hat{\Lambda}_{22} &= \Lambda_{22} + \mu_2 \begin{bmatrix} I_{2n\times 2n} & 0 \\ 0 & 0 \end{bmatrix}, \\
S_1(k) &= \begin{bmatrix} \sqrt{2}(R_{11} + R_{21})\hat{G}_{01}(k) \\ \sqrt{2} R_{21} \hat{F}_{01}(k) \\ \sqrt{\sigma_\alpha} R_{21} \hat{W}_{01} \\ \sqrt{2}(R_{12} + R_{22})\hat{G}_{11}(k) \\ \sqrt{2}(R_{12} + R_{22})\hat{S}_{01}(k) \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 & \Xi_{15} & 0 & 0 \\ \Xi_{21} & -\sqrt{2} R_{21} \mathbb{A}_x & \Xi_{23} & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha} R_{21} \mathbb{B}_x M & 0 & \sqrt{\sigma_\alpha} R_{21} \mathbb{B}_x M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \Xi_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Xi_{55} & 0 \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
\Xi_{15} &= -\sqrt{2}(R_{11} + R_{21})\bigl(\mathbb{A} + \Delta\mathbb{A}(k)\bigr); \\
\Xi_{21} &= \sqrt{2} R_{21}(\mathbb{A}_x - \alpha_0 \mathbb{B}_x M); \\
\Xi_{23} &= -\sqrt{2} R_{21}(1 - \alpha_0)\mathbb{B}_x M; \\
\Xi_{44} &= \sqrt{2}(R_{12} + R_{22})\bigl(\mathbb{D} + \Delta\mathbb{D}(k)\bigr); \\
\Xi_{55} &= \sqrt{2}(R_{12} + R_{22})\bigl(F + \Delta F(k)\bigr),
\end{aligned}$$

$$S_2(k) = \begin{bmatrix} \sqrt{2}(R_{12}+R_{22})\hat{G}_{02}(k) \\ \sqrt{2}R_{22}\hat{F}_{02}(k) \\ \sqrt{\sigma_\beta}R_{22}\hat{W}_{02} \\ \sqrt{2}(R_{11}+R_{21})\hat{G}_{12}(k) \\ \sqrt{2}(R_{11}+R_{21})\hat{S}_{02}(k) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \Theta_{16} & 0 & 0 & 0 \\ \Theta_{21} & -\sqrt{2}R_{22}\mathbb{A}_y & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta}R_{22}\mathbb{B}_yN & 0 & \sqrt{\sigma_\beta}R_{22}\mathbb{B}_yN & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Theta_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Theta_{57} & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} &\Theta_{16} = -\sqrt{2}(R_{12}+R_{22})\big(\mathbb{C}+\Delta\mathbb{C}(k)\big); \qquad \Theta_{21} = \sqrt{2}R_{22}(\mathbb{A}_y-\beta_0\mathbb{B}_yN); \\ &\Theta_{23} = -\sqrt{2}R_{22}(1-\beta_0)\mathbb{B}_yN; \qquad \Theta_{45} = \sqrt{2}(R_{11}+R_{21})\big(\mathbb{B}+\Delta\mathbb{B}(k)\big); \\ &\Theta_{57} = \sqrt{2}(R_{11}+R_{21})\big(E+\Delta E(k)\big), \\ &J_1 = \operatorname{diag}\big\{-(R_{11}+R_{21}), -R_{21}, -R_{21}, -(R_{12}+R_{22}), -(R_{12}+R_{22})\big\}, \\ &J_2 = \operatorname{diag}\big\{-(R_{12}+R_{22}), -R_{22}, -R_{22}, -(R_{11}+R_{21}), -(R_{11}+R_{21})\big\}. \end{aligned}$$

Note that $S_1(k)$ and $S_2(k)$ can be decomposed as

$$\begin{aligned} &S_1(k) = S_1 + \Delta S_1(k), \\ &S_2(k) = S_2 + \Delta S_2(k), \end{aligned} \tag{24}$$

where

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \bar{\Xi}_{15} & 0 & 0 \\ \Xi_{21} & -\sqrt{2}R_{21}\mathbb{A}_x & \Xi_{23} & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_xM & 0 & \sqrt{\sigma_\alpha}R_{21}\mathbb{B}_xM & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\Xi}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\Xi}_{55} & 0 \end{bmatrix},$$

$$\bar{\Xi}_{15} = -\sqrt{2}(R_{11}+R_{21})\mathbb{A}; \qquad \bar{\Xi}_{44} = \sqrt{2}(R_{12}+R_{22})\mathbb{D};$$

$$\bar{\Xi}_{55} = \sqrt{2}(R_{12}+R_{22})F,$$

$$\Delta S_1(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & -\sqrt{2}(R_{11}+R_{21})\Delta\mathbb{A}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}(R_{12}+R_{22})\Delta\mathbb{D}(k) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{12}+R_{22})\Delta\mathbb{F}(k) & 0 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{16} & 0 & 0 & 0 \\ \Theta_{21} & -\sqrt{2}R_{22}\mathbb{A}_y & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta}R_{22}\mathbb{B}_yN & 0 & \sqrt{\sigma_\beta}R_{22}\mathbb{B}_yN & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\Theta}_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{57} & 0 & 0 \end{bmatrix},$$

where

$$\bar{\Theta}_{16} = -\sqrt{2}(R_{12}+R_{22})\mathbb{C}; \qquad \bar{\Theta}_{45} = \sqrt{2}(R_{11}+R_{21})\mathbb{B}; \qquad \bar{\Theta}_{57} = \sqrt{2}(R_{11}+R_{21})E,$$

$$\Delta S_2(k) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \kappa_{16} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \kappa_{57} & 0 & 0 \end{bmatrix},$$

where $\kappa_{16} = -\sqrt{2}(R_{12}+R_{22})\Delta\mathbb{C}(k)$, $\kappa_{45} = \sqrt{2}(R_{11}+R_{21})\Delta\mathbb{B}(k)$, $\kappa_{57} = \sqrt{2}(R_{11}+R_{21})\Delta E(k)$.

From Assumption 1, it follows readily that

$$\Delta S_1(k) = \bar{T}_1 N(k)\bar{W}_1, \qquad \Delta S_2(k) = \bar{T}_2 N(k)\bar{W}_2, \tag{25}$$

where

$$\bar{T}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & -\sqrt{2}(R_{11}+R_{21})T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}(R_{12}+R_{22})T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{12}+R_{22})T & 0 \end{bmatrix},$$

$$\bar{T}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\sqrt{2}(R_{12}+R_{22})T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}(R_{11}+R_{21})T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}(R_{11}+R_{21})T & 0 & 0 \end{bmatrix},$$

$$\bar{W}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & W_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W_6 & 0 \end{bmatrix},$$

$$\bar{W}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & W_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_5 & 0 & 0 \end{bmatrix}.$$

Note that $\Lambda_3(k)$ and $\Lambda_4(k)$ can be decomposed as follows:

$$\Lambda_3(k) = \Lambda_3 + \Delta\Lambda_3(k), \qquad \Lambda_4(k) = \Lambda_4 + \Delta\Lambda_4(k), \tag{26}$$

where

$$\Lambda_3 = \begin{bmatrix} \hat{\Lambda}_{11} & * \\ S_1 & J_1 \end{bmatrix} < 0, \qquad \Delta\Lambda_3(k) = \begin{bmatrix} 0 & * \\ \Delta S_1(k) & 0 \end{bmatrix},$$

$$\Lambda_4 = \begin{bmatrix} \hat{\Lambda}_{22} & * \\ S_2 & J_2 \end{bmatrix} < 0 \quad \text{and} \quad \Delta\Lambda_4(k) = \begin{bmatrix} 0 & * \\ \Delta S_2(k) & 0 \end{bmatrix}.$$

Let

$$\tilde{T}_1^T = [0, \bar{T}_1^T], \qquad \tilde{W}_1 = [\bar{W}_1, 0],$$
$$\tilde{T}_2^T = [0, \bar{T}_2^T], \qquad \tilde{W}_2 = [\bar{W}_2, 0].$$

Using Lemma 2.3(i), $\Delta\Lambda_3(k)$ and $\Delta\Lambda_4(k)$ can be rewritten as

$$\begin{aligned}\Delta\Lambda_3(k) &= \tilde{T}_1 N(k)\tilde{W}_1 + \tilde{W}_1^T N^T(k)\tilde{T}_1^T \le \varepsilon_1^{-1}\tilde{T}_1\tilde{T}_1^T + \varepsilon_1\tilde{W}_1^T\tilde{W}_1,\\ \Delta\Lambda_4(k) &= \tilde{T}_2 N(k)\tilde{W}_2 + \tilde{W}_2^T N^T(k)\tilde{T}_2^T \le \varepsilon_2^{-1}\tilde{T}_2\tilde{T}_2^T + \varepsilon_2\tilde{W}_2^T\tilde{W}_2.\end{aligned} \tag{27}$$

It is clear from equations (26) and (27) that

$$\Lambda_3(k) \le \Lambda_3' + \varepsilon_1^{-1}\tilde{T}_1\tilde{T}_1^T, \qquad \Lambda_4(k) \le \Lambda_4' + \varepsilon_2^{-1}\tilde{T}_2\tilde{T}_2^T, \tag{28}$$

where

$$\Lambda_3' = \begin{bmatrix} \Lambda_{11}' + \mu_1 \left[\begin{smallmatrix} I_{2n\times 2n} & 0\\ 0 & 0\end{smallmatrix}\right] & * \\ S_1 & J_1 \end{bmatrix},$$
$$\Lambda_4' = \begin{bmatrix} \Lambda_{22}' + \mu_2 \left[\begin{smallmatrix} I_{2n\times 2n} & 0\\ 0 & 0\end{smallmatrix}\right] & * \\ S_2 & J_2 \end{bmatrix}.$$

It follows from Lemma 2.4 that equation (22) is equivalent to the case that the right-hand side of equation (28) is negative definite. Hence, we come to the conclusion that $\Lambda_3(k) < 0$ and $\Lambda_4(k) < 0$, and therefore equation (22) holds. Moreover, the combination of equations (20) and (22) leads to

$$\mathbb{E}\{\Delta\mathbb{V}(k)\} \le \mu_1\mathbb{E}\{|\bar{x}(k)|^2\} - \mu_2\mathbb{E}\{|\bar{y}(k)|^2\}. \tag{29}$$

We are in a position to prove the stability of system (8). First, from equation (10), it is easily verified that

$$\begin{aligned}\mathbb{E}\{\Delta V(k)\} \le{}& \varepsilon_{11}\mathbb{E}\{|\bar{x}(k)|^2\} + \varepsilon_{21}\sum_{i=k-\tau_M}^{k-1}\mathbb{E}\{|\bar{x}(i)|^2\}\\ &+ \varepsilon_{12}\mathbb{E}\{|\bar{y}(k)|^2\} + \varepsilon_{22}\sum_{i=k-\delta_M}^{k-1}\mathbb{E}\{|\bar{y}(i)|^2\},\end{aligned} \tag{30}$$

where

$$\begin{aligned}\varepsilon_{11} &= \max\{\lambda_{\max}(R_{11}), \lambda_{\max}(R_{21}), \lambda_{\max}(R_{52})\},\\ \varepsilon_{21} &= (\tau_M - \tau_m + 1)(\lambda_{\max}(R_{31}) + \lambda_{\max}(R_{41})),\\ \varepsilon_{12} &= \max\{\lambda_{\max}(R_{12}), \lambda_{\max}(R_{22}), \lambda_{\max}(R_{51})\},\\ \varepsilon_{22} &= (\delta_M - \delta_m + 1)(\lambda_{\max}(R_{32}) + \lambda_{\max}(R_{42})).\end{aligned}$$

For any scalar $\zeta > 1$, the above inequality, combined with equation (29), indicates that

$$\begin{aligned}
&\zeta^{k+1}\mathbb{E}\{\mathbb{V}(k+1)\} - \zeta^{k}\mathbb{E}(\mathbb{V}(k)) = \zeta^{k+1}\mathbb{E}\{\Delta\mathbb{V}(k)\} + \zeta^{k}(\zeta-1)\mathbb{E}\{\mathbb{V}(k)\} \\
&\qquad \leq -\zeta^{k+1}\big(\mu_1\mathbb{E}\{|\bar{x}(k)|^2\} - \mu_2\mathbb{E}\{|\bar{y}(k)|^2\}\big) + \zeta^{k}(\zeta-1), \\
&\left(\varepsilon_{11}\mathbb{E}\{|\bar{x}(k)|^2\} + \varepsilon_{21}\sum_{i=k-\tau_M}^{k-1}\mathbb{E}\{|\bar{x}(i)|^2\} + \varepsilon_{12}\mathbb{E}\{|\bar{y}(k)|^2\} + \varepsilon_{22}\mathbb{E}\{|\bar{y}(i)|^2\}\right) \\
&\quad = \zeta^{k}\eta_{11}(\zeta)\mathbb{E}\{|\bar{x}(k)|^2\} + \zeta^{k}\eta_{21}(\zeta)\sum_{i=k-\tau_M}^{k-1}\mathbb{E}\{|\bar{x}(i)|^2\} \\
&\qquad + \zeta^{k}\eta_{12}(\zeta)\mathbb{E}\{|\bar{y}(k)|^2\} + \zeta^{k}\eta_{22}(\zeta)\sum_{i=k-\delta_M}^{k-1}\mathbb{E}\{|\bar{y}(i)|^2\},
\end{aligned} \tag{31}$$

where

$$\begin{aligned}
&\eta_{11}(\zeta) = -\zeta\mu_1 + (\zeta-1)\varepsilon_{11}, \qquad \eta_{21}(\zeta) = (\zeta-1)\varepsilon_{21}, \\
&\eta_{12}(\zeta) = -\zeta\mu_2 + (\zeta-1)\varepsilon_{12} \quad \text{and} \quad \eta_{22}(\zeta) = (\zeta-1)\varepsilon_{22}.
\end{aligned}$$

In addition, for any integer $N \geq \max\{\delta_M, \tau_M\} + 1$, summing both sides of equation (31) from 0 to $N-1$ with respect to k, we have

$$\begin{aligned}
&\zeta^{N}\mathbb{E}\{\mathbb{V}(N)\} - \mathbb{E}\{\mathbb{V}(0)\} \\
&\quad \leq \eta_{11}(\zeta)\sum_{k=0}^{N-1}\zeta^{k}\mathbb{E}\{|\bar{x}(k)|^2\} + \eta_{21}(\zeta)\sum_{k=0}^{N-1}\sum_{i=k-\tau_M}^{k-1}\zeta^{k}\mathbb{E}\{|\bar{x}(i)|^2\} \\
&\qquad + \eta_{12}(\zeta)\sum_{k=0}^{N-1}\zeta^{k}\mathbb{E}\{|\bar{y}(k)|^2\} + \eta_{22}(\zeta)\sum_{k=0}^{N-1}\sum_{i=k-\delta_M}^{k-1}\zeta^{k}\mathbb{E}\{|\bar{y}(i)|^2\}.
\end{aligned} \tag{32}$$

Note that, for $\tau_M, \delta_M \geq 1$,

$$\begin{aligned}
&\sum_{k=0}^{N-1}\sum_{i=k-\tau_M}^{k-1}\zeta^{k}\mathbb{E}\{|\bar{x}(i)|^2\} \leq \tau_M\zeta^{\tau_M}\max_{-\tau_M\leq i\leq 0}\mathbb{E}\{|\Omega(i)|^2\} + \tau_M\zeta^{\tau_M}\sum_{i=0}^{N-1}\zeta^{i}\mathbb{E}\{|\bar{x}(k)|^2\}, \\
&\sum_{k=0}^{N-1}\sum_{i=k-\delta_M}^{k-1}\zeta^{k}\mathbb{E}\{|\bar{y}(i)|^2\} \leq \delta_M\zeta^{\delta_M}\max_{-\delta_M\leq i\leq 0}\mathbb{E}\{|\Pi(i)|^2\} + \delta_M\zeta^{\delta_M}\sum_{i=0}^{N-1}\zeta^{i}\mathbb{E}\{|\bar{y}(k)|^2\}.
\end{aligned} \tag{33}$$

Then, from equations (32) and (33), one has

$$\begin{aligned}
\zeta^{N}\mathbb{E}\{\mathbb{V}(N)\} &\leq \mathbb{E}\{\mathbb{V}(0)\} + \big[\eta_{11}(\zeta) + \tau_M\zeta^{\tau_M}\eta_{21}(\zeta)\big]\sum_{k=0}^{N-1}\zeta^{k}\mathbb{E}\{|\bar{x}(k)|^2\} \\
&\quad + \tau_M\zeta^{\tau_M}\eta_{21}(\zeta)\max_{-\tau_M\leq i\leq 0}\mathbb{E}\{|\Omega(i)|^2\} \\
&\quad + \big[\eta_{12}(\zeta) + \delta_M\zeta^{\delta_M}\eta_{22}(\zeta)\big]\sum_{k=0}^{N-1}\zeta^{k}\mathbb{E}\{|\bar{y}(k)|^2\} \\
&\quad + \delta_M\zeta^{\delta_M}\eta_{22}(\zeta)\max_{-\delta_M\leq i\leq 0}\mathbb{E}\{|\Pi(i)|^2\}.
\end{aligned} \tag{34}$$

Let

$$\varepsilon_{01} = \min\{\lambda_{\min}(R_{11}), \lambda_{\min}(R_{21}), \lambda_{\min}(R_{52})\}, \qquad \tilde{\varepsilon}_1 = \max\{\varepsilon_{11}, \varepsilon_{21}\},$$
$$\varepsilon_{02} = \min\{\lambda_{\min}(R_{12}), \lambda_{\min}(R_{22}), \lambda_{\min}(R_{51})\}, \qquad \tilde{\varepsilon}_2 = \max\{\varepsilon_{12}, \varepsilon_{22}\}.$$

It is clear that

$$\mathbb{E}\{\mathbb{V}(N)\} \geq \varepsilon_{01}\mathbb{E}\{|\bar{x}(N)|^2\} + \varepsilon_{02}\mathbb{E}\{|\bar{y}(N)|^2\}. \tag{35}$$

It follows readily from equation (30) that

$$\mathbb{E}\{\mathbb{V}(0)\} \leq \tilde{\varepsilon}_1 \max_{-\tau_M \leq i \leq 0} \mathbb{E}\{|\Omega(i)|^2\} + \tilde{\varepsilon}_2 \max_{-\delta_M \leq i \leq 0} \mathbb{E}\{|\Pi(i)|^2\}. \tag{36}$$

Additionally, it can be verified that there exists a scalar $\zeta_0 > 1$ such that

$$\begin{aligned} &\eta_{11}(\zeta_0) + \tau_M \zeta_0^{\tau_M} \eta_{21}(\zeta_0) = 0, \\ &\eta_{12}(\zeta_0) + \delta_M \zeta_0^{\delta_M} \eta_{22}(\zeta_0) = 0. \end{aligned} \tag{37}$$

Substituting equations (35)–(37) into equation (34), we can get

$$\begin{aligned} \varepsilon_{01}\mathbb{E}\{|\bar{x}(N)|^2\} + \varepsilon_{02}\mathbb{E}\{|\bar{y}(N)|^2\} &\leq (\tilde{\varepsilon}_1 + \tau_M \zeta_0^{\tau_M} \eta_{21}(\zeta_0)) \max_{-\tau_M \leq i \leq 0} \mathbb{E}\{|\Omega(i)|^2\} \\ &\quad + (\tilde{\varepsilon}_2 + \delta_M \zeta_0^{\delta_M} \eta_{22}(\zeta_0)) \max_{-\delta_M \leq i \leq 0} \mathbb{E}\{|\Pi(i)|^2\}. \end{aligned} \tag{38}$$

The above equation (38) completes the proof of exponential stability with $v_x(k) = 0$ and $v_y(k) = 0$. □

Remark 3.1 In this paper, we have considered the time-varying delays $\delta(k)$, $\tau(k)$ and the leakage delays ρ_1, ρ_2 in the negative feedback term of the GRNs which lead to the instability of the systems with small amount of leakage delay. This paper is to establish techniques to accord with the robust H_∞ state estimation concern for uncertain discrete stochastic GRNs (equation (1)) with leakage delays, distributed delays, and probabilistic measurement delays.

Consider that the H_∞ attainment of the estimation error system (8) is robustly stochastically stable with non-zero exogenous disturbance signals $v_x(k), v_y(k) \in L_2[0,\infty)$.

Theorem 3.2 *Let Assumptions* 1 *and* 2 *hold. Let the leakage delays* ρ_1, ρ_2 *and the estimation parameters* $\mathbb{A}_x$, $\mathbb{B}_x$, $\mathbb{A}_y$, $\mathbb{B}_y$, *and* $\gamma > 0$ *be given. Then the estimation error system* (8) *is robustly stochastically stable with disturbance attenuation* γ, *if there exist positive definite matrices* R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}, R_{51}, R_{52} *and three positive constant scalars* λ, ε_1, *and* ε_2 *such that the following LMI holds*:

$$\Lambda_1 = \begin{bmatrix} \Lambda'_{11} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ S_1 & 0 & J_1 & * \\ 0 & 0 & \bar{T}_1^T & -\varepsilon_1 I \end{bmatrix} < 0, \qquad \Lambda_2 = \begin{bmatrix} \Lambda'_{22} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ S_2 & 0 & J_2 & * \\ 0 & 0 & \bar{T}_2^T & -\varepsilon_2 I \end{bmatrix} < 0, \tag{39}$$

and the other variables are described in Theorem 3.1.

Proof Choose the Lyapunov–Krasovskii function (equation (10)) as in Theorem 3.1. For given $\gamma > 0$, we define

$$T(n) = \mathbb{E}\sum_{k=0}^{n}\big[\bar{x}^T(k)\bar{x}(k) + \bar{y}^T(k)\bar{y}(k) - \gamma^2 v_x^T(k)v_x(k) - \gamma^2 v_y^T(k)v_y(k)\big]. \tag{40}$$

Here, n is a nonnegative integer. Our aim is to show $T(n) < 0$. Under the zero initial condition, we have

$$\begin{aligned}
T(n) &= \mathbb{E}\sum_{k=0}^{n}\big[\bar{x}^T(k)\bar{x}(k) + \bar{y}^T(k)\bar{y}(k) - \gamma^2 v_x^T(k)v_x(k) - \gamma^2 v_y^T(k)v_y(k) + \Delta\mathbb{V}(k)\big] \\
&\quad - \mathbb{E}\mathbb{V}(n+1) \\
&\leq T(n) + \sum_{k=0}^{n}\mathbb{E}\big(\Delta\mathbb{V}(k)\big) \\
&= \sum_{k=0}^{n}\mathbb{E}\big\{\varpi^T(k)\big[\tilde{\Lambda}_{11} + \sigma_\alpha\tilde{W}_{01}^T R_{21}\tilde{W}_{01} + 2\tilde{G}_{01}^T(k)(R_{11}+R_{21})\tilde{G}_{01}(k) \\
&\quad + 2\tilde{F}_{01}^T(k)R_{21}\tilde{F}_{01}(k) + 2\tilde{G}_{11}^T(k)(R_{12}+R_{22})\tilde{G}_{11}(k) \\
&\quad + 2\tilde{S}_{01}^T(k)(R_{12}+R_{22})\tilde{S}_{01}(k)\big]\varpi(k) \\
&\quad + \Gamma^T(k)\big[\tilde{\Lambda}_{22} + \sigma_\beta\tilde{W}_{02}^T R_{22}\tilde{W}_{02} + 2\tilde{G}_{02}^T(k)(R_{12}+R_{22})\tilde{G}_{02}(k) \\
&\quad + 2\tilde{F}_{02}^T(k)R_{22}\tilde{F}_{02}(k) + 2\tilde{G}_{12}^T(k)(R_{11}+R_{21})\tilde{G}_{12}(k) \\
&\quad + 2\tilde{S}_{02}^T(k)(R_{11}+R_{21})\tilde{S}_{02}(k)\big]\Gamma(k)\big\},
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
&\varpi(k) = \big[\varpi_0(k), v_x(k)\big]^T, \qquad \Gamma(k) = \big[\Gamma_0(k), v_y(k)\big]^T, \qquad \tilde{W}_{01} = \big[\hat{W}_{01}^T, 0\big], \\
&\tilde{G}_{01}(k) = \big[\hat{G}_{01}^T(k), 0\big], \qquad \tilde{F}_{01}(k) = \big[\hat{F}_{01}^T(k), 0\big], \qquad \tilde{G}_{11}(k) = \big[\hat{G}_{11}^T(k), 0\big], \\
&\tilde{W}_{02} = \big[\hat{W}_{02}^T, 0\big], \qquad \tilde{G}_{02}(k) = \big[\hat{G}_{02}^T(k), 0\big], \qquad \tilde{F}_{02}(k) = \big[\hat{F}_{02}^T(k), 0\big], \\
&\tilde{G}_{12}(k) = \big[\hat{G}_{12}^T(k), 0\big], \qquad \tilde{S}_{01}(k) = \big[\hat{S}_{01}^T(k), 0\big], \qquad \tilde{S}_{02}(k) = \big[\hat{S}_{02}^T(k), 0\big], \\
&\tilde{\Lambda}_{11} = \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \quad \text{and} \quad \tilde{\Lambda}_{22} = \begin{bmatrix} \Lambda_{22} & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.
\end{aligned}$$

By equation (41), in order to assure $T(n) < 0$, we just need to show

$$\begin{aligned}
&\tilde{\Lambda}_{11} + \sigma_\alpha\tilde{W}_{01}^T R_{21}\tilde{W}_{01} + 2\tilde{G}_{01}^T(k)(R_{11}+R_{21})\tilde{G}_{01}(k) + 2\tilde{F}_{01}^T(k)R_{21}\tilde{F}_{01}(k) \\
&\quad + 2\tilde{G}_{11}^T(k)(R_{12}+R_{22})\tilde{G}_{11}(k) + 2\tilde{S}_{01}^T(k)(R_{12}+R_{22})\tilde{S}_{01}(k) < 0, \\
&\tilde{\Lambda}_{22} + \sigma_\beta\tilde{W}_{02}^T R_{22}\tilde{W}_{02} + 2\tilde{G}_{02}^T(k)(R_{12}+R_{22})\tilde{G}_{02}(k) + 2\tilde{F}_{02}^T(k)R_{22}\tilde{F}_{02}(k) \\
&\quad + 2\tilde{G}_{12}^T(k)(R_{11}+R_{21})\tilde{G}_{12}(k) + 2\tilde{S}_{02}^T(k)(R_{11}+R_{21})\tilde{S}_{02}(k) < 0,
\end{aligned} \tag{42}$$

which, by Lemma 2.4, is equivalent to

$$\tilde{\Lambda}_3(k) = \begin{bmatrix} \tilde{\Lambda}_{11} & * \\ \bar{S}_1(k) & J_1 \end{bmatrix} < 0 \quad \text{and} \quad \tilde{\Lambda}_4(k) = \begin{bmatrix} \Lambda_{22} & * \\ \bar{S}_2(k) & J_2 \end{bmatrix} < 0, \tag{43}$$

where

$$\bar{S}_1(k) = \bar{S}_1 + \Delta\bar{S}_1(k) = [S_1, 0] + [\Delta S_1(k), 0],$$
$$\bar{S}_2(k) = \bar{S}_2 + \Delta\bar{S}_2(k) = [S_2, 0] + [\Delta S_2(k), 0]$$

and J_1 and J_2 are defined in Theorem 3.1. Note that $\tilde{\Lambda}_3(k)$ and $\tilde{\Lambda}_4(k)$ can be rearranged as follows:

$$\tilde{\Lambda}_3(k) = \tilde{\Lambda}_3 + \Delta\tilde{\Lambda}_3(k), \qquad \tilde{\Lambda}_4(k) = \tilde{\Lambda}_4 + \Delta\tilde{\Lambda}_4(k), \tag{44}$$

where

$$\tilde{\Lambda}_3 = \begin{bmatrix} \tilde{\Lambda}_{11} & * \\ \bar{S}_1 & J_1 \end{bmatrix} < 0 \quad \text{and} \quad \Delta\tilde{\Lambda}_3(k) = \begin{bmatrix} 0 & * \\ \Delta\bar{S}_1(k) & 0 \end{bmatrix},$$
$$\tilde{\Lambda}_4 = \begin{bmatrix} \tilde{\Lambda}_{22} & * \\ \bar{S}_2 & J_2 \end{bmatrix} < 0 \quad \text{and} \quad \Delta\tilde{\Lambda}_4(k) = \begin{bmatrix} 0 & * \\ \Delta\bar{S}_2(k) & 0 \end{bmatrix}.$$

Let

$$\breve{T}_1^T = [0, \tilde{T}_1^T], \qquad \breve{W}_1 = [\tilde{W}_1, 0], \qquad \breve{T}_2^T = [0, \tilde{T}_2^T], \qquad \breve{W}_2 = [\tilde{W}_2^T, 0],$$
$$\breve{T}_1^T = [0, 0, \bar{T}_1^T], \qquad \breve{W}_1 = [\bar{W}_1, 0, 0], \qquad \breve{T}_2^T = [0, 0, \bar{T}_2^T] \quad \text{and} \quad \breve{W}_2 = [\bar{W}_2, 0, 0].$$

Using Lemma 2.3(i), $\Delta\Lambda_3(k)$ and $\Delta\Lambda_4(k)$ can be rewritten as

$$\Delta\tilde{\Lambda}_3(k) = \breve{T}_1 N(k)\breve{W}_1 + \breve{W}_1^T N^T(k)\breve{T}_1^T \le \epsilon_1^{-1}\breve{T}_1\breve{T}_1^T + \epsilon_1\breve{W}_1^T\breve{W}_1,$$
$$\Delta\tilde{\Lambda}_4(k) = \breve{T}_2 N(k)\breve{W}_2 + \breve{W}_2^T N^T(k)\breve{T}_2^T \le \epsilon_2^{-1}\breve{T}_2\breve{T}_2^T + \epsilon_1\breve{W}_2^T\breve{W}_2. \tag{45}$$

It is implied from equations (44) and (45) that

$$\tilde{\Lambda}_3(k) \le \begin{bmatrix} \Lambda'_{11} & * & * \\ 0 & -\gamma^2 I & * \\ S_1 & 0 & J_1 \end{bmatrix} + \epsilon_1^{-1}\breve{T}_1\breve{T}_1^T,$$
$$\tilde{\Lambda}_4(k) \le \begin{bmatrix} \Lambda'_{22} & * & * \\ 0 & -\gamma^2 I & * \\ S_2 & 0 & J_2 \end{bmatrix} + \epsilon_2^{-1}\breve{T}_2\breve{T}_2^T. \tag{46}$$

Using Lemma 2.4, the above inequality (45) holds if and only if the right-hand side of (45) is negative definite, which implies $T(n) < 0$. Letting $n \to \infty$, we have

$$\mathbb{E}\sum_{k=0}^{\infty}\{|\bar{x}(k)|^2 + |\bar{y}(k)|^2\} \le \gamma^2\mathbb{E}\sum_{k=0}^{\infty}(|v_x(k)|^2 + |v_y(k)|^2).$$

Hence the proof of Theorem 3.2 is complete. □

Theorem 3.3 *With the help of the assumptions, system* (7) *becomes a robust H_∞ state estimator of GRNs* (1) *with leakage delays, distributed delays, and probabilistic measurement delays* (5) *if there exist positive definite matrices X_1, X_2, Y_1, Y_2, R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}, R_{51}, and R_{52} and three positive constant scalars λ, ε_1, and ε_2 such that the following LMIs hold*:

$$\Lambda_1 = \begin{bmatrix} \Lambda'_{11} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ S'_1 & 0 & J_1 & * \\ 0 & 0 & \bar{T}_1^T & -\varepsilon_1 I \end{bmatrix} < 0, \qquad \Lambda_2 = \begin{bmatrix} \Lambda'_{22} & * & * & * \\ 0 & -\gamma^2 I & * & * \\ S'_2 & 0 & J_2 & * \\ 0 & 0 & \bar{T}_2^T & -\varepsilon_2 I \end{bmatrix} < 0,$$

where

$$S'_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \Sigma_{15} & 0 & 0 \\ \sqrt{2}(X_1 - \alpha_0 X_2 M) & -\sqrt{2}X_1 & -\sqrt{2}(1-\alpha_0)X_2 M & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha}X_2 M & 0 & \sqrt{\sigma_\alpha}X_2 M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}(R_{12}+R_{22})\mathbb{D} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \Sigma_{56} & 0 \end{bmatrix},$$

$$\Sigma_{15} = -\sqrt{2}(R_{11}+R_{21})A; \qquad \Sigma_{56} = \sqrt{2}(R_{12}+R_{22})F,$$

$$S'_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \Upsilon_{16} & 0 & 0 & 0 \\ \sqrt{2}(Y_1 - \beta_0 Y_2 N) & -\sqrt{2}Y_1 & -\sqrt{2}(1-\beta_0)Y_2 N & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta}Y_2 N & 0 & \sqrt{\sigma_\beta}Y_2 N & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}(R_{11}+R_{21})\mathbb{B} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Upsilon_{57} & 0 & 0 \end{bmatrix},$$

$$\Upsilon_{16} = -\sqrt{2}(R_{12}+R_{22})\mathbb{C}; \qquad \Upsilon_{57} = \sqrt{2}(R_{11}+R_{21})E,$$

and the other variables are described in Theorem 3.1. *Furthermore, the state estimator gain matrices can be described as follows*:

$$\mathbb{A}_x = R_{21}^{-1}X_1, \qquad \mathbb{B}_x = R_{21}^{-1}X_2, \qquad \mathbb{A}_y = R_{22}^{-1}Y_1 \quad \textit{and} \quad \mathbb{B}_y = R_{22}^{-1}Y_2.$$

Proof The rest of the proof of this theorem is the same as that of Theorem 3.2. Due to the limitation of the length of this paper, we omit it here. Then the proof of Theorem 3.3 is completed. □

Consider the discrete-time genetic regulatory network system:

$$\begin{aligned} x(k+1) &= -\mathbb{A}x(k-\rho_1) + \mathbb{B}\hat{g}\big(y(k-\delta(k))\big) + E\sum_{s=1}^{\infty}\mu_s h\big(y(k-s)\big) \\ &\quad + \sigma\big(k, x(k-\rho_1)\big)\omega(k) + L_x v_x(k), \\ y(k+1) &= -\mathbb{C}y(k-\rho_2) + \mathbb{D}x\big(k-\tau(k)\big) + F\sum_{n=1}^{\infty}\xi_n x(k-n) + L_y v_y(k). \end{aligned} \tag{47}$$

Corollary 3.1 *Let the leakage delays ρ_1, ρ_2 and the estimation parameters $\mathbb{A}_x$, $\mathbb{B}_x$, $\mathbb{A}_y$, and $\mathbb{B}_y$ be given and also the acceptable conditions hold. Then the estimation error system* (8) *with $v_x(k) = 0$ and $v_y(k) = 0$ is robustly exponentially stable in the mean square if there* *exist positive definite matrices $R_{11}, R_{12}, R_{21}, R_{22}, R_{31}, R_{32}, R_{41}, R_{42}, R_{51}, R_{52}$ and the positive constant scalar λ such that the following LMI holds*:

$$\Lambda_1 = \begin{bmatrix} \Lambda'_{11} & * & * \\ S_1 & J_1 & * \\ 0 & 0 & I \end{bmatrix} < 0, \qquad \Lambda_2 = \begin{bmatrix} \Lambda'_{22} & * & * \\ S_2 & J_2 & * \\ 0 & 0 & I \end{bmatrix} < 0, \tag{48}$$

where

$$\Lambda'_{11} = \begin{bmatrix} \psi_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{31} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{41} + \varepsilon_1 W_4^T W_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & HR_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I(R_{12} + R_{22}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\xi} R_{52} \end{bmatrix},$$

$$\Lambda'_{22} = \begin{bmatrix} \psi_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -R_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -R_{32} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -R_{42} - \lambda\tilde{N}_1 + \varepsilon_2 W_2^T W_2 & -\lambda\tilde{N}_2^T & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\lambda\tilde{N}_2 & -\lambda I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I(R_{11} + R_{21}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{\mu} R_{51} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{\mu} R_{51} \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \bar{\Xi}_{15} & 0 & 0 \\ \Xi_{21} & -\sqrt{2} R_{21}\mathbb{A}_x & \Xi_{23} & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\alpha} R_{21}\mathbb{B}_x M & 0 & \sqrt{\sigma_\alpha} R_{21}\mathbb{B}_x M & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\Xi}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\Xi}_{55} & 0 \end{bmatrix},$$

where

$$\bar{\Xi}_{15} = -\sqrt{2}(R_{11} + R_{21})\mathbb{A}; \qquad \bar{\Xi}_{44} = \sqrt{2}(R_{12} + R_{22})\mathbb{D}; \qquad \bar{\Xi}_{55} = \sqrt{2}(R_{12} + R_{22})F;$$

$$\Xi_{21} = \sqrt{2} R_{21}(\mathbb{A}_x - \alpha_0 \mathbb{B}_x M); \qquad \Xi_{23} = -\sqrt{2} R_{21}(1 - \alpha_0)\mathbb{B}_x M,$$

$$S_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{16} & 0 & 0 & 0 \\ \Theta_{21} & -\sqrt{2} R_{22}\mathbb{A}_y & \Theta_{23} & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\sigma_\beta} R_{22}\mathbb{B}_y N & 0 & \sqrt{\sigma_\beta} R_{22}\mathbb{B}_y N & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\Theta}_{45} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \bar{\Theta}_{57} & 0 & 0 \end{bmatrix},$$

where

$$\bar{\Theta}_{16} = -\sqrt{2}(R_{12} + R_{22})\mathbb{C}; \qquad \bar{\Theta}_{45} = \sqrt{2}(R_{11} + R_{21})\mathbb{B}; \qquad \bar{\Theta}_{57} = \sqrt{2}(R_{11} + R_{21})E;$$

$$\Theta_{21} = \sqrt{2} R_{22}(A_y - \beta_0 \mathbb{B}_y N); \qquad \Theta_{23} = -\sqrt{2} R_{22}(1 - \beta_0)\mathbb{B}_y N,$$

$$J_1 = \operatorname{diag}\bigl\{-(R_{11} + R_{21}), -R_{21}, -R_{21}, -(R_{12} + R_{22}), -(R_{12} + R_{22})\bigr\},$$

$$J_2 = \operatorname{diag}\bigl\{-(R_{12} + R_{22}), -R_{22}, -R_{22}, -(R_{11} + R_{21}), -(R_{11} + R_{21})\bigr\},$$

$$\bar{\mu} = \sum_{s=1}^{\infty} \mu_s, \qquad \bar{\xi} = \sum_{n=1}^{\infty} \xi_n,$$

$$\psi_{11} = -R_{11} + R_{31} + (\tau_M - \tau_m + 1)R_{41} + \bar{\xi} R_{52}; \qquad \psi_{12} = -R_{12} + R_{32} + (\delta_M - \delta_m + 1)R_{42},$$
$$\tilde{N}_1 = \frac{(N_1^T N_2 + N_2^T N_1)}{2}; \qquad \tilde{N}_2 = -\frac{(N_1^T + N_2^T)}{2}.$$

4 Numerical examples

In this part, two mathematical examples with simulations are provided to show the effectiveness of the proposed robust state estimator.

Example 4.1 Consider the discrete-time GRN (1) with parameters given as follows:

$$A = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \qquad B = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.2 \end{pmatrix},$$
$$C = D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad E = \begin{pmatrix} 0.36 & 0 \\ 0 & 0.1 \end{pmatrix},$$
$$d_1 = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad d_2 = \begin{pmatrix} 0.28 & 0 \\ 0 & 0.135 \end{pmatrix},$$
$$L_x = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.5 \end{pmatrix}, \qquad L_y = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix},$$
$$W_1 = W_2 = W_3 = W_4 = W_5 = W_6 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.3 \end{pmatrix}, \qquad \mu = \xi = \exp(-2),$$
$$F = 0.4I, \qquad R = 0.2I, \qquad G = \begin{pmatrix} \sin(k) & 0 \\ 0 & \cos(k) \end{pmatrix},$$

and the leakage delays $\rho_1 = \rho_2 = 1$. The regulatory function is taken as $g(s) = \frac{s^2}{1+s^2}$. The time-varying delays are chosen as $\delta(k) = 3 + (2 * \sin(k * \pi/2))$ and $\tau(k) = 3 + (2 * \cos(k * \pi/2))$, and the exogenous disturbance inputs are selected as $v_x(k) = \sin(6k)\exp(-0.1k)$ and $v_y(k) = \cos(2k)\exp(-0.2k)$.

Now consider the estimation error system (8) with parameters given by

$$A = 0.1I, \qquad B = \begin{pmatrix} -0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \qquad E = F = 0.3I, \qquad C = D = 0.2I,$$
$$M = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad N = \begin{pmatrix} 0.4 & 0 \\ 0.3 & 0.5 \end{pmatrix}, \qquad R = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.3 \end{pmatrix},$$
$$N(k) = \begin{pmatrix} \sin(k * \pi/2) & 0 \\ 0 & \cos(k * \pi/2) \end{pmatrix}, \qquad \alpha = 0.001, \qquad \beta = 0.003,$$
$$d_1 = \begin{pmatrix} 0.2 \times (\cos(\pi/2) - 2) & 0 \\ 0 & 0.1 \times (\sin(\pi/2) - 1) \end{pmatrix}, \qquad d_2 = \begin{pmatrix} 0.28 & 0 \\ 0 & 0.135 \end{pmatrix},$$
$$L_x = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix}, \quad L_y = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},$$
$$W_1 = W_2 = W_3 = W_4 = W_5 = W_6 = 0.1I, \qquad \mu = \xi = \exp(-1),$$

and the leakage delays $\rho_1 = \rho_2 = 1$. The exogenous disturbance inputs are selected as

$$\nu_x(k) = (\sin 6k)\exp(-0.1k), \qquad \nu_y(k) = (\cos 2k)\exp(-0.1k).$$

The regulatory function is taken as $g(s) = \frac{s^2}{1+s^2}$. The time-varying delays are chosen as $\delta(k) = 3 + (2 * \sin(k * \pi/2))$ and $\tau(k) = 3 + (2 * \cos(k * \pi/2))$. By using the Matlab LMI toolbox, LMIs (40) and (41) are solved and a set of feasible solutions is obtained as follows:

$$X_1 = \begin{pmatrix} 0.4338 & -0.0041 \\ -0.0041 & 0.2852 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0.0210 & -0.0260 \\ -0.0260 & 0.0533 \end{pmatrix},$$
$$R_{11} = \begin{pmatrix} 9.3434 & 0.0025 \\ 0.0025 & 6.9265 \end{pmatrix}, \qquad R_{21} = \begin{pmatrix} 0.2169 & -0.4861 \\ -0.4861 & 0.9363 \end{pmatrix},$$
$$Y_1 = \begin{pmatrix} 1.1918 & -0.0128 \\ -0.0128 & 0.5832 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 1.0836 & -0.2279 \\ -0.2279 & 0.1073 \end{pmatrix}.$$

The state estimator gain matrices can be determined as follows:

$$A_x = \begin{pmatrix} 1.2173 & 0.4060 \\ 0.6324 & 0.1804 \end{pmatrix}, \qquad A_y = \begin{pmatrix} 0.2203 & 0.0032 \\ 0.0063 & 0.2096 \end{pmatrix},$$
$$B_x = \begin{pmatrix} 2.1102 & 0.4831 \\ 1.3729 & 0.3185 \end{pmatrix}, \qquad B_y = \begin{pmatrix} 0.2005 & 0.4226 \\ 0.8342 & 0.3887 \end{pmatrix}.$$

The concentration of mRNA and protein and their estimation error are illustrated in Figs. 1 and 2 with the initial conditions $\phi_1(k) = \{1, 0.1\}$, $\psi_1(k) = \{0.9, 0.7\}$, $\phi_2(k) = \{0.9, 0.8\}$, and $\psi_2(k) = \{0.15, 0.9\}$.

Example 4.2 Consider the discrete-time GRN (47) with parameters given by

$$A = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.2 \end{pmatrix}, \qquad B = \begin{pmatrix} -0.5 & 0 \\ 2.5 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.2 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.08 & 0 \\ 0 & 0.2 \end{pmatrix},$$

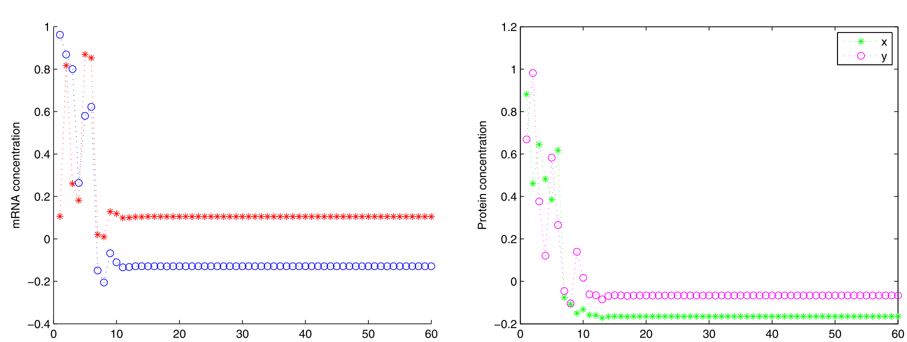

Figure 1 mRNA and protein concentration

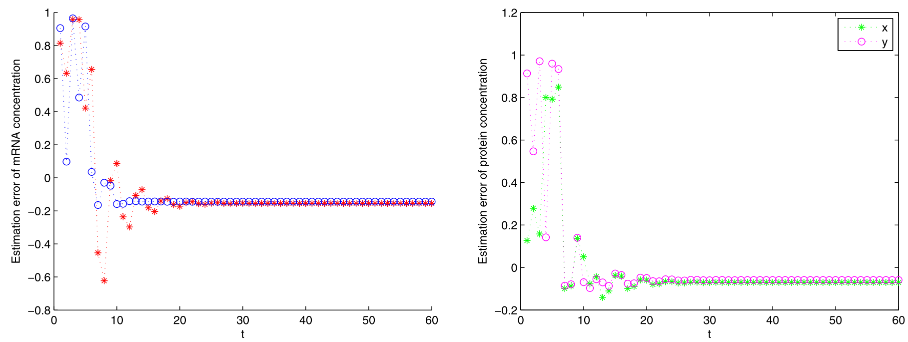

Figure 2 Estimation error for mRNA and protein concentration

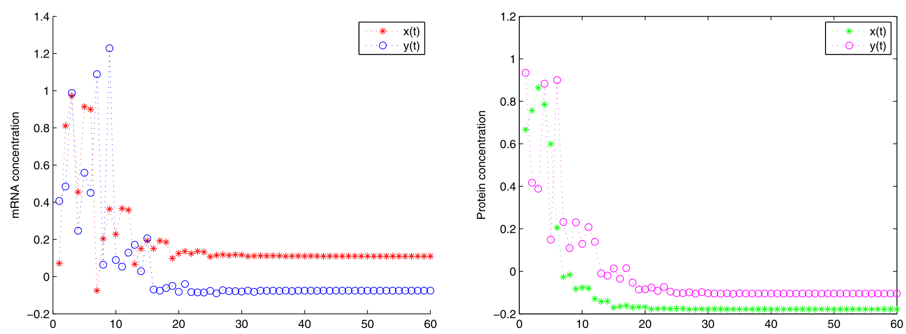

Figure 3 The state response *x*(*t*), *y*(*t*) of equation (47) with the mRNA and protein concentrations

$$E = \begin{pmatrix} 0.36 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad F = \begin{pmatrix} 0.4 & 0 \\ 0 & 0.4 \end{pmatrix},$$

$$d_1 = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.1 \end{pmatrix}, \qquad d_2 = \begin{pmatrix} 0.28 & 0 \\ 0 & 0.135 \end{pmatrix},$$

$$L_x = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}, \qquad L_y = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.2 \end{pmatrix},$$

and the leakage delays $\rho_1 = \rho_2 = 1$. The regulatory function is taken as $g(s) = \frac{s^2}{1+s^2}$. The time-varying delays are chosen as $\delta(k) = 2$ and $\tau(k) = 1$, and the exogenous disturbance inputs are selected as $v_x(k) = \sin(6k)\exp(-0.1k)$ and $v_y(k) = \cos(2k)\exp(-0.2k)$. The the state responses $x(t)$ and $y(t)$ are shown in Fig. 3.

5 Conclusions

In this paper, we have studied the approximation concern for the discrete-time stochastic GRNs with the leakage delays, distributed delays, and probabilistic measurement delays into the problem and modeled the robust H_∞ state estimator for a class of discrete-time stochastic GRNs. Here, the probabilistic measurement delays, which narrate the binary shifting sequence, are satisfied by the conditional probability distribution. So, the crisis of parameter uncertainties, including errors, stochastic disturbance, leakage delays, distributed delays, and the activation function of the addressed GRNs, is identified by sector-bounded nonlinearities. By applying the Lyapunov stability theory and stochastic analysis techniques, sufficient conditions are first entrenched to assure the presence of the desired estimators in terms of a linear matrix inequality (LMI). These circumstances are reliant on both the lower and upper bounds of time-varying delays. Again, the absolute expression of the desired estimator is demonstrated to assure the estimation error dynamics to be robustly exponentially stable in the mean square for the consigned system. Lastly, numerical simulations have been utilized to illustrate the suitability and usefulness of our advanced theoretical results.

Acknowledgements
This work was jointly supported by the National Natural Science Foundation of China under Grant No. 61573096, the Jiangsu Provincial Key Laboratory of Networked Collective Intelligence under Grant No. BM2017002, the Thailand research grant fund No. RSA5980019 and Maejo University.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Alagappa University, Karaikudi, India. [2]Ramanujan Centre for Higher Mathematics, Alagappa University, Karaikudi, India. [3]School of Mathematics, Southeast University, Nanjing, China. [4]Department of Mathematics, Faculty of Science, Maejo University, Chiang Mai, Thailand. [5]Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia.

Publisher's Note

Received: 19 July 2017 Accepted: 20 March 2018 Published online: 03 April 2018

References

1. Wang, T., Ding, Y., Zhang, L., Hao, K.: Robust state estimation for discrete-time stochastic genetic regulatory networks with probabilistic measurement delays. Neurocomputing **111**, 1–12 (2013)
2. Wang, Z., Liu, Y., Liu, X., Shi, Y.: Robust state estimation for discrete-time stochastic neural networks with probabilistic measurement delays. Neurocomputing **74**, 256–264 (2010)
3. Cao, J., Ren, F.: Exponential stability of discrete-time genetic regulatory networks with delays. IEEE Trans. Neural Netw. **19**(3), 520–523 (2008)
4. Lv, B., Liang, J., Cao, J.: Robust distributed state estimation for genetic regulatory networks with Markovian jumping parameters. Commun. Nonlinear Sci. Numer. Simul. **16**, 4060–4078 (2011)
5. Shen, B., Wang, Z., Liang, J., Liu, X.: Sampled-data H_∞ filtering for stochastic genetic regulatory networks. Int. J. Robust Nonlinear Control **21**(15), 1759–1777 (2011)
6. Wang, L., Luo, Z., Yang, H., Cao, J.: Stability of genetic regulatory networks based on switched systems and mixed time-delays. Math. Biosci. **278**, 94–99 (2016)
7. Hu, J., Liang, J., Cao, J.: Stabilization of genetic regulatory networks with mixed time-delays: an adaptive control approach. IMA J. Math. Control Inf. **32**, 343–358 (2015)
8. Xiao, M., Zheng, W., Cao, J.: Stability and bifurcation of genetic regulatory networks with small RNAs and multiple delays. Int. J. Comput. Math. **91**, 907–927 (2014)
9. Xiao, M., Cao, J.: Genetic oscillation deduced from Hopf bifurcation in a genetic regulatory network with delays. Math. Biosci. **215**(1), 55–63 (2008)
10. Chesi, G.: Robustness analysis of genetic regulatory networks affected by model uncertainty. Automatica **47**, 1131–1138 (2011)
11. Hu, J., Liang, J., Cao, J.: Stability analysis for genetic regulatory networks with delays: the continuous-time case and the discrete-time case. Appl. Math. Comput. **220**, 507–517 (2013)
12. Huang, C., Cao, J., Xiao, M.: Hybrid control on bifurcation for a delayed fractional gene regulatory network. Chaos Solitons Fractals **87**, 19–29 (2016)
13. Chesi, G.: On the steady states of uncertain genetic regulatory networks. IEEE Trans. Syst. Man Cybern., Part A, Syst. Hum. **42**(4), 1020–1024 (2012)
14. Ma, L., Da, F., Zhang, K.: Exponential H_∞ filter design for discrete time-delay stochastic systems with Markovian jump parameters and missing measurements. IEEE Trans. Circuits Syst. **58**(5), 994–1007 (2011)
15. Wang, Z., Ho, D.W.C., Liu, X.: Variance-constrained filtering for uncertain stochastic systems with missing measurements. IEEE Trans. Autom. Control **48**(7), 1254–1258 (2003)
16. Bao, H., Cao, J.: Exponential stability for stochastic BAM networks with discrete and distributed delays. Appl. Math. Comput. **218**(11), 6188–6199 (2012)

17. Wang, Z., Yang, F., Ho, D.W.C., Liu, X.: Robust H_∞ filtering for stochastic time-delay systems with missing measurements. IEEE Trans. Signal Process. **54**(7), 2579–2587 (2006)
18. Sun, Y., Feng, G., Cao, J.: Robust stochastic stability analysis of genetic regulatory networks with disturbance attenuation. Neurocomputing **79**, 39–49 (2012)
19. Sakthivel, R., Raja, R., Marshal Anthoni, S.: Asymptotic stability of delayed stochastic genetic regulatory networks with impulses. Phys. Scr. **82**(5), 055009 (2010)
20. Wei, G., Wang, Z., Shen, B., Li, M.: Probability-dependent gain-scheduled filtering for stochastic systems with missing measurements. IEEE Trans. Circuits Syst. II **58**(11), 753–757 (2011)
21. Yang, X., Ho, D.W.C.: Synchronization of delayed memristive neural networks: robust analysis approach. IEEE Trans. Cybern. **46**(12), 3377–3387 (2016)
22. Yang, X., Lu, J.: Finite-time synchronization of coupled networks with Markovian topology and impulsive effects. IEEE Trans. Autom. Control **61**(8), 2256–2261 (2016)
23. Liu, J., Tian, E., Gu, Z., Zhang, Y.: State estimation for Markovian jumping genetic regulatory networks with random delays. Commun. Nonlinear Sci. Numer. Simul. **19**(7), 2479–2492 (2013)
24. Wan, X., Xu, L., Fang, H., Ling, G.: Robust non-fragile H_∞ state estimation for discrete-time genetic regulatory networks with Markov jump delays and uncertain transition probabilities. Neurocomputing **154**, 162–173 (2015)
25. Liang, J., Lam, J., Wang, Z.: State estimation for Markov-type genetic regulatory networks with delays and uncertain mode transition rates. Phys. Lett. A **373**, 4328–4337 (2009)
26. Wang, Z., Lam, J., Wei, G., Fraser, K., Liu, X.: Filtering for nonlinear genetic regulatory networks with stochastic disturbances. IEEE Trans. Autom. Control **53**(10), 2448–2457 (2008)